Chin. Ann. Math.
37B(4), 2016, 585–594
DOI: 10.1007/s11401-016-0956-6

Chinese Annals of Mathematics, Series B

Order Bounded Weighted Composition Operators Mapping into the Dirichlet Type Spaces*

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Abstract The authors characterize the order boundedness of weighted composition operators acting between Dirichlet type spaces.

Keywords Weighted composition operator, Order boundedness, Dirichlet type spaces, Positive Borel measure
2000 MR Subject Classification 47B38, 47B33

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$. Suppose that φ and h are analytic functions defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $W_{h,\varphi}$ is defined as

$$W_{h,\varphi} f = h(f \circ \varphi) \quad \text{for all } f \in H(\mathbb{D}).$$

When φ is identity on $\mathbb{D}$, the operator is the multiplication operator M_h. When $h \equiv 1$, it is the composition operator C_φ.

Weighted composition operators are general class of operators and they appear naturally in the study of surjective isometries on most of the function spaces, semigroup theory, dynamical systems, Brennan's conjecture, etc. Recently, there has been an increasing interest in studying weighted composition operators acting on different spaces of analytic functions (see [1–3, 12, 16–17, 20–22] and the references therein).

Let X be a Banach space of analytic functions in $\mathbb{D}$ and let $q > 0$. Let μ be a positive Borel measure on the unit circle. The operator $T : X \to L^q(\mu)$ is said to be order bounded if there exists $h \in L^q(\mu)$, $h \geq 0$ such that for each $f \in X$ with $\|f\|_X \leq 1$,

$$|T(f)(\mathrm{e}^{\mathrm{i}\theta})| \leq h(\mathrm{e}^{\mathrm{i}\theta}), \qquad \text{a.e. } [\mu].$$

Manuscript received May 10, 2014. Revised January 13, 2015.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11371276, 11301373, 11201331).

In particular, let φ be an analytic self-map of $\mathbb{D}$ such that $\varphi^* \in L^{p\beta},\ p \geq 1,\ \beta > 0$. In [10], H. Hunziker characterized the self-maps φ of the unit disk for which the composition operator $C_\varphi : H^p \to L^{p\beta}$ is order bounded. In this context, $C_\varphi f$ is supposed the boundary function $(f \circ \varphi)^*$.

In this paper, we focus on weighted composition operators acting on the Dirichlet type spaces. In contrast, the Dirichlet type spaces include functions that have no boundary values. Thus a discussion of order bounded weighted composition operators on Dirichlet type spaces will require the assumption that $|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})| < 1$ a.e. with respect to the normalized Lebesgue measure m. The composition operator C_φ on Hardy space or weighted Bergman spaces was investigated in [4]. Within setting of composition operators the order boundedness is connected with the boundedness or the compactness of them. For instance, H. Hunziker and H. Jarchow [11] proved that for $\beta \geq 1$, the order boundedness of $C_\varphi : H^p \to L^{\beta p}(\partial\mathbb{D}, m)$ implies the compactness of $C_\varphi : H^p \to L^{\beta p}$, where m is the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$. So the order boundedness is also an interesting subject in the study of composition operators. R. A. Hibschweiler [9] studied the order bounded weighted composition operators mapping into $L^{\beta p}(\partial\mathbb{D}, m)$. Recently Ueki [18] has studied the order boundedness of weighted composition operators on Bergman spaces. In this article, we characterize the order bounedeness of weighted composition operators on Dirichlet type spaces.

Recall that for $1 \leq p < \infty$, the classical Hardy space $\mathcal{H}^p$ consists of analytic functions f on $\mathbb{D}$, for which the norm

$$\|f\|_{H^p} = \Big(\sup_{0\leq r<1} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\frac{\mathrm{d}\theta}{2\pi}\Big)^{\frac{1}{p}} \quad \text{is finite.}$$

If $p = \infty$, $\mathcal{H}^\infty$ is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Fatou's theorem asserts that any Hardy function f has radial limit at $\mathrm{e}^{\mathrm{i}\theta} \in \partial\mathbb{D}$ except on a set Lebesgue measure zero. Throughout this work, $f(\mathrm{e}^{\mathrm{i}\theta})$ will denote the radial limit of f at $\mathrm{e}^{\mathrm{i}\theta}$, i.e., $f(\mathrm{e}^{\mathrm{i}\theta}) = \lim\limits_{r\to 1^-} f(r\mathrm{e}^{\mathrm{i}\theta})$. Let $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ denote the normalised Lebesgue area measure on $\mathbf{D}$. Also, let $\mathrm{d}A_\alpha(z) = (1+\alpha)\,(1-|z|^2)^\alpha\,\mathrm{d}A(z)$ denote the weighted