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Augmentation Quotients for Complex Representation Rings of Generalized Quaternion Groups*

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Abstract Denote by $\mathcal{Q}_m$ the generalized quaternion group of order $4m$. Let $\mathcal{R}(\mathcal{Q}_m)$ be its complex representation ring, and $\Delta(\mathcal{Q}_m)$ its augmentation ideal. In this paper, the author gives an explicit $\mathbb{Z}$-basis for the $\Delta^n(\mathcal{Q}_m)$ and determines the isomorphism class of the n-th augmentation quotient $\frac{\Delta^n(\mathcal{Q}_m)}{\Delta^{n+1}(\mathcal{Q}_m)}$ for each positive integer n.

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1 Introduction

Let G be a finite group. A complex matrix representation (for convenience, we use "representation" in the sequel) of G is a group homomorphism

$$\rho : G \to GL_d(\mathbb{C}), \tag{1.1}$$

where $GL_d(\mathbb{C})$ is the complex general linear group of rank d ($d \in \mathbb{N}$). We also say that d is the degree of ρ (here we set $GL_0(\mathbb{C})$ to be the trivial group consisting of the empty matrix). Two representations ρ and η are said to be similar (denoted by $\rho \sim \eta$), if there exists an invertible square matrix P such that

$$\eta(g) = P^{-1}\rho(g)P, \quad \forall g \in G. \tag{1.2}$$

It is easy to see that similarity of representations is an equivalence relation. The equivalence classes are called similarity classes. The similarity class of ρ is denoted by $\overline{\rho}$. The direct sum $\overline{\rho} \oplus \overline{\eta}$ of two similarity classes $\overline{\rho}$ and $\overline{\eta}$ is defined by $\overline{\rho} \oplus \overline{\eta} = \overline{\rho \oplus \eta}$, where

$$\rho \oplus \eta : G \to GL_d(\mathbb{C}) \times GL_{d'}(\mathbb{C}) \rightarrowtail GL_{d+d'}(\mathbb{C}). \tag{1.3}$$

The complex representation ring $\mathcal{R}(G)$ is the group completion of the monoid (under direct sum $\oplus$) of similarity classes of representations of G. Its addition and multiplication are induced by the direct sum and the tensor product of matrices, respectively. By [1], $\mathcal{R}(G)$ is a commutative ring with an identity element. Its underlying group is a finitely generated free abelian

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group with a basis consists of the similarity classes of irreducible representations. Hence, by [2], its free rank is equal to the number of conjugacy classes of G.

The notion of degree of a representation induces a ring homomorphism

$$\phi : \mathcal{R}(G) \to \mathbb{Z}. \tag{1.4}$$

This homomorphism is called the augmentation map. Its kernel $\Delta(G)$ is called the augmentation ideal of $\mathcal{R}(G)$. Let $\Delta^n(G)$ and $Q_n(G)$ denote the n-th power of $\Delta(G)$ and the n-th consecutive quotient group $\frac{\Delta^n(G)}{\Delta^{n+1}(G)}$, respectively.

It is an interesting problem to determine the structures of $\Delta^n(G)$ and $Q_n(G)$ since they have many connections with other algebraic branches. The problem has been tackled in some papers [3–5]. In particular, the author and the collaborators proved in [3] that, for any finite abelian group G,

$$Q_n(G) \cong \frac{I^n}{I^{n+1}}, \tag{1.5}$$

where I is the augmentation ideal of the integral group ring $\mathbb{Z}G$. Karpilovsky raised the problem of determining the isomorphism type of the groups $\frac{I^n}{I^{n+1}}$ in [6]. The author and Tang Guoping solved it in [7] and thereby solved the problem for the groups $Q_n(G)$.

The goal of this article is to give an explicit $\mathbb{Z}$-basis for each $\Delta^n(\mathcal{Q}_m)$ and determine the isomorphism class of each $Q_n(\mathcal{Q}_m)$ for each positive integer n, where $\mathcal{Q}_m$ is the generalized quaternion group of order $4m$, $m \geqslant 2$.

The result also computes $\mathrm{Tor}_1^{\mathcal{R}(\mathcal{Q}_m)}\big(\frac{\mathcal{R}(\mathcal{Q}_m)}{\Delta^n(\mathcal{Q}_m)}, \frac{\mathcal{R}(\mathcal{Q}_m)}{\Delta(\mathcal{Q}_m)}\big)$ because for any finite group G, $Q_n(G) \cong \mathrm{Tor}_1^{\mathcal{R}(G)}\big(\frac{\mathcal{R}(G)}{\Delta^n(G)}, \frac{\mathcal{R}(G)}{\Delta(G)}\big)$.

2 Preliminaries

In this section, we provide some useful results about $Q_n(G)$ and the finite generated free abelian groups.

The following theorem and corollary (proved in [3]) are simple but useful.

Theorem 2.1 (cf. [3]) *For any natural number n, $Q_n(G)$ is a finite abelian $|G|$-torsion group.*

Corollary 2.1 (cf. [3]) *For each positive integer n, $\Delta^n(G)$ has free rank $c(G) - 1$, where $c(G)$ is the number of conjugacy classes of G.*

It is well known that any two representations with the same character are similar. Moreover, there is an injective ring homomorphism

$$\chi : \mathcal{R}(G) \to \mathbb{C}^G, \tag{2.1}$$

which sends $\overline{\rho}$ to its character χ_ρ for each representation ρ of G.

At last, we recall a classical result about the finite generated free abelian groups.

