Chin. Ann. Math.
37B(4), 2016, 503–514
DOI: 10.1007/s11401-016-0974-4

Chinese Annals of Mathematics, Series B

Exact Controllability with Internal Controls for First-Order Quasilinear Hyperbolic Systems with Zero Eigenvalues

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Abstract For first-order quasilinear hyperbolic systems with zero eigenvalues, the author establishes the local exact controllability in a shorter time-period by means of internal controls acting on suitable domains. In particular, under certain special but reasonable hypotheses, the local exact controllability can be realized only by internal controls, and the control time can be arbitrarily small.

Keywords First-order quasilinear hyperbolic system, Zero eigenvalue, Exact controllability, Internal control
2000 MR Subject Classification 35L50, 35L20, 93B05, 93C20

1 Introduction

Consider the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $A(u)$ is a given $n \times n$ matrix with suitably smooth components $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, $F(u) = (f_1(u), \cdots, f_n(u))^{\mathrm{T}}$ is a smooth vector function of u and

$$F(0) = 0. \tag{1.2}$$

By hyperbolicity, on the domain under consideration, the matrix $A(u)$ has n real eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ $(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{1.3}$$

with

$$\det |l_{ij}(u)| \neq 0. \tag{1.4}$$

Suppose that there are no zero eigenvalues, namely, on the domain under consideration we have

$$\lambda_r(u) < 0 < \lambda_s(u), \quad r = 1, \cdots, m;\ s = m + 1, \cdots, n. \tag{1.5}$$

Manuscript received January 7, 2015. Revised April 8, 2015.
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For general first-order quasilinear hyperbolic systems with general nonlinear boundary conditions, Li Tatsien et al. [1–4] proposed a constructive method to establish the local exact boundary controllability by means of boundary controls. Since the speed of wave propagation is finite, the control time $T > 0$ can not be too short. However, in many practical problems, we always hope to reduce the control time. For this purpose, Zhuang Kaili, Li Tatsien and Rao Bopeng [7] added some suitable internal controls, and established the local exact controllability in a shorter time by using the combined effect of boundary controls and internal controls.

In this paper, we will discuss the quasilinear hyperbolic system (1.1) with zero eigenvalues. Assume that on the domain under consideration, the eigenvalues of $A(u)$ satisfy the following condition:

$$\lambda_p(u) < \lambda_q(u) \equiv 0 < \lambda_r(u), \quad p = 1, \cdots, l;\ q = l+1, \cdots, m;\ r = m+1, \cdots, n. \tag{1.6}$$

Consider the simplest equation with zero eigenvalue

$$\frac{\partial u}{\partial t} = 0. \tag{1.7}$$

It is easy to see that the exact controllability cannot be achieved only by boundary controls. Therefore, different from the situation that the eigenvalues satisfy (1.5), in order to realize the exact controllability for quasilinear hyperbolic systems with zero eigenvalues, we should use not only suitable boundary controls but also suitable internal controls. For general first-order quasilinear hyperbolic systems with zero eigenvalues together with the general nonlinear boundary conditions, by using boundary controls and adding internal controls to a part of equations corresponding to zero eigenvalues, Li Tatsien and Yu Lixin [5], Zhang Qi [6] established the corresponding local exact controllability. At this time, the control time $T > 0$ should be suitably large, too.

In the present paper, we try to input some suitable internal controls to reduce the control time. To this end, it is necessary to rewrite the system (1.1) into the corresponding characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = \widetilde{F}_i(u) \overset{\triangle}{=} l_i(u)F(u), \quad i = 1, \cdots, n, \tag{1.8}$$

in which the i-th equation consists of only the directional derivative of the unknown function u with respect to t along the i-th characteristic direction $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(u)$, and

$$\widetilde{F}_i(0) = 0, \quad i = 1, \cdots, n. \tag{1.9}$$

Adding suitable internal controls $c_i(t,x)$ $(i = 1, \cdots, n)$ to (1.8), we have

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = \widetilde{F}_i(u) + c_i(t,x), \quad i = 1, \cdots, n, \tag{1.10}$$

where

$$c_i(t,x) = l_i(\vartheta)\Big(\frac{\partial \vartheta}{\partial t} + \lambda_i(\vartheta)\frac{\partial \vartheta}{\partial x}\Big) + k_i(t,x), \quad i = 1, \cdots, n, \tag{1.11}$$

in which $\vartheta = \vartheta(t,x)$ is a C^1 vector function of (t,x) and $k_i(t,x)$ $(i = 1, \cdots, n)$ are C^1 functions of (t,x).

