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Sharp Distortion Theorems for a Subclass of Biholomorphic Mappings Which Have a Parametric Representation in Several Complex Variables*

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Abstract In this paper, the sharp distortion theorems of the Fréchet-derivative type for a subclass of biholomorphic mappings which have a parametric representation on the unit ball of complex Banach spaces are established, and the corresponding results of the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ are also given. Meanwhile, the sharp distortion theorems of the Jacobi determinant type for a subclass of biholomorphic mappings which have a parametric representation on the unit ball with an arbitrary norm in $\mathbb{C}^n$ are obtained, and the corresponding results of the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ are got as well. Thus, some known results in prior literatures are generalized.

Keywords Distortion theorem, A zero of order $k+1$, Fréchet-derivative, Jacobi determinant, Parametric representation
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1 Introduction

In the case of one complex variable, it is well known that any univalent function has a parametric representation. However, Poreda [1], Kohr [2] and Graham, Hamada and Kohr [3] all pointed out that there are differences between one complex variable and several complex variables.

For the case of several complex variables, Poreda [1] first discussed biholomorphic mappings of the unit polydisk in $\mathbb{C}^n$ which have a parametric representation. After that, Kuba and Poreda [4] studied the parametric representation of starlike mappings defined on the Euclidean unit ball in $\mathbb{C}^n$. Further results in the case of the Euclidean unit ball were obtained by Kohr [2]. The case of a unit ball with any arbitrary norm in $\mathbb{C}^n$ is due to Graham, Hamada and Kohr [3]. Kohr and Liczberski [5] extended the above results to the general case. On the other hand, some of the above results were extended to the case of bounded complete circular domains whose Minkowski functional is C^1 on $\mathbb{C}^n\setminus\{0\}$. As to the case of the unit ball in Banach spaces, Hamada, Honda and Kohr [6] and Hamada and Honda [7] derived general versions of the prior results. Recently, Xu and Liu [8] and Xu, Liu and Xu [9] discussed the distortion theorems of a subclass of biholomorphic mappings which have a parametric representation from different

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aspects, respectively. We also mention that the definition of the biholomorphic mappings with a parametric representation in [8] is slightly different to that in [6], and their proofs were given without applying the Loewner chain.

Up to now, there have been a lot of significant results which cope with the distortion theorems for convex mappings. For instance, Barnard, FitzGerald and Gong [10] first established the estimates of the Jacobi-determinant type for convex mappings defined on the Euclidean unit ball in $\mathbb{C}^2$ in 1994, and after that, Liu and Zhang [11] extended the above result to the general case. With respect to the distortion theorem of the Fréchet-derivative type for convex mappings, the related results for convex mappings were first studied by Gong, Wang and Yu [12]. Consequently, many distinct versions of the distortion theorem for convex mappings on different unit balls in complex Banach spaces were discussed by Gong and Liu [13], Liu and Zhang [14], Zhu and Liu [15], as well as Chu, Hamada, Honda and Kohr [16]. In particular, a strong version of the upper bounds estimate of the distortion theorem for convex mappings on the unit ball of a complex Hilbert space was obtained, due to Hamada and Kohr [17]. However, compared with the distortion theorems for convex mappings, there are only the following two results directly concerning the distortion theorem for starlike mappings by now. One is the distortion theorem for starlike mappings on the unit polydisk along a unit direction which was derived by Liu, Wang and Lu [18], the another is the sharp distortion theorem for a subclass of starlike mappings in several complex variables, due to Liu and Liu [19].

Let X denote a complex Banach space with the norm $\|\cdot\|$, X^* be the dual space of X, B be the open unit ball in X, and U be the Euclidean open unit disk in $\mathbb{C}$. Also, let U^n be the open unit polydisk in $\mathbb{C}^n$, and $\mathbb{N}$ be the set of all positive integers. Let ∂U^n denote the boundary of U^n, $\partial_0 U^n$ be the distinguished boundary of U^n. Let the symbol $'$ mean the transpose. For each $x \in X\backslash\{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from B into X. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n)$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

We say that a holomorphic mapping $f : B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X.

We say that a normalized biholomorphic mapping $f : B \to X$ is a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Now we recall some definitions as follows.

Definition 1.1 (cf. [8]) *Suppose that $g \in H(U)$ is a biholomorphic function such that $g(0) = 1$, $g(\overline{\xi}) = \overline{g(\xi)}$, $\Re e\, g(\xi) > 0$, $\xi \in U$ (so, g has real coefficients in its power series expansion), and assume that g satisfies the conditions:*

$$\begin{cases} \min\limits_{|\xi|=r} |g(\xi)| = \min\limits_{|\xi|=r} \Re e g(\xi) = g(-r), \\ \max\limits_{|\xi|=r} |g(\xi)| = \max\limits_{|\xi|=r} \Re e g(\xi) = g(r). \end{cases}$$

.

We denote by $\mathcal{M}_g$ the set

$$\mathcal{M}_g = \Big\{ p \in H(B) : p(0) = 0, Dp(0) = I, \frac{\|x\|}{T_x(p(x))} \in g(U), x \in B \setminus \{0\}, T_x \in T(x) \Big\}.$$

Definition 1.2 (cf. [20]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in (0, 1)$ and*

$$\Big| \frac{1}{\|x\|} T_x[(Df(x))^{-1} f(x)] - \frac{1}{2\alpha} \Big| < \frac{1}{2\alpha}, \quad x \in B \setminus \{0\},$$

then we say that f is a starlike mapping of order α on B.

We denote by $S^*_\alpha(B)$ the set of all starlike mappings of order α on B.

Definition 1.3 (cf. [21]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in (0, 1]$ and*

$$\Big| \arg \Big\{ \frac{1}{\|x\|} T_x[(Df(x))^{-1} f(x)] \Big\} \Big| < \frac{\pi}{2}\alpha, \quad x \in B \setminus \{0\},$$

then we say that f is a strongly starlike mapping of order α on B.

Definition 1.3 in the case of the Euclidean unit ball B^n was originally introduced by Curt [22].

Let $SS^*_\alpha(B)$ be the set of all strongly starlike mappings of order α on B.

Definition 1.4 (cf. [23]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in [0, 1)$ and*

$$\Re e\{T_x[(Df(x))^{-1} f(x)]\} \geq \alpha \|x\|, \quad x \in B \setminus \{0\},$$

then we say that f is an almost starlike mapping of order α on B.

We denote by $AS^*_\alpha(B)$ the set of all almost starlike mappings of order α on B.

Definition 1.5 (cf. [21]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $c \in (0, 1)$ and*

$$\Big| \frac{1}{\|x\|} T_x[(Df(x))^{-1} f(x)] - \frac{1 + c^2}{1 - c^2} \Big| < \frac{2c}{1 - c^2}, \quad x \in B \setminus \{0\},$$

then we say that f is a strongly starlike mapping on B.

Definition 1.5 in the case of the Euclidean unit ball B^n was first introduced by Chuaqui [24].

Let $SS^*(B)$ be the set of all strongly starlike mappings on B.

Definition 1.6 (cf. [25]) *Let $f \in H(B)$. It is said that f is k-fold symmetric if*

$$\mathrm{e}^{-\frac{2\pi \mathrm{i}}{k}} f(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k}} x) = f(x)$$

for all $x \in B$, where $k \in \mathbb{N}$ and $\mathrm{i} = \sqrt{-1}$.