Lemma 2.1 *Let H be a finite generated free abelian group of rank N. If the N elements $g_1, \cdots, g_N$ generate H, then they form a basis of H.*

3 Necessary Tools

In this section, we construct some tools which are needed to prove the main results of this paper. They include some basic properties of $\Delta^n(\mathcal{Q}_m)$ and $Q_n(\mathcal{Q}_m)$. Recall that the generalized quaternion group of order $4m$ is defined as

$$\mathcal{Q}_m = \langle\, g, h \mid g^{2m} = h^4 = 1,\ g^m = h^2,\ h^{-1}gh = g^{-1}\rangle. \tag{3.1}$$

Hence, each representation ρ of $\mathcal{Q}_m$ depends only on its values at g and h. Therefore, we use

$$\begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(h) \end{pmatrix} \tag{3.2}$$

to denote ρ.

The following theorem found in [1] classifies the similarity classes of all irreducible representations of $\mathcal{Q}_m$

Theorem 3.1 *Let*

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.3}$$

$$\begin{cases} \rho_3 = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \rho_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \textit{if } m \textit{ is odd,} \\ \rho_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_4 = \begin{pmatrix} -1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, & \textit{if } m \textit{ is even,} \end{cases} \tag{3.4}$$

$$\eta_k = \begin{pmatrix} \xi^k & 0 & 0 & 0 \\ 0 & \xi^{-k} & 0 & 0 \\ 0 & 0 & 0 & (-1)^k \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{3.5}$$

where $\xi = \mathrm{e}^{\pi \frac{\mathrm{i}}{m}}$. *Then all distinct similarity classes of irreducible representations of* $\mathcal{Q}_m$ *are* $\overline{\rho}_1$, $\overline{\rho}_2$, $\overline{\rho}_3$, $\overline{\rho}_4$, $\overline{\eta}_k$, $1 \leqslant k \leqslant m-1$.

For later use, we remind the readers that by Corollary 2.1 and Theorem 3.1, $\Delta^n(\mathcal{Q}_m)$ has free rank $m+2$ for each natural number n.

In the rest of this section, we construct a basis of $\Delta(\mathcal{Q}_m)$ and show some of its basic properties. For convenience, we fix the following notation. Throughout, n and N are natural numbers.

(1) $E = \overline{\rho}_1 - \overline{\rho}_2$, $F = \overline{\rho}_1 - \overline{\rho}_3$, $Y_k = \overline{\eta}_k - \overline{\eta}_{k-1}$, $k \in \mathbb{Z}$.

(2) $\mathcal{S}_{n,j}(N) = \{F^j Y_k Y_1^{n-j-1} \mid 1 \leqslant k \leqslant N\}$, $n \geqslant 2$, $0 \leqslant j \leqslant n-1$.

(3) For any subset $\mathcal{S} \subset \mathcal{R}(\mathcal{Q}_m)$, denote by $\mathbb{Z}\mathcal{S}$ the set of all $\mathbb{Z}$-linear combinations of elements of $\mathcal{S}$.

(4) Denote by C_d the cyclic group of order d.

It is easy to see that $E, F, Y_k \in \Delta(\mathcal{Q}_m)$. Hence $\mathcal{S}_{n,j}(N) \subset \Delta^n(\mathcal{Q}_m)$.

Lemma 3.1 $\Delta(\mathcal{Q}_m)$ *is the free abelian group based on*

$$\mathcal{B}_m = \{E,\ F,\ Y_1, \cdots, Y_m\}. \tag{3.6}$$

Proof Note that the cardinality of $\mathcal{B}_m$ is $m+2$. So by Lemma 2.1, we only need to show that it generates $\Delta(\mathcal{Q}_m)$. Let

$$\omega = a_1\overline{\rho}_1 + a_2\overline{\rho}_2 + a_3\overline{\rho}_3 + a_4\overline{\rho}_4 + \sum_{k=1}^{m-1} c_k\overline{\eta}_k \in \Delta(\mathcal{Q}_m). \tag{3.7}$$

Then $a_1+a_2+a_3+a_4+2\sum\limits_{k=1}^{m-1} c_k = 0$. Short calculations show that $\overline{\eta}_0 = \overline{\rho}_1+\overline{\rho}_2$, $\overline{\eta}_m = \overline{\rho}_3+\overline{\rho}_4$. So

$$\begin{aligned}\omega &= a_1\overline{\rho}_1 + a_2\overline{\rho}_2 - (a_4-a_3)\overline{\rho}_3 + \sum_{k=1}^{m-1} c_k\overline{\eta}_k + a_4\overline{\eta}_m \\ &= a_1\overline{\rho}_1 + a_2\overline{\rho}_2 - (a_4-a_3)\overline{\rho}_3 + \Big(a_4 + \sum_{j=1}^{m-1} c_j\Big)\overline{\eta}_0 + X \\ &= \Big(a_1 + a_3 + \sum_{j=1}^{\frac{m}{2}-1} c_j\Big)\overline{\rho}_1 + \Big(a_2 + a_4 + \sum_{j=1}^{m-1} c_j\Big)\overline{\rho}_2 + (a_4-a_3)F + X \\ &= \Big(a_1 + a_3 + \sum_{j=1}^{\frac{m}{2}-1} c_j\Big)E + (a_4-a_3)F + X \\ &\in \mathbb{Z}\mathcal{B}_m,\end{aligned} \tag{3.8}$$

where $X = \sum\limits_{k=1}^{m-1}\big(a_4 + \sum\limits_{j=k}^{m-1} c_j\big)Y_k + a_4Y_m$.

