Chin. Ann. Math.
37B(4), 2016, 523–532
DOI: 10.1007/s11401-016-1022-0

Chinese Annals of Mathematics, Series B

Geometrical Realization of Low-Dimensional Complete Intersections*

Jianbo WANG[1] Jianpeng DU[2]

Abstract This paper aims to give some examples of diffeomorphic (or homeomorphic) low-dimensional complete intersections, which can be considered as a geometrical realization of classification theorems about complete intersections. A conjecture of Libgober and Wood (1982) will be confirmed by one of the examples.

Keywords Complete intersection, Realization, Classification
2000 MR Subject Classification 14M10, 57R19

1 Introduction

Let $X_n(\underline{d}) \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of multidegree $\underline{d} := (d_1, \cdots, d_r)$ and the product $d_1 d_2 \cdots d_r$ is called the total degree, denoted by d. The celebrated Lefschetz hyperplane section theorem asserts that the pair $(\mathbb{C}P^{n+r}, X_n(\underline{d}))$ is n-connected. Thom showed that the complex dimension n and the multidegree $\underline{d}$ determine the diffeomorphism type of $X_n(\underline{d})$. However, the multidegree is not a topological invariant. An interesting and challenging problem is to classify complete intersections by numerical topological invariants, such as the total degree, the Pontrjagin classes and the Euler characteristics, which usually are polynomials on the multidegree (see Section 2).

In dimension 1, the classification of complete intersections follows from the classical theory of Riemann surfaces. In dimension 3, appealing to general classification theorems in differential topology, the classification was established by Jupp [1] and Wall [2], while in dimension 2, the homeomorphism classification has been settled by Freedman [3]. Ebeling [4] and Libgober-Wood [5] independently found examples of homeomorphic complex 2-dimensional complete intersections which are not diffeomorphic. In [6], Fang and Klaus proved that, in dimension $n = 4$, two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are homeomorphic if and only if they have the same total degree, Pontrjagin classes and Euler characteristics. Furthermore, Fang and the first author [7] generalized this homeomorphism result to dimensions $n = 5, 6, 7$.

Theorem 1.1 (see [6–7]) *Two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are homeomorphic if and only if they have the same total degree, Pontrjagin classes and Euler characteristics, provided that $n = 4, 5, 6, 7$.*

Manuscript received November 30, 2014. Revised August 30, 2015.
[1]Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, China.
E-mail: wjianbo@tju.edu.cn
[2]Graduate School, China Academy of Engineering Physics, Beijing 100088, China.
E-mail: 1261708156@qq.com
*This work was supported by the National Natural Science Foundation of China (No. 11001195), the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry and Beiyang Elite Scholar Program of Tianjin University (No. 0903061016).

With the help of Kreck's modified surgery theory (see [8]), Traving [9] obtained partial classification results in higher dimensions under some restrictions on the total degree. Particularly, to the prime factorization of total degree $d = \prod_{p \text{ primes}} p^{\nu_p(d)}$, under the assumption that $\nu_p(d) \geqslant \frac{2n+1}{2(p-1)} + 1$ for all primes p with $p(p-1) \leqslant n+1$, Traving proved the following result (see [8, Theorem A] or [9]).

Theorem 1.2 (see [9]) *Two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ of a complex dimension $n > 2$ fulfilling the assumption above for the total degree are diffeomorphic if and only if the total degrees, the Pontrjagin classes and the Euler characteristics agree.*

It is well known all complete intersections of a fixed multidegree are diffeomorphic. On the other hand, there exist diffeomorphic complete intersections with different multi-degrees. For lower dimensions, such as complex dimensions $2, 3, 4, 5$, the diffeomorphic examples can be found in [10–13].

The aim of this paper is to give examples of diffeomorphic (or homeomorphic) complex n-dimensional complete intersections, mainly $n \leqslant 7$, which can be considered as a geometrical realization of the above Theorems 1.1–1.2. On the other hand, Libgober and Wood [13, §9] conjectured that there exist the diffeomorphic complex 3-dimensional complete intersections with different first Chern classes, which give an example of a disconnected moduli space. In this paper, the above conjecture will be confirmed by one of the listed examples.