Let

$$\upsilon_i = l_i(u)u, \quad i = 1, \cdots, n. \tag{1.12}$$

We consider the mixed initial-boundary value problem for the quasilinear hyperbolic system (1.10) with the initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L \tag{1.13}$$

and the following boundary conditions:

$$x = 0: \quad \upsilon_r = G_r(t, \upsilon_1, \cdots, \upsilon_l, \upsilon_{l+1}, \cdots, \upsilon_m) + H_r(t), \qquad r = m+1, \cdots, n, \tag{1.14}$$

$$x = L: \quad \upsilon_p = G_p(t, \upsilon_{l+1}, \cdots, \upsilon_m, \upsilon_{m+1}, \cdots, \upsilon_n) + H_p(t), \quad p = 1, \cdots, l. \tag{1.15}$$

Without loss of generality, we assume that

$$G_p(t, 0, \cdots, 0) \equiv 0, \quad G_r(t, 0, \cdots, 0) \equiv 0, \quad p = 1, \cdots, l;\ r = m+1, \cdots, n. \tag{1.16}$$

For any given initial data φ and final data ψ with small $C^1[0, L]$ norm, if there exists a $T > 0$ such that, taking H_p, H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ or a part of H_p, H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ as boundary controls and $c_i(t,x)$ $(i = 1, \cdots, n)$ as internal controls, the corresponding mixed initial-boundary value problem (1.10) and (1.13)–(1.15) admits a unique C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$, which satisfies exactly the final condition

$$t = T: \quad u = \psi(x), \quad 0 \le x \le L \tag{1.17}$$

or

$$t = T: \quad u = 0, \quad 0 \le x \le L, \tag{1.18}$$

then we say that the mixed initial-boundary value problem (1.10) and (1.13)–(1.15) possesses the local exact controllability or the local exact null controllability, respectively.

2 Local Exact Controllability with Boundary Controls and Internal Controls

Theorem 2.1 (Local Two-Sided Exact Controllability) *Assume that* $\lambda_i(u), l_i(u), \widetilde{F}_i(u)$, $G_p(t,\cdot)$, *and* $G_r(t,\cdot)$ $(i = 1, \cdots, n;\ p = 1, \cdots, l;\ r = m+1, \cdots, n)$ *are all* C^1 *functions with respect to their arguments. Assume furthermore that* (1.6), (1.9) *and* (1.16) *hold. For any given* δ $(0 < \delta < \frac{L}{2})$, *if*

$$T > \Big(\frac{L}{2} - \delta\Big) \max_{\substack{p=1,\cdots,l \\ r=m+1,\cdots,n}} \Big(\frac{1}{|\lambda_p(0)|}, \frac{1}{\lambda_r(0)}\Big), \tag{2.1}$$

then for any given initial data φ and final data ψ with small $C^1[0,L]$ norm, there exist boundary controls H_p, H_r $(p=1,\cdots,l;\ r=m+1,\cdots,n)$ with small $C^1[0,T]$ norm and internal controls $c_i(t,x)$ $(i=1,\cdots,n)$ given by (1.11)*, in which the $C^1[R(T)]$ norm of ϑ and k_i $(i=1,\cdots,n)$ is suitably small, such that the mixed initial-boundary value problem* (1.10) *and* (1.13)–(1.15) *admits a unique C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)$, which satisfies exactly the final condition* (1.17).

To prove Theorem 2.1, we construct the following system of the characteristic form:

$$\begin{cases} l_p(u)\Big(\dfrac{\partial u}{\partial t}+\lambda_p(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_p(u)+c_p(t,x), \quad p=1,\cdots,l,\\ l_q(u)\Big(\dfrac{\partial u}{\partial t}+\overline{\lambda}_q(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_q(u)+\widetilde{c}_q(t,x), \quad q=l+1,\cdots,m,\\ l_r(u)\Big(\dfrac{\partial u}{\partial t}+\lambda_r(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_r(u)+c_r(t,x), \quad r=m+1,\cdots,n,\end{cases} \tag{2.2}$$

where $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n)$ are the corresponding internal controls and

$$\overline{\lambda}_q(u)=\lambda_1(u), \quad q=l+1,\cdots,m. \tag{2.3}$$

Obviously, (2.2) is a system without zero eigenvalues. Corresponding to (2.3), we give the following artificial boundary conditions:

$$x=L: \quad v_q=H_q(t), \quad q=l+1,\cdots,m, \tag{2.4}$$

in which $H_q(t)$ $(q=l+1,\cdots,m)$ are C^1 functions of t.

For the mixed initial-boundary value problem (2.2), (1.13)–(1.15) and (2.4), according to the result on local two-sided exact controllability in [7], we have the lemma.

Lemma 2.1 *Under the hypotheses of Theorem* 2.1*, suppose furthermore that* (2.3) *holds. Let $T>0$ be defined by* (2.1)*. For any given initial data φ and final data ψ with small $C^1[0,L]$ norm, there exist boundary controls H_i $(i=1,\cdots,n)$ with small $C^1[0,T]$ norm and internal controls*

$$\widetilde{c}_i(t,x)=\begin{cases} l_i(\widetilde{\vartheta})\Big(\dfrac{\partial\widetilde{\vartheta}}{\partial t}+\lambda_i(\widetilde{\vartheta})\dfrac{\partial\widetilde{\vartheta}}{\partial x}\Big)+\widetilde{k}_i(t,x), \quad i=1,\cdots,l;\ m+1,\cdots,n,\\ l_i(\widetilde{\vartheta})\Big(\dfrac{\partial\widetilde{\vartheta}}{\partial t}+\overline{\lambda}_i(\widetilde{\vartheta})\dfrac{\partial\widetilde{\vartheta}}{\partial x}\Big)+\widetilde{k}_i(t,x), \quad i=l+1,\cdots,m,\end{cases} \tag{2.5}$$

in which the $C^1[R(T)]$ norm of $\widetilde{\vartheta}$ and $\widetilde{k}_i$ $(i=1,\cdots,n)$ is suitably small, such that the mixed initial-boundary value problem (2.2), (1.13)–(1.15) *and* (2.4) *admits a unique C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)$, which satisfies exactly the final condition* (1.17).