Definition 1.7 (cf. [26]) *Suppose that Ω is a domain (a connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.*

We readily see that $x = 0$ is a zero of order $k+1$ ($k \in \mathbb{N}$) of $f(x) - x$ if f is a k-fold symmetric normalized holomorphic mapping ($f(x) \not\equiv x$) defined on B, but the converse fails.

Let $S_g^*(B)$ be the subset of $S^*(B)$ consisting of normalized locally biholomorphic mappings f which satisfy $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, and let $S_{g,k+1}^*(B)$ be a subset of $S_g^*(B)$ such that $x = 0$ is a zero of order $k+1$ of $f(x) - x$. We denote by $S_{k+1}^*(B)$ (resp. $S_{\alpha,k+1}^*(B)$, $SS_{\alpha,k+1}^*(B)$, $AS_{\alpha,k+1}^*(B)$, $SS_{k+1}^*(B)$) the subset of $S^*(B)$ (resp. $S_\alpha^*(B)$, $SS_\alpha^*(B)$, $AS_\alpha^*(B)$, $SS^*(B)$) which satisfies that $x = 0$ is a zero of order $k+1$ of $f(x) - x$.

In this paper, we organize the contents as follows. In Section 2, we shall establish the sharp distortion theorems of the Fréchet-derivative type for a subclass of biholomorphic mappings which have a parametric representation on the unit ball of complex Banach spaces and the corresponding results of the above generalized mappings on the unit polydisk in $\mathbb{C}^n$. In Section 3, we shall establish the sharp distortion theorems of the Jacobi-determinant type for a subclass of biholomorphic mappings which have a parametric representation on the unit ball with an arbitrary norm in $\mathbb{C}^n$ and the corresponding results of the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ as well. Our derived conclusions are generalizations of some known results in the prior literatures.

2 Sharp Distortion Theorems of the Fréchet-Derivative Type for a Subclass of Biholomorphic Mappings Which Have a Parametric Representation

In order to prove the desired theorems in this section, the following lemmas are necessary.

Lemma 2.1 *If $h \in \mathcal{M}_g$ and $x = 0$ is a zero of order $k+1$ of $h(x) - x$, then*

$$\frac{\|x\|}{g(\|x\|^k)} \leq \Re e[T_x(h(x))] \leq |T_x(h(x))| \leq \frac{\|x\|}{g(-\|x\|^k)}, \quad x \in B.$$

.

Proof Fix $x \in B \setminus \{0\}$, and we write $x_0 = \frac{x}{\|x\|}$. Consider

$$p(\xi) = \begin{cases} \dfrac{\xi}{T_x(h(\xi x_0))}, & \xi \in U \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(U)$, $p(0) = g(0) = 1$, and it is shown that

$$p(\xi) = \frac{\xi}{T_x(h(\xi x_0))} = \frac{\|\xi x_0\|}{T_{\xi x_0}(h(\xi x_0))} \in g(U), \quad \xi \in U$$

from $h \in \mathcal{M}_g$. Also, since $x = 0$ is a zero of order $k+1$ of $h(x) - x$, then $\xi = 0$ is at least a zero of order k of $\varphi(\xi) - 1$, where $\varphi(\xi) = \frac{1}{p(\xi)}$. Let

$$\widetilde{\varphi}(\xi) = \begin{cases} \dfrac{\varphi(\xi) - 1}{\xi^k}, & \xi \in U \setminus \{0\}, \\ \dfrac{\varphi^k(0)}{k!}, & \xi = 0, \end{cases}$$

and then $\widetilde{\varphi} \in H(U)$. Note that

$$\varphi(\xi) \in \psi(U), \quad \xi \in U, \tag{2.1}$$

where $\psi(\xi) = \frac{1}{g(\xi)}$. In view of $\psi^{-1}(1) = 0$,

$$G(w) = \begin{cases} \dfrac{\psi^{-1}(w)}{w-1}, & w \neq 1, \\ \dfrac{1}{\psi'(0)}, & w = 1 \end{cases}$$

is well defined, and G is a holomorphic function in $\mathbb{C}$. Hence

$$\psi^{-1}(\varphi(\xi)) = \xi^k \widetilde{\varphi}(\xi) G(\varphi(\xi)), \quad \xi \in U$$

and

$$|\psi^{-1}(\varphi(\xi))| \leq |\xi|^k, \quad \xi \in U$$

from the Schwarz lemma and (2.1). Therefore,

$$\varphi(\xi) = \psi(\xi^k H(\xi)), \quad |H(\xi)| \leq 1, \quad \xi \in U,$$

where $H(\xi) = \widetilde{\varphi}(\xi) G(\varphi(\xi))$. Finally, taking into account the minimum principle of harmonic functions and the maximum modulus principle of holomorphic functions, it yields that

$$\frac{1}{g(|\xi|^k |H(\xi)|)} \leq \Re e \varphi(\xi) \leq |\varphi(\xi)| \leq \frac{1}{g(-|\xi|^k |H(\xi)|)}, \quad \xi \in U.$$

We mention that $\frac{1}{g(r)}$ $\left(\text{resp. } \frac{1}{g(-r)}\right)$ is a decreasing function (resp. an increasing function) on the interval $[0, 1]$ from the maximum and minimum modulus principles of holomorphic functions. Thus

$$\frac{1}{g(|\xi|^k)} \leq \Re e \varphi(\xi) \leq |\varphi(\xi)| \leq \frac{1}{g(-|\xi|^k)}, \quad \xi \in U. \tag{2.2}$$

Taking $\xi = \|x\|$ in (2.2), the result follows, as desired. This completes the proof.

Remark 2.1 Lemma 2.1 provides more details of the proof than those in [9, Lemma 2.4], and our proof does not apply the subordinate principle.

Lemma 2.2 (cf. [9, Theorem 2.1]) *If $F \in S^*_{g,k+1}(B)$, then*

$$\|x\| \exp \int_0^{\|x\|} [g(-\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda} \leq \|F(x)\| \leq \|x\| \exp \int_0^{\|x\|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad x \in B$$

and the above estimates are sharp.

We now begin to present the following distortion theorems in this section.

Theorem 2.1 *Let $g : U \to C$ satisfy the condition of Definition* 1.1, $f : B \to \mathbb{C} \in H(B)$, *and $F(x) = xf(x) \in S^*_{g,k+1}(B)$. Then*

$$\|x\|g(-\|x\|^k)\exp\int_0^{\|x\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$
$$\le \|DF(x)x\| \le \|x\|g(\|x\|^k)\exp\int_0^{\|x\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}, \quad x \in B$$

and the above estimates are sharp.

Proof In view of $F(x) = xf(x)$, it is shown that

$$DF(x)x = xf(x) + (Df(x)x)x, \quad x \in B. \tag{2.3}$$

Also $F(x) = xf(x) \in S^*_{g,k+1}(B)$, and then it yields that

$$\|x\|\exp\int_0^{\|x\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda} \le \|F(x)\| \le \|x\|\exp\int_0^{\|x\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}, \quad x \in B$$

from Lemma 2.2. Hence

$$|f(x)| \ge \exp\int_0^{\|x\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda} > 0, \quad x \in B.$$

Namely, $f(x) \ne 0$ for $x \in B$.

