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Positivity of Fock Toeplitz Operators via the Berezin Transform*

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Abstract This paper deals with the relationship between the positivity of the Fock Toeplitz operators and their Berezin transforms. The author considers the special case of the bounded radial function $\varphi(z) = a + b\mathrm{e}^{-\alpha|z|^2} + c\mathrm{e}^{-\beta|z|^2}$, where a, b, c are real numbers and α, β are positive numbers. For this type of φ, one can choose these parameters such that the Berezin transform of φ is a nonnegative function on the complex plane, but the corresponding Toeplitz operator T_φ is not positive on the Fock space.

Keywords Positive Toeplitz operators, Fock space, Berezin transform
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1 Introduction

Let $\mathrm{d}\mu$ be the Gaussian measure on the complex plane $\mathbb{C}$. It is well-known that, in terms of the standard area measure $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y = \frac{r}{\pi}\mathrm{d}r\mathrm{d}\theta$ on $\mathbb{C}$, we have

$$\mathrm{d}\mu(z) = \frac{1}{2}\mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{d}A(z).$$

Recall that the Fock space $\mathcal{F}^2$ is defined to be the subspace

$$\Big\{f \text{ is analytic on } \mathbb{C} : \|f\|^2 := \int_{\mathbb{C}} |f(z)|^2 \mathrm{d}\mu(z) < +\infty\Big\}.$$

It is easy to check that the functions z^n $(n \geqslant 0)$ are orthogonal in $\mathcal{F}^2$ and their linear span is dense in $\mathcal{F}^2$. Using polar coordinates we get that $\|z^n\|^2 = n!2^n$. Thus

$$\{e_n(z)\}_{n=0}^{\infty} = \Big\{\frac{z^n}{\sqrt{n!2^n}}\Big\}_{n=0}^{\infty}$$

forms an orthonormal basis of the Fock space $\mathcal{F}^2$.

Given φ in $L^\infty(\mathbb{C})$, the Fock Toeplitz operator with symbol φ is defined by

$$T_\varphi f = P(\varphi f),$$

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where $P : L^2(\mathbb{C}, \mathrm{d}\mu) \to \mathcal{F}^2$ is the orthogonal projection. Using the reproducing kernel $K_z(w) = \mathrm{e}^{\frac{\overline{w}z}{2}}$, we express the Toeplitz operator as an integral operator:

$$\begin{aligned} T_\varphi f(z) &= \int_{\mathbb{C}} \varphi(w) f(w) \overline{K_z(w)} \mathrm{d}\mu(w) \\ &= \int_{\mathbb{C}} \varphi(w) f(w) \mathrm{e}^{\frac{\overline{w}z}{2}} \mathrm{d}\mu(w). \end{aligned}$$

For more information on the topics of the Fock space and Fock Toeplitz operators, we refer to [2–3, 9].

As usual, let k_z denote the normalized reproducing kernel for $\mathcal{F}^2$. That is,

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \mathrm{e}^{\frac{\overline{z}w}{2} - \frac{|z|^2}{4}}.$$

For a bounded operator A on the Fock space $\mathcal{F}^2$, the Berezin transform of A is the function $\widetilde{A}$ on the complex plane defined by

$$\widetilde{A}(z) = \langle A k_z, k_z \rangle \quad (z \in \mathbb{C}).$$

For $\varphi \in L^\infty(\mathbb{C})$, $\widetilde{\varphi}$ is called the Berezin transform of φ given by

$$\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = \langle T_\varphi k_z, k_z \rangle = \langle \varphi k_z, k_z \rangle \quad (z \in \mathbb{C}).$$

The Berezin transform is a very useful tool in studying Toeplitz operators on the Bergman space and the Fock space. For instance, the compactness, boundedness, positivity, invertibility and Fredholmness of the Toeplitz operators are partially or completely characterized by their Berezin transforms (please see [1, 4–5, 7–8]).