These examples were partially found by computer searching. Note that, according to [13, Proposition 7.3], if $X_n(d_1, \cdots, d_r) \subset \mathbb{C}P^{n+r}$ satisfies $n > 2$ and $r \leqslant \frac{n+2}{2}$, then the total degree and Pontrjagin classes of $X_n(d_1, \cdots, d_r)$ determine the multidegree. Thus, it is impossible to find out such a homeomorphic or diffeomorphic example, in which the codimension r is small relative to the complex dimension n. It is worth stating that the degrees and codimensions in our examples will be as small as possible.

2 Characteristic Classes of Complete Intersections

For a complete intersection $X_n(\underline{d})$, let H be the restriction of the dual bundle of the canonical line bundle over $\mathbb{C}P^{n+r}$ to $X_n(\underline{d})$, and $x = c_1(H) \in H^2(X_n(\underline{d}); \mathbb{Z})$. Associate the multidegree $\underline{d} = (d_1, d_2, \cdots, d_r)$ and define the power sums $s_i = \sum_{j=1}^{r} d_j^i$ for $1 \leqslant i \leqslant n$. Then the Chern classes, Pontrjagin classes and Euler characteristic are presented as follows (see [13]):

$$c_k = \frac{1}{k!} g_k(n+r+1-s_1, \cdots, n+r+1-s_k)x^k, \quad 1 \leqslant k \leqslant n;$$
$$p_k = \frac{1}{k!} g_k(n+r+1-s_2, \cdots, n+r+1-s_{2k})x^{2k}, \quad 1 \leqslant k \leqslant \left[\frac{n}{2}\right];$$
$$e = c_n(X_n(\underline{d})) \cap [X_n(\underline{d})] = d\frac{1}{n!} g_n(n+r+1-s_1, \cdots, n+r+1-s_n),$$

where g_k $(k \geqslant 1)$ are the polynomials with k indeterminates $(s_1, s_2, \cdots, s_k)$ that can be iteratively computed from the Newton formula:

$$s_k - g_1(s_1)s_{k-1} + \frac{1}{2} g_2(s_1, s_2)s_{k-2} + \cdots + (-1)^k \frac{1}{k!} g_k(s_1, s_2, \cdots, s_k)k = 0.$$

For example, here are the first seven g_k's.

$$g_1(s_1) = s_1,$$

$$
\begin{aligned}
g_2(s_1,s_2) &= s_1^2 - s_2,\\
g_3(s_1,s_2,s_3) &= s_1^3 - 3s_1s_2 + 2s_3,\\
g_4(s_1,\cdots,s_4) &= s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4,\\
g_5(s_1,\cdots,s_5) &= s_1^5 - 10s_1^3s_2 + 20s_1^2s_3 - 30s_1s_4 + 15s_1s_2^2 - 20s_2s_3 + 24s_5,\\
g_6(s_1,\cdots,s_6) &= s_1^6 - 15s_1^4s_2 + 40s_1^3s_3 - 90s_1^2s_4 + 45s_1^2s_2^2 - 120s_1s_2s_3\\
&\quad + 144s_1s_5 - 15s_2^3 + 90s_2s_4 + 40s_3^2 - 120s_6,\\
g_7(s_1,\cdots,s_7) &= s_1^7 - 21s_1^5s_2 + 70s_1^4s_3 - 210s_1^3s_4 + 105s_1^3s_2^2 - 420s_1^2s_2s_3\\
&\quad + 504s_1^2s_5 - 105s_1s_2^3 + 630s_1s_2s_4 + 280s_1s_3^2 - 840s_1s_6\\
&\quad + 210s_2^2s_3 - 504s_2s_5 - 420s_3s_4 + 720s_7.
\end{aligned}
$$

Note that the k^{th} Chern class c_k and the Pontrjagin class p_k are integral multiples of x^k and x^{2k}, so we can compare these invariants for different complete intersections. For convenience, throughout the rest of the paper, the Chern class c_k and the Pontrjagin class p_k of $X_n(\underline{d})$ are viewed as the multiples of x^k and x^{2k}, respectively.