It is shown that

$$[DF(x)]^{-1}F(x) = \frac{f(x)x}{f(x)+Df(x)x}, \quad x \in B \tag{2.4}$$

by a direct calculation. We mention that $\Re e\{T_x[(DF(x))^{-1}F(x)]\} > 0,\ x \in B\setminus\{0\}$ if $F \in S^*_{g,k+1}(B) \subset S^*(B)$. Consequently, by (2.4), we see that

$$f(x) + Df(x)x \ne 0,\ x \in B.$$

According to Lemma 2.1, it is shown that

$$\frac{\|x\|}{g(\|x\|^k)} \le |T_x[(DF(x))^{-1}F(x)]| \le \frac{\|x\|}{g(-\|x\|^k)}, \quad x \in B.$$

On the other hand, it yields that

$$DF(x)x = xf(x) + (Df(x)x)x = xf(x)\Big(1 + \frac{Df(x)x}{f(x)}\Big) = F(x)\frac{\|x\|}{T_x[(DF(x))^{-1}F(x)]}$$

from (2.3)–(2.4). Therefore the desired result follows from Lemmas 2.1–2.2.

Let $b \in S^*_g(U)$ such that $b(0) = b'(0) - 1 = 0$ and

$$\frac{\xi b'(\xi)}{b(\xi)} = g(\xi), \quad \xi \in U.$$

We define

$$b_k(\xi) = \xi(\varphi(\xi^k))^{\frac{1}{k}},$$

where $\varphi(\xi)=\frac{b(\xi)}{\xi}$, and $k=1,2,\cdots$. The branches of the above functions are chosen, which satisfy

$$(\varphi(\xi^k))^{\frac{1}{k}}|_{\xi=0}=1,\quad k=1,2,\cdots.$$

Also, consider

$$F_u(x)=\frac{b_k(T_u(x))}{T_u(x)}x,\quad x\in B, \tag{2.5}$$

where $\|u\|=1$. We derive the following equivalent formulation of Theorem 2.1:

$$\frac{\|x\|\widetilde{b_k'}(\|x\|)}{\widetilde{b_k}(\|x\|)}\exp\int_0^{\|x\|}\Big[\frac{\lambda\widetilde{b_k'}(\lambda)}{\widetilde{b_k}(\lambda)}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}\le\frac{\|DF(x)x\|}{\|x\|}\le\frac{\|x\|b_k'(\|x\|)}{b_k(\|x\|)}\exp\int_0^{\|x\|}\Big[\frac{\lambda b_k'(\lambda)}{b_k(\lambda)}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}$$

for $x\in B\setminus\{0\}$, where $\widetilde{b_k}(\xi)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{k}}b_k(\mathrm{e}^{\frac{\pi\mathrm{i}}{k}}\xi)$, $\mathrm{i}=\sqrt{-1}$. Note that $b_k(\lambda),\widetilde{b_k}(\lambda)>0$ for $\lambda>0$ and $b_k'(0)=\widetilde{b_k'}(0)=1$. Then it is shown that

$$\begin{aligned}&\frac{\|x\|\widetilde{b_k'}(\|x\|)}{\widetilde{b_k}(\|x\|)}\exp\Big[\log\frac{\widetilde{b_k}(\|x\|)}{\|x\|}-\log\widetilde{b_k'}(0)\Big]\\ \le&\frac{\|DF(x)x\|}{\|x\|}\le\frac{\|x\|b_k'(\|x\|)}{b_k(\|x\|)}\exp\Big[\log\frac{b_k(\|x\|)}{\|x\|}-\log b_k'(0)\Big]\end{aligned}$$

for $x\in B\setminus\{0\}$. This implies that

$$\|x\|b_k'(\mathrm{e}^{\frac{\pi\mathrm{i}}{k}}\|x\|)\le\|DF(x)x\|\le\|x\|b_k'(\|x\|).$$

Let $F_u\in S_g^*(B)$ be given by (2.5). A straightforward computation shows that $\|DF_u(ru)ru\|=rb_k'(r)$ and $\|DF_u(\mathrm{e}^{\frac{\pi\mathrm{i}}{k}}ru)\mathrm{e}^{\frac{\pi\mathrm{i}}{k}}ru\|=rb_k'(\mathrm{e}^{\frac{\pi\mathrm{i}}{k}}r)$. Then it is shown that the estimates of Theorem 2.1 are sharp. This completes the proof.

Remark 2.2 Taking $k=1$ in Theorems 2.1, it is easy to see that Theorems 2.1 generalizes [8, Theorem 5].

Let $g(\xi)=\frac{1+\xi}{1-\xi}$ in Theorem 2.1. Then we derive the following corollary.

Corollary 2.1 *Let $f:B\to\mathbb{C}\in H(B)$, and $F(x)=xf(x)\in S_{k+1}^*(B)$. Then*

$$\frac{\|x\|(1-\|x\|^k)}{(1+\|x\|^k)^{1+\frac{2}{k}}}\le\|DF(x)x\|\le\frac{\|x\|(1+\|x\|^k)}{(1-\|x\|^k)^{1+\frac{2}{k}}},\quad x\in B$$

and the above estimates are sharp.

Let $g(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorem 2.1, where $\alpha\in(0,1)$. Then we get the following corollary.

Corollary 2.2 *Let $f:B\to\mathbb{C}\in H(B)$, $\alpha\in(0,1)$, and $F(x)=xf(x)\in S_{\alpha,k+1}^*(B)$. Then*

$$\frac{\|x\|(1-(1-2\alpha)\|x\|^k)}{(1+\|x\|^k)^{1+\frac{2(1-\alpha)}{k}}}\le\|DF(x)x\|\le\frac{\|x\|(1+(1-2\alpha)\|x\|^k)}{(1-\|x\|^k)^{1+\frac{2(1-\alpha)}{k}}},\quad x\in B$$

and the above estimates are sharp.

Let $g(\xi)=\frac{(1+\xi)^\alpha}{(1-\xi)^\alpha}$ in Theorem 2.1, where $\alpha\in(0,1]$. Then we obtain the following corollary.

Corollary 2.3 *Let* $f : B \to \mathbb{C} \in H(B)$, $\alpha \in (0,1]$, *and* $F(x) = xf(x) \in SS^*_{\alpha,k+1}(B)$. *Then*

$$\|x\|\Big(\frac{1-\|x\|^k}{1+\|x\|^k}\Big)^{\alpha}\exp\int_0^{\|x\|}\Big[\Big(\frac{1-\lambda^k}{1+\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda} \le \|DF(x)x\|$$
$$\le \|x\|\Big(\frac{1+\|x\|^k}{1-\|x\|^k}\Big)^{\alpha}\exp\int_0^{\|x\|}\Big[\Big(\frac{1+\lambda^k}{1-\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}, \quad x \in B$$

and the above estimates are sharp.

Let $g(\xi) = \frac{1+\xi}{1-(1-2\alpha)\xi}$ in Theorem 2.1, where $\alpha \in [0,1)$. Then we derive the following corollary.

Corollary 2.4 *Let* $f : B \to \mathbb{C} \in H(B)$, *and* $F(x) = xf(x) \in AS^*_{\alpha,k+1}(B)$.

