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Recognizing the Automorphism Groups of Mathieu Groups Through Their Orders and Large Degrees of Their Irreducible Characters*

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Abstract It is a well-known fact that characters of a finite group can give important information about the structure of the group. It was also proved by the third author that a finite simple group can be uniquely determined by its character table. Here the authors attempt to investigate how to characterize a finite almost-simple group by using less information of its character table, and successfully characterize the automorphism groups of Mathieu groups by their orders and at most two irreducible character degrees of their character tables.

Keywords Finite group, Character degrees, Irreducible characters, Simple groups, Mathieu groups
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1 Introduction

All groups considered are finite groups and all characters are complex characters. Let G be a group and $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G. Also, we denote the set of character degrees of G by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. In this paper, we will refer to character degrees as degrees. We use $\mathrm{cd}^*(G)$ to denote the multi-set of degrees of irreducible characters, i.e., each element of this set $\mathrm{cd}^*(G)$ can occur many times upon the number of characters of the same degree. In particular, $|\mathrm{cd}^*(G)| = |\mathrm{Irr}(G)|$. $H \cdot M$ denotes the non-split extension of H by M and $H : M$ the split extension of H by M. For any group G, $L_1(G)$ and $L_2(G)$ denote the largest and the second-largest irreducible character degrees of G, respectively. All the other notations and terminologies are standard (cf. [1]).

It is a well-known fact that characters of a group can give some important information about the group's structure. For example, Chen [2] proved that a non-abelian simple group can be uniquely determined by its character table. In [3], Huppert posed the following conjecture.

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Huppert's Conjecture Let H be any non-abelian simple group, and G a group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$, where A is an abelian group.

Huppert conjectured that each non-abelian simple group G is characterized by $\mathrm{cd}(G)$, the set of degrees of its complex irreducible characters. In [3–5], he confirmed that the conjecture holds for the simple groups, such as $L_2(q)$ and $S_z(q)$. Moreover, he also proved that the conjecture follows for 19 out of 26 sporadic simple groups, and a few others (cf. [3–5]). In [6–7], Daneshkhah, et al. showed that the conjecture holds for another three sporadic simple groups Co_1, Co_2 and Co_3. Xu, et al. attempted to characterize the finite simple groups by less information of its characters, and for the first time successfully characterized the simple K_3-groups and sporadic simple groups by their orders and one or both of its largest and second-largest irreducible character degrees (cf. [8–10]). For convenience, we summarize some results of these articles which will be used later in the following Proposition 1.1.

Proposition 1.1 (cf. [9]) *Let G be a finite group and M a Mathieu group. Then the following assertions hold:*

(i) *If M is one of M_{11}, M_{12} and M_{23}, then $G \cong M$ if and only if $|G| = |M|$ and $L_1(G) = L_1(M)$.*

(ii) *If $M = M_{24}$, then $G \cong M_{24}$ if and only if $|G| = |M_{24}|$ and $L_2(G) = L_2(M_{24})$.*

(iii) *If $M = M_{22}$, then $G \cong M_{22}$ or $H \times M_{11}$, where H is a Frobenius group with an elementary kernel of order 8 and a cyclic complement of order 7, if and only if $|G| = |M_{22}|$ and $L_1(G) = L_1(M_{22})$.*

In this article, we continue this investigation, and show that the automorphism groups of Mathieu groups can also be characterized by their orders and at most two irreducible character degrees of their character tables.

We obtain the following main Theorem 1.1.