3 Examples of Diffeomorphic (Homeomorphic) Low-Dimensional Complete Intersections

In this section, some examples of diffeomorphic or homeomorphic complex n-dimensional complete intersections will be listed for $n = 2, 3, \cdots, 7$, and one of the examples will confirm the conjecture of Libgober and Wood [13, §9].

3.1 Complex 2-dimensional complete intersections

For the complex 2-dimensional complete intersection $X_2(d_1,\cdots,d_r)$, the total degree, the first Chern class, the Pontrjagin class and the Euler characteristic are as follows:

$$
\begin{aligned}
d &= d_1 \times \cdots \times d_r,\\
c_1 &= 3 + r - s_1,\\
p_1 &= 3 + r - s_2,\\
e &= \frac{d}{2}[(3+r-s_1)^2 - (3+r-s_2)].
\end{aligned}
$$

Theorem 3.1 (see [4]) *Two complete intersections $X_2(d_1,\cdots,d_r)$ and $X_2(d_1',\cdots,d_s')$ are homeomorphic, if and only if*

$$
\begin{aligned}
d\cdot p_1 &= d'\cdot p_1',\\
e &= e',\\
c_1 &\equiv c_1' \pmod 2.
\end{aligned}
$$

Theorem 3.1 is basically due to Freedman's celebrated work on the topology of 4 manifolds (see [3]), which implies that the homeomorphism type of complex 2-dimensional complete intersections is determined by its intersection form. A more detailed description can be found in [4].

Two homeomorphic complete intersections $X_2(d_1,\cdots,d_r)$ and $X_2(d_1',\cdots,d_s')$ with $c_1 \neq c_1'$ will give an example of homeomorhpic but non-diffeomorphic complex 2-dimensional complete intersections (see [4–5]). Note that Ebeling [4] and Libgober-Wood [5] independently found that $X_2(10,7,7,6,6,3,3)$ and $X_2(9,5,3,3,3,3,3,2,2)$ are homeomorphic but not diffeomorphic. Here, some other examples with lower codimensions are listed.

Example 3.1 The following invariants give seven examples of pairs of complex 2-dimensional complete intersections which are homeomorphic but not diffeomorphic. The two smoothings in each pair have distinct c_1's.

Table 1 Homeomorphic but non-diffeomorphic 2-dimensional complete intersections

$X_2(\underline{d})$	$d \cdot p_1$	e	c_1
$X_2(6,5,3)$	-5760	5760	-8
$X_2(5,2,2,2,2,2)$	-5760	5760	-6
$X_2(14,14,8)$	-705600	1058400	-30
$X_2(14,7,5,5)$	-705600	1058400	-24
$X_2(15,11,10,3)$	-2217600	3643200	-32
$X_2(11,10,5,2,2,2,2)$	-2217600	3643200	-24
$X_2(10,9,7,7,7)$	-9878400	20744640	-32
$X_2(7,7,7,5,2,2,2,2,2)$	-9878400	20744640	-24
$X_2(10,9,9,6,2)$	-2857680	5239080	-28
$X_2(10,7,7,3,3,3)$	-2857680	5239080	-24
$X_2(15,13,7,3,2)$	-3669120	6027840	-32
$X_2(13,7,5,2,2,2,2,2)$	-3669120	6027840	-24
$X_2(10,7,7,6,3,3)$	-6429780	12859560	-27
$X_2(9,5,3,3,3,3,3,2,2)$	-6429780	12859560	-21

3.2 Complex 3-dimensional complete intersections

For a complex 3-dimensional complete intersection $X_3(d_1,\cdots,d_r)$, the related topological characteristic classes are as follows:

$$
\begin{aligned}
d &= d_1 \times \cdots \times d_r,\\
p_1 &= 4 + r - s_2,\\
e &= \frac{d}{6}[(4+r-s_1)^3 - 3(4+r-s_1)(4+r-s_2) + 2(4+r-s_3)],\\
c_1 &= 4 + r - s_1.
\end{aligned}
$$