(i) *If* $\alpha \in [0,1)$ *and* $\alpha \neq \frac{1}{2}$, *then*

$$\frac{\|x\|(1-\|x\|^k)}{(1+(1-2\alpha)\|x\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}} \le \|DF(x)x\| \le \frac{\|x\|(1+\|x\|^k)}{(1-(1-2\alpha)\|x\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}}, \quad x \in B$$

and the above estimates are sharp.

(ii) *If* $\alpha = \frac{1}{2}$, *then*

$$\|x\|(1-\|x\|^k)\exp\Big(-\frac{1}{k}\|x\|^k\Big) \le \|DF(x)x\| \le \|x\|(1+\|x\|^k)\exp\Big(\frac{1}{k}\|x\|^k\Big), \quad x \in B$$

and the above estimates are sharp.

Let $g(\xi) = \frac{1+c\xi}{1-c\xi}$ in Theorem 2.1, where $c \in (0,1)$. Then we get the following corollary.

Corollary 2.5 *Let* $f : B \to \mathbb{C} \in H(B)$, $c \in (0,1)$, *and* $F(x) = xf(x) \in SS^*_{k+1}(B)$. *Then*

$$\frac{\|x\|(1-c\|x\|^k)}{(1+c\|x\|^k)^{1+\frac{2}{k}}} \le \|DF(x)x\| \le \frac{\|x\|(1+c\|x\|^k)}{(1-c\|x\|^k)^{1+\frac{2}{k}}}, \quad x \in B$$

and the above estimates are sharp.

In the following Theorem 2.2 and Corollaries 2.6–2.10, each m_l is a non-negative integer, $N = m_1 + m_2 + \cdots + m_n \in \mathbb{N}$, $m_l = 0$ means that the corresponding components in $Z = (Z_1, \cdots, Z_l, \cdots, Z_n)'$ and $F(Z) = (F_1(Z_1), \cdots, F_l(Z_l), \cdots, F_n(Z_n))'$ are vanished.

Theorem 2.2 *Let* $g : U \to C$ *satisfy the condition of Definition* 1.1, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $l = 1, 2, \cdots, n$, *and* $F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \cdots, Z_nf_n(Z_n))' \in S^*_{g,k+1}(U^N)$. *Then*

$$\|Z\|g(-\|Z\|^k)\exp\int_0^{\|Z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda} \le \|DF(Z)Z\| \le \|Z\|g(\|Z\|^k)\exp\int_0^{\|Z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

Proof Let $F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))'$. In view of the hypothesis of Theorem 2.2, for any $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, we see that

$$(DF(Z))^{-1}F(Z) = ((DF_1(Z_1))^{-1}F_1(Z_1), (DF_2(Z_2))^{-1}F_2(Z_2), \cdots, (DF_n(Z_n))^{-1}F_n(Z_n))'$$

by a direct calculation. We mention that

$$(DF(Z))^{-1}F(Z) = (0, \cdots, (DF_l(Z_l))^{-1}F_l(Z_l), \cdots, 0)'$$

if $Z = (0, \cdots, Z_l, \cdots, 0)' \in U^n,\ l = 1, 2, \cdots, n$. Let

$$\begin{aligned} W(Z) = (W_1, W_2, \cdots, W_n)' &= (W_{11}, \cdots, W_{1m_1}, W_{21}, \cdots, W_{2m_2}, \cdots, W_{n1}, \cdots, W_{nm_n})' \\ &= (DF(Z))^{-1}F(Z). \end{aligned}$$

Then it is shown that

$$F \in S_g^*(U^N) \Leftrightarrow F_l \in S_g^*(U^{m_l}), \quad l = 1, 2, \cdots, n$$

from the definition of $S_g^*(U^n)$. Also we readily see that $Z_l = 0$ is at least a zero of order $k+1$ of each $F_l(Z_l) - Z_l$ $(l = 1, 2, \cdots, n)$ if $Z = 0$ is a zero of order $k+1$ of $F(Z) - Z$. In view of the fact that $g(\lambda^k) > 1$ for $\lambda > 0$, it is not difficult to know that $tg(t^k)\exp\int_0^t [g(\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda}$ is an increasing function on the interval $[0, 1)$ with respect to t. Noticing that $\|DF(Z)Z\| = \max\limits_{1\le l\le n}\{\|DF_l(Z_l)Z_l\|\}$ and

$$\begin{aligned} &\|Z_l\|g(-\|Z_l\|^k)\exp\int_0^{\|Z_l\|}[g(-\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda} \\ \le{}& \|DF(Z_l)Z_l\| \le \|Z_l\|g(\|Z_l\|^k)\exp\int_0^{\|Z_l\|}[g(\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda} \end{aligned}$$

for $Z_l \in U^{m_l},\ l = 1, 2, \cdots, n$ (the case $B = U^{m_l}$ of Theorem 2.1), it is shown that

$$\begin{aligned} \|DF(Z)Z\| &= \max_{1\le l\le n}\{\|DF_l(Z_l)Z_l\|\} \le \|Z\|g(\|Z\|^k)\exp\int_0^{\|Z\|}[g(\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda}, \quad Z \in U^N, \\ \|DF(Z)Z\| &\ge \|DF_j(Z_j)Z_j\| \ge \|Z_j\|g(-\|Z_j\|^k)\exp\int_0^{\|Z_j\|}[g(-\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda} \\ &= \|Z\|g(-\|Z\|^k)\exp\int_0^{\|Z\|}[g(-\lambda^k) - 1]\frac{\mathrm{d}\lambda}{\lambda}, \quad Z \in U^N, \end{aligned}$$

where $\|Z_l\|_{m_l}$ (resp. $\|Z\|_N$) is briefly denoted by $\|Z_l\|$ (resp. $\|Z\|$), and j satisfies $\|Z_j\| = \|Z\| = \max\limits_{1\le l\le n}\{\|Z_l\|\}$. The sharpness of the estimates of Theorem 2.2 is analogous to that in the proof of Theorem 2.1, so we only point out that the two equalities of the estimates of Theorem 2.2 hold for $Z_l = (\mathrm{e}^{\frac{\pi \mathrm{i}}{k}}r, 0, \cdots, 0)'$ $(0 \le r < 1)$ $(l = 1, 2, \cdots, n)$ and $Z_l = (r, 0, \cdots, 0)'$ $(0 \le r < 1)$ $(l = 1, 2, \cdots, n)$, where $\mathrm{i} = \sqrt{-1}$. We omit the details here. This completes the proof.

Letting $g(\xi) = \frac{1+\xi}{1-\xi}$ in Theorem 2.2, we get the following corollary.

Corollary 2.6 *Let $g : U \to C$ satisfy the condition of Definition* 1.1*, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l}),\ l = 1, 2, \cdots, n$, and $F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \cdots, Z_nf_n(Z_n))' \in S_{k+1}^*(U^N)$. Then*

$$\frac{\|Z\|(1 - \|Z\|^k)}{(1 + \|Z\|^k)^{1+\frac{2}{k}}} \le \|DF(Z)Z\| \le \frac{\|Z\|(1 + \|Z\|^k)}{(1 - \|Z\|^k)^{1+\frac{2}{k}}}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, and the above estimates are sharp.

Letting $g(\xi) = \frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorem 2.2, where $\alpha \in (0, 1)$, we derive the following corollary.