Theorem 1.1 *Let G be a finite group and $|G| = |\mathrm{Aut}(M)|$, where M is a Mathieu group. Then the following assertions hold:*

(1) *If $M = M_{11}$ or M_{23}, then $G \cong \mathrm{Aut}(M)$ if and only if $L_1(G) = L_1(\mathrm{Aut}(M))$.*

(2) *If $M = M_{24}$, then $G \cong \mathrm{Aut}(M)$ if and only if $L_2(G) = L_2(\mathrm{Aut}(M))$.*

(3) *If $M = M_{12}$, then G is isomorphic to one of the groups $\mathrm{Aut}(M_{12})$, $2 \cdot M_{12}$ and $2 \times M_{12}$ if and only if $L_1(G) = L_1(\mathrm{Aut}(M))$ and $L_2(G) = L_2(\mathrm{Aut}(M))$.*

(4) *If $M = M_{22}$, then $G \cong \mathrm{Aut}(M)$ or $2 \cdot M$ if and only if $L_1(G) = L_1(\mathrm{Aut}(M))$ and $L_2(G) = L_2(\mathrm{Aut}(M))$.*

2 Preliminaries

In this section, we consider some results which will be applied for our further investigations.

Lemma 2.1 *Let G be a finite solvable group of order $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1, q_2, \cdots, q_s$ are distinct primes. If $(kq_s + 1) \nmid q_i^{\alpha_i}$ for each $i \leq s-1$ and $k > 0$, then the Sylow q_s-subgroup is normal in G.*

Proof Let N be a minimal normal subgroup of G. Since G is solvable, then we have $|N| = q^m$. If $q = q_s$, by induction on G/N, it is easy to see the normality of the Sylow

q_s-subgroup in G. Now, assume that $q = q_i$ for some $i < s$. Considering the factor group G/N, by induction, one has that the Sylow q_s-subgroup Q/N of G/N is normal in G/N. Thus $Q \trianglelefteq G$. Let P be a Sylow q_s-subgroup of Q. Then $Q = NP$. By Sylow's theorem, we have $|Q : N_Q(P)| = q_i^l$ $(l \leq m \leq \alpha_i)$ and $q_s \mid (q_i^l - 1)$. But this implies that $(kq_s + 1) \mid q^{\alpha_i}$, and then $k = 0$ by assumption. Hence $P \trianglelefteq Q$. Since $Q \trianglelefteq G$, we have $P \trianglelefteq G$.

Lemma 2.2 (cf. [8]) *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 2.3 *Let G be a non-solvable group. Suppose that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong M$ is a non-abelian simple group with $\mathrm{Mult}(M) = 1$ and H has a normal series: $H = H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_t = 1$ such that*

(1) $H_i \trianglelefteq K$;

(2) H_{i-1}/H_i is abelian and $\mathrm{Aut}(H_{i-1}/H_i)$ does not contain any simple section isomorphic to M, where $i = 2, 3, \cdots, t$.

Then K has a normal series $1 \trianglelefteq H_1 \trianglelefteq K$ such that $H_1 \cong M$. Moreover, if $|\mathrm{Out}(M)| = 1$, then $G \cong M \times T$ for some subgroup $T \leq G$.

Proof By hypotheses, we have $H_1/H_2 \trianglelefteq K/H_2$. Considering the conjugate action of K/H_2 on H_1/H_2, we can obtain that $K/H_2/C_{K/H_2}(H_1/H_2) \lesssim \mathrm{Aut}(H_1/H_2)$. Since H_{i-1}/H_i is abelian, we have $H_1/H_2 \leq C_{K/H_2}(H_1/H_2)$. Hence the factor group $K/H_2 / C_{K/H_2}(H_1/H_2) \cong M$ or 1. By assumption, we have that $K/H_2 = C_{K/H_2}(H_1/H_2)$, i.e., $H_1/H_2 = Z(K/H_2)$. Since $\mathrm{Mult}(M) = 1$, one has that $K/H_2 = M \times H_1/H_2$. Therefore, K has a normal series $H_2 \trianglelefteq K_1 \trianglelefteq K$ such that $K_1/H_2 \cong M$ and $K/K_1 \cong H_1/H_2$.