Theorem 3.2 (see [1–2]) *Two complex 3-dimensional complete intersections are diffeomorphic, if and only if they have the same total degree d, the first Pontrjagin class p_1 and the Euler characteristic e.*

In [13, §9], Libgober and Wood conjectured that there exist such examples of diffeomorphic complex 3-dimensional complete intersections with different first Chern classes, which would give an example of a disconnected moduli space. Brückmann [10] showed that there exist the diffeomorphic complex 3-dimensional complete intersections belonging to components of the moduli space of different dimensions, but with the same first Chern classes. In the following, some examples of diffeomorphic complex 3-dimensional complete intersections will be listed, and the conjecture in [13] is confirmed by Table 3.

Example 3.2 The following invariants give some pairs of diffeomorphic complex 3-dimensional complete intersections, where the ones with different first Chern classes c_1's give an example of a disconnected moduli space.

Table 2 Diffeomorphic 3-dimensional complete intersections with the same c_1

$X_3(\underline{d})$	d	p_1	e	c_1
$X_3(20,20,11,7,4)$	123200	−977	−6974721600	−53
$X_3(22,16,14,5,5)$	123200	−977	−6974721600	−53
$X_3(14,14,5,4,4,4)$	62720	−455	−1068748800	−35
$X_3(16,10,7,7,2,2,2)$	62720	−455	−1068748800	−35

Table 3 Diffeomorphic 3-dimensional complete intersections with different c_1's

$X_3(\underline{d})$	d	p_1	e	c_1
$X_3(70,16,16,14,7,6)$	10536960	−5683	−7767425433600	−119
$X_3(56,49,8,6,5,4,4)$	10536960	−5683	−7767425433600	−121
$X_3(88,28,19,14,6,6)$	23595264	−9147	−35445749391360	−151
$X_3(76,56,11,7,6,6,2)$	23595264	−9147	−35445749391360	−153
$X_3(84,29,25,25,18,7)$	191835000	−9510	−384536710530000	−178
$X_3(60,58,49,9,5,5,5)$	191835000	−9510	−384536710530000	−180

3.3 Complex 4-dimensional complete intersections

For a complex 4-dimensional complete intersection $X_4(d_1,\cdots,d_r)$, we have the following invariants:

$$\begin{aligned} d &= d_1\times\cdots\times d_r,\\ p_1 &= 5+r-s_2,\\ p_2 &= \frac{1}{2}[(5+r-s_2)^2-(5+r-s_4)],\\ e &= \frac{d}{24}[(5+r-s_1)^4-6(5+r-s_1)^2(5+r-s_2)+8(5+r-s_1)(5+r-s_3)\\ &\quad +3(5+r-s_2)^2-6(5+r-s_4)]. \end{aligned}$$

According to Theorem 1.2, to construct homeomorphic complex 4-dimensional complete intersections, we need to find out two different multi-degrees to satisfy that the above invariants all agree. Furthermore, by Theorem 1.2, if $\nu_2(d)>5.5$, then the homeomorphic 4-dimensional complete intersections are diffeomorphic.

Example 3.3 Table 4 gives two pairs of homeomorphic complex 4-dimensional complete intersections.

Table 4 Homeomorphic 4-dimensional complete intersections

$X_4(\underline{d})$	d	p_1	p_2	e
$X_4(66,63,29,23,6,4)$	66561264	−9736	65253028	118373538335553248
$X_4(69,58,36,14,11,3)$	66561264	−9736	65253028	118373538335553248
$X_4(46,44,33,27,17,15,10)$	4598629200	−6472	25986916	546159737882484000
$X_4(45,45,34,23,22,12,11)$	4598629200	−6472	25986916	546159737882484000