Corollary 2.7 *Let $g : U \to C$ satisfy the condition of Definition* 1.1, $\alpha \in (0,1)$, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $l = 1, 2, \cdots, n$, *and* $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*_{\alpha,k+1}(U^N)$. *Then*

$$\frac{\|Z\|(1-(1-2\alpha)\|Z\|^k)}{(1+\|Z\|^k)^{1+\frac{2(1-\alpha)}{k}}} \le \|DF(Z)Z\| \le \frac{\|Z\|(1+(1-2\alpha)\|Z\|^k)}{(1-\|Z\|^k)^{1+\frac{2(1-\alpha)}{k}}}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

Letting $g(\xi) = \frac{(1+\xi)^\alpha}{(1-\xi)^\alpha}$ in Theorem 2.2, where $\alpha \in (0,1]$, we obtain the following corollary.

Corollary 2.8 *Let $g : U \to C$ satisfy the condition of Definition* 1.1, $\alpha \in (0,1]$, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $l = 1, 2, \cdots, n$, *and* $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in SS^*_{\alpha,k+1}(U^N)$. *Then*

$$\|Z\|\Big(\frac{1-\|Z\|^k}{1+\|Z\|^k}\Big)^\alpha \exp\int_0^{\|Z\|}\Big[\Big(\frac{1-\lambda^k}{1+\lambda^k}\Big)^\alpha - 1\Big]\frac{\mathrm{d}\lambda}{\lambda} \le \|DF(Z)Z\|$$
$$\le \|Z\|\Big(\frac{1+\|Z\|^k}{1-\|Z\|^k}\Big)^\alpha \exp\int_0^{\|Z\|}\Big[\Big(\frac{1+\lambda^k}{1-\lambda^k}\Big)^\alpha - 1\Big]\frac{\mathrm{d}\lambda}{\lambda}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

Letting $g(\xi) = \frac{1+\xi}{1-(1-2\alpha)\xi}$ in Theorem 2.2, where $\alpha \in [0,1)$, we derive the following corollary.

Corollary 2.9 *Let $g : U \to C$ satisfy the condition of Definition* 1.1, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $l = 1, 2, \cdots, n$, *and* $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in AS^*_{\alpha,k+1}(U^N)$.

(i) *If* $\alpha \in [0,1)$ *and* $\alpha \ne \frac{1}{2}$, *then*

$$\frac{\|Z\|(1-\|Z\|^k)}{(1+(1-2\alpha)\|Z\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}} \le \|DF(Z)Z\| \le \frac{\|Z\|(1+\|Z\|^k)}{(1-(1-2\alpha)\|Z\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

(ii) *If* $\alpha = \frac{1}{2}$, *then*

$$\|Z\|(1-\|Z\|^k)\exp\Big(-\frac{1}{k}\|Z\|^k\Big) \le \|DF(Z)Z\| \le \|Z\|(1+\|Z\|^k)\exp\Big(\frac{1}{k}\|Z\|^k\Big)$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

Letting $g(\xi) = \frac{1+c\xi}{1-c\xi}$ in Theorem 2.2, where $c \in (0,1)$, we get the following corollary.

Corollary 2.10 *Let $g : U \to C$ satisfy the condition of Definition* 1.1, $c \in (0,1)$, $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $l = 1, 2, \cdots, n$, *and* $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in SS^*_{k+1}(U^N)$. *Then*

$$\frac{\|Z\|(1-c\|Z\|^k)}{(1+c\|Z\|^k)^{1+\frac{2}{k}}} \le \|DF(Z)Z\| \le \frac{\|Z\|(1+c\|Z\|^k)}{(1-c\|Z\|^k)^{1+\frac{2}{k}}}$$

for $Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N$, *and the above estimates are sharp.*

Theorem 2.3 *Suppose that $g : U \to C$ satisfies the condition of Definition* 1.1, $F(z) = (F_1(z), F_2(z), \cdots, F_n(z)) \in H(U^n)$, *and $z = 0$ is a zero of order $k+1$ ($k \in \mathbb{N}$) of $F(z) - z$. If* $\frac{DF_j(z)z}{F_j(z)} \in g(U)$, $z \in U^n \setminus \{0\}$, *where j satisfies the condition* $|z_j| = \|z\| = \max\limits_{1\le l\le n} |z_l|$, *then*

$$\|z\|g(-\|z\|^k)\exp\int_0^{\|z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda} \le \|DF(z)z\| \le \|z\|g(\|z\|^k)\exp\int_0^{\|z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$

for $z = (z_1, z_2, \cdots, z_n)' \in U^n$, *and the above estimates are sharp.*

Proof Fix $z \in U^n \setminus \{0\}$, and we write $z_0 = \frac{z}{\|z\|}$. Define

$$h_j(\xi) = \frac{\|z\|}{z_j} F_j(\xi z_0), \quad \xi \in U, \tag{2.6}$$

where j satisfies the condition $|z_j| = \|z\| = \max\limits_{1 \le l \le n}\{|z_l|\}$. Taking into account the hypothesis of Theorem 2.3 and (2.6), we see that

$$\frac{h_j'(\xi)\xi}{h_j(\xi)} = \frac{DF_j(\xi z_0)\xi z_0}{F_j(\xi z_0)} \in g(U), \quad \xi \in U \setminus \{0\}$$

by a simple calculation. Thus it is shown that $h_j \in S_g^*(U)$, and $\xi = 0$ is at least a zero of order $k+1$ of $h_j(\xi) - \xi$.

On the other hand, for $z_0 \in \partial U^n$, it yields that

$$\begin{aligned} &|\xi| g(-|\xi|^k) \exp \int_0^{|\xi|} [g(-\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda} \le |DF_j(\xi z_0)\xi z_0| \\ &= |\xi h_j'(\xi)| \le |\xi| g(|\xi|^k) \exp \int_0^{|\xi|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda} \end{aligned}$$

from Theorem 2.1 (the case of $X = \mathbb{C},\ B = U$). It is readily known that

$$|\xi| g(-|\xi|^k) \exp \int_0^{|\xi|} [g(-\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda} \le |DF_j(\xi z_0)\xi z_0| \le \|DF(\xi z_0)\xi z_0\|, \quad z_0 \in \partial U^n. \tag{2.7}$$

We set $\xi = \|z\|$. Then

$$\|DF(z)z\| \ge \|z\| g(-\|z\|^k) \exp \int_0^{\|z\|} [g(-\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad z \in U^n. \tag{2.8}$$

For $z_0 \in \partial_0 U^n$, we know that

$$|DF_l(\xi z_0)\xi z_0| \le |\xi| g(|\xi|^k) \exp \int_0^{|\xi|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad l = 1, 2, \cdots, n$$

from (2.7). We mention that $w(z) = DF_l(\xi z)\xi z$ is a holomorphic function on $\overline{U^n}$. According to the maximum modulus theorem of holomorphic functions on the unit polydisk, it is shown that

$$|DF_l(\xi z_0)\xi z_0| \le |\xi| g(|\xi|^k) \exp \int_0^{|\xi|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad z_0 \in \partial U^n,\ l = 1, 2, \cdots, n.$$

This implies that

$$\|DF(\xi z_0)\xi z_0\| \le |\xi| g(|\xi|^k) \exp \int_0^{|\xi|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad z_0 \in \partial U^n.$$

Taking $\xi = \|z\|$, it yields that

$$\|DF(z)z\| \le \|z\| g(\|z\|^k) \exp \int_0^{\|z\|} [g(\lambda^k) - 1] \frac{\mathrm{d}\lambda}{\lambda}, \quad z \in U^n. \tag{2.9}$$

We derive the desired results from (2.8)–(2.9). The sharpness of the estimates of Theorem 2.3 is similar to that in the proof of Theorem 2.2, so the details are omitted here. This completes the proof.