Recently, the author and Zheng [6] studied the positive Toeplitz operators on the Bergman space via their Berezin transforms. They showed that the positivity of a Toeplitz operator on the Bergman space is not completely determined by the positivity of the Berezin transform of its symbol. Indeed, they constructed a quadratic polynomial of $|z|$ on the unit disk and showed that even if the minimal value of the Berezin transform of the polynomial is positive, the Toeplitz operator with the function as the symbol may not be positive.

Motivated by this result, we try to study the positivity of the Toeplitz operators on the Fock space in the present paper. Observe that from the definition of the Berezin transform we see that the function $\widetilde{\varphi}$ is nonnegative on $\mathbb{C}$ provided that $T_\varphi \geqslant 0$, and $T_\varphi \geqslant 0$ if the function $\varphi(z) \geqslant 0$ for all $z \in \mathbb{C}$. Thus, it is natural to ask the following question.

Question 1.1 Is the positivity of a Fock Toeplitz operator completely determined by the positivity of the Berezin transform of its symbol? If not, then how to construct the "simplest" function φ which satisfies that $\widetilde{\varphi}$ is positive on the complex plane but the Toeplitz operator T_φ is not positive?

As we mentioned above, one can find an example from the set

$$\{\varphi(z) = a|z|^2 + b|z| + c : a, b, c \in \mathbb{R}\}$$

such that T_φ is not positive on the Bergman space but $\widetilde{\varphi}$ is strictly positive on the unit disk. So, it is natural to try to use the radial functions (i.e., $\varphi(z) = \varphi(|z|)$ for all $z \in \mathbb{C}$) of the form $\frac{|z|^n}{\sqrt{n!2^n}}$ $(n \geqslant 0)$ to construct a suitable example and give a negative answer to the above question. However, these simple radial functions are not bounded on the complex plane $\mathbb{C}$. Our main idea in this paper is to simplify our calculations and estimations by the bounded radial functions $\mathrm{e}^{-\lambda|z|^2}$ $(\lambda > 0)$. More precisely, we will consider the bounded radial functions of the following form:

$$\{\varphi(z) = a + b\mathrm{e}^{-\alpha|z|^2} + c\mathrm{e}^{-\beta|z|^2} : a, b, c \in \mathbb{R};\ \alpha, \beta > 0,\ \alpha \neq \beta\}. \tag{1.1}$$

Using the above functions we will give a negative answer to Question 1.1 by selecting the parameters carefully. In fact, if we choose $2\alpha = \beta = 1$, then we can find the real numbers a, b and c such that the Berezin transform $\widetilde{\varphi}$ is positive on $\mathbb{C}$, but the Toeplitz operator T_φ is not positive. The examples and details will be contained in Section 3. Noting that $2\alpha = \beta = 1$ is an isolated case, we still want to use this result to construct a family of examples to answer Question 1.1, so we consider the following question:

Question 1.2 Is there a function φ in (1.1) such that $\widetilde{\varphi}(z) \geqslant 0$ $(\forall z \in \mathbb{C})$ but T_φ is not positive on $\mathcal{F}^2$ for all $2\alpha = \beta > 0$?

Based on the answer to Question 1.1, we will give a negative answer to Question 1.2 by taking $2\alpha = \beta = \frac{1}{2}$ and showing that T_φ is positive if and only if its Berezin transform is a nonnegative function in this case. That is, for this type of α and β, there do not exist a, b, c such that $\widetilde{\varphi}$ is positive but T_φ is not. The proofs will be given in the last section.

2 Preliminaries

It is difficult to study the positivity of the Toeplitz operators on function spaces in the general case even if the symbols are continuous functions. However, if φ is a radial function on $\mathbb{C}$, we can find the relationship between the positivity of T_φ and the Berezin transform $\widetilde{\varphi}$ by its matrix, since the matrix representation of this type of Toeplitz operator is a diagonal matrix under the orthonormal basis.