Repeating the process of the above reasoning for the normal series $H_3 \trianglelefteq H_2 \trianglelefteq K_1$, we can get that K_1 has a normal series $H_3 \trianglelefteq K_2 \trianglelefteq K_1$ such that $K_2/H_3 \cong M$, $K_1/K_2 \cong H_2/H_3$ and $K_1/K_3 = M \times H_2/H_3$. Repeating the process of the above argument, we can get that K has a normal series $1 = K_{t+1} \trianglelefteq K_t \trianglelefteq \cdots \trianglelefteq K_1 \trianglelefteq K$, where $K_t = M$.

It is easy to see that H is solvable by (2), and therefore, K has M as its unique simple normal factor, which implies that H_1 is a characteristic subgroup of K, so $H_1 \trianglelefteq G$. Therefore, $G \cong H_1 \times T$ since $|\mathrm{Out}(H_1)| = 1$, as desired.

Remark 2.1 Let S be a Mathieu group, and then S is isomorphic to one of M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. By [1], we can obtain $|S|$, $|\mathrm{Out}(S)|$, $\mathrm{Mult}(S)$ (the Schur multiplier) and $|\mathrm{Aut}(S)|$. Applying the software Magma (V2.11-1) (cf. [11]), it is easy to compute the values of $L_1(\mathrm{Aut}(S))$ and $L_2(\mathrm{Aut}(S))$. For convenience, we have tabulated the results in Table 1.

Table 1

S	$\lvert S\rvert$	$\mathrm{Mult}(S)$	$\lvert\mathrm{Out}(S)\rvert$	$\lvert\mathrm{Aut}(S)\rvert$	$L_1(\mathrm{Aut}(S))$	$L_2(\mathrm{Aut}(S))$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$5 \cdot 11$	
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	2	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	$2^4 \cdot 11$	$2^4 \cdot 3^2$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	2	$2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$2^4 \cdot 5 \cdot 7$	$5 \cdot 7 \cdot 11$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2^3 \cdot 11 \cdot 23$	
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1	1	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3^3 \cdot 5 \cdot 7 \cdot 11$	$2^2 \cdot 3^2 \cdot 7 \cdot 11$

Remark 2.2 Let S be a K_3-simple group, and then S is one of A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. By [1], we can get $|S|$, $|\mathrm{Out}(S)|$, $\mathrm{Mult}(S)$ (the Schur multiplier) and

$|\mathrm{Aut}(S)|$. Using the software Magma (V2.11-1), we can compute the values of $L_1(\mathrm{Aut}(S))$ and $L_2(\mathrm{Aut}(S))$, and we have tabulated them in Table 2.

Table 2

S	$\lvert S\rvert$	Mult(S)	$\lvert\mathrm{Out}(S)\rvert$	$\lvert\mathrm{Aut}(S)\rvert$	$L_1(\mathrm{Aut}(S))$	$L_2(\mathrm{Aut}(S))$
A_5	$2^2\cdot 3\cdot 5$	2	2	$2^3\cdot 3\cdot 5$	6	5
$L_2(7)$	$2^3\cdot 3\cdot 7$	2	2	$2^4\cdot 3\cdot 7$	8	7
A_6	$2^3\cdot 3^2\cdot 5$	6	4	$2^5\cdot 3^2\cdot 5$	20	16
$L_2(8)$	$2^3\cdot 3^2\cdot 7$	1	3	$2^3\cdot 3^3\cdot 7$	27	21
$L_2(17)$	$2^4\cdot 3^2\cdot 17$	2	2	$2^5\cdot 3^2\cdot 17$	18	17
$L_3(3)$	$2^4\cdot 3^3\cdot 13$	1	2	$2^5\cdot 3^3\cdot 13$	52	39
$U_4(2)$	$2^6\cdot 3^4\cdot 5$	2	2	$2^7\cdot 3^4\cdot 5$	90	81
$U_3(3)$	$2^5\cdot 3^3\cdot 7$	1	2	$2^6\cdot 3^3\cdot 7$	64	56

3 Proof of Main Theorem

Remark 3.1 A group G is called an almost-simple group related to S if $S \le G \le \mathrm{Aut}(S)$, where S is a non-abelian simple group.