Proposition 3.1 *Regarding elements of $\mathcal{R}(\mathcal{Q}_m)$, we have the following assertions:*

(1) *Y_k depends only on the residue class of k modulo $2m$, and $Y_k = -Y_{2m+1-k}$. Thus the set $\{Y_k \mid k\in\mathbb{Z}\}$ is equal to $\{Y_1,\cdots,Y_m\}$.*

(2) *$E^2 = 2E$, $EF = 2F + \sum\limits_{k=1}^{m} Y_k$ and $F^2 = \begin{cases} 2F-E, & \text{if } m \text{ is odd},\\ 2F, & \text{if } m \text{ is even}.\end{cases}$*

(3) *$EY_k = 0$ and $FY_k = Y_k - Y_{m+k} = Y_k + Y_{m+1-k}$.*

(4) *$Y_kY_l = (Y_{k+l} - Y_{k+l-1}) - (Y_{k-l+1} - Y_{k-l}) = \sum\limits_{j=-l+1}^{l-1} Y_{k+j}Y_1$.*

(5) *$\mathbb{Z}\mathcal{S}_{n,j}(N) = \mathbb{Z}\{F^j(Y_k-(2k-1)Y_1)Y_1^{n-j-2} \mid 2\leqslant k\leqslant N+1\}$, when $0\leqslant j\leqslant n-2$.*

Proof One can easily verify (1)–(3) by calculating the characters of relative representations. For (4), a short calculation shows that $\chi_{\eta_k}\chi_{\eta_l} = \chi_{\eta_{k+l}} + \chi_{\eta_{k-l}}$. So

$$\overline{\eta}_k\overline{\eta}_l = \overline{\eta}_{k+l} + \overline{\eta}_{k-l}. \tag{3.9}$$

Hence

$$\begin{aligned}Y_kY_l &= (\overline{\eta}_k - \overline{\eta}_{k-1})(\overline{\eta}_l - \overline{\eta}_{l-1}) \\ &= \overline{\eta}_{k+l} + 2\overline{\eta}_{k-l} - 2\overline{\eta}_{k+l-1} - \overline{\eta}_{k-l+1} - \overline{\eta}_{k-l-1} + \overline{\eta}_{k+l-2} \\ &= (Y_{k+l} - Y_{k+l-1}) - (Y_{k-l+1} - Y_{k-l}) \\ &= \sum_{j=-l+1}^{l-1}[(Y_{k+j+1} - Y_{k+j}) - (Y_{k+j} - Y_{k+j-1})]\end{aligned}$$

$$= \sum_{j=-l+1}^{l-1} Y_{k+j}Y_1. \tag{3.10}$$

We now consider (5). Due to (4), we get

$$Y_kY_1 = (Y_{k+1} - Y_k) - (Y_k - Y_{k-1}). \tag{3.11}$$

Recall that $Y_0 = -Y_1$. So for each natural number N,

$$O_N^2 \begin{pmatrix} Y_1Y_1 \\ Y_2Y_1 \\ \vdots \\ Y_NY_1 \end{pmatrix} = \begin{pmatrix} Y_2 - 3Y_1 \\ Y_3 - 5Y_1 \\ \vdots \\ Y_{N+1} - (2N+1)Y_1 \end{pmatrix}, \tag{3.12}$$

where

$$O_N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{N\times N} \in GL_N(\mathbb{Z}). \tag{3.13}$$

Therefore

$$\mathbb{Z}\mathcal{S}_{2,0}(N) = \mathbb{Z}\{Y_k - (2k-1)Y_1 \mid 2 \leqslant k \leqslant N+1\}. \tag{3.14}$$

From this, (5) follows.

4 Structure of $Q_n(\mathcal{Q}_m)$

This section is divided into two subsections, according to when m is even or odd.

4.1 m is an even number

We first give a basis of $\Delta^n(\mathcal{Q}_m)$ as a free abelian group. The following lemma is simple but useful.

Lemma 4.1 *For any* $1 \leqslant j \leqslant n-2$, $\mathcal{S}_{n,j+1}(m) \subset \mathbb{Z}\mathcal{S}_{n,j}(m)$.

Proof It is easy to see that we only need to show $\mathcal{S}_{3,2}(m) \subset \mathbb{Z}\mathcal{S}_{3,1}(m)$. By Proposition 3.1, we get

$$F(Y_k - (2k-1)Y_1) \in \mathbb{Z}\mathcal{S}_{3,1}(m), \quad 2 \leqslant k \leqslant m+1. \tag{4.1}$$

Note that $FY_{\frac{m}{2}} = FY_{\frac{m}{2}+1}$, so

$$2FY_1 = F(Y_{\frac{m}{2}} - (m-1)Y_1) - F(Y_{\frac{m}{2}+1} - (m+1)Y_1) \tag{4.2}$$

lies in $\mathbb{Z}\mathcal{S}_{3,1}(m)$. So does $2FY_k$ $(2 \leqslant k \leqslant m)$, since it equals

$$2F(Y_k - (2k-1)Y_1) + 2(2k-1)FY_1. \tag{4.3}$$

From this, the lemma follows since $F^2 = 2F$.

Theorem 4.1 *For any $n \geqslant 2$, $\Delta^n(\mathcal{Q}_m)$ is the free abelian group based on*

$$\{2^{n-1}E, 2^{n-1}F\} \cup \mathcal{S}_{n,0}\Big(\frac{m}{2}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m}{2}\Big). \tag{4.4}$$

Proof Note that (4.4) has cardinality $m+2$. So by Lemma 2.1, we only need to show that it generates $\Delta^n(\mathcal{Q}_m)$. Due to Lemma 3.1 and the fact that $EY_k = 0$, we get, for any $n \geqslant 2$, $\Delta^n(\mathcal{Q}_m)$ is generated by

$$\{E^j F^{n-j} \mid 0 \leqslant j \leqslant n\} \tag{4.5}$$

and all products

$$F^j \underbrace{Y_* \cdots Y_*}_{n-j}, \quad 0 \leqslant j \leqslant n-1. \tag{4.6}$$