Lemma 2.1 *Suppose that φ is a bounded radial function on $\mathbb{C}$. Then the matrix representation of the Toeplitz operator T_φ under the basis $\big\{\frac{z^n}{\sqrt{n!2^n}}\big\}_{n=0}^{\infty}$ is*

$$\mathrm{diag}\Big(\Big\{\frac{1}{n!2^n}\int_0^{+\infty} \varphi(r)\mathrm{e}^{-\frac{r^2}{2}} r^{2n+1}\mathrm{d}r\Big\}_{n=0}^{\infty}\Big).$$

In particular, if

$$\varphi(z) = \mathrm{e}^{-\lambda|z|^2}$$

with $\lambda > 0$, then the matrix representation of T_φ is given by

$$\mathrm{diag}\Big(\Big\{\frac{1}{(2\lambda+1)^{n+1}}\Big\}_{n=0}^{\infty}\Big).$$

Proof For each $n \geqslant 0$, we have

$$\begin{aligned} T_\varphi e_n(z) &= \langle T_\varphi e_n, K_z \rangle = \langle \varphi e_n, K_z \rangle \\ &= \int_{\mathbb{C}} \varphi(w) e_n(w) \overline{K_z(w)} \mathrm{d}\mu(w) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{n!2^n}} \int_{\mathbb{C}} \varphi(w) w^n \mathrm{e}^{\frac{\overline{w}z}{2}} \mathrm{e}^{-\frac{|w|^2}{2}} \mathrm{d}A(w) \\ &= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{n!2^n}} \int_0^{2\pi} \int_0^{+\infty} \varphi(r) r^n \mathrm{e}^{\mathrm{i}n\theta} \sum_{n=0}^{\infty} \frac{z^m r^m}{m!2^m} \mathrm{e}^{-\mathrm{i}m\theta} \mathrm{e}^{-\frac{r^2}{2}} r \mathrm{d}r \mathrm{d}\theta \\ &= \frac{1}{\sqrt{n!2^n}} \int_0^{+\infty} \varphi(r) \frac{z^n}{n!2^n} \mathrm{e}^{-\frac{r^2}{2}} r^{2n+1} \mathrm{d}r \\ &= \Big(\frac{1}{n!2^n} \int_0^{+\infty} \varphi(r) \mathrm{e}^{-\frac{r^2}{2}} r^{2n+1} \mathrm{d}r \Big) e_n(z). \end{aligned}$$

If $\varphi(z) = \mathrm{e}^{-\lambda |z|^2}$ ($\lambda > 0$) and $n \geqslant 0$, then we have

$$\begin{aligned} \int_0^{+\infty} \varphi(r) \mathrm{e}^{-\frac{r^2}{2}} r^{2n+1} \mathrm{d}r &= \int_0^{+\infty} \mathrm{e}^{-(\lambda+\frac{1}{2})r^2} r^{2n+1} \mathrm{d}r \\ &= \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-(\lambda+\frac{1}{2})x} x^n \mathrm{d}x \\ &= \frac{1}{2(\lambda+\frac{1}{2})^{n+1}} \int_0^{+\infty} \mathrm{e}^{-x} x^n \mathrm{d}x \\ &= \frac{n!}{2(\lambda+\frac{1}{2})^{n+1}}. \end{aligned}$$

Thus,

$$\begin{aligned} T_\varphi e_n(z) &= \Big(\frac{1}{n!2^n} \times \frac{n!}{2(\lambda+\frac{1}{2})^{n+1}} \Big) e_n(z) \\ &= \frac{1}{(2\lambda+1)^{n+1}} e_n(z) \quad (n \geqslant 0). \end{aligned}$$

This completes the proof of Lemma 2.1.

Note that for the Bergman space case, the Berezin transforms of the functions $|z|^l$ ($l \geqslant 0$) are so complicated even if these functions are very simple (see [6, Lemma 3.3]). However, the Berezin transform of the function $\mathrm{e}^{-\lambda|z|^2}$ ($\lambda > 0$) on $\mathbb{C}$ has a good expression, that is the following lemma, which is very useful for proving our main results.