Proof of Theorem 1.1 By Table 1, if M is isomorphic to one of M_{11}, M_{23} and M_{24}, then $|\mathrm{Out}(M)| = 1$, and hence $\mathrm{Aut}(M) = M$. By Proposition 1.1, we see that conclusions (1) and (2) hold in this case. In the following, we only need to discuss that the remaining conclusions hold while $M = M_{12}$ or M_{22}. Obviously, it is enough to prove the sufficiency. We write the proof by what M is.

Case 1 We are to prove that the theorem follows if $M = M_{12}$.

In this case, one has that $|G| = 2^7\cdot 3^3\cdot 5\cdot 11$, $L_1(G) = 2^4\cdot 11$ and $L_2(G) = 2^4\cdot 3^2$ by hypotheses and Table 1. Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = 2^4\cdot 11$ and $\beta(1) = 2^4\cdot 3^2$.

We first assert that G is non-solvable. If G is solvable, then by Lemma 2.1, G_{11} is normal in G, where $G_{11} \in \mathrm{Syl}_{11}(G)$. Hence, $\chi(1) \mid |G : G_{11}| = 2^7\cdot 3^3\cdot 5$, a contradiction. Therefore, G is non-solvable, so the assertion is true.

By Lemma 2.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of non-abelian simple groups which are pairwise isomorphic to each other and $|G/K| \mid |\mathrm{Out}(K/H)|$. Since $|G| = 2^7\cdot 3^3\cdot 5\cdot 11$, we deduce that K/H can only be isomorphic to one of A_5, A_6, $L_2(11)$, M_{11} and M_{12}.

Subcase 1.1 $K/H \ncong A_5$.

Otherwise, by Table 2, $|G : K| = 1$ or 2. In this case, we have $|H| = 2^u\cdot 3^2\cdot 11$, where $4 \le u \le 5$. Since H is solvable, by Lemma 2.1, we have that H_{11} is normal in H. Hence, H_{11} Char H. Since $H \trianglelefteq G$, one has that $H_{11} \trianglelefteq G$. Therefore, $\chi(1) \mid |G : H_{11}| = 2^7\cdot 3^3\cdot 5$, a contradiction.

By the similar arguments as before, we can prove that $K/H \ncong A_6$.

Subcase 1.2 $K/H \ncong L_2(11)$.

If $K/H \cong L_2(11)$, then we have $|G : K| = 1$ or 2.

If $|G : K| = 1$, then $|H| = 2^5\cdot 3^2$. Let $\varphi \in \mathrm{Irr}(H)$ such that $[\beta_H, \varphi] \ne 0$. Then $\beta(1)/\varphi(1) \mid |G : H| = 2^2\cdot 3\cdot 5\cdot 11$, and thus $12 \mid \varphi(1)$. If $\varphi(1) > 12$, then we have $\varphi(1)^2 > |H|$, a contradiction. Hence $\varphi(1) = 12$. Using the software Magma (V2.11-1), it is easy to check that there are only 1045 such groups of order $2^5 \cdot 3^2$ in small groups $(2^5 \cdot 3^2)$ up to isomorphism (cf. [11]). Moreover, we also know that the set of all irreducible character degrees of H, i.e., $\mathrm{cd}(H)$ can only be equal to one of the following sets:

$$\{1,2\}, \quad \{1\}, \quad \{1,2,4\}, \quad \{1,2,3,6\}, \quad \{1,2,3,4,6\}, \quad \{1,2,4,8\}, \quad \{1,3,4\},$$
$$\{1,3,4\}, \quad \{1,3\}, \quad \{1,8\}, \quad \{1,3,6\}, \quad \{1,2,3,6\}, \quad \{1,4\}, \quad \{1,2,3,4\}, \quad \{1,3\}, \quad \{1,2,3\}.$$

Hence, $12 \notin \mathrm{cd}(H)$, a contradiction.