A short calculation shows that (4.5) equals $\{2^{n-1}E, 2^{n-1}F, 2^{n-2}EF\}$. In addition, the generator $2^{n-2}EF$ can be omitted since

$$\begin{aligned} 2^{n-2}EF &= 2^{n-2}\Big(2F + \sum_{k=1}^{m} Y_k\Big) \\ &= 2^{n-1}F + 2^{n-2}\sum_{k=1}^{\frac{m}{2}} FY_k \\ &= 2^{n-1}F + \sum_{k=1}^{\frac{m}{2}} F^{n-1}Y_k. \end{aligned} \tag{4.7}$$

By Proposition 3.1, $Y_kY_l \in \mathbb{Z}\mathcal{S}_{2,0}(m)$. Hence, $\Delta^n(\mathcal{Q}_m)$ is generated by

$$\{2^{n-1}E, 2^{n-1}F\} \cup \Big(\bigcup_{j=0}^{n-1} \mathcal{S}_{n,j}(m)\Big). \tag{4.8}$$

Moreover, Lemma 4.1 implies that $\Delta^n(\mathcal{Q}_m)$ is generated by

$$\{2^{n-1}E,\ 2^{n-1}F\} \cup \mathcal{S}_{n,0}(m) \cup \mathcal{S}_{n,1}(m). \tag{4.9}$$

Recall that

$$FY_k = Y_k + Y_{\frac{m}{2}+1-k}. \tag{4.10}$$

So

$$\mathcal{S}_{n,0}(m) \subset \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m}{2}\Big) + \mathbb{Z}\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big). \tag{4.11}$$

In addition, one can easily verify that

$$\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big) \subset \mathbb{Z}\mathcal{S}_{n,1}(m), \quad \mathcal{S}_{n,1}(m) = \mathcal{S}_{n,1}\Big(\frac{m}{2}\Big). \tag{4.12}$$

It follows that $\Delta^n(\mathcal{Q}_m)$ is generated by (4.4).

Now we come to the main result of this subsection.

Theorem 4.2 *When m is an even number, we have*

$$Q_n(\mathcal{Q}_m) \cong \begin{cases} C_2^2 \oplus C_{2m}, & \text{if } n = 1, \\ C_2^3 \oplus C_{2m}, & \text{if } n \geqslant 2. \end{cases} \tag{4.13}$$

Proof By Proposition 3.1 and Theorem 4.1, we get

$$\begin{aligned} \Delta^{n+1}(\mathcal{Q}_m) &= 2^n\mathbb{Z}E + 2^n\mathbb{Z}F + \mathbb{Z}\mathcal{S}_{n+1,0}\Big(\frac{m}{2}\Big) + \mathbb{Z}\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big) \\ &= 2^n\mathbb{Z}E + 2^n\mathbb{Z}F + \mathbb{Z}\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big) \\ &\quad + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m}{2} + 1\Big\}. \end{aligned} \tag{4.14}$$

Note that

$$(Y_{\frac{m}{2}} - (m-1)Y_1) + (Y_{\frac{m}{2}+1} - (m+1)Y_1) = FY_{\frac{m}{2}} - 2mY_1. \tag{4.15}$$

So

$$\begin{aligned} \Delta^{n+1}(\mathcal{Q}_m) &= 2^n\mathbb{Z}E + 2^n\mathbb{Z}F + \mathbb{Z}\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big) + 2m\mathbb{Z}Y_1^n \\ &\quad + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\}. \end{aligned} \tag{4.16}$$

We consider the case $Q_1(\mathcal{Q}_m)$ first. It is easy to see that $\Delta(\mathcal{Q}_m)$ has the basis

$$\{E, F, Y_1\} \cup \Big\{Y_k - (2k-1)Y_1 \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\} \cup \mathcal{S}_{2,1}\Big(\frac{m}{2}\Big). \tag{4.17}$$

Thus

$$Q_1(\mathcal{Q}_m) \cong \frac{\mathbb{Z}E}{2\mathbb{Z}E} \oplus \frac{\mathbb{Z}F}{2\mathbb{Z}F} \oplus \frac{\mathbb{Z}Y_1}{2m\mathbb{Z}Y_1} \cong C_2^2 \oplus C_{2m}. \tag{4.18}$$

Secondly, by Proposition 3.1, for any $n \geqslant 2$,

$$\mathbb{Z}\mathcal{S}_{n+1,1}\Big(\frac{m}{2}\Big) = \mathbb{Z}\Big\{F(Y_k - (2k-1)Y_1)Y_1^{n-2} \mid 2 \leqslant k \leqslant \frac{m}{2} + 1\Big\}. \tag{4.19}$$

Applying (4.2), we get that (4.19) equals

$$2\mathbb{Z}FY_1^{n-1} + \mathbb{Z}\Big\{F(Y_k - (2k-1)Y_1)Y_1^{n-2} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\}. \tag{4.20}$$

Therefore

$$\begin{aligned} \Delta^{n+1}(\mathcal{Q}_m) &= 2^n\mathbb{Z}E + 2^n\mathbb{Z}F + 2m\mathbb{Z}Y_1^n + 2\mathbb{Z}FY_1^{n-1} \\ &\quad + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\} \\ &\quad + \mathbb{Z}\Big\{F(Y_k - (2k-1)Y_1)Y_1^{n-2} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\}. \end{aligned} \tag{4.21}$$

Like in (4.17), $\Delta^n(\mathcal{Q}_m)$ has the basis

$$\begin{aligned} &\{2^{n-1}E, 2^{n-1}F, Y_1^n, FY_1^{n-1}\} \cup \Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\} \\ &\cup \Big\{F(Y_k - (2k-1)Y_1)Y_1^{n-2} \mid 2 \leqslant k \leqslant \frac{m}{2}\Big\}. \end{aligned} \tag{4.22}$$

Thus

$$Q_n(\mathcal{Q}_m) \cong \frac{2^{n-1}\mathbb{Z}E}{2^n\mathbb{Z}E} \oplus \frac{2^{n-1}\mathbb{Z}F}{2^n\mathbb{Z}F} \oplus \frac{\mathbb{Z}Y_1^n}{2m\mathbb{Z}Y_1^n} \oplus \frac{\mathbb{Z}FY_1^{n-1}}{2\mathbb{Z}FY_1^{n-1}} \cong C_2^3 \oplus C_{2m}. \tag{4.23}$$

(4.18) and (4.23) together finish the proof.