Lemma 2.2 *Suppose that φ is a bounded radial function on $\mathbb{C}$. Then the Berezin transform of φ is given by*

$$\widetilde{\varphi}(z) = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!2^n)^2} \int_0^{+\infty} \varphi(r) \mathrm{e}^{-\frac{r^2}{2}} r^{2n+1} \mathrm{d}r.$$

In particular, if $\varphi(z) = \mathrm{e}^{-\lambda|z|^2}$ with $\lambda > 0$, then

$$\widetilde{\varphi}(z) = \frac{1}{2\lambda+1} \mathrm{e}^{(\frac{1}{4\lambda+2} - \frac{1}{2})|z|^2}$$

for all $z \in \mathbb{C}$.

Proof By the definition of the Berezin transform, we have

$$\begin{aligned}\widetilde{\varphi}(z) &= \langle \varphi k_z, k_z\rangle = \frac{1}{\|K_z\|^2}\langle \varphi K_z, K_z\rangle \\ &= \mathrm{e}^{-\frac{|z|^2}{2}}\int_{\mathbb{C}}\varphi(w)|K_z(w)|^2\mathrm{d}\mu(w) \\ &= \frac{1}{2}\mathrm{e}^{-\frac{|z|^2}{2}}\int_{\mathbb{C}}\varphi(w)\Big(\sum_{n=0}^{\infty}\frac{\overline{z}^n w^n}{n!2^n}\Big)\Big(\sum_{m=0}^{\infty}\frac{\overline{w}^m z^m}{m!2^m}\Big)\mathrm{e}^{-\frac{|w|^2}{2}}\mathrm{d}A(w) \\ &= \frac{1}{2\pi}\mathrm{e}^{-\frac{|z|^2}{2}}\int_0^{2\pi}\int_0^{+\infty}\varphi(r)\Big(\sum_{n=0}^{\infty}\frac{\overline{z}^n r^n \mathrm{e}^{\mathrm{i}n\theta}}{n!2^n}\Big)\Big(\sum_{m=0}^{\infty}\frac{r^m \mathrm{e}^{-\mathrm{i}m\theta} z^m}{m!2^m}\Big)\mathrm{e}^{-\frac{r^2}{2}}r\mathrm{d}r\mathrm{d}\theta \\ &= \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(n!2^n)^2}\int_0^{+\infty}\varphi(r)\mathrm{e}^{-\frac{r^2}{2}}r^{2n+1}\mathrm{d}r.\end{aligned}$$

For the special case, we have the following simple calculations:

$$\begin{aligned}\int_0^{+\infty}\varphi(r)\mathrm{e}^{-\frac{r^2}{2}}r^{2n+1}\mathrm{d}r &= \int_0^{+\infty}\mathrm{e}^{-(\lambda+\frac{1}{2})r^2}r^{2n+1}\mathrm{d}r \\ &= \frac{n!}{2(\lambda+\frac{1}{2})^{n+1}},\end{aligned}$$

where the last "=" comes from the proof of Lemma 2.1. Therefore,

$$\begin{aligned}\widetilde{\varphi}(z) &= \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(n!2^n)^2}\int_0^{+\infty}\varphi(r)\mathrm{e}^{-\frac{r^2}{2}}r^{2n+1}\mathrm{d}r \\ &= \frac{1}{2\lambda+1}\mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{|z|^2}{4\lambda+2}\Big)^n \\ &= \frac{1}{2\lambda+1}\mathrm{e}^{-\frac{|z|^2}{2}}\cdot \mathrm{e}^{\frac{|z|^2}{4\lambda+2}},\end{aligned}$$

as desired.

Before studying the answer to Question 1.1, we first consider the function φ in (1.1) with $c=0$. Combining the above two lemmas, we get the following result.