If $|G:K| = 2$, then $|K| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$ and $|H| = 2^4 \cdot 3^2$. Let $\Lambda \in \mathrm{Irr}(H)$ such that $[\beta_H, \Lambda] \neq 0$. Then $\beta(1)/\Lambda(1) \mid |G:H| = 2^3 \cdot 3 \cdot 5 \cdot 11$. Thus, we have $6 \mid \Lambda(1)$. If $\Lambda(1) > 12$, then $\Lambda(1)^2 > |H|$, a contradiction. If $\Lambda(1) = 12$, then $\Lambda(1)^2 > |H|$, a contradiction, too. Therefore, $\Lambda(1) = 6$. Let $\lambda \in \mathrm{Irr}(H)$ such that $[\chi_H, \lambda] \neq 0$, and one has that $\chi(1)/\lambda(1) \mid |G:H| = 2^3 \cdot 3 \cdot 5 \cdot 11$, so $2 \mid \lambda(1)$. Set $e = [\chi_H, \lambda]$, $t = |G : I_G(\lambda)|$. Then we have $et\lambda(1) = 2^4 \cdot 11$. Again, using the Magma, there are only 197 such groups of order $2^4 \cdot 3^2$ in small groups $(2^4 \cdot 3^2)$ up to isomorphism. Moreover, if $\lambda \in \mathrm{cd}^*(H)$, then $\mathrm{cd}^*(H)$ can only be equal to one of the following sets:

$$\{1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,4,4,4,6\},$$
$$\{1,1,1,1,2,2,2,2,2,2,2,2,3,3,3,3,6,6\},$$
$$\{1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,6,6\},$$
$$\{1,1,2,2,2,2,2,2,3,3,4,4,4,4,6\}, \quad \{1,1,1,1,2,2,2,2,3,3,3,3,4,6,6\}.$$

By the structure of $\mathrm{cd}^*(H)$, one has that $\lambda(1) = 1$ or 2 or 4. Moreover, the following conclusions hold:

(i) If $\lambda(1) = 1$, then $t \leq 11$ by the above $\mathrm{cd}^*(H)$. Thus $e \geq 2^4$. But $[\chi_H, \chi_H] = e^2 t \geq 2^8 \cdot 11 > |G:H| = 2^3 \cdot 3 \cdot 5 \cdot 11$, a contradiction to [12, Lemma 2.29].

(ii) If $\lambda(1) = 2$ or 4, then $t \leq 9$ by the above $\mathrm{cd}^*(H)$. But $t = |G : I_G(\lambda)|$, and we have $t = 1$ as any maximal subgroup has an index in $G \geq 11$ (cf. [1]). Hence $e = 2^3 \cdot 11$. But $[\chi_H, \chi_H] = e^2 t = 2^6 \cdot 11^2 > |G:H| = 2^3 \cdot 3 \cdot 5 \cdot 11$, which by [12, Lemma 2.29], is a contradiction.

Subcase 1.3 $K/H \ncong M_{11}$.

Assume that $K/H \cong M_{11}$. Since $|\mathrm{Out}(M_{11})| = 1$, then $|H| = 2^3 \cdot 3 \cdot 7$. If H is non-solvable, then $H \cong L_2(7)$. In this case, we have $G \cong M_{11} \times L_2(7)$. On the other hand, since $L_1(M_{11}) = 55$ and $L_1(L_2(7)) = 8$ by Table 2, then by the structure of G, we can get that the largest irreducible degree $L_1(G) = 2^3 \cdot 5 \cdot 11$, a contradiction to $\chi(1) = 2^4 \cdot 11$.

If H is solvable, since $|\mathrm{Out}(M_{11})| = \mathrm{Mult}(M_{11}) = 1$, then by Lemma 2.3 we obtain that $G \cong M_{11} \times H$, where $|H| = 2^3 \cdot 3 \cdot 7$. By Table 1, we see that the largest irreducible character degree of M_{11} is 55, and $22 \notin \mathrm{cd}(M_{11})$ by [1]. According to the structure of $G \cong M_{11} \times H$, one has that there exists no irreducible character in G of degree $2^4 \cdot 11$, a contradiction.