By Lemma 2.1, we can prove Theorem 2.1.

In fact, substituting the C^1 solution $u=u(t,x)$ given by Lemma 2.1 into boundary conditions (1.14)–(1.15), we get the desired boundary controls:

$$H_p(t)=(v_p-G_p(t,v_{l+1},\cdots,v_m,v_{m+1},\cdots,v_n))|_{x=L}, \quad p=1,\cdots,l, \tag{2.6}$$

$$H_r(t)=(v_r-G_r(t,v_1,\cdots,v_l,v_{l+1},\cdots,v_m))|_{x=0}, \quad r=m+1,\cdots,n, \tag{2.7}$$

where υ_i $(i = 1, \cdots, n)$ are defined by (1.12): $\upsilon_i = l_i(u(t,x))u(t,x)$ $(i = 1, \cdots, n)$. Noting (1.16), the C^1 norm of H_p and H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ is suitably small. On the other hand, substituting $u = u(t,x)$ into system (1.10), we get the desired internal controls

$$c_i(t,x) = l_i(u(t,x))\Big(\frac{\partial u(t,x)}{\partial t} + \lambda_i(u(t,x))\frac{\partial u(t,x)}{\partial x}\Big) - \widetilde{F}_i(u(t,x)), \quad i = 1, \cdots, n, \tag{2.8}$$

which correspond to (1.11) with

$$\vartheta(t,x) = u(t,x), \quad k_i(t,x) = -\widetilde{F}_i(u(t,x)), \quad i = 1, \cdots, n. \tag{2.9}$$

Noting (1.9), the $C^1[R(T)]$ norm of ϑ and k_i $(i = 1, \cdots, n)$ is also small. Thus, we obtain the desired exact controllability.

Theorem 2.2 (Local One-Sided Exact Controllability) *Under the hypotheses of Theorem* 2.1*, suppose furthermore that the number of the positive eigenvalues is not greater than that of negative ones:*

$$\overline{m} \overset{\triangle}{=} n - m \le l, \quad \textit{i.e.}, \ n \le l + m. \tag{2.10}$$

Suppose finally that in a neighborhood of $u = 0$, the boundary conditions (1.14) *on $x = 0$ can be equivalently rewritten as*

$$x = 0: \quad \upsilon_{\overline{p}} = \overline{G}_{\overline{p}}(t, \upsilon_{l+1}, \cdots, \upsilon_m, \upsilon_{m+1}, \cdots, \upsilon_n) + \overline{H}_{\overline{p}}(t), \quad \overline{p} = 1, \cdots, \overline{m} \tag{2.11}$$

with

$$\overline{G}_{\overline{p}}(t, 0, \cdots, 0) \equiv 0, \quad \overline{p} = 1, \cdots, \overline{m}. \tag{2.12}$$

Then

$$\|\overline{H}_{\overline{p}}\|_{C^1[0,T]} \ (\overline{p} = 1, \cdots, \overline{m}) \ \textit{small} \Leftrightarrow \|H_r\|_{C^1[0,T]} \ (r = m+1, \cdots, n) \ \textit{small}. \tag{2.13}$$

For any given δ $(0 < \delta < \frac{L}{2})$, if

$$T > \Big(\frac{L}{2} - \delta\Big)\Big(\max_{p=1,\cdots,l} \frac{1}{|\lambda_p(0)|} + \max_{r=m+1,\cdots,n} \frac{1}{\lambda_r(0)}\Big), \tag{2.14}$$

then for any given initial data φ and final data ψ with small $C^1[0,L]$ norm, and any given H_r $(r = m+1, \cdots, n)$ with small $C^1[0,T]$ norm, such that the conditions of C^1 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(T,0)$, respectively, there exist boundary controls H_p $(p = 1, \cdots, l)$ with small $C^1[0,T]$ norm and internal controls $c_i(t,x)$ $(i = 1, \cdots, n)$ given by (1.11)*, in which the $C^1[R(T)]$ norm of ϑ and k_i $(i = 1, \cdots, n)$ is suitably small, such that the conclusion of Theorem* 2.1 *holds.*

To prove Theorem 2.2, we construct the characteristic system (2.2), where $\widetilde{c}_i(t,x)$ $(i = 1, \cdots, n)$ are the corresponding internal controls and $\overline{\lambda}_q(u)$ $(q = l+1, \cdots, m)$ are given by (2.3). Obviously, (2.2) is a system without zero eigenvalues. Corresponding to (2.3), we give the artificial boundary conditions (2.4) on $x = L$, in which $H_q(t)$ $(q = l+1, \cdots, m)$ are C^1 functions of t.