Proposition 2.1 *Let $\varphi(z) = a - \mathrm{e}^{-\lambda|z|^2}$, where $a \in \mathbb{R}$ and $\lambda > 0$. Then the following conditions are equivalent:*

(1) *The Fock Toeplitz operator T_φ is positive;*

(2) $a \geqslant \frac{1}{2\lambda+1}$;

(3) *The Berezin transform $\widetilde{\varphi}(z) \geqslant 0$ for all $z \in \mathbb{C}$.*

Proof By Lemma 2.1, we get that the matrix representation of T_φ is

$$\mathrm{diag}\Big(\Big\{a - \frac{1}{(2\lambda+1)^{n+1}}\Big\}_{n=0}^{\infty}\Big).$$

Thus, T_φ is positive if and only if $a \geqslant \frac{1}{(2\lambda+1)^{n+1}}$ for all $n \geqslant 0$, which gives that $a \geqslant \frac{1}{2\lambda+1}$. This proves (1) $\Leftrightarrow$ (2). Now Lemma 2.2 implies

$$\widetilde{\varphi}(z) = a - \frac{1}{2\lambda+1}\mathrm{e}^{-\frac{\lambda}{2\lambda+1}|z|^2}.$$

So we obtain that $\widetilde{\varphi}$ is nonnegative on $\mathbb{C}$ if and only if

$$a - \frac{1}{2\lambda+1}\mathrm{e}^{-\frac{\lambda}{2\lambda+1}|z|^2} \geqslant 0 \quad (\forall z \in \mathbb{C}).$$

This gives that $a \geqslant \frac{1}{2\lambda+1}$, so (3) $\Leftrightarrow$ (2). This completes the proof.

Remarks 2.1 The above proposition tells us that there exists a function $\varphi \in L^\infty(\mathbb{C})$ such that the Fock Toeplitz operator $T_\varphi \geqslant 0$ but φ is not a nonnegative function on $\mathbb{C}$. To see this, we only need to take $\frac{1}{2\lambda+1} \leqslant a < 1$ with $\lambda > 0$ in Proposition 2.1. Furthermore, this result also implies that to give a negative answer to Question 1.1, we only need to consider the case of $b \neq 0$ and $c \neq 0$ in (1.1).

3 A Negative Answer to Question 1.1

In this section, we construct a function φ such that the Berezin transform of φ is positive on the complex plane but the corresponding Fock Toeplitz operator is not positive. To do so, we first consider the case $2\alpha = \beta = 1$. Without loss of generality, we may assume $c = 1$. Then we have the following theorem.

Theorem 3.1 *Suppose that $\varphi(z) = a + b\mathrm{e}^{-\frac{|z|^2}{2}} + \mathrm{e}^{-|z|^2}$, where $a, b \in \mathbb{R}$. Then*

(1) *if $b \leqslant -\frac{8}{9}$ or $b \geqslant 0$, then T_φ is positive if and only if $\widetilde{\varphi}(z)$ is a nonnegative function on the complex plane;*

(2) *for each $b \in (-\frac{8}{9}, -\frac{1}{2}]$, there exists a real number a such that $\widetilde{\varphi}(z) \geqslant 0$ for all $z \in \mathbb{C}$, but T_φ is not a positive Toeplitz operator.*

Proof Use Lemmas 2.1–2.2 and let $\lambda = \frac{1}{2}, 1$, respectively. Then we obtain that the matrix representation of T_φ is given by

$$\mathrm{diag}\Big(\Big\{a + \frac{b}{2^{n+1}} + \frac{1}{3^{n+1}}\Big\}_{n=0}^{\infty}\Big)$$

and the Berezin transform of φ is

$$\widetilde{\varphi}(z) = a + \frac{b}{2}\mathrm{e}^{-\frac{|z|^2}{4}} + \frac{1}{3}\mathrm{e}^{-\frac{|z|^2}{3}} \quad (z \in \mathbb{C}).$$