Subcase 1.4 If $K/H \cong M_{12}$, then (3) follows.

If $K/H \cong M_{12}$, by Table 1, we have $|G:K| = 1$ or 2.

If $|G:K| = 1$, then $|H| = 2$, so $H \leq Z(G)$. In this case, we can get that $G/H \cong M_{12}$. Therefore G is a central extension of Z_2 by M_{12}. Since $\mathrm{Mult}(M_{12}) = 2$, G is isomorphic to one of $2 \cdot M_{12} \cong Z_2 \cdot M_{12}$ (a non-split extension of Z_2 by M_{12}) and $2 : M_{12} \cong Z_2 \times M_{12}$ (a split extension of Z_2 by M_{12}).

By [1], it is easy to check that both $2 \cdot M_{12}$ and 2: M_{12} satisfy the conditions $|G| = |\mathrm{Aut}(M_{12})|$, $L_1(G) = L_1(\mathrm{Aut}(M_{12}))$ and $L_2(G) = L_2(\mathrm{Aut}(M_{12}))$.

If $|G:K|=2$, then $K\cong M_{12}$. Therefore $G\cong M_{12}\cdot 2=\mathrm{Aut}(M_{12})$. Hence, (3) follows. This concludes Case 1.

Case 2 We are to prove that the theorem follows if $M=M_{22}$.

By Table 1, we have $|G|=2^8\cdot 3^2\cdot 5\cdot 7\cdot 11$. Let $\chi,\beta\in\mathrm{Irr}(G)$ such that $\chi(1)=L_1(G)=2^4\cdot 5\cdot 7$ and $\beta(1)=L_2(G)=5\cdot 7\cdot 11$.

We first prove that G is not a solvable group. Assume the contrary, and by Lemma 2.1, we have that the Sylow 11-subgroup G_{11} of G is normal in G. Thus $\chi(1)\mid |G:G_{11}|=2^8\cdot 3^2\cdot 5\cdot 7$, a contradiction. Therefore, G is non-solvable. By Lemma 2.2, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that K/H is a direct product of non-abelian simple groups which are pairwise isomorphic to each other and $|G/K|\mid|\mathrm{Out}(K/H)|$. As $|G|=2^8\cdot 3^2\cdot 5\cdot 7\cdot 11$. We deduce that K/H can only be isomorphic to one of A_5, A_6, $L_2(7)$, $L_2(8)$, A_7, $L_2(11)$, M_{11}, $L_3(4)$, A_8 and M_{22}.

Subcase 2.1 $K/H\ncong A_5$.

If $K/H\cong A_5$, by Table 2, we have $|G:K|=1$ or 2. Thus $|H|=2^v\cdot 3\cdot 7\cdot 11$, where $5\le v\le 6$. If H is solvable, then by Lemma 2.1, the Sylow-11 subgroup H_{11} of H is normal in H, and hence H_{11} Char H. Since $H\trianglelefteq G$, one has that $H_{11}\trianglelefteq G$. Therefore, $\chi(1)\mid|G:H_{11}|=2^8\cdot 3^2\cdot 5\cdot 7$, a contradiction.

If H is non-solvable, by Lemma 2.2, we can get a normal series of H: $1\trianglelefteq N\trianglelefteq M\trianglelefteq H$ such that $M/N\cong L_2(7)$ and $|H/M|\mid|\mathrm{Out}(A_5)|=2$. Thus $|H:M|=1$ or 2. In this case, we have $|N|=2^m\cdot 11$, where $1\le m\le 3$. Let $\Delta\in\mathrm{Irr}(N)$ such that $[\beta_N,\Delta]\neq 0$, and then $\beta(1)/\Delta(1)\mid|G:N|=2^{8-m}\cdot 3^2\cdot 5\cdot 7$. Hence $\Delta(1)=11$. But we have that $\Delta(1)^2>|N|$, a contradiction.