4.2 m is an odd number

We study this case by using the same method as in the above subsection.

Lemma 4.2 *For any $n \geqslant 2$,*

$$\bigcup_{j=0}^{n-1} \mathcal{S}_{n,j}(m) \subset \mathbb{Z}F^{n-1}Y_{\frac{m+1}{2}} + \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big) + \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big). \tag{4.24}$$

Proof Recall that $EF = 2F + Y_{\frac{m+1}{2}} + \sum\limits_{k=1}^{\frac{m-1}{2}} FY_k$, $F^2 = 2F - E$. So

$$\begin{aligned} FY_k &= FY_{\frac{m+1}{2}} + FY_k - FY_{\frac{m+1}{2}} \\ &= FY_{\frac{m+1}{2}} + Y_k + Y_{m+1-k} - 2Y_{\frac{m+1}{2}} \\ &= FY_{\frac{m+1}{2}} + (\overline{\eta}_{\frac{m+1}{2}-k} - \overline{\eta}_0)Y_{\frac{m+1}{2}} \\ &= (F + \overline{\eta}_{\frac{m+1}{2}-k} - \overline{\eta}_0)\Big(EF - F^2 - E - \sum_{k=1}^{\frac{m-1}{2}} FY_k\Big) \\ &\in F(EF - F^2 - E) + \mathbb{Z}(\mathcal{S}_{3,1}(m) \cup \mathcal{S}_{3,2}(m)). \end{aligned} \tag{4.25}$$

Short calculations show that

$$\begin{aligned} \bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) &\subset \mathbb{Z}F^{n-1}(EF - F^2 - E) + \mathbb{Z}\Big(\bigcup_{j=1}^{n} \mathcal{S}_{n+1,j}(m)\Big) \\ &= \mathbb{Z}F^{n-1}Y_{\frac{m+1}{2}} + \mathbb{Z}\Big(\bigcup_{j=1}^{n} \mathcal{S}_{n+1,j}(m)\Big). \end{aligned} \tag{4.26}$$

Hence, by the fact that $FY_{\frac{m+1}{2}} = 2Y_{\frac{m+1}{2}}$, we get

$$\bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) \subset \mathbb{Z}F^{n-1}Y_{\frac{m+1}{2}} + \mathbb{Z}\Big(\bigcup_{j=1}^{2n-3} \mathcal{S}_{2n-2,j}(m)\Big). \tag{4.27}$$

Note that $F^2Y_k = (2F - E)Y_k = 2FY_k$. From this, it follows that

$$\mathcal{S}_{2n-2,j}(m) \subset \mathbb{Z}\mathcal{S}_{n,1}(m), \quad 1 \leqslant j \leqslant n-1. \tag{4.28}$$

For $n \leqslant j \leqslant 2n-3$, we have

$$\mathcal{S}_{2n-2,j}(m) = 2^{n-2}\mathcal{S}_{n,j-n+2}(m) \subset \mathbb{Z}\mathcal{S}_{n,1}(m), \tag{4.29}$$

since $2\mathcal{S}_{n,*+1}(m) \subset \mathcal{S}_{n,*}(m)$ in this case (as Lemma 4.1). So

$$\bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) \subset \mathbb{Z}F^{n-1}Y_{\frac{m+1}{2}} + \mathbb{Z}\mathcal{S}_{n,1}(m). \tag{4.30}$$

Now consider $\mathcal{S}_{n,0}(m)$. Since $Y_k = FY_k - Y_{\frac{m}{2}+1-k}$ and

$$Y_{\frac{m+1}{2}}Y_1 = FY_{\frac{m-1}{2}} - FY_{\frac{m+1}{2}} \in \mathbb{Z}\mathcal{S}_{2,1}(m), \tag{4.31}$$

we have

$$\mathcal{S}_{n,0}(m) \subset \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big) + \mathbb{Z}\mathcal{S}_{n,1}(m). \tag{4.32}$$

Then the lemma follows from the fact that $\mathcal{S}_{n,1}(m) = \mathcal{S}_{n,1}\big(\frac{m+1}{2}\big)$.

Lemma 4.3 *For any $n \geqslant 2$,*

$$\{E^j F^{n-j} \mid 0 \leqslant j \leqslant n\} \subset \mathbb{Z}\{E^2F^{n-2}, EF^{n-1}, F^n\}. \tag{4.33}$$

Proof The lemma is trivial for $n = 2$. Assume $n \geqslant 3$ in the sequel. Then for any $2 \leqslant j \leqslant n-1$, we have

$$\begin{aligned} E^jF^{n-j} - E^{j-1}F^{n-j+1} &= E^{j-1}F^{n-j-1}(EF - F^2) \\ &= E^{j-1}F^{n-j-1}\Big(2F + \sum_{k=1}^{m} Y_k - 2F + E\Big) \\ &= E^jF^{n-j-1}. \end{aligned} \tag{4.34}$$

Hence the lemma follows from $E^{j+1}F^{n-j-1} = 2E^jF^{n-j-1}$ since $E^2 = 2E$.