These imply that $T_\varphi \geqslant 0$ if and only if

$$a + \frac{b}{2^{n+1}} + \frac{1}{3^{n+1}} \geqslant 0$$

for all $n \geqslant 0$, and $\widetilde{\varphi}(z) \geqslant 0$ ($\forall z \in \mathbb{C}$) if and only if

$$a + \frac{b}{2}\mathrm{e}^{-\frac{|z|^2}{4}} + \frac{1}{3}\mathrm{e}^{-\frac{|z|^2}{3}} \geqslant 0$$

for all $z \in \mathbb{C}$. Therefore, $T_\varphi \geqslant 0$ is equivalent to

$$a \geqslant -\inf_{n\geqslant 1}\Big\{\frac{b}{2^n} + \frac{1}{3^n}\Big\}.$$

Letting $t = \mathrm{e}^{-\frac{|z|^2}{12}} \in [0, 1]$, then $\widetilde{\varphi}(z) \geqslant 0$ ($\forall z \in \mathbb{C}$) is equivalent to

$$a \geqslant -\min_{0\leqslant t\leqslant 1}\Big(\frac{bt^3}{2} + \frac{t^4}{3}\Big).$$

First, we prove part (1) of the above theorem. Now we are going to determine the minimal value of the function $\frac{bt^3}{2}+\frac{t^4}{3} := f(t)$ $(0 \leqslant t \leqslant 1)$ and the minimal term of the sequence $\{\frac{b}{2^n}+\frac{1}{3^n}\}_{n\geqslant 1}$. To do this, we consider the following two cases.

Case I Suppose that $b \geqslant 0$. It is easy to see that $f(t)$ is increasing on $[0,1]$ and $\frac{b}{2^n}+\frac{1}{3^n}$ is decreasing for $n \geqslant 1$. So we have

$$\min_{0\leqslant t\leqslant 1} f(t) = f(0) = 0$$

and

$$\inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\frac{1}{3^n}\right\} = 0.$$

Case II Suppose $b \leqslant -\frac{8}{9}$. A simple calculation gives that $f(t)$ is decreasing if $0 \leqslant t < -\frac{9b}{8}$ and $f(t)$ is increasing if $t > -\frac{9b}{8}$. Note that $1 \leqslant -\frac{9b}{8}$, so

$$\min_{0\leqslant t\leqslant 1} f(t) = f(1) = \frac{b}{2}+\frac{1}{3}.$$

To find the minimal term of the sequence, observe that $\frac{b}{2}+\frac{1}{3} < 0$ and we claim that $\frac{b}{2}+\frac{1}{3}$ is the minimal term. Indeed, if there exists some $N > 1$ such that

$$\frac{b}{2}+\frac{1}{3} > \frac{b}{2^N}+\frac{1}{3^N},$$

then we obtain

$$-\frac{8}{9} \geqslant b > -\frac{\frac{1}{3}-\frac{1}{3^N}}{\frac{1}{2}-\frac{1}{2^N}},$$

which shows that

$$2^{3-N} - 3^{2-N} > 1$$

for some $N \geqslant 2$. However, this is impossible. The contradiction shows that

$$\inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\frac{1}{3^n}\right\} = \frac{b}{2}+\frac{1}{3}.$$

Combining the above two cases gives the conclusion in part (1).

For the second part, we turn to consider $b \in (-\frac{8}{9}, -\frac{1}{2}]$. First, observe that T_φ is not positive if and only if

$$a < -\inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\frac{1}{3^n}\right\}.$$

To finish the proof, it suffices to show that for each $b \in (-\frac{8}{9}, -\frac{1}{2}]$, there exists a real constant a such that

$$a \geqslant -\min_{0\leqslant t\leqslant 1}\left(\frac{bt^3}{2}+\frac{t^4}{3}\right)$$

and T_φ is not positive. As shown above, we need the following inequality:

$$-\min_{0\leqslant t\leqslant 1}\left(\frac{bt^3}{2}+\frac{t^4}{3}\right) \leqslant a < -\inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\frac{1}{3^n}\right\}.$$

Indeed, we will show that

$$\min_{0\leqslant t\leqslant 1}\left(\frac{bt^3}{2}+\frac{t^4}{3}\right) > \inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\frac{1}{3^n}\right\}$$

for each $b \in (-\frac{8}{9}, -\frac{1}{2}]$. To do this, we first find the minimal value of $f(t)$. Note that $0 < -\frac{9b}{8} < 1$ if $-\frac{8}{9} < b \leqslant -\frac{1}{2}$. Using the monotonicity of the function f (see the argument in Case II), we get that

$$\min_{0 \leqslant t \leqslant 1} f(t) = f\Big(-\frac{9b}{8}\Big) = -\frac{b^4}{8} \cdot \Big(\frac{9}{8}\Big)^3.$$

