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Symmetries and Their Lie Algebra of a Variable Coefficient Korteweg-de Vries Hierarchy*

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Abstract Isospectral and non-isospectral hierarchies related to a variable coefficient Painlevé integrable Korteweg-de Vries (KdV for short) equation are derived. The hierarchies share a formal recursion operator which is not a rigorous recursion operator and contains t explicitly. By the hereditary strong symmetry property of the formal recursion operator, the authors construct two sets of symmetries and their Lie algebra for the isospectral variable coefficient Korteweg-de Vries (vcKdV for short) hierarchy.

Keywords vcKdV hierarchies, Symmetries, Lie algebra
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1 Introduction

Nonlinear evolution equations with variable coefficients play important roles in applications, as in inhomogeneous plasmas, optical fibers, viscous fluids and Bose-Einstein condensates. Usually, these equations are not integrable, or are only nearly integrable. Although there is no exact definition for what the integrability is, there are many approaches to getting clues, such as integrable characteristics, which link a nonlinear system to being integrable. These integrable characteristics, including passing the Painlevé test, having a Lax pair, having multi-Hamiltonian structures, infinitely many symmetries, infinitely many conserved quantities, having bilinear forms, multi-soliton solutions, and so on, are deeply linked to each other. Let us take the following vcKdV equation:

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0 \tag{1.1}$$

as an example. This equation was first proposed by Grimshaw [1] in 1979 and has been widely studied. As far as the integrability is concerned, the vcKdV (1.1) can pass the Painlevé test under the condition, given by Joshi [2] in 1987,

$$g(t) = af(t)S(t), \quad S(t) = b + \int^t f(t)\mathrm{d}t, \tag{1.2}$$

where a and b are real constants and $a \neq 0$. This is also the condition when the vcKdV equation (1.1) has a Lax pair, a bilinear form, N-soliton like solutions and infinitely many conservation

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laws (see [3–5]). There are also many results (see [6–7]) on the integrability of (1.1) with special forms of $f(t)$ and $g(t)$ which agree with (1.2).

In this paper, we would like to construct the infinitely many symmetries of the vcKdV (1.1) under the condition (1.2). We will first start from the Lax pair of (1.1) to derive isospectral and non-isospectral hierarchies and the related formal recursion operator (which is not a rigorous recursion operator and contains t explicitly). Then we discuss the hereditary and strong symmetry properties of the formal recursion operator. By these properties we can construct a Lie algebraic structure of adjoint flows, and finally we get, for the isospectral vcKdV hierarchy, two sets of symmetries, which also form a Lie algebra.

The paper is organized as follows. In Section 2, we introduce some basic notions. In Section 3, we derive isospectral and non-isospectral vcKdV hierarchies. Finally, we investigate the symmetries and their Lie algebra for the isospectral vcKdV hierarchy.

2 Basic Notions

Let us first recall some basic notions and properties related to symmetries.

Suppose that V is a function space consisting of scalar functions $f(t,x)$ which are C^∞ differentiable with respect to t and x, the functions $u =: u(t,x), K(u) =: K(t,x,u,u_x,u_{xx},\cdots) \in V$, and $\Phi =: \Phi(t,x,u)$ is an operator living on V. For the functions $f(u), g(u) \in V$ and the operator Φ, the Gâteaux derivatives of f and Φ in the direction h w.r.t. u are defined as

$$f'[h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(u+\epsilon h)\Big|_{\epsilon=0}, \quad \Phi'[h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(u+\epsilon h)\Big|_{\epsilon=0}.$$

For $f(u), g(u) \in V$, we define the product (a commutator)

$$[\![f,g]\!] = f'[g] - g'[f]. \tag{2.1}$$

By the commutator we define the symmetry, $\tau =: \tau(t,x,u)$, of the nonlinear evolution equation

$$u_t = K(u), \tag{2.2}$$

if

$$\frac{\widetilde{\partial}\tau}{\widetilde{\partial}t} = [\![K,\tau]\!], \tag{2.3}$$

where by $\frac{\widetilde{\partial}\tau}{\widetilde{\partial}t}$ we specially denote the derivative of τ with respect to t explicitly included in τ. If τ_1 and τ_2 are symmetries of (2.2), then $[\![\tau_1,\tau_2]\!]$ is also a symmetry for (2.2).