Lemma 4.4 *For any $n \geqslant 2$, $\Delta^n(\mathcal{Q}_m)$ is generated by*

$$\{EF^{n-1}, F^n, F^{n-1}Y_{\frac{m+1}{2}}\} \cup \mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big) \tag{4.35}$$

as an abelian group.

Proof Due to Lemmas 4.2–4.3, $\Delta^n(Q_m)$ is generated by

$$\{E^2F^{n-2}, EF^{n-1}, F^n\} \cup \{F^{n-1}Y_{\frac{m+1}{2}}\} \cup \mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big). \tag{4.36}$$

So we only need to show that (4.35) generates E^2F^{n-2}. Short calculations show

$$\begin{aligned} EF^{n-1} - F^n &= F^{n-2}(EF - F^2) \\ &= F^{n-2}\Big(E + Y_{\frac{m+1}{2}} + \sum_{k=1}^{\frac{m-1}{2}} FY_k\Big) \\ &= EF^{n-2} + F^{n-2}Y_{\frac{m+1}{2}} + \sum_{k=1}^{\frac{m-1}{2}} F^{n-1}Y_k. \end{aligned} \tag{4.37}$$

Hence, by the identities $E^2 = 2E$ and $FY_{\frac{m+1}{2}} = 2Y_{\frac{m+1}{2}}$, we get

$$\begin{aligned} E^2F^{n-2} &= 2(EF^{n-2} + F^{n-2}Y_{\frac{m+1}{2}}) - F^{n-1}Y_{\frac{m+1}{2}} \\ &= 2\Big(EF^{n-1} - F^n - \sum_{k=1}^{\frac{m-1}{2}} F^{n-1}Y_k\Big) - F^{n-1}Y_{\frac{m+1}{2}} \end{aligned} \tag{4.38}$$

as required.

Corollary 4.1 *For any $n \geqslant 3$, $\Delta^{n+1}(\mathcal{Q}_m)$ is generated by*

$$\{EF^n, F^{n+1}, F^n Y_{\frac{m+1}{2}}\} \cup \mathcal{S}_{n+1,0}\Big(\frac{m-1}{2}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big) \tag{4.39}$$

as an abelian group.

Proof It is easy to see that

$$\mathcal{S}_{n+1,1}\Big(\frac{m+1}{2}\Big) \subset \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big). \tag{4.40}$$

So by Lemma 4.4, we just need to show

$$\mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big) \subset \Delta^{n+1}(\mathcal{Q}_m). \tag{4.41}$$

Due to (4.25), we get

$$EF + FY_k \in \Delta^3(\mathcal{Q}_m). \tag{4.42}$$

Hence

$$FY_kY_1 = (EF + FY_k)Y_1 \in \Delta^4(\mathcal{Q}_m). \tag{4.43}$$

From this, the corollary follows.

Theorem 4.3 *For any $n \geqslant 2$, there exist three integers a_n, b_n, d_n with $2^{n-2}b_n + a_nd_n = 1$ such that $\Delta^n(\mathcal{Q}_m)$ is the free abelian group based on*

$$\{EF^{n-1}, F^n\} \cup \mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m-1}{2}\Big) \cup \{X_n\}, \tag{4.44}$$

where $X_n = a_nF^{n-1}Y_{\frac{m+1}{2}} + b_nFY_{\frac{m+1}{2}}Y_1^{n-2}$.

Proof Note that (4.44) has cardinality $m+2$. So we just need to show that (4.44) generates $\Delta^n(Q_m)$. Moreover, we only need to show that it generates $F^{n-1}Y_{\frac{m+1}{2}}$ and $FY_{\frac{m+1}{2}}Y_1^{n-2}$ by comparing it with (4.35). The theorem is trivial for $n = 2$ by setting $b_2 = 0$, $a_2 = d_2 = 1$.

Assume $n \geqslant 3$. By (3.12) and a short calculation, we get

$$2\begin{pmatrix} F^2Y_1 \\ F^2Y_2 \\ \vdots \\ F^2Y_{\frac{m+1}{2}} \end{pmatrix} = A_m O^2_{\frac{m+1}{2}} \begin{pmatrix} FY_1Y_1 \\ FY_2Y_1 \\ \vdots \\ FY_{\frac{m+1}{2}}Y_1 \end{pmatrix}, \tag{4.45}$$

where

$$A_m = \begin{cases} \begin{pmatrix} 0 & -1 \\ 4 & -3 \end{pmatrix}, & \text{if } m = 3, \\ \begin{pmatrix} 0 & 0 \\ 4I_{\frac{m-1}{2}} & 0 \end{pmatrix} + \begin{pmatrix} 0 \cdots 0 & 1 & 0 & -1 \\ 0 \cdots 0 & 3 & 0 & -3 \\ \vdots \quad \vdots & \vdots & \vdots & \vdots \\ 0 \cdots 0 & m & 0 & -m \end{pmatrix}, & \text{if } m \geqslant 5. \end{cases} \tag{4.46}$$

From this, it follows that

$$2^{n-2}\begin{pmatrix} F^{n-1}Y_1 \\ F^{n-1}Y_2 \\ \vdots \\ F^{n-1}Y_{\frac{m+1}{2}} \end{pmatrix} = (A_m O^2_{\frac{m+1}{2}})^{n-2} \begin{pmatrix} FY_1Y_1^{n-2} \\ FY_2Y_1^{n-2} \\ \vdots \\ FY_{\frac{m+1}{2}}Y_1^{n-2} \end{pmatrix}. \tag{4.47}$$