The above argument is effective for the subcases that $11\nmid|G/K|$. Hence, we can prove that K/H is not isomorphic to one of A_6, $L_2(7)$, $L_2(8)$, A_7, $L_3(4)$ and A_8.

Subcase 2.2 $K/H\ncong L_2(11)$.

If not, we have $|G:K|=1$ or 2. In the following, we only discuss the condition of $|G:K|=1$, and we omit the details for $|G:K|=2$ because the arguments are similar to those proofs for $|G:K|=1$. Hence, we may assume that $|G:K|=1$, so $|H|=2^6\cdot 3\cdot 7$ in this case.

If H is solvable, then there exists a subgroup T of H such that $|H:T|=3$. Considering the permutation representation of H on the right cosets of T with the kernel T_H, the core of T in H, we get that $H/T_H\lesssim S_3$. Then $|T_H|=2^6\cdot 7$ or $2^5\cdot 7$.

If $|T_H|=2^6\cdot 7$, let $\theta\in\mathrm{Irr}(T_H)$ such that $[\chi_{T_H},\theta]\neq 0$, and then $\chi(1)/\theta(1)\mid|G:T_H|=2^2\cdot 3^2\cdot 5\cdot 11$. Hence, $2^2\cdot 7\mid\theta(1)$. But $\theta(1)^2>|T_H|$, a contradiction.

If $|T_H|=2^5\cdot 7$, let $\vartheta\in\mathrm{Irr}(T_H)$ such that $[\chi_{T_H},\vartheta]\neq 0$, and then $\chi(1)/\vartheta(1)\mid|G:T_H|=2^3\cdot 3^2\cdot 5\cdot 11$. Thus $14\mid\vartheta(1)$. If $\vartheta(1)>14$, then $\vartheta(1)^2>|T_H|$, a contradiction. Assume that $\vartheta(1)=14$. Now, applying the software Magma (V2.11-1), it is easy to check that there are only 197 such groups of order $2^5\cdot 7$ in small groups $(2^5\cdot 7)$ up to isomorphism (cf. [11]). Moreover, we can calculate that the sets of all irreducible character degrees of the groups of order $2^5\cdot 7$, i.e., $\mathrm{cd}(T_H)$, can only be equal to one of $\{1,2\}$, $\{1\}$, $\{1,2,4\}$, $\{1,7\}$ and $\{1,4\}$.

However, by checking each set of $\mathrm{cd}(T_H)$ above, we know that there exists no such irreducible character of degree 14 of T_H, i.e., $14\notin\mathrm{cd}(T_H)$, a contradiction.

Therefore, H is non-solvable. Since $|H|=2^6\cdot 3\cdot 7$, by Lemma 2.2, H has a normal series: $1\trianglelefteq N\trianglelefteq M\trianglelefteq H$ such that $M/N\cong L_2(7)$. Hence, there exist two composite factors of G, i. e.,