Remark 2.3 It is shown that Theorem 2.1 (the case $X=\mathbb{C}^n$ $B=U^n$) is a special case of Theorem 2.3, and Theorem 2.2 (the case $m_1=n,\ m_l=0,\ l=2,\cdots,n$ or $m_l=1,\ l=1,2,\cdots,n$) is also a special case of Theorem 2.3.

Set $g(\xi)=\frac{1+\xi}{1-\xi}$ in Theorem 2.3. Then we get the following corollary.

Corollary 2.11 *Suppose that $g:U\to C$ satisfies the condition of Definition 1.1, $F(z)=(F_1(z),F_2(z),\cdots,F_n(z))\in H(U^n)$, and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$. If $\Re e\frac{DF_j(z)z}{F_j(z)}>0,\ z\in U^n\setminus\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, then*

$$\frac{\|z\|(1-\|z\|^k)}{(1+\|z\|^k)^{1+\frac{2}{k}}}\le\|DF(z)z\|\le\frac{\|z\|(1+\|z\|^k)}{(1-\|z\|^k)^{1+\frac{2}{k}}}$$

for $z=(z_1,z_2,\cdots,z_n)'\in U^n$, and the above estimates are sharp.

Set $g(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorem 2.3, where $\alpha\in(0,1)$. Then we derive the following corollary.

Corollary 2.12 *Suppose that $g:U\to C$ satisfies the condition of Definition 1.1, $\alpha\in(0,1),\ F(z)=(F_1(z),F_2(z),\cdots,F_n(z))\in H(U^n)$, and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$. If $\Re e\frac{DF_j(z)z}{F_j(z)}>\alpha,\ z\in U^n\setminus\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, then*

$$\frac{\|z\|(1-(1-2\alpha)\|z\|^k)}{(1+\|z\|^k)^{1+\frac{2(1-\alpha)}{k}}}\le\|DF(z)z\|\le\frac{\|z\|(1+(1-2\alpha)\|z\|^k)}{(1-\|z\|^k)^{1+\frac{2(1-\alpha)}{k}}}$$

for $z=(z_1,z_2,\cdots,z_n)'\in U^n$, and the above estimates are sharp.

Set $g(\xi)=\frac{(1+\xi)^\alpha}{(1-\xi)^\alpha}$ in Theorem 2.3, where $\alpha\in(0,1]$. Then we obtain the following corollary.

Corollary 2.13 *Suppose that $g:U\to C$ satisfies the condition of Definition 1.1, $\alpha\in(0,1],\ F(z)=(F_1(z),F_2(z),\cdots,F_n(z))\in H(U^n)$, and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$. If $\big|\arg\big(\frac{F_j(z)}{DF_j(z)z}\big)\big|<\frac{\pi}{2}\alpha,\ z\in U^n\setminus\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, then*

$$\begin{aligned}&\|z\|\Big(\frac{1-\|z\|^k}{1+\|z\|^k}\Big)^\alpha\exp\int_0^{\|z\|}\Big[\Big(\frac{1-\lambda^k}{1+\lambda^k}\Big)^\alpha-1\Big]\frac{\mathrm{d}\lambda}{\lambda}\le\|DF(z)z\|\\ \le\ &\|z\|\Big(\frac{1+\|z\|^k}{1-\|z\|^k}\Big)^\alpha\exp\int_0^{\|z\|}\Big[\Big(\frac{1+\lambda^k}{1-\lambda^k}\Big)^\alpha-1\Big]\frac{\mathrm{d}\lambda}{\lambda}\end{aligned}$$

for $z=(z_1,z_2,\cdots,z_n)'\in U^n$, and the above estimates are sharp.

Set $g(\xi)=\frac{1+\xi}{1-(1-2\alpha)\xi}$ in Theorem 2.3, where $\alpha\in[0,1)$. Then we get the following corollary.

Corollary 2.14 *Suppose that $g:U\to C$ satisfies the condition of Definition 1.1, $\alpha\in[0,1),\ F(z)=(F_1(z),F_2(z),\cdots,F_n(z))\in H(U^n)$, and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$. If $\Re e\frac{F_j(z)}{DF_j(z)z}>\alpha,\ z\in U^n\setminus\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}|z_l|$, then*

(i) *If $\alpha\in[0,1)$ and $\alpha\ne\frac{1}{2}$, then*

$$\frac{\|z\|(1-\|z\|^k)}{(1+(1-2\alpha)\|z\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}}\le\|DF(z)z\|\le\frac{\|z\|(1+\|z\|^k)}{(1-(1-2\alpha)\|z\|^k)^{1+\frac{2(1-\alpha)}{k(1-2\alpha)}}}$$

for $z = (z_1, z_2, \cdots, z_n)' \in U^n$*, and the above estimates are sharp.*

(ii) *If* $\alpha = \frac{1}{2}$*, then*

$$\|z\|(1-\|z\|^k)\exp\Big(-\frac{1}{k}\|z\|^k\Big) \le \|DF(z)z\| \le \|z\|(1+\|z\|^k)\exp\Big(\frac{1}{k}\|z\|^k\Big)$$

for $z = (z_1, z_2, \cdots, z_n)' \in U^n$*, and the above estimates are sharp.*

Set $g(\xi) = \frac{1+c\xi}{1-c\xi}$ in Theorem 2.3, where $c \in (0,1)$. Then we derive the following corollary.

Corollary 2.15 *Suppose that* $g: U \to C$ *satisfies the condition of Definition* 1.1*,* $c \in (0,1)$*,* $F(z) = (F_1(z), F_2(z), \cdots, F_n(z)) \in H(U^n)$*, and* $z = 0$ *is a zero of order* $k+1$ $(k \in \mathbb{N})$ *of* $F(z) - z$*. If* $\big|\frac{F_j(z)}{DF_j(z)z} - \frac{1+c^2}{1-c^2}\big| > \frac{2c}{1-c^2}$*,* $z \in U^n \setminus \{0\}$*, where* j *satisfies the condition* $|z_j| = \|z\| = \max\limits_{1\le l\le n} |z_l|$*, then*

$$\frac{\|z\|(1-c\|z\|^k)}{(1+c\|z\|^k)^{1+\frac{2}{k}}} \le \|DF(z)z\| \le \frac{\|z\|(1+c\|z\|^k)}{(1-c\|z\|^k)^{1+\frac{2}{k}}}$$

for $z = (z_1, z_2, \cdots, z_n)' \in U^n$*, and the above estimates are sharp.*

Remark 2.4 From Theorems 2.1 and 2.2, we see that the forms of distortion theorems of the Fréchet-derivative type for a subclass of starlike mappings are similar to each other. We may propose the following open problem.

Open Problem 2.1 *Suppose that* $g: U \to C$ *satisfies the condition of Definition* 1.1*. If* $F \in S^*_{g,k+1}(B)$*, then*

$$\begin{aligned}&\|x\|g(-\|x\|^k)\exp\int_0^{\|x\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\\ &\le \|DF(x)x\| \le \|x\|g(\|x\|^k)\exp\int_0^{\|x\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}, \quad x \in B,\end{aligned}$$

and the above estimates are sharp.