Consider the matrix $A_m O^2_{\frac{m+1}{2}}$. By direct calculation, we get

$$A_m O^2_{\frac{m+1}{2}} \equiv \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{M_{\frac{m+1}{2}}(2\mathbb{Z})}. \tag{4.48}$$

Hence

$$(A_m O^2_{\frac{m+1}{2}})^{n-2} \equiv \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{M_{\frac{m+1}{2}}(2\mathbb{Z})}. \tag{4.49}$$

Denote by d_n the integer in the lower right-hand corner of $\left(A_m O^2_{\frac{m+1}{2}}\right)^{n-2}$. Set

$$Z_n = 2^{n-2}F^{n-1}Y_{\frac{m+1}{2}} - d_n FY_{\frac{m+1}{2}}Y_1^{n-2}.$$

Then

(1) d_n is odd, so there exist two integers a_n, b_n such that $2^{n-2}b_n + a_n d_n = 1$,

(2) $Z_n \in \mathbb{Z}\mathcal{S}_{n,1}\left(\frac{m-1}{2}\right)$.

Therefore, either

$$F^{n-1}Y_{\frac{m+1}{2}} = d_n X_n + b_n Z_n \tag{4.50}$$

or

$$FY_{\frac{m+1}{2}}Y_1^{n-2} = 2^{n-2}X_n - a_n Z_n \tag{4.51}$$

is generated by (4.44), as required.

Theorem 4.4 *When m is an odd number, we have, for any natural number n,*

$$Q_n(\mathcal{Q}_m) \cong C_4 \oplus C_m. \tag{4.52}$$

Proof We compute $Q_1(\mathcal{Q}_m)$ first. By Lemma 4.4,

$$\begin{aligned} \Delta^2(\mathcal{Q}_m) &= \mathbb{Z}\{EF, F^2, FY_{\frac{m+1}{2}}\} + \mathbb{Z}\mathcal{S}_{2,0}\left(\frac{m-1}{2}\right) + \mathbb{Z}\mathcal{S}_{2,1}\left(\frac{m+1}{2}\right) \\ &= \mathbb{Z}\Big\{2F + Y_{\frac{m+1}{2}} + \sum_{k=1}^{\frac{m-1}{2}} FY_k, 2F - E, 2Y_{\frac{m+1}{2}}\Big\} \\ &\quad + \mathbb{Z}\mathcal{S}_{2,1}\left(\frac{m-1}{2}\right) + \mathbb{Z}\Big\{Y_k - (2k-1)Y_1 \mid 2 \leqslant k \leqslant \frac{m+1}{2}\Big\} \end{aligned}$$

$$= \mathbb{Z}\{2F + Y_{\frac{m+1}{2}}, 2F - E, 2Y_{\frac{m+1}{2}}, Y_{\frac{m+1}{2}} - mY_1\}$$
$$+ \mathbb{Z}\mathcal{S}_{2,1}\Big(\frac{m-1}{2}\Big) + \mathbb{Z}\Big\{Y_k - (2k-1)Y_1 \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\}. \tag{4.53}$$

The first part of the right-hand side of (4.53) equals

$$\mathbb{Z}\{2F - E, 2F + Y_{\frac{m+1}{2}}, 4F, m(2F + Y_1)\} \tag{4.54}$$

since

$$\begin{pmatrix} 2F - E \\ 2F + Y_{\frac{m+1}{2}} \\ 4F \\ m(2F + Y_1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ m & 0 & \frac{1-m}{2} & -1 \end{pmatrix} \begin{pmatrix} 2F + Y_{\frac{m+1}{2}} \\ 2F - E \\ 2Y_{\frac{m+1}{2}} \\ Y_{\frac{m+1}{2}} - mY_1 \end{pmatrix} \tag{4.55}$$

and the square matrix lies in $GL_4(\mathbb{Z})$. It is easy to verify that $\Delta(\mathcal{Q}_m)$ has the basis

$$\{2F - E, 2F + Y_{\frac{m+1}{2}}, F, 2F + Y_1\} \cup \mathcal{S}_{2,1}\Big(\frac{m-1}{2}\Big)$$
$$\cup \Big\{Y_k - (2k-1)Y_1 \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\}. \tag{4.56}$$

So

$$Q_1(\mathcal{Q}_m) \cong \frac{\mathbb{Z}F}{4\mathbb{Z}F} \oplus \frac{\mathbb{Z}(2F + Y_1)}{m\mathbb{Z}(2F + Y_1)} \cong C_4 \oplus C_m. \tag{4.57}$$

Secondly, by (4.31), we get

$$\begin{aligned}
\Delta^3(\mathcal{Q}_m) &= \mathbb{Z}\{EF^2, F^3, F^2Y_{\frac{m+1}{2}}\} + \mathbb{Z}\mathcal{S}_{3,0}\Big(\frac{m-1}{2}\Big) + \mathbb{Z}\mathcal{S}_{3,1}\Big(\frac{m+1}{2}\Big) \\
&= \mathbb{Z}\{2F^2 + FY_{\frac{m+1}{2}}, 2F^2 - EF, 2FY_{\frac{m+1}{2}}, Y_{\frac{m+1}{2}}Y_1 - mY_1^2, 4FY_1\} \\
&\quad + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1 \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\} \\
&\quad + \mathbb{Z}\Big\{F(Y_k - (2k-1)Y_1) \mid 2 \leqslant k \leqslant \frac{m+1}{2}\Big\} \\
&= \mathbb{Z}\{2F^2 - EF, 2F^2 + FY_{\frac{m+1}{2}}, 2FY_{\frac{m+1}{2}}, 2FY_1 + mY_1^2, 4FY_1, \\
&\quad FY_{\frac{m+1}{2}} - mFY_1\} + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1 \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\} \\
&\quad + \mathbb{Z}\Big\{F(Y_k - (2k-1)Y_1) \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\}.
\end{aligned} \tag{4.58}$$