Example 3.4 Table 5 gives two pairs of diffeomorphic complex 4-dimensional complete intersections, since the total degrees satisfy

$$488980800 = 2^6 \times 3^4 \times 5^2 \times 7^3 \times 11,$$
$$3953664000 = 2^{13} \times 3^3 \times 5^3 \times 11 \times 13.$$

Table 5 Diffeomorphic 4-dimensional complete intersections

$X_4(\underline{d})$	d	p_1	p_2	e
$X_4(36,33,30,20,14,7,7)$	488980800	−3967	9807916	19704249035856000
$X_4(35,35,28,22,12,9,6)$	488980800	−3967	9807916	19704249035856000
$X_4(52,44,36,25,20,12,8)$	3953664000	−7157	32268711	546278189783040000
$X_4(50,48,32,26,22,10,9)$	3953664000	−7157	32268711	546278189783040000

3.4 Complex 5-dimensional complete intersections

For a complex 5-dimensional complete intersection $X_5(d_1,\cdots,d_r)$, we have the following invariants:

$$\begin{aligned}
d &= d_1 \times \cdots \times d_r,\\
p_1 &= 6 + r - s_2,\\
p_2 &= \frac{1}{2}[(6+r-s_2)^2 - (6+r-s_4)],\\
e &= \frac{1}{5!}d[(6+r-s_1)^5 - 10(6+r-s_1)^3(6+r-s_2) + 20(6+r-s_1)^2(6+r-s_3)\\
&\quad - 30(6+r-s_1)(6+r-s_4) + 15(6+r-s_1)(6+r-s_2)^2\\
&\quad - 20(6+r-s_2)(6+r-s_3) + 24(6+r-s_5)].
\end{aligned}$$

Two complex 5-dimensional complete intersections with the same d, p_1, p_2, e are homeomorphic. Furthermore, by Theorem 1.2, if $\nu_2(d) \geqslant 6.5$ and $\nu_3(d) \geqslant 3.75$, then they will be diffeomorphic.

Example 3.5 Consider the following Table 6.

If the two pairs of multi-degrees have the same power sums $s_1, \cdots, s_5$ and the total degree d, then it is easily deduced that the corresponding complex 5-dimensional complete intersections have the same Pontrjagin classes and the Euler characteristics. Since the total degrees have the following prime factorization,

$$767763360000 = 2^8 \times 3^5 \times 5^4 \times 7^2 \times 13 \times 31,$$
$$14677977600 = 2^9 \times 3^6 \times 5^2 \times 11^2 \times 13,$$

the corresponding two pairs of complete intersections are diffeomorphic.

Table 6 Diffeomorphic 5-dimensional complete intersections

$\underline{d}$	s_1	s_2	s_3	s_4	s_5	d
$(112,93,91,50,45,20,18)$	429	34723	3192813	311347699	31322739669	767763360000
$(108,105,78,62,35,25,16)$	429	34723	3192813	311347699	31322739669	767763360000
$(54,48,30,30,13,11,11,4)$	201	7447	326979	15489571	763263411	14677977600
$(55,44,39,18,18,16,6,5)$	201	7447	326979	15489571	763263411	14677977600

Example 3.6 The following table gives diffeomorphic complete intersections with distinct codimensions, where the diffeomorphism comes from the prime factorization of the total degree: $d = 104626080000 = 2^8 \times 3^7 \times 5^4 \times 13 \times 23$.

Table 7 Diffeomorphic 5-dimensional complete intersections with distinct codimensions

$X_5(\underline{d})$ (codim=9, 10)	d	p_1	p_2	$\frac{e}{d}$
$X_5(52, 50, 30, 27, 23, 18, 6, 5, 4)$	104626080000	−7748	37660770	−12876778992
$X_5(54, 46, 36, 25, 20, 15, 13, 3, 2, 2)$	104626080000	−7748	37660770	−12876778992