For the mixed initial-boundary value problem (2.2), (1.13)–(1.15) and (2.4), according to the result on local one-sided exact controllability in [7], we have Lemma 2.2.

Lemma 2.2 *Under the hypotheses of Theorem* 2.2*, suppose furthermore that* (2.3) *holds. Let* $T > 0$ *be defined by* (2.14)*. For any given initial data* φ *and final data* ψ *with small* $C^1[0, L]$ *norm, and any given* H_r $(r = m+1, \cdots, n)$ *with small* $C^1[0, T]$ *norm, such that the conditions of* C^1 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(T, 0)$*, respectively, there exist boundary controls* H_p *and* H_q $(p = 1, \cdots, l;\ q = l + 1, \cdots, m)$ *with small* $C^1[0, T]$ *norm and internal controls* $\widetilde{c}_i(t, x)$ $(i = 1, \cdots, n)$ *given by* (2.5)*, in which the* $C^1[R(T)]$ *norm of* $\widetilde{\vartheta}$ *and* $\widetilde{k}_i$ $(i = 1, \cdots, n)$ *is suitably small, such that the conclusion of Lemma* 2.1 *holds.*

By Lemma 2.2, we can prove Theorem 2.2.

In fact, substituting the C^1 solution $u = u(t, x)$ given by Lemma 2.2 into boundary conditions (1.15), we get the desired boundary controls $H_p(t)$ $(p = 1, \cdots, l)$ given by (2.6), where υ_i $(i = 1, \cdots, n)$ are defined by (1.12). Noting (1.16), the C^1 norm of H_p $(p = 1, \cdots, l)$ is suitably small. On the other hand, substituting $u = u(t, x)$ into the system (1.10), we get the desired internal controls $c_i(t, x)$ $(i = 1, \cdots, n)$ given by (2.8), which correspond to (1.11) with (2.9). Noting (1.9), the $C^1[R(T)]$ norm of ϑ and k_i $(i = 1, \cdots, n)$ is also small. Thus, we obtain the desired exact controllability.

Theorem 2.3 (Local Two-Sided Exact Controllability with Less Controls) *Under the hypotheses of Theorem* 2.1*, suppose furthermore that the number of positive eigenvalues is less than that of negative ones:*

$$\overline{m} \stackrel{\triangle}{=} n - m < l, \quad \text{i.e., } n < l + m. \tag{2.15}$$

Suppose finally that in a neighborhood of $u = 0$*, without loss of generality, the first* $\overline{m}$ *boundary conditions in* (1.15)*, namely,*

$$x = L: \quad \upsilon_{\overline{p}} = G_{\overline{p}}(t, \upsilon_{l+1}, \cdots, \upsilon_m, \upsilon_{m+1}, \cdots, \upsilon_n) + H_{\overline{p}}(t), \quad \overline{p} = 1, \cdots, \overline{m} \tag{2.16}$$

can be equivalently rewritten as

$$x = L: \quad \upsilon_r = \overline{G}_r(t, \upsilon_1, \cdots, \upsilon_{\overline{m}}, \upsilon_{l+1}, \cdots, \upsilon_m) + \overline{H}_r(t), \quad r = m + 1, \cdots, n \tag{2.17}$$

with

$$\overline{G}_r(t, 0, \cdots, 0) \equiv 0, \quad r = m + 1, \cdots, n. \tag{2.18}$$

Then

$$\|\overline{H}_r\|_{C^1[0,T]} \ (r = m + 1, \cdots, n) \text{ small} \Leftrightarrow \|H_{\overline{p}}\|_{C^1[0,T]} \ (\overline{p} = 1, \cdots, \overline{m}) \text{ small}. \tag{2.19}$$

For any given δ $\left(0 < \delta < \frac{L}{2}\right)$*, if* $T > 0$ *satisfies* (2.14)*, then for any given initial data* φ *and final data* ψ *with small* $C^1[0, L]$ *norm, and for any given* $H_{\overline{p}}$ $(\overline{p} = 1, \cdots, \overline{m})$ *with small* $C^1[0, T]$ *norm, such that the corresponding conditions of* C^1 *compatibility are satisfied at the points* $(t, x) = (0, L)$ *and* (T, L)*, respectively, there exist boundary controls* $H_{\overline{\overline{p}}}$ *and* H_r $(\overline{\overline{p}} = \overline{m} + 1, \cdots, l;\ r = m+1, \cdots, n)$ *with small* $C^1[0, T]$ *norm and internal controls* $c_i(t, x)$ $(i = 1, \cdots, n)$ *given by* (1.11)*, in which the* $C^1[R(T)]$ *norm of* ϑ *and* k_i $(i = 1, \cdots, n)$ *is suitably small, such that the conclusion of Theorem* 2.1 *holds.*

To prove Theorem 2.3, we also construct the characteristic system (2.2) without zero eigenvalues, where $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n)$ are the corresponding internal controls and $\overline{\lambda}_q(u)$ $(q=l+1,\cdots,m)$ are given by (2.3). Corresponding to (2.3), we give the artificial boundary conditions (2.4) on $x=L$, in which $H_q(t)$ $(q=l+1,\cdots,m)$ are C^1 functions of t.