The first part of the right-hand side of (4.58) equals

$$\mathbb{Z}\{2F^2 - EF, 2F^2 + FY_{\frac{m+1}{2}}, 2F^2 + FY_1, 4F^2, mY_1^2\} \tag{4.59}$$

since

$$\begin{pmatrix} 2F^2 + FY_{\frac{m+1}{2}} \\ 2F^2 + FY_1 \\ 4F^2 \\ mY_1^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -\frac{m-1}{2} & 0 & \frac{(m-1)^2}{4} & m-2 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & \frac{m-1}{2} & 2 \end{pmatrix} \begin{pmatrix} 2F^2 + FY_{\frac{m+1}{2}} \\ 2FY_{\frac{m+1}{2}} \\ 2FY_1 + mY_1^2 \\ 4FY_1 \\ FY_{\frac{m+1}{2}} - mFY_1 \end{pmatrix}, \tag{4.60}$$

$$\begin{pmatrix} 2F^2 + FY_{\frac{m+1}{2}} \\ 2FY_{\frac{m+1}{2}} \\ 2FY_1 + mY_1^2 \\ 4FY_1 \\ FY_{\frac{m+1}{2}} - mFY_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 4 & -2 & 0 \\ 1 & -m & \frac{m-1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2F^2 + FY_{\frac{m+1}{2}} \\ 2F^2 + FY_1 \\ 4F^2 \\ mY_1^2 \end{pmatrix}. \tag{4.61}$$

Due to Theorem 4.3, one can easily verify that $\Delta^2(\mathcal{Q}_m)$ has the basis

$$\begin{aligned} &\{2F^2 - EF, 2F^2 + FY_{\frac{m+1}{2}}, 2F^2 + FY_1, F^2, Y_1^2\} \\ &\cup \Big\{(Y_k - (2k-1)Y_1)Y_1 \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\} \\ &\cup \Big\{F(Y_k - (2k-1)Y_1) \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\}. \end{aligned} \tag{4.62}$$

Thus

$$Q_2(\mathcal{Q}_m) \cong \frac{\mathbb{Z}F^2}{4\mathbb{Z}F^2} \oplus \frac{\mathbb{Z}Y_1^2}{m\mathbb{Z}Y_1^2} \cong C_4 \oplus C_m. \tag{4.63}$$

Finally, by Corollary 4.1 and (4.31), we get, for any $n \geqslant 3$,

$$\begin{aligned} \Delta^{n+1}(\mathcal{Q}_m) &= \mathbb{Z}\{EF^n, F^{n+1}, F^n Y_{\frac{m+1}{2}}\} + \mathbb{Z}\mathcal{S}_{n+1,0}\Big(\frac{m-1}{2}\Big) + \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m+1}{2}\Big) \\ &= \mathbb{Z}\{2F^n - F^{n-1}Y_{\frac{m+1}{2}}, 2F^n - EF^{n-1}, 2F^{n-1}Y_{\frac{m+1}{2}}, mY_1^n, FY_{\frac{m+1}{2}}Y_1^{n-2}\} \\ &\quad + \mathbb{Z}\Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\} + \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m-1}{2}\Big). \end{aligned} \tag{4.64}$$

Thanks to (4.50)–(4.51) and the fact that Z_n lies in $\mathbb{Z}\mathcal{S}_{n,1}\big(\frac{m-1}{2}\big)$, the first part of the right-hand side of (4.64) can be replaced by

$$\mathbb{Z}\{2F^n - d_nX_n, 2F^n - EF^{n-1}, 2d_nX_n, mY_1^n, 2^{n-2}X_n\}. \tag{4.65}$$

Recall that d_n is an odd number, which implies $\gcd(2d_n, 2^{n-2}) = 2$. From this, it follows that (4.65) equals

$$\mathbb{Z}\{2F^n - d_nX_n, 2F^n - EF^{n-1}, 2X_n, mY_1^n\}, \tag{4.66}$$

and hence it equals

$$\mathbb{Z}\{2F^n - EF^{n-1}, 2F^n - X_n, 4F^n, mY_1^n\}, \tag{4.67}$$

since

$$\begin{pmatrix} 2F^n - EF^{n-1} \\ 2F^n - X_n \\ 4F^n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \frac{d_n-1}{2} \\ 2 & 0 & d_n \end{pmatrix} \begin{pmatrix} 2F^n - d_nX_n \\ 2F^n - EF^{n-1} \\ 2X_n \end{pmatrix} \tag{4.68}$$

and the square matrix lies in $GL_3(\mathbb{Z})$. Due to Theorem 4.3, $\Delta^n(\mathcal{Q}_m)$ has the basis

$$\begin{aligned}&\{2F^n - EF^{n-1}, 2F^n - X_n, F^n, Y_1^n\}\\ &\cup \Big\{(Y_k - (2k-1)Y_1)Y_1^{n-1} \mid 2 \leqslant k \leqslant \frac{m-1}{2}\Big\} \cup \mathcal{S}_{n,1}\Big(\frac{m-1}{2}\Big).\end{aligned} \tag{4.69}$$

Therefore, for any $n \geqslant 3$,

$$Q_n(\mathcal{Q}_m) \cong \frac{\mathbb{Z}F^n}{4\mathbb{Z}F^n} \oplus \frac{\mathbb{Z}Y_1^n}{m\mathbb{Z}Y_1^n} \cong C_4 \oplus C_m. \tag{4.70}$$

(4.57), (4.63) and (4.70) together finish the proof.

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