M_1/N_1 and M_2/N_2, respectively, such that $M_1/N_1 \cong L_2(11)$ and $M_2/N_2 \cong L_2(7)$. Considering the action of G/N on M/N, we have that the factor group $G/N/C_{G/N}(M/N) \lesssim \mathrm{Aut}(M/N)$. Let $C_{G/N}(M/N) = W/N$. Obviously, $\overline{W}/N$ has a section isomorphic to the simple group $L_2(11)$. Let $N \trianglelefteq T \trianglelefteq S \trianglelefteq \overline{W}$ such that $S/T \cong L_2(11)$. Since $\overline{W}$ has exactly one simple section, we have $S/T \trianglelefteq G/T$, so $S \trianglelefteq G$. Moreover, it is easy to see that $T = T^G \trianglelefteq G$. Hence, G/T has a maximal subgroup $S/T \times TM/T \cong L_2(11) \times L_2(7)$. Consequently, we can obtain the following composite group series of G: $1 \trianglelefteq Q \trianglelefteq P \trianglelefteq G$ such that $G/P \cong L_2(7)$ and $P/Q \cong L_2(11)$. In this case, $|Q| = 8$. Let $\eta \in \mathrm{Irr}(P)$ such that $[\beta_P, \eta] \neq 0$, and then $\beta(1)/\eta(1) \mid |G : P| = 2^3 \cdot 3 \cdot 7$. Thus $\eta(1) = 55$. Let $\lambda \in \mathrm{Irr}(Q)$ such that $[\eta_Q, \lambda] \neq 0$. Let $e = [\eta_Q, \lambda]$ and $t = |P : I_P(\lambda)|$, and then we have $\lambda(1) = 1$ and $et = 55$. By checking the maximal subgroups of $L_2(11)$ (cf. [1]), we can get that $t = 1$. Hence, $e = 55$. But $[\eta_Q, \eta_Q] = e^2 t = 55^2 > |P : Q| = 2^2 \cdot 3 \cdot 5 \cdot 11$, a contradiction to [12, Lemma 2.29].

Subcase 2.3 $K/H \ncong M_{11}$.

Since $|\mathrm{Out}(M_{11})| = 1$, one has that $|H| = 2^4 \cdot 7$ if $K/H \cong M_{11}$. Since H is solvable and $|\mathrm{Out}(M_{11})| = \mathrm{Mult}(M_{11}) = 1$, then by Lemma 2.3, we have that $G \cong M_{11} \times H$. Again, using the software magma (V2.11-1), it is easy to check that there are only 43 such groups of order $2^4 \cdot 7$ in small groups $(2^4 \cdot 7)$ up to isomorphism (cf. [11]). Moreover, we can get the sets of all irreducible character degrees of the groups of order $2^4 \cdot 7$. In other words, $\mathrm{cd}(H)$ can only be equal to one of $\{1\}$, $\{1, 2\}$, $\{1, 2, 4\}$, $\{1, 7\}$. By [1], we know that the largest irreducible character degree of M_{11} is 55, i.e., $L_1(M_{11}) = 55$ and $22 \notin \mathrm{cd}(M_{11})$. On the other hand, by the structure of $G \cong M_{11} \times H$ and as the largest irreducible character degree in the above sets is 7, we have $L_1(G) = 5 \cdot 7 \cdot 11$, a contradiction to $L_1(G) = 2^4 \cdot 5 \cdot 7$.

Subcase 2.4 If $K/H \cong M_{22}$, then (4) follows.

In this case, we can get that $|G : K| = 1$ or 2.

If $|G : K| = 1$, then $|H| = 2$ and $H \leq Z(G)$. In this case, we have $G/H \cong M_{22}$. Therefore G is a central extension of Z_2 by M_{22} and G is isomorphic to one of $2 \cdot M_{22} \cong Z_2 \cdot M_{22}$ (a non-split extension of Z_2 by M_{22}) and $2 : M_{22} \cong Z_2 \times M_{22}$ (a split extension of Z_2 by M_{22}).

If $G \cong 2 \cdot M_{22}$, by [1], it is easy to check that $|G| = |\mathrm{Aut}(M_{22})|$, $L_1(G) = L_1(\mathrm{Aut}(M_{22}))$ and $L_2(G) = L_2(\mathrm{Aut}(M_{22}))$.

If $G \cong 2 : M_{22}$, by [1], we get that $L_1(G) = L_1(2 : M_{22}) = 5 \cdot 7 \cdot 11$. But $L_1(G) = 2^4 \cdot 5 \cdot 7 = 560$, a contradiction.

If $|G : K| = 2$, then $K \cong M_{12}$, and thus $G \cong M_{12} \cdot 2 = \mathrm{Aut}(M_{22})$. Therefore, (4) follows.

This completes the proof of the main Theorem 1.1.

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