The operator Φ is called a strong symmetry (see [8]) of the evolution equation (2.2), if

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = [K',\Phi] = K'\Phi - \Phi K'. \tag{2.4}$$

That Φ is a strong symmetry of (2.2) means that if τ is a symmetry of (2.2), so is $\Phi\tau$. If Φ satisfies

$$\Phi'[\Phi f]g - \Phi'[\Phi g]f = \Phi(\Phi'[f]g - \Phi'[g]f), \quad \forall f,g \in V, \tag{2.5}$$

then Φ is called a hereditary operator (see [8–9]). For a operator Φ which satisfies $\Phi 0 = 0$ and does not explicitly contain t $\big($that is, $\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} = 0$, and then in (2.4) $\frac{\mathrm{d}\Phi}{\mathrm{d}t}$ goes to $\Phi'[K(u)]\big)$, if Φ is hereditary and is a strong symmetry of (2.2), then it is a strong symmetry for all the equations (see [8])

$$u_t = \Phi^n K(u), \quad n = 0, 1, \cdots. \tag{2.6}$$

Such Φ is referred to as a strong hereditary symmetry (see [8]) for the hierarchy (2.6). For convenience of later use, we redescribe the property by the following proposition.

Theorem 2.1 *If Φ is a hereditary operator satisfying $\Phi 0 = 0$ and also*

$$\Phi'[K(u)] = [K(u)', \Phi], \tag{2.7}$$

then

$$\Phi'[\Phi^n K(u)] = [\{\Phi^n K(u)\}', \Phi], \quad n = 0, 1, 2, \cdots. \tag{2.8}$$

3 Isospectral and Non-isospectral vcKdV Hierarchies

In this section, we derive the isospectral and non-isospectral vcKdV hierarchies. We start from the spectral problem (see [5])

$$\phi_{xx} = \frac{1}{S^2(t)}\Big(\lambda + \frac{x}{6a} - \frac{S(t)}{6a}u\Big)\phi \tag{3.1a}$$

with the time evolution

$$\phi_t = A\phi + B\phi_x. \tag{3.1b}$$

The compatibility condition $\phi_{xxt} = \phi_{txx}$ yields

$$\begin{aligned}
&2A_x + B_{xx} = 0,\\
&\Big(\frac{A}{S^2(t)} - 2\frac{f(t)}{S^3(t)}\Big)\Big(\lambda + \frac{x}{6a} - \frac{S(t)}{6a}u\Big) + \frac{1}{S^2(t)}\Big(\lambda_t - \frac{f(t)}{6a}u - \frac{S(t)}{6a}u_t\Big)\\
= {}& A_{xx} + \frac{A + 2B_x}{S^2(t)}\Big(\lambda + \frac{x}{6a} - \frac{S(t)}{6a}u\Big) + \frac{B}{S^2(t)}\Big(\frac{1}{6a} - \frac{S(t)}{6a}u_x\Big).
\end{aligned}$$

Further, we have

$$u_t = \frac{3a}{S(t)}\Psi B - \frac{2f(t)}{S^2(t)}(x - S(t)u) - \frac{uf(t)}{S(t)} - \frac{12a(S(t)B_x + f(t))}{S^2(t)}\lambda + \frac{6a}{S(t)}\lambda_t, \tag{3.2}$$

where Ψ is an operator defined by

$$\Psi = S^2(t)\partial^3 + \frac{2(S(t)u - x)\partial}{3a} + \frac{S(t)u_x - 1}{3a}. \tag{3.3}$$

Substituting the expansion

$$B = \sum_{j=0}^{n} b_j \lambda^{n-j}, \quad n = 1, 2, \cdots \tag{3.4}$$

into (3.2) and comparing the coefficients of the same powers of λ yield

$$u_t = \frac{3a}{S(t)}\Psi b_n + \frac{2f(t)}{S^2(t)}(S(t)u - x) - \frac{uf(t)}{S(t)}, \tag{3.5a}$$

$$b_{n,x} = \frac{1}{4}\Psi b_{n-1} - \frac{f(t)}{S(t)}, \tag{3.5b}$$

$$b_{j+1,x} = \frac{1}{4}\Psi b_j, \quad j = 0,\ 1,\ 2,\ \cdots,\ n-2, \tag{3.5c}$$

$$b_{0,x} = 0. \tag{3.5d}$$

Taking $\lambda_t = 0$ and

$$b_0 = \frac{-4^n a\, f(t)}{S(t)},$$

we have

$$b_{j+1,x} = \Big(\frac{-af(t)}{S(t)}\Big)\frac{4^{n-j}}{12a}\Phi^j(S(t)u_x - 1), \quad j = 0, 1, \cdots, n-2, \tag{3.6a}$$

$$b_{n,x} = \Big(\frac{-af(t)}{S(t)}\Big)\frac{1}{3a}\Phi^{n-1}(S(t)u_x - 1) - \frac{f(t)}{S(t)}, \tag{3.6b}$$

where

$$\Phi = S^2(t)\partial^2 + \frac{2(S(t)u - x)}{3a} + \frac{S(t)u_x - 1}{3a}\partial^{-1}. \tag{3.7}$$