3 Sharp Distortion Theorems of the Jacobi-Determinant Type for a Subclass of Biholomorphic Mappings Which Have a Parametric Representation

In this section, we denote by $J_F(z)$ the Jacobi matrix of the holomorphic mapping $F(z)$, and let $\det J_F(z)$ be the Jacobi determinant of the holomorphic mapping $F(z)$. Also B is denoted by the unit ball of $\mathbb{C}^n$ with an arbitrary norm $\|\cdot\|$, and I_n is denoted by the unit matrix of $\mathbb{C}^n$. We usually write vectors in $\mathbb{C}^n$ as column vectors in this section.

Theorem 3.1 *Let* $g: U \to C$ *satisfy the condition of Definition* 1.1*,* $f: B \to \mathbb{C} \in H(B)$*, and* $F(z) = zf(z) \in S^*_g(B)$*. Then*

$$\begin{aligned}&g(-\|z\|^k)\exp n\int_0^{\|z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\\ &\le |\det J_F(z)| \le g(\|z\|^k)\exp n\int_0^{\|z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}, \quad z \in B,\end{aligned}$$

and the above estimates are sharp.

Proof Since $F(z)=zf(z)\in S_g^*(B)$, it is shown that

$$J_F(z)z=zf(z)+(J_f(z)z)z,\quad z\in B \tag{3.1}$$

by a direct computation. With the same arguments as in the proof of Theorem 2.1, it yields that $f(z)\neq 0$ for $z\in B$, and

$$f(z)+J_f(z)z\neq 0,\quad z\in B.$$

We see that

$$[J_F(z)]^{-1}F(z)=\frac{f(z)z}{f(z)+J_f(z)z},\quad z\in B \tag{3.2}$$

by a simple calculation. According to Lemma 2.1 (the case of $X=\mathbb{C}^n$), we see that

$$\frac{\|z\|}{g(\|z\|^k)}\leq |T_z[(J_F(z))^{-1}F(z)]|\leq \frac{\|z\|}{g(-\|z\|^k)},\quad z\in B. \tag{3.3}$$

Also we deduce that

$$J_F(z)=f(z)I_n+z(J_f(z))'=f(z)\Big(I_n+\frac{z(J_f(z))'}{f(z)}\Big)$$

from (3.1). In view of (3.2), it yields that

$$|\det J_F(z)|=|f(z)|^n\Big|1+\frac{J_f(z)z}{f(z)}\Big|=|f(z)|^n\frac{\|z\|}{|T_z[(J_F(z))^{-1}F(z)]|}.$$

Consequently, it is shown that

$$\begin{aligned}&g(-\|z\|^k)\exp n\int_0^{\|z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\\ &\leq|\det J_F(z)|\leq g(\|z\|^k)\exp n\int_0^{\|z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda},\quad z\in B\end{aligned}$$

from Theorem 2.1 (the case $X=\mathbb{C}^n$) and (3.3). The sharpness of Theorem 3.1 is similar to that in the proof of Theorem 2.1. We omit the details here. This completes the proof.

Remark 3.1 Taking $k=1$ in Theorem 3.1, it is easy to see that Theorem 3.1 generalizes [8, Theorem 4].

Setting $g(\xi)=\frac{1+\xi}{1-\xi}$ in Theorem 3.1, then we get the following corollary.

Corollary 3.1 *Let $f:B\to\mathbb{C}\in H(B)$, and $F(z)=zf(z)\in S_{k+1}^*(B)$. Then*

$$\frac{1-\|z\|^k}{(1+\|z\|^k)^{\frac{2n}{k}+1}}\leq|\det J_F(z)\|\leq\frac{1+\|z\|^k}{(1-\|z\|^k)^{\frac{2n}{k}+1}},\quad z\in B$$

and the above estimates are sharp.

Letting $g(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorem 3.1, where $\alpha\in(0,1)$, we derive the following corollary.

Corollary 3.2 *Let $f:B\to\mathbb{C}\in H(B)$, $\alpha\in(0,1)$, and $F(z)=zf(z)\in S_{\alpha,k+1}^*(B)$. Then*

$$\frac{1-(1-2\alpha)\|z\|^k}{(1+\|z\|^k)^{\frac{2(1-\alpha)n}{k}+1}}\leq|\det J_F(z)\|\leq\frac{1+(1-2\alpha)\|z\|^k}{(1-\|z\|^k)^{\frac{2(1-\alpha)n}{k}+1}},\quad z\in B$$

and the above estimates are sharp.

Setting $g(\xi)=\frac{(1+\xi)^{\alpha}}{(1-\xi)^{\alpha}}$ in Theorem 3.1, where $\alpha\in(0,1]$, we obtain the following corollary.

Corollary 3.3 *Let $f:B\to\mathbb{C}\in H(B)$, $\alpha\in(0,1]$, and $F(z)=zf(z)\in SS^*_{\alpha,k+1}(B)$. Then*

$$\Big(\frac{1-\|z\|^k}{1+\|z\|^k}\Big)^{\alpha}\exp n\int_0^{\|z\|}\Big[\Big(\frac{1-\lambda^k}{1+\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}\le|\det J_F(z)|$$
$$\le\Big(\frac{1+\|z\|^k}{1-\|z\|^k}\Big)^{\alpha}\exp n\int_0^{\|z\|}\Big[\Big(\frac{1+\lambda^k}{1-\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda},\quad z\in B$$

and the above estimates are sharp.

Let $g(\xi)=\frac{1+\xi}{1-(1-2\alpha)\xi}$ in Theorem 3.1, where $\alpha\in[0,1)$. Then we derive the following corollary.

Corollary 3.4 *Let $f:B\to\mathbb{C}\in H(B)$, and $F(z)=zf(z)\in AS^*_{\alpha,k+1}(B)$.*
(i) *If $\alpha\in(0,1)$ and $\alpha\neq\frac{1}{2}$, then*

$$\frac{1-\|z\|^k}{(1+(1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{(1-2\alpha)k}n+1}}\le|\det J_F(z)|\le\frac{1+\|z\|^k}{(1-(1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{(1-2\alpha)k}n+1}},\quad z\in B$$

and the above estimates are sharp.
(ii) *If $\alpha=\frac{1}{2}$, then*

$$(1-\|z\|^k)\exp\Big(-n\frac{\|z\|^k}{k}\Big)\le|\det J_F(z)|\le(1+\|z\|^k)\exp\Big(n\frac{\|z\|^k}{k}\Big),\quad z\in B$$

and the above estimates are sharp.

Setting $g(\xi)=\frac{1+c\xi}{1-c\xi}$ in Theorem 3.1, where $c\in(0,1)$, we obtain the following corollary.

Corollary 3.5 *Let $f:B\to\mathbb{C}\in H(B)$, $c\in(0,1)$, and $F(z)=zf(z)\in SS^*_{k+1}(B)$. Then*

$$\frac{1-c\|z\|^k}{(1+c\|z\|^k)^{\frac{2}{k}n+1}}\le|\det J_F(z)\|\le\frac{1+c\|z\|^k}{(1-c\|z\|^k)^{\frac{2}{k}n+1}},\quad z\in B$$

and the above estimates are sharp.