3.5 Complex 6-dimensional complete intersections

For a complex 6-dimensional complete intersection $X_6(d_1, \cdots, d_r)$, the related topological characteristic classes are as follows:

$$
\begin{aligned}
d &= d_1 \times \cdots \times d_r, \\
p_1 &= 7 + r - s_2, \\
p_2 &= \frac{1}{2}[(7 + r - s_2)^2 - (7 + r - s_4)], \\
p_3 &= \frac{1}{6}[(7 + r - s_1)^3 - 3(7 + r - s_1)(7 + r - s_2) + 2(7 + r - s_3)], \\
e &= \frac{d}{6!}[(7 + r - s_1)^6 - 15(7 + r - s_1)^4(7 + r - s_2) + 40(7 + r - s_1)^3(7 + r - s_3) \\
&\quad - 90(7 + r - s_1)^2(7 + r - s_4) + 45(7 + r - s_1)^2(7 + r - s_2)^2 \\
&\quad - 120(7 + r - s_1)(7 + r - s_2)(7 + r - s_3) + 144(7 + r - s_1)(7 + r - s_5) \\
&\quad - 15(7 + r - s_2)^3 + 90(7 + r - s_2)(7 + r - s_4) \\
&\quad + 40(7 + r - s_3)^2 - 120(7 + r - s_6)].
\end{aligned}
$$

Two complex 6-dimensional complete intersections with the same d, p_1, p_2, p_3, e are homeomorphic. Furthermore, by Theorem 1.2, if $\nu_2(d) \geqslant 7.5$ and $\nu_3(d) \geqslant 4.25$, then they will be diffeomorphic.

Example 3.7 Consider the following two multi-degrees[1]:

$$
\begin{aligned}
\underline{d} &= (116, 114, 96, 78, 59, 55, 50, 40, 32, 22, 13, 9), \\
\underline{d}' &= (118, 110, 100, 72, 64, 57, 48, 39, 29, 26, 11, 10).
\end{aligned}
$$

They have the same total degree d and power sums $s_1, \cdots, s_7$:

$$
\begin{aligned}
d &= 52\,933\,656\,400\,035\,840\,000, \\
s_1 &= 684, \\
s_2 &= 54\,116, \\
s_3 &= 5\,008\,824, \\
s_4 &= 503\,305\,604, \\
s_5 &= 52\,970\,710\,824,
\end{aligned}
$$

[1] The multi-degrees in Examples 3.7 and 3.9 were introduced by Guo Xianqiang on a mathematical BBS: http://bbs.emath.ac.cn/thread-5853-1-1.html.

$$s_6 = 5\,730\,100\,991\,396,$$
$$s_7 = 630\,552\,267\,588\,024.$$

Then the corresponding two complete intersections have the same total degree, Pontrjagin classes and Euler characteristics, so they are homeomorphic. Since the total degree has the prime factorization $d = 2^{19} \times 3^5 \times 5^4 \times 11^2 \times 13^2 \times 19 \times 29 \times 59$, the corresponding two complete intersections

$$X_6(116, 114, 96, 78, 59, 55, 50, 40, 32, 22, 13, 9),$$
$$X_6(118, 110, 100, 72, 64, 57, 48, 39, 29, 26, 11, 10)$$

are diffeomorphic.

3.6 Complex 7-dimensional complete intersections

For a complex 7-dimensional complete intersection $X_7(d_1, \cdots, d_r)$, the related topological characteristic classes are as follows:

$$\begin{aligned}
p_1 &= 8 + r - s_2,\\
p_2 &= \frac{1}{2}[(8 + r - s_2)^2 - (8 + r - s_4)],\\
p_3 &= \frac{1}{6}[(8 + r - s_1)^3 - 3(7 + r - s_1)(8 + r - s_2) + 2(8 + r - s_3)],\\
e &= \frac{d}{7!}[(8 + r - s_1)^7 - 21(8 + r - s_1)^5(8 + r - s_2) + 70(8 + r - s_1)^4(8 + r - s_3)\\
&\quad - 210(8 + r - s_1)^3(8 + r - s_4) + 105(8 + r - s_1)^3(8 + r - s_2)^2\\
&\quad - 420(8 + r - s_1)^2(8 + r - s_2)(8 + r - s_3) + 504(8 + r - s_1)^2(8 + r - s_5)\\
&\quad - 105(8 + r - s_1)(8 + r - s_2)^3 + 630(8 + r - s_1)(8 + r - s_2)(8 + r - s_4)\\
&\quad + 280(8 + r - s_1)(8 + r - s_3)^2 - 840(8 + r - s_1)(8 + r - s_6)\\
&\quad + 210(8 + r - s_2)^2(8 + r - s_3) - 504(8 + r - s_2)(8 + r - s_5)\\
&\quad - 420(8 + r - s_3)(8 + r - s_4) + 720(8 + r - s_7)].
\end{aligned}$$