Then we get the isospectral vcKdV hierarchy

$$u_t = G_n = Z(t)\Phi^n K_0 + \Delta, \tag{3.8}$$

where

$$K_0 = S(t)u_x - 1, \quad Z(t) = \frac{-af(t)}{S^2(t)}, \quad \Delta = -\frac{uf(t)}{S(t)} - \frac{xf(t)(S(t)u_x - 1)}{S^2(t)}. \tag{3.9}$$

This hierarchy can be formally extended to starting from $n = 0$. When $n = 1$, it just gives the vcKdV equation (1.1). Besides, the Lax pair of vcKdV (1.1) is provided by (3.1) with

$$A = \frac{f(t)}{6S(t)}(S(t)u_x - 1), \quad B = \frac{-f(t)}{S(t)}\Big(\frac{2x}{3} + \frac{S(t)u}{3} + 4a\lambda\Big). \tag{3.10}$$

In the non-isospectral case, we suppose that

$$\lambda_t = \frac{1}{2}\Big(\frac{-af(t)}{S^2(t)}\Big)(4\lambda)^{n+1}. \tag{3.11}$$

In this case, still using the expansion (3.4) but taking $b_0 = \frac{-af(t)}{S^2(t)}(4\lambda)^n x$, similar to the isospectral case, we can have the following non-isospectral vcKdV hierarchy:

$$u_t = W_n = Z(t)\Phi^n\sigma_0 + \Delta, \quad n = 0, 1, \cdots, \tag{3.12}$$

where

$$\sigma_0 = \frac{2(S(t)u - x) + x(S(t)u_x - 1)}{S(t)}. \tag{3.13}$$

Now we have obtained the isospectral vcKdV hierarchy (3.8) and the non-isospectral hierarchy (3.12). Besides, as by-products, we get two sets of flows,

$$K_n = \Phi^n K_0, \quad \sigma_n = \Phi^n \sigma_0, \quad n = 0, 1, \cdots, \tag{3.14}$$

which we call adjoint flows in this paper. The recursion operator Φ for the adjoint flows is not a rigorous recursion operator, but a formal one for the vcKdV hierarchies.

4 Symmetries and Their Lie Algebraic Structures

In this section, we will derive two sets of symmetries for the isospectral vcKdV hierarchy (3.8), and we will also prove that these two sets of symmetries form a Lie algebra.

Our tactic goes as follows. First, we prove that the formal recursion operator (3.7) is a hereditary operator and is further a strong symmetry for the isospectral vcKdV hierarchy (3.8). Next we show that the adjoint flows $\{K_n\}$ and $\{\sigma_n\}$ form a Lie algebra with respect to the commutator (2.1). Then we prove that the arbitrary member $u_t = G_l$ in the hierarchy (3.8) has two ground symmetries K_0 and τ_0^l, and also we can get two sets of symmetries by acting Φ. Finally we show that the obtained symmetries form a Lie algebra. In fact, there are many ways for deriving two sets of symmetries (usually called K-symmetries and τ-symmetries) starting from a Lax pair (see [10–17]). Our tactic copies these ideas more or less, while the procedure contains some generalization and specialization since Φ contains t explicitly and is not a rigorous recursion operator.

4.1 The strong hereditary symmetry Φ

Let us start from the following lemmas related to the operator Φ given by (3.7).

Lemma 4.1 Φ *is a strong symmetry of the first equation in* (3.8), *that is,*

$$u_t = G_0 = -\frac{af(t)}{S^2(t)}(S(t)u_x - 1) - \frac{uf(t)}{S(t)} - \frac{xf(t)(S(t)u_x - 1)}{S^2(t)}. \tag{4.1}$$

In fact, by a direct calculation we can find that Φ and G_0 satisfy

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = [{G_0}', \Phi], \tag{4.2}$$

where we should make use of the fact $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = f(t)$ and the expression of u_t given by (4.1).