From now on, let each m_l be a non-negative integer, $N=m_1+m_2+\cdots+m_n\in\mathbb{N}$, and $m_l=0$ means that the corresponding components in $Z=(Z_1,\cdots,Z_l,\cdots,Z_n)'$ and $F(Z)=(F_1(Z_1),\cdots,F_l(Z_l),\cdots,F_n(Z_n))'$ are vanished.

Theorem 3.2 *Let $g:U\to C$ satisfy the condition of Definition 1.1, $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, and $F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in S^*_{g,k+1}(U^N)$. Then*

$$g^n(-\|Z\|^k)\exp N\int_0^{\|Z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\le|\det J_F(Z)|\le g^n(\|Z\|^k)\exp N\int_0^{\|Z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, and the above estimates are sharp.

Proof According to the conditions of Theorem 3.2, for any $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, it yields that

$$J_F(Z)=\begin{pmatrix} J_{F_1}(Z_1) & 0 & \cdots & 0\\ 0 & J_{F_2}(Z_2) & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & J_{F_n}(Z_n)\end{pmatrix}. \tag{3.4}$$

With the same arguments as in the proof of Theorem 2.2, we have that

$$F\in S_g^*(U^n)\Leftrightarrow F_l\in S_g^*(U^{m_l}),\quad l=1,2,\cdots,n,$$

and $Z_l=0$ is at least a zero of order $k+1$ of each $F_l(Z_l)-Z_l$ $(l=1,2,\cdots,n)$ if $Z=0$ is a zero of order $k+1$ of $F(Z)-Z$. Note that $g(-t^k)\exp n\int_0^t[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$ (resp. $g(t^k)\exp n\int_0^t[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$ $(l=1,2,\cdots,n)$) is a decreasing function (resp. an increasing function) on the interval $[0,1)$ with respect to t. Also,

$$g(-\|Z_l\|^k)\exp m_l\int_0^{\|Z_l\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\leq|\det J_{F_l}(Z_l)|\leq g(\|Z_l\|^k)\exp m_l\int_0^{\|Z_l\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$

for $Z_l\in U^{m_l}$ $(l=1,2,\cdots,n)$. Hence the desired results follow from (3.4). The sharpness of Theorem 3.2 is similar to that in the proof of Theorem 2.2. The details are omitted here. This completes the proof.

Letting $g(\xi)=\frac{1+\xi}{1-\xi}$ in Theorem 3.2, we get the following corollary.

Corollary 3.6 *Let $g:U\to C$ satisfy the condition of Definition* 1.1, *$f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, and $F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in S_{k+1}^*(U^N)$. Then*

$$\frac{(1-\|Z\|^k)^n}{(1+\|Z\|^k)^{\frac{2N}{k}+n}}\leq|\det J_F(Z)\|\leq\frac{(1+\|Z\|^k)^n}{(1-\|Z\|^k)^{\frac{2N}{k}+n}}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, and the above estimates are sharp.

Let $g(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorem 3.2, where $\alpha\in(0,1)$. Then we derive the following corollary.

Corollary 3.7 *Let $g:U\to C$ satisfy the condition of Definition* 1.1, *$\alpha\in(0,1)$, $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, and*

$$F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in S_{\alpha,k+1}^*(U^N).$$

Then

$$\frac{(1-(1-2\alpha)\|Z\|^k)^n}{(1+\|Z\|^k)^{\frac{2(1-\alpha)N}{k}+n}}\leq|\det J_F(Z)\|\leq\frac{(1+(1-2\alpha)\|Z\|^k)^n}{(1-\|Z\|^k)^{\frac{2(1-\alpha)N}{k}+n}}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, and the above estimates are sharp.

We set $g(\xi)=\frac{(1+\xi)^\alpha}{(1-\xi)^\alpha}$ in Theorem 3.2, where $\alpha\in(0,1]$. Then it is obvious to show that the following corollary holds.

Corollary 3.8 *Let $g:U\to C$ satisfy the condition of Definition* 1.1, *$\alpha\in(0,1]$, $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, and $F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in$*

$SS^*_{\alpha,k+1}(U^N)$. *Then*

$$\Big(\frac{1-\|Z\|^k}{1+\|Z\|^k}\Big)^{n\alpha}\exp N\int_0^{\|Z\|}\Big[\Big(\frac{1-\lambda^k}{1+\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}\le|\det J_F(Z)|$$
$$\le\Big(\frac{1+\|Z\|^k}{1-\|Z\|^k}\Big)^{n\alpha}\exp N\int_0^{\|Z\|}\Big[\Big(\frac{1+\lambda^k}{1-\lambda^k}\Big)^{\alpha}-1\Big]\frac{\mathrm{d}\lambda}{\lambda}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, *and the above estimates are sharp.*

Let $g(\xi)=\frac{1+\xi}{1-(1-2\alpha)\xi}$ in Theorem 3.2, where $\alpha\in[0,1)$. Then we derive the following corollary.

Corollary 3.9 *Let* $g:U\to C$ *satisfy the condition of Definition* 1.1, $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, *and* $F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in AS^*_{\alpha,k+1}(U^N)$.

(i) *If* $\alpha\in(0,1)$ *and* $\alpha\ne\frac{1}{2}$, *then*

$$\frac{(1-\|Z\|^k)^n}{(1+(1-2\alpha)\|Z\|^k)^{\frac{2(1-\alpha)}{(1-2\alpha)k}N+n}}\le|\det J_F(Z)|\le\frac{(1+\|Z\|^k)^n}{(1-(1-2\alpha)\|Z\|^k)^{\frac{2(1-\alpha)}{(1-2\alpha)k}N+n}}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, *and the above estimates are sharp.*

(ii) *If* $\alpha=\frac{1}{2}$, *then*

$$(1-\|z\|^k)^n\exp\Big(-N\frac{\|z\|^k}{k}\Big)\le|\det J_F(z)|\le(1+\|z\|^k)^n\exp\Big(N\frac{\|z\|^k}{k}\Big)$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, *and the above estimates are sharp.*

Let $g(\xi)=\frac{1+c\xi}{1-c\xi}$ in Theorem 3.2, where $c\in(0,1)$. Then we get the following corollary.

Corollary 3.10 *Let* $g:U\to C$ *satisfy the condition of Definition* 1.1, $c\in(0,1)$, $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l})$, $l=1,2,\cdots,n$, *and* $F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in SS^*_{k+1}(U^N)$. *Then*

$$\frac{(1-c\|Z\|^k)^n}{(1+c\|Z\|^k)^{\frac{2}{k}N+n}}\le|\det J_F(Z)\|\le\frac{(1+c\|Z\|^k)^n}{(1-c\|Z\|^k)^{\frac{2}{k}N+n}}$$

for $Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, *and the above estimates are sharp.*

Remark 3.2 In view of Theorem 3.2 (the case $m_l=1,\ l=1,2,\cdots,n$), we may propose the following open problem.

Open Problem 3.1 *Let* $g:U\to C$ *satisfy the condition of Definition* 1.1, *and* $F\in S^*_{g,k+1}(U^n)$. *Then*

$$g^n(-\|z\|^k)\exp n\int_0^{\|z\|}[g(-\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}\le|\det J_F(z)|\le g^n(\|z\|^k)\exp n\int_0^{\|z\|}[g(\lambda^k)-1]\frac{\mathrm{d}\lambda}{\lambda}$$

for $z\in U^n$, *and the above estimates are sharp.*

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