For the mixed initial-boundary value problem (2.2), (1.13)–(1.15) and (2.4), according to the result on local two-sided exact controllability with less controls in [7], we have the following lemma.

Lemma 2.3 *Under the hypotheses of Theorem* 2.3*, suppose furthermore that* (2.3) *holds. Let $T>0$ be defined by* (2.14)*. For any given initial data φ and final data ψ with small $C^1[0,L]$ norm, and any given $H_{\overline{p}}$ $(\overline{p}=1,\cdots,\overline{m})$ with small $C^1[0,T]$ norm, such that the conditions of C^1 compatibility are satisfied at the points $(t,x)=(0,L)$ and (T,L), respectively, there exist boundary controls $H_{\overline{q}}, H_q$ and H_r $(\overline{q}=\overline{m}+1,\cdots,l;\ q=l+1,\cdots,m;\ r=m+1,\cdots,n)$ with small $C^1[0,T]$ norm and internal controls $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n)$ given by* (2.5)*, in which the $C^1[R(T)]$ norm of $\widetilde{\vartheta}$ and $\widetilde{k}_i$ $(i=1,\cdots,n)$ is suitably small, such that the conclusion of Lemma* 2.1 *holds.*

By Lemma 2.3, we can prove Theorem 2.3.

In fact, substituting the C^1 solution $u=u(t,x)$ given by Lemma 2.3 into boundary conditions (1.14) and the last $l-\overline{m}=l+m-n$ boundary conditions in (1.15), we get the desired boundary controls:

$$H_{\overline{\overline{p}}}(t)=(\upsilon_{\overline{\overline{p}}}-G_{\overline{\overline{p}}}(t,\upsilon_{l+1},\cdots,\upsilon_m,\upsilon_{m+1},\cdots,\upsilon_n))|_{x=L},\quad \overline{\overline{p}}=\overline{m}+1,\cdots,l, \tag{2.20}$$

and H_r $(r=m+1,\cdots,n)$ given by (2.7), where υ_i $(i=1,\cdots,n)$ are defined by (1.12). Noting (1.16), the C^1 norm of $H_{\overline{\overline{p}}}$ and H_r $(\overline{\overline{p}}=\overline{m}+1,\cdots,l;\ r=m+1,\cdots,n)$ is suitably small. On the other hand, substituting $u=u(t,x)$ into system (1.10), we get the desired internal controls $c_i(t,x)$ $(i=1,\cdots,n)$ given by (2.8), which corresponds to (1.11) with (2.9). Noting (1.9), the $C^1[R(T)]$ norm of ϑ and k_i $(i=1,\cdots,n)$ is also small. Thus, we obtain the desired exact controllability.

3 Local Exact Null Controllability with Boundary Controls and Internal Controls

For the local exact null controllability, we can get the same conclusion (especially Theorem 2.2) as in the previous section under much less hypotheses.

Theorem 3.1 (Local One-Sided Exact Null Controllability) *For any given δ $(0<\delta<\frac{L}{2})$, let $T>0$ satisfy* (2.14)*.*

(A) *Suppose that in boundary conditions* (1.15) *on $x=L$, we have*

$$H_p(t)\equiv 0,\quad p=1,\cdots,l. \tag{3.1}$$

Then, for any given initial data φ with small $C^1[0,L]$ norm, such that the conditions of C^1 compatibility are satisfied at the point $(t,x)=(0,L)$, there exist boundary controls H_r $(r=$

$m+1,\cdots,n$) with small $C^1[0,T]$ norm and internal controls $c_i(t,x)$ $(i=1,\cdots,n)$ given by (1.11)*, in which the $C^1[R(T)]$ norm of ϑ and k_i $(i=1,\cdots,n)$ is suitably small, such that the mixed initial-boundary value problem* (1.10) *and* (1.13)–(1.15) *admits a unique C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)$, which satisfies exactly the zero final condition* (1.18).

(B) *Suppose that in boundary conditions* (1.14) *on $x=0$, we have*

$$H_r(t)\equiv 0,\quad r=m+1,\cdots,n. \tag{3.2}$$

Then, for any given initial data φ with small $C^1[0,L]$ norm, such that the conditions of C^1 compatibility are satisfied at the point $(t,x)=(0,0)$, there exist boundary controls H_p $(p=1,\cdots,l)$ with small $C^1[0,T]$ norm and internal controls $c_i(t,x)$ $(i=1,\cdots,n)$ given by (1.11)*, in which the $C^1[R(T)]$ norm of ϑ and k_i $(i=1,\cdots,n)$ is suitably small, such that the conclusion* (A) *of Theorem* 3.1 *holds.*

We only prove the first part of Theorem 3.1. The proof of the second part is similar. To this end, we also construct the characteristic system (2.2), where $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n)$ are the corresponding internal controls and

$$\overline{\lambda}_q(u)=\lambda_n(u),\quad q=l+1,\cdots,m. \tag{3.3}$$

Corresponding to (3.3), we give the following artificial boundary conditions:

$$x=0:\quad v_q=\overline{H}_q(t),\quad q=l+1,\cdots,m, \tag{3.4}$$

in which $\overline{H}_q(t)$ $(q=l+1,\cdots,m)$ are C^1 functions of t.