By the similar direct verification (but here we skip the tedious process), we find the following results.

Lemma 4.2 Φ *is a hereditary operator satisfying* (2.5).

Lemma 4.3 Φ *satisfies*

$$\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} - \Phi\,\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} = [(\Delta - \Phi\Delta)', \Phi] - \Phi'[\Delta - \Phi\Delta], \tag{4.3}$$

where Δ *is given in* (3.9).

With these lemmas in hand, we can reach the final results of this subsection.

Theorem 4.1 *The operator* Φ *given by* (3.7) *is a strong hereditary symmetry for the isospectral vcKdV hierarchy* (3.8).

Proof We prove the theorem by the reductive method. By virtue of Lemma 4.1, we suppose that Φ is a strong symmetry of the equation $u_t = G_n$ satisfying

$$\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} + \Phi'[G_n] = [G_n', \Phi], \tag{4.4}$$

and we next go to prove

$$\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} + \Phi'[G_{n+1}] = [G_{n+1}', \Phi]. \tag{4.5}$$

Since

$$G_{n+1} = \Phi G_n + \Delta - \Phi\Delta, \tag{4.6}$$

(4.5) becomes

$$\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} + \Phi'[\Phi G_n + \Delta - \Phi\Delta] = [(\Phi G_n + \Delta - \Phi\Delta)',\, \Phi].$$

Further,

$$\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} + \Phi'[\Phi G_n] - [(\Phi G_n)',\, \Phi] = [(\Delta - \Phi\Delta)',\, \Phi] - \Phi'[\Delta - \Phi\Delta].$$

So by virtue of (4.3), we only need to prove

$$\Big(\Phi\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t} + \Phi'[\Phi G_n] - [(\Phi G_n)',\, \Phi]\Big)g = 0, \tag{4.7}$$

where g is an arbitrary function. Using (4.4) to replace $\frac{\widetilde{\partial}\Phi}{\widetilde{\partial}t}$ and then making use of the formula

$$(\Phi a)'[b] = \Phi'[b]\, a + \Phi \circ a'[b], \tag{4.8}$$

(4.7) becomes

$$\Phi(\Phi'[G_n]g - \Phi'[g]G_n) - \Phi'[\Phi G_n]g + \Phi'[\Phi g]G_n = 0,$$

which is true due to the hereditariness of Φ. We note that in (4.8), $\Phi \circ a'[b]$ specially means that Φ applies to the function $a'[b]$. The proof is completed.

4.2 Lie algebra of the adjoint flows

In this subsection, we discuss the algebra relationship of the adjoint flows $\{K_n\}$ and $\{\sigma_m\}$ which were given in (3.14).

Let us start from the relations of Φ and the flows $\{K_n\}$ and $\{\sigma_m\}$.

Lemma 4.4 *For the adjoint flows* $\{K_n\}$ *and* $\{\sigma_m\}$*, we have*

$$\Phi'[K_n] = [K_n', \Phi] \tag{4.9}$$

and

$$\Phi'[\sigma_n] = [\sigma_n', \Phi] + 2\Phi^{n+1}. \tag{4.10}$$

Proof We only prove (4.10). From

$$\sigma_0 = \frac{2(S(t)u - x) + x(S(t)u_x - 1)}{S(t)},$$

it is easy to see that

$$\Phi'[\sigma_0] = [\sigma_0', \Phi] + 2\Phi. \tag{4.11}$$

Then (4.10) holds for $n = 0$. Now we suppose that (4.10) holds for $n = k$, as in the following form

$$\Phi'[\sigma_k] = [\sigma_k', \Phi] + 2\Phi^{k+1}. \tag{4.12}$$

Then, by the formula (4.8), for arbitrary $\nu \in V$, we have

$$\begin{aligned}
&(\Phi'[\sigma_{k+1}] - [\sigma_{k+1}', \Phi])\nu \\
&= \Phi'[\Phi\sigma_k]\nu - \Phi'[\Phi\nu]\sigma_k - \Phi \circ \sigma_k'[\Phi\nu] + \Phi \circ \Phi'[\nu]\sigma_k + \Phi \circ \Phi\sigma_k'[\nu] \\
&= \Phi(\Phi'[\sigma_k]\nu - \Phi'[\nu]\sigma_k) - \Phi \circ \sigma_k'[\Phi\nu] + \Phi \circ \Phi'[\nu]\sigma_k + \Phi \circ \Phi\sigma_k'[\nu] \\
&= \Phi(\Phi'[\sigma_k] - [\sigma_k', \Phi])\nu \\
&= 2\Phi^{k+2}\nu.
\end{aligned}$$

Thus we have completed the proof.