Two complex 7-dimensional complete intersections with the same d, p_1, p_2, p_3, e are homeomorphic. Furthermore, by Theorem 1.2, if $\nu_2(d) \geqslant 8.5$ and $\nu_3(d) \geqslant 4.75$, then they will be diffeomorphic.

Example 3.8 Consider the multi-degrees in Example 3.7, it is evident that the corresponding two 7-dimensional complete intersections have the same total degree, Pontrjagin classes and Euler characteristics. Thus complete intersections

$$X_7(116, 114, 96, 78, 59, 55, 50, 40, 32, 22, 13, 9),$$
$$X_7(118, 110, 100, 72, 64, 57, 48, 39, 29, 26, 11, 10)$$

are diffeomorphic by Theorem 1.2.

Example 3.9 Consider the following two multi-degrees:

$$(596, 592, 556, 520, 480, 450, 438, 423, 408, 404, 381, 369, 327, 312, 300)$$
$$(600, 584, 564, 508, 492, 447, 436, 417, 416, 400, 390, 360, 333, 306, 303).$$

They have the same total degree d and power sums: $s_1, \cdots, s_7$:

$$\begin{aligned}
d &= 2\,895\,548\,222\,951\,602\,337\,839\,765\,820\,276\,736\,000\,000,\\
s_1 &= 6\,556,\\
s_2 &= 2\,994\,164,\\
s_3 &= 1\,424\,846\,116,\\
s_4 &= 703\,565\,690\,996,\\
s_5 &= 358\,728\,296\,599\,276,\\
s_6 &= 187\,921\,032\,698\,324\,444,\\
s_7 &= 100\,667\,444\,609\,447\,734\,036.
\end{aligned}$$

Furthermore, $\nu_2(d) = 28$ and $\nu_3(d) = 13$, so the corresponding two complete intersections

$$\begin{aligned}
&X_7(596, 592, 556, 520, 480, 450, 438, 423, 408, 404, 381, 369, 327, 312, 300),\\
&X_7(600, 584, 564, 508, 492, 447, 436, 417, 416, 400, 390, 360, 333, 306, 303)
\end{aligned}$$

are diffeomorphic.

Example 3.10 For the following two multi-degrees[2]:

$$\begin{aligned}
\underline{d} = (&12, 16, 22, 26, 28, 45, 58, 59, 65, 69, 81, 85, 86, 91, 105, 106,\\
&108, 128, 132, 134, 144, 156, 168, 192, 200, 214, 242, 250, 272, 274),\\
\underline{d}' = (&13, 14, 24, 25, 29, 43, 54, 64, 66, 72, 78, 84, 88, 90, 96, 107, 121,\\
&125, 130, 136, 137, 162, 170, 182, 210, 212, 236, 256, 268, 276),
\end{aligned}$$

they have the same total degree d and power sums $s_1, \cdots, s_{11}$:

$$\begin{aligned}
d &= 43\,548\,968\,602\,421\,704\,369\,217\,485\,857\,781\,603\,627\,739\,564\,212\,224\,000\,000\,000,\\
s_1 &= 3\,568,\\
s_2 &= 601\,432,\\
s_3 &= 120\,991\,744,\\
s_4 &= 26\,887\,338\,904,\\
s_5 &= 6\,339\,294\,665\,608,\\
s_6 &= 1\,550\,802\,794\,333\,392,\\
s_7 &= 388\,678\,376\,991\,878\,944,\\
s_8 &= 99\,057\,950\,894\,851\,518\,184,\\
s_9 &= 25\,552\,911\,575\,591\,712\,680\,248,\\
s_{10} &= 6\,651\,777\,099\,183\,876\,642\,569\,152,\\
s_{11} &= 1\,743\,797\,004\,813\,063\,555\,915\,251\,344.
\end{aligned}$$