On the other hand, we have the following inequality:

$$-\frac{b^4}{8} \cdot \Big(\frac{9}{8}\Big)^3 > \frac{b}{4} + \frac{1}{9} \geqslant \inf_{n \geqslant 1} \Big\{\frac{b}{2^n} + \frac{1}{3^n}\Big\}$$

for each $b \in \big(-\frac{8}{9}, -\frac{1}{2}\big]$. Indeed, if we consider the function

$$F(x) := 2x + \Big(\frac{9}{8}\Big)^3 x^4 \quad \Big(-\frac{8}{9} < x \leqslant -\frac{1}{2}\Big),$$

then the first inequality follows immediately from the mean value theorem. This finishes the proof of Theorem 3.1.

One may ask that if we remove the condition $2\alpha = \beta$, can we easily construct an example from (1.1) to give a negative answer to Question 1.1? Actually, we will show that the answer is yes by the following corollary, which can be proved by the same method as the one in the above theorem, so we omit its proof here.

Corollary 3.1 *Let $\varphi(z) = a + b\mathrm{e}^{-\frac{|z|^2}{2}} + \mathrm{e}^{-3\frac{|z|^2}{2}}$, where $a, b \in \mathbb{R}$. Then for each $b \in \big(-\frac{3}{4}, -\frac{1}{2}\big]$, there exists a real number a such that $\widetilde{\varphi}(z) \geqslant 0$ for all $z \in \mathbb{C}$, but T_φ is not a positive Toeplitz operator on $\mathcal{F}^2$.*

4 A Negative Answer to Question 1.2

In Section 3, we study the radial function $\varphi(z) = a + b\mathrm{e}^{-\alpha|z|^2} + c\mathrm{e}^{-\beta|z|^2}$ $(a, b, c \in \mathbb{R},\ \alpha, \beta > 0)$ and choose real numbers a, b and c such that $\widetilde{\varphi}$ is nonnegative on $\mathbb{C}$ and T_φ is not positive under the assumption $2\alpha = \beta = 1$. In the final section, we will show that this is not true for all $2\alpha = \beta > 0$. More precisely, we have the following theorem, which gives a negative answer to Question 1.2 with the condition $2\alpha = \beta = \frac{1}{2}$.

Theorem 4.1 *Suppose that $\varphi(z) = a + b\mathrm{e}^{-\frac{|z|^2}{2}} + \mathrm{e}^{-\frac{|z|^2}{4}}$, where $a, b \in \mathbb{R}$. Then T_φ is positive if and only if $\widetilde{\varphi}(z)$ is a nonnegative function on $\mathbb{C}$.*

Proof By Lemma 2.1, we see that the matrix representation of T_φ is

$$\mathrm{diag}\Big(\Big\{a + \frac{b}{2^{n+1}} + \Big(\frac{2}{3}\Big)^{n+1}\Big\}_{n=0}^{\infty}\Big).$$

So, $T_\varphi \geqslant 0$ if and only if

$$a \geqslant -\inf_{n \geqslant 1} \Big\{\frac{b}{2^n} + \Big(\frac{2}{3}\Big)^n\Big\}.$$

On the other hand, using Lemma 2.2 we obtain that $\widetilde{\varphi}(z) \geqslant 0$ for all $z \in \mathbb{C}$ if and only if

$$a \geqslant -\frac{b}{2}\mathrm{e}^{-\frac{|z|^2}{4}} - \frac{2}{3}\mathrm{e}^{-\frac{|z|^2}{6}}$$

for all $z \in \mathbb{C}$.