Now we can check some relations between simple adjoint flows. By calculations we find

$$\begin{aligned}
&[\![K_0, K_1]\!] = 0, \quad [\![K_0, \sigma_0]\!] = K_0, \quad [\![K_0, \sigma_1]\!] = K_1, \\
&[\![K_1, \sigma_0]\!] = 3K_1, \quad [\![K_1, \sigma_1]\!] = 3K_2, \quad [\![\sigma_1, \sigma_0]\!] = 2\sigma_1.
\end{aligned}$$

Starting with these relations and using Lemma 4.4, by the reductive method, we can prove the following general relations.

Theorem 4.2 *The adjoint flows* $\{K_n\}$ *and* $\{\sigma_n\}$ *form a Lie algebra with respect to the commutator* (2.1) *of the following structure:*

$$[\![K_m, K_n]\!] = 0, \tag{4.13a}$$

$$[\![K_m, \sigma_n]\!] = (2m+1)K_{m+n}, \tag{4.13b}$$

$$[\![\sigma_m, \sigma_n]\!] = 2(m-n)\sigma_{m+n} \tag{4.13c}$$

for $m, n = 0, 1, 2, \cdots$.

4.3 Symmetries and Lie algebra

Now we consider symmetries for the arbitrary isospectral equation

$$u_t = G_l = Z(t)K_l + \Delta. \tag{4.14}$$

First, it is easy to check that

$$\frac{\widetilde{\partial} K_0}{\widetilde{\partial} t} = f(t)u_x = [\![\Delta, K_0]\!] = [\![G_0, K_0]\!], \tag{4.15}$$

which means that K_0 is a symmetry of the equation $u_t = G_0$. Further, by virtue of (4.13a), we have

$$[\![G_l, K_0]\!] = [\![\Delta, K_0]\!]. \tag{4.16}$$

This means that K_0 is a symmetry for the equation (4.14). Since we have shown that Φ is a strong symmetry of (4.14) in Theorem 4.1, all the flows

$$K_n = \Phi^n K_0, \quad n = 0, 1, \cdots \tag{4.17}$$

are symmetries of (4.14). This further leads to

$$\frac{\widetilde{\partial} K_n}{\widetilde{\partial} t} = [\![G_l, K_n]\!], \tag{4.18}$$

and from (4.13a),

$$\frac{\widetilde{\partial} K_n}{\widetilde{\partial} t} = [\![\Delta, K_n]\!]. \tag{4.19}$$

Next we derive another set of symmetries of (4.14).

Lemma 4.5

$$\tau_n^l = (2l+1)\frac{a}{S(t)} K_{n+l} + \Phi^n \sigma_0, \quad n = 0, 1, 2, \cdots \tag{4.20}$$

are symmetries of (4.14) *for* $l = 0, 1, 2, \cdots$.

Proof Since Φ is a strong symmetry of (4.14), we only need to prove

$$\tau_0^l = (2l+1)\frac{a}{S(t)} K_l + \sigma_0 \tag{4.21}$$

is a symmetry of (4.14). Noting that

$$\frac{\widetilde{\partial} \sigma_0}{\widetilde{\partial} t} = [\![\Delta,\, \sigma_0]\!] = 3x\frac{f(t)}{S^2(t)},$$

together with (4.19) and (4.13b), it is easy to get

$$\frac{\widetilde{\partial} \tau_0^l}{\widetilde{\partial} t} = [\![G_l,\, \tau_0^l]\!],$$

which means that τ_0^l is a symmetry of $u_t = G_l$.

As a by-product, we have

$$\frac{\widetilde{\partial} \sigma_n}{\widetilde{\partial} t} = [\![\Delta,\, \sigma_n]\!]. \tag{4.22}$$

Thus we already have two sets of symmetries for the equation (4.14), i.e., $\{K_n\}$ and $\{\tau_n^l\}$, which are usually referred to as K-symmetries and τ-symmetries, respectively. These symmetries can form a Lie algebra by the algebra relation (4.13). We conclude these by the following theorem.