For the mixed initial-boundary value problem (2.2), (1.13)–(1.15) and (3.4), according to the result on local one-sided exact null controllability in [7], we have the following lemma.

Lemma 3.1 *Under the hypotheses of Theorem* 3.1(A)*, suppose furthermore that* (3.3) *holds. Let $T>0$ be defined by* (2.14)*. For any given initial data φ with small $C^1[0,L]$ norm, such that the conditions of C^1 compatibility are satisfied at the point $(t,x)=(0,L)$, there exist boundary controls $\overline{H}_q$ and H_r $(q=l+1,\cdots,m;\ r=m+1,\cdots,n)$ with small $C^1[0,T]$ norm and internal controls $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n)$ given by* (2.5)*, in which the $C^1[R(T)]$ norm of $\widetilde{\vartheta}$ and $\widetilde{k}_i$ $(i=1,\cdots,n)$ is suitably small, such that the mixed initial-boundary value problem* (2.2) *and* (1.13)–(1.15) *and* (3.4) *admits a unique C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)$, which satisfies exactly the zero final condition* (1.18).

By Lemma 3.1, we can prove Theorem 3.1.

In fact, substituting the C^1 solution $u=u(t,x)$ given by Lemma 3.1 into boundary conditions (1.14), we get the desired boundary controls H_r $(r=m+1,\cdots,n)$ given by (2.7), where v_i $(i=1,\cdots,n)$ are defined by (1.12). Noting (1.16), the C^1 norm of H_r $(r=m+1,\cdots,n)$ is suitably small. On the other hand, substituting $u=u(t,x)$ into the system (1.10), we get the desired internal controls $c_i(t,x)$ $(i=1,\cdots,n)$ given by (2.8), which correspond to (1.11) with (2.9). Noting (1.9), the $C^1[R(T)]$ norm of ϑ and k_i $(i=1,\cdots,n)$ is also small. Thus, we obtain the desired exact controllability.

4 Local Exact Internal Controllability

Under some special but meaningful assumptions, we may realize the local exact controllability only by internal controls. Suppose that the number of positive eigenvalues is equal to that of negative ones, i.e.,

$$n = l + m. \tag{4.1}$$

Suppose furthermore that in a neighborhood of $u = 0$, the boundary conditions (1.14)–(1.15) can be equivalently rewritten as

$$x = 0: \quad \upsilon_p = \overline{G}_p(t, \upsilon_{l+1}, \cdots, \upsilon_m, \upsilon_{m+1}, \cdots, \upsilon_n) + \overline{H}_p(t), \quad p = 1, \cdots, l, \tag{4.2}$$

$$x = L: \quad \upsilon_r = \overline{G}_r(t, \upsilon_1, \cdots, \upsilon_l, \upsilon_{l+1}, \cdots, \upsilon_m) + \overline{H}_r(t), \qquad r = m+1, \cdots, n, \tag{4.3}$$

respectively. Without loss of generality, we may assume that

$$\overline{G}_p(t, 0, \cdots, 0) \equiv 0, \quad \overline{G}_r(t, 0, \cdots, 0) \equiv 0, \quad p = 1, \cdots, l;\ r = m+1, \cdots, n. \tag{4.4}$$

Then

$$\|\overline{H}_p\|_{C^1[0,T]}\ (p = 1, \cdots, l) \text{ small} \Leftrightarrow \|H_r\|_{C^1[0,T]}\ (r = m+1, \cdots, n) \text{ small}, \tag{4.5}$$

$$\|\overline{H}_r\|_{C^1[0,T]}\ (r = m+1, \cdots, n) \text{ small} \Leftrightarrow \|H_p\|_{C^1[0,T]}\ (p = 1, \cdots, l) \text{ small}. \tag{4.6}$$

Theorem 4.1 (Local Exact Internal Controllability) *Under the hypotheses of Theorem* 2.1*, suppose furthermore that* (4.1)–(4.4) *hold. For any given* δ $(0 < \delta < \frac{L}{2})$*, if* $T > 0$ *satisfies* (2.14)*, then for any given initial data* φ *and final data* ψ *with small* $C^1[0, L]$ *norm, and for any given* H_p *and* H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ *with small* $C^1[0, T]$ *norm, such that the conditions of* C^1 *compatibility are satisfied at the points* $(t, x) = (0, 0), (0, L), (T, 0)$ *and* (T, L)*, respectively, there exist internal controls* $c_i(t, x)$ $(i = 1, \cdots, n)$ *given by* (1.11)*, in which the* $C^1[R(T)]$ *norm of* ϑ *and* k_i $(i = 1, \cdots, n)$ *is suitably small, such that the conclusion of Theorem* 2.1 *holds.*

To prove Theorem 4.1, we introduce the following unknown vector function of (t, x):

$$\widetilde{u} = (u_1, \cdots, u_n, u_{n+1}, \cdots, u_{n+m-l})^{\mathrm{T}} = (u^{\mathrm{T}}, u_{n+1}, \cdots, u_{n+m-l})^{\mathrm{T}} \tag{4.7}$$

and a set of $(n + m - l)$-D row vectors

$$\widetilde{l}_i(\widetilde{u}) = (l_{i1}(u), \cdots, l_{in}(u), 0, \cdots, 0), \quad i = 1, \cdots, n, \tag{4.8}$$

$$\widetilde{l}_{n+j}(\widetilde{u}) = (0, \cdots, 0, \overset{(n+j)}{1}, 0 \cdots, 0), \qquad j = 1, \cdots, m - l. \tag{4.9}$$

Obviously, $\widetilde{l}_1(\widetilde{u}), \cdots, \widetilde{l}_{n+m-l}(\widetilde{u})$ is a set of linearly independent vectors.

We construct the following system of characteristic form:

$$\begin{cases}\widetilde{l}_p(\widetilde{u})\Big(\dfrac{\partial\widetilde{u}}{\partial t}+\lambda_p(u)\dfrac{\partial\widetilde{u}}{\partial x}\Big)=\widetilde{F}_p(u)+\widetilde{c}_p(t,x), & p=1,\cdots,l,\\ \widetilde{l}_q(\widetilde{u})\Big(\dfrac{\partial\widetilde{u}}{\partial t}+\overline{\lambda}_q(u)\dfrac{\partial\widetilde{u}}{\partial x}\Big)=\widetilde{F}_q(u)+\widetilde{c}_q(t,x), & q=l+1,\cdots,m,\\ \widetilde{l}_r(\widetilde{u})\Big(\dfrac{\partial\widetilde{u}}{\partial t}+\lambda_r(u)\dfrac{\partial\widetilde{u}}{\partial x}\Big)=\widetilde{F}_r(u)+\widetilde{c}_r(t,x), & r=m+1,\cdots,n,\\ \widetilde{l}_s(\widetilde{u})\Big(\dfrac{\partial\widetilde{u}}{\partial t}+\overline{\lambda}_s(u)\dfrac{\partial\widetilde{u}}{\partial x}\Big)=\widetilde{c}_s(t,x), & s=n+1,\cdots,n+m-l,\end{cases}\tag{4.10}$$

where $\widetilde{c}_i(t,x)$ $(i=1,\cdots,n+m-l)$ are the corresponding internal controls and

$$\overline{\lambda}_q(u)=\lambda_1(u),\overline{\lambda}_s(u)=\lambda_n(u),\quad q=l+1,\cdots,m;\ s=n+1,\cdots,n+m-l.\tag{4.11}$$

Noting (4.8)–(4.9), the system (4.10) can be simplified into

$$\begin{cases}l_p(u)\Big(\dfrac{\partial u}{\partial t}+\lambda_p(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_p(u)+\widetilde{c}_p(t,x), & p=1,\cdots,l,\\ l_q(u)\Big(\dfrac{\partial u}{\partial t}+\overline{\lambda}_q(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_q(u)+\widetilde{c}_q(t,x), & q=l+1,\cdots,m,\\ l_r(u)\Big(\dfrac{\partial u}{\partial t}+\lambda_r(u)\dfrac{\partial u}{\partial x}\Big)=\widetilde{F}_r(u)+\widetilde{c}_r(t,x), & r=m+1,\cdots,n,\\ \dfrac{\partial u_s}{\partial t}+\overline{\lambda}_s(u)\dfrac{\partial u_s}{\partial x}=\widetilde{c}_s(t,x), & s=n+1,\cdots,n+m-l.\end{cases}\tag{4.12}$$

Obviously, (4.10) is a system without zero eigenvalues.

Let

$$\begin{aligned}\widetilde{v}_i&=\widetilde{l}_i(\widetilde{u})\widetilde{u},\quad i=1,\cdots,n+m-l\\ &=\begin{cases}l_i(u)u=v_i, & i=1,\cdots,n,\\ u_i, & i=n+1,\cdots,n+m-l.\end{cases}\end{aligned}\tag{4.13}$$

We give the initial condition

$$t=0:\quad \widetilde{u}=(\varphi(x)^{\mathrm{T}},0,\cdots,0)^{\mathrm{T}},\quad 0\le x\le L\tag{4.14}$$

and the final condition

$$t=0:\quad \widetilde{u}=(\psi(x)^{\mathrm{T}},0,\cdots,0)^{\mathrm{T}},\quad 0\le x\le L.\tag{4.15}$$

Corresponding to (4.11), we give the following artificial boundary conditions:

$$x=0:\quad u_s=v_{s-(n-l)},\quad s=n+1,\cdots,n+m-l,\tag{4.16}$$

$$x=L:\quad v_q=u_{q+(n-l)},\quad q=l+1,\cdots,m.\tag{4.17}$$

For the mixed initial-boundary value problem (4.10), (4.14), (1.14)–(1.15) and (4.16)–(4.17), according to the result on local exact internal controllability in [7], we have the following lemma.

Lemma 4.1 *Under the hypotheses of Theorem* 4.1*, suppose furthermore that* (4.11) *holds. Let $T > 0$ be defined by* (2.14)*. For any given φ and ψ with small $C^1[0, L]$ norm, and for any given H_p and H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ with small $C^1[0, T]$ norm, such that the conditions of C^1 compatibility are satisfied at the points $(t, x) = (0, 0), (0, L), (T, 0)$ and (T, L), respectively, there exist internal controls*

$$\widetilde{c}_i(t,x) = \begin{cases} \widetilde{l}_i(\widetilde{\widetilde{\vartheta}})\Big(\dfrac{\partial\widetilde{\widetilde{\vartheta}}}{\partial t} + \lambda_i(\widetilde{\widetilde{\vartheta}})\dfrac{\partial\widetilde{\widetilde{\vartheta}}}{\partial x}\Big) + \widetilde{\widetilde{k}}_i(t,x), & i = 1, \cdots, l;\ m+1, \cdots, n, \\ \widetilde{l}_i(\widetilde{\widetilde{\vartheta}})\Big(\dfrac{\partial\widetilde{\widetilde{\vartheta}}}{\partial t} + \overline{\lambda}_i(\widetilde{\widetilde{\vartheta}})\dfrac{\partial\widetilde{\widetilde{\vartheta}}}{\partial x}\Big) + \widetilde{\widetilde{k}}_i(t,x), & i = l+1, \cdots, m;\ n+1, \cdots, n+m-l, \end{cases} \tag{4.18}$$

in which $\lambda_i(\widetilde{\widetilde{\vartheta}})$ and $\overline{\lambda}_i(\widetilde{\widetilde{\vartheta}})$ depend only on the first n components of $\widetilde{\widetilde{\vartheta}}$, and the $C^1[R(T)]$ norm of $\widetilde{\widetilde{\vartheta}} = \widetilde{\widetilde{\vartheta}}(t, x)$ and $\widetilde{\widetilde{k}}_i$ $(i = 1, \cdots, n+m-l)$ is suitably small, such that the mixed initial-boundary value problem (4.10)*,* (4.14)*,* (1.14)*–*(1.15) *and* (4.16)*–*(4.17) *admits a unique C^1 solution $\widetilde{u} = \widetilde{u}(t, x)$ with small C^1 norm on the domain $R(T)$, which satisfies exactly the final condition* (4.15)*.*

By Lemma 4.1, we can prove Theorem 4.1.

In fact, let $u = u(t, x) = (u_1, \cdots, u_n)^{\mathrm{T}}$ be the column vector composed of the first n components of the C^1 solution $\widetilde{u} = \widetilde{u}(t, x)$ given by Lemma 4.1. Obviously, $u = u(t, x)$ verifies boundary conditions (1.14)–(1.15), the initial condition (1.13) and the final condition (1.17). Substituting $u = u(t, x)$ into the system (1.10), we get the desired internal controls $c_i(t, x)$ $(i = 1, \cdots, n)$ given by (2.8), which correspond to (1.11) with (2.9). Noting (1.9), the $C^1[R(T)]$ norm of ϑ and k_i $(i = 1, \cdots, n)$ is also small. Thus, we obtain the desired exact controllability.

If we consider only the exact null controllability and assume that (3.1)–(3.2) hold, then we can still get the local exact internal controllability without assumptions (4.1)–(4.4).

Theorem 4.2 (Local Exact Internal Null Controllability) *Under the hypotheses of Theorem* 2.1*, suppose furthermore that* (3.1)*–*(3.2) *hold. For any given δ $(0 < \delta < \frac{L}{2})$, if $T > 0$ satisfies* (2.14)*, then for any given initial data φ with small $C^1[0, L]$ norm, such that the conditions of C^1 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively, there exist internal controls $c_i(t, x)$ $(i = 1, \cdots, n)$ given by* (1.11)*, in which the $C^1[R(T)]$ norm of ϑ and k_i $(i = 1, \cdots, n)$ is suitably small, such that the conclusion of Theorem* 4.1 *holds for the zero final condition* (1.18)*.*

The proof of Theorem 4.2 is similar to that of Theorem 4.1.

5 Remarks

Remark 5.1 The estimate given by (2.1) and (2.14) on the control time in Theorems 2.1–4.2 is sharp. By (2.1) and (2.14), the larger the value δ, the smaller the control time T. In particular, when $\delta \to \frac{L}{2}$, the right-hand sides of (2.1) and (2.14) tend to zero. It shows that by means of internal controls one can realize the local exact controllability almost immediately in principle.

Remark 5.2 By [7], for first-order quasilinear hyperbolic systems without zero eigenvalues, the internal controls are acting only on the rectangle $[\varepsilon_0, T-\varepsilon_0]\times[\frac{L}{2}-\delta, \frac{L}{2}+\delta]$, where $\varepsilon_0>0$ is suitably small. Thus, by (2.2) and the first n equations in (4.12), in the domains $[0,T]\times[0,\frac{L}{2}-\delta]$ and $[0,T]\times[\frac{L}{2}+\delta, L]$, the internal controls in Theorems 2.1–4.2 are added only to those equations corresponding to zero eigenvalues.

Remark 5.3 The boundary controls and the internal controls given in Theorems 2.1–4.2 are not unique.

Acknowledgement The author would like to express her sincere gratitude to Professor Li Tatsien for his effective instruction.

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