Letting $t = \mathrm{e}^{-\frac{|z|^2}{12}} \in [0,1]$ and $g(t) := \frac{bt^3}{2} + \frac{2t^2}{3}$, we see that the above condition on the positivity of $\widetilde{\varphi}$ is equivalent to

$$a \geqslant -\min_{0\leqslant t\leqslant 1} g(t).$$

Using the same idea as the one in the proof of Theorem 3.1, we divide the proof into the following four cases.

Case I We first consider the case $b \geqslant 0$. It is easy to see that

$$\min_{0\leqslant t\leqslant 1} g(t) = g(0) = 0 = \inf_{n\geqslant 1}\Big\{\frac{b}{2^n} + \Big(\frac{2}{3}\Big)^n\Big\},$$

which shows that T_φ is positive if and only if $\widetilde{\varphi}$ is a nonnegative function on $\mathbb{C}$.

Case II Suppose that $-\frac{8}{9} \leqslant b < 0$. It is easy to check that $g(t)$ is increasing if $0 \leqslant t \leqslant -\frac{8}{9b}$ and decreasing if $t > -\frac{8}{9b}$. Observe that $-\frac{8}{9b} \geqslant 1$, so we have

$$\min_{0\leqslant t\leqslant 1} g(t) = g(0) = 0.$$

On the other hand, we have

$$\begin{aligned}\Big[\frac{b}{2^n} + \Big(\frac{2}{3}\Big)^n\Big] - \Big[\frac{b}{2^{n+1}} + \Big(\frac{2}{3}\Big)^{n+1}\Big] &= \frac{b}{2^{n+1}} + \frac{1}{3}\cdot\Big(\frac{2}{3}\Big)^n \\ &\geqslant \frac{1}{3}\cdot\Big(\frac{2}{3}\Big)^n - \frac{8}{9}\cdot\frac{1}{2^{n+1}} \\ &= \frac{1}{3}\cdot\Big(\frac{2}{3}\Big)^n\Big[1 - \Big(\frac{3}{4}\Big)^{n-1}\Big] \\ &\geqslant 0\end{aligned}$$

for all $n \geqslant 1$ provided that $b \geqslant -\frac{8}{9}$. This implies that the sequence $\{\frac{b}{2^n} + (\frac{2}{3})^n\}_{n\geqslant 1}$ is decreasing, so the minimal term is $\lim_{n\to\infty}(\frac{b}{2^n} + (\frac{2}{3})^n) = 0$, as desired.

Case III In this case, we consider $-\frac{4}{3} \leqslant b < -\frac{8}{9}$. Note that $0 < -\frac{8}{9b} < 1$ and

$$0 \leqslant \frac{b}{2} + \frac{2}{3} = g(1).$$

It follows from the argument in Case II that

$$\min_{0\leqslant t\leqslant 1} g(t) = g(0) = 0.$$

Also, it is easy to see that $\frac{b}{2^n} + \Big(\frac{2}{3}\Big)^n \geqslant 0$ for all $n \geqslant 1$ if $b \geqslant -\frac{4}{3}$. This gives that

$$\inf_{n\geqslant 1}\Big\{\frac{b}{2^n} + \Big(\frac{2}{3}\Big)^n\Big\} = 0 = \min_{0\leqslant t\leqslant 1} g(t).$$

Case IV Finally, we deal with the case $b < -\frac{4}{3}$. Then $0 < -\frac{8}{9b} < 1$ and $\frac{b}{2} + \frac{2}{3} < 0 = g(0)$. It follows that

$$\min_{0\leqslant t\leqslant 1} g(t) = g(1) = \frac{b}{2} + \frac{2}{3}.$$

Now using the same techniques as the one in Case II of Theorem 3.1, we see that the minimal term of the above sequence is the first term $\frac{b}{2}+\frac{2}{3}$. We conclude that

$$\min_{0\leqslant t\leqslant 1} g(t)=\inf_{n\geqslant 1}\left\{\frac{b}{2^n}+\left(\frac{2}{3}\right)^n\right\}$$

for all $b\in\mathbb{R}$. This completes the whole proof.

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