Then the corresponding two complete intersections are homeomophic. Since $\nu_2(d) = 51$ and $\nu_3(d) = 18$, the corresponding two complete intersections

$$X_7(12, 16, 22, 26, 28, 45, 58, 59, 65, 69, 81, 85, 86, 91, 105, 106,$$

[2] These two multi-degrees firstly appeared in http://www.emath.ac.cn/florilegium/r011n30.htm by Guo Xianqiang.

$$108, 128, 132, 134, 144, 156, 168, 192, 200, 214, 242, 250, 272, 274),$$
$$X_7(13, 14, 24, 25, 29, 43, 54, 64, 66, 72, 78, 84, 88, 90, 96, 107, 121,$$
$$125, 130, 136, 137, 162, 170, 182, 210, 212, 236, 256, 268, 276)$$

are diffeomorphic.

Remark 3.1 Note that, just like Examples 3.7–3.8, for the multi-degrees that appeared in Subsection 3.6, the corresponding complex 6-dimensional complete intersections are also diffeomorphic.

Remark 3.2 For higher-dimensional complete intersections, e.g., $n = 8, 9, 10, 11$, multi-degrees in Example 3.10 are still valid to satisfy that the n-dimensional complete intersections have the same total degree, Pontrjagin classes and Euler characteristic. However, because of the unknown classifications of complete intersections for higher dimensions, we can not give more examples of diffeomorphic or homeomorphic complete intersections.

Acknowledgement The first author would like to thank Guo Xianqiang for his valuable and helpful discussions.

References

[1] Jupp, P., Classification of certain 6-manifolds, *Proc. Cambridge Philos. Soc.,* **73**, 1973, 293–300.

[2] Wall, C. T. C., Classification problems in differential topology. On certain 6-manifolds, *Invent. Math.,* **1**, 1966, 355–374.

[3] Freedman, M., The topology of four-dimensional manifolds, *J. Differential Geom.,* **17**, 1982, 357–453.

[4] Ebeling, W., An example of two homeomorphic, nondiffeomorphic complete intersection surfaces, *Invent. Math.,* **99**(3), 1990, 651–654.

[5] Libgober, A. S. and Wood, J. W., Uniqueness of the complex structure on Kähler manifolds of certain homotopy types, *J. Differential Geom.,* **32**(1), 1990, 139–154.

[6] Fang, F. Q. and Klaus, S., Topological classification of 4-dimensional complete intersections, *Manuscript Math.,* **90**, 1996, 139–147.

[7] Fang, F. Q. and Wang, J. B., Homeomorphism classification of complex projective complete intersections of dimensions 5, 6 and 7, *Math. Z.,* **266**, 2010, 719–746.

[8] Kreck, M., Surgery and duality, *Ann. of Math.,* **149**(3), 1999, 707–754.

[9] Traving, C., Klassification vollständiger Durchschnitte, Diplomarbeit, Mainz, 1985, http://www.info.de /Staff/traving.pdf

[10] Brückmann, P., A remark on moduli spaces of complete intersections, *J. Reine Angew. Math.,* **476**, 1996, 209–215.

[11] Brückmann, P., A remark on moduli spaces of 4-dimensional complete intersections, *J. Reine Angew. Math.,* **525**, 2000, 213–217.

[12] Wang, J. B., Remarks on 5-dimensional complete intersections, *Electron. Res. Announc. Math. Sci.,* **21**, 2014, 28–40.

[13] Libgober, A. S. and Wood, J. W., Differentiable structures on complete intersections, I, *Topology,* **21**, 1982, 469–482.