Theorem 4.3 *The equation* (4.14) $u_t = G_l$ *has* K*-symmetries* $\{K_n\}$ *and* τ*-symmetries* $\{\tau_n^l\}$*, which form a Lie algebra with the structure*

$$[\![K_m, K_n]\!] = 0, \tag{4.23a}$$

$$[\![K_m, \tau_n^l]\!] = (2m+1)K_{m+n}, \tag{4.23b}$$

$$[\![\tau_m^l, \tau_n^l]\!] = 2(m-n)\tau_{m+n}^l \tag{4.23c}$$

for $m, n = 0, 1, 2, \cdots$.

5 Conclusion

Under the Painlevé-integrable condition (1.2), we have derived isospectral and non-isospectral vcKdV hierarchies. We proved that the formal recursion operator Φ is a strong hereditary symmetry of the isospectral hierarchy, although it contains t explicitly and is not a rigorous recursion operator. By the relation between Φ and the adjoint flows $\{K_n\}$ and $\{\sigma_m\}$, we proved that $\{K_n\}$ and $\{\sigma_m\}$ form a Lie algebra. Then, by constructing ground symmetries, we got two sets of symmetries for the isospectral vcKdV hierarchy. Finally, the two sets of symmetries are shown to form a Lie algebra. During the above procedure, the adjoint flows $\{\sigma_m\}$ play the role of master symmetries (see [18]).

References

[1] Grimshaw, R., Slowly varying solitary waves, I. Korteweg-de Vries equation, *Proc. Royal Soc. Lon. A-Math. Phys. Engin. Sci.,* **368**, 1979, 359–375.

[2] Joshi, N., Painlevé property of general variable-coefficient versions of the Korteweg-de Vries and nonlinear Schrödinger equations, *Phys. Lett. A,* **125**, 1987, 456–460.

[3] Zhang, Y. C., Yao, Z. Z., Zhu, H. W., et al., Exact analytic N-soliton-like solution in Wronskian form for a generalized variable-coefficient Korteweg-de Vries model from plasmas and fluid dynamics, *Chin. Phys. Lett.,* **24**, 2007, 1173–1176.

[4] Zhang, Y. C., Li, J., Meng, X. H., et al., Existence of formal conservation laws of a variable-coefficient Korteweg-de Vries equation from fluid dynamics and plasma physics via symbolic computation, *Chin. Phys. Lett.,* **25**, 2008, 878–880.

[5] Zhang, D. J., Conservation laws and Lax pair of the variable coefficient KdV equation, *Chin. Phys. Lett.,* **24**, 2007, 3021–3023.

[6] Hong, W. and Jung, Y. D., Auto-Bäcklund transformation and analytic solutions for general variable-coefficient KdV equation, *Phys. Lett. A*, **257**, 1999, 149–152.

[7] Fan, E. G., Auto-Bäcklund transformation and similarity reductions for general variable coefficient KdV equations, *Phys. Lett. A,* **294**, 2002, 26–30.

[8] Fuchssteiner, B. and Fokas, A., Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4**, 1981, 47–66.

[9] Fuchssteiner, B., Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Analysis, TMA*, **3**, 1979, 849–862.

[10] Ma, W. X., K-symmetries and τ-symmetries of evolution equations and their Lie algebras, *J. Phys. A: Math. Gen.,* **23**, 1990, 2707–2716.

[11] Ma, W. X., The algebraic structures of isospectral Lax operators and applications to integrable equations, *J. Phys. A: Math. Gen.,* **25**, 1992, 5329–5343.

[12] Ma, W. X., Lax representations and Lax operator algebras of isospectral and nonisospectral hierarchies of evolution equations, *J. Math. Phys.,* **33**, 1992, 2464–2476.

[13] Ma, W. X. and Fuchssteiner, B., Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations, *J. Math. Phys.,* **40**, 1999, 2400–2418.

[14] Tamizhmani, K. and Ma, W. X., Master symmetries from Lax operators for certain lattice soliton hierarchies, *J. Phys. Soc. Jpn.,* **69**, 2000, 351–361.

[15] Chen, D. Y. and Zhang, H. W., Lie algebraic structure for the AKNS system, *J. Phys. A: Gen. Math. Phys.,* **24**, 1991, 377–383.

[16] Chen, D. Y. and Zhang, D. J., Lie algebraic structures of (1+1)-dimensional Lax integrable systems, *J. Math. Phys.,* **37**, 1996, 5524–5538.

[17] Tian, C., Symmetries, in Soliton Theory and Its Applications, Gu C. H. ed., Springer-Verlag, Berlin, 1996.

[18] Fuchssteiner, B., Master symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Prog. Theo. Phys.,* **70**, 1983, 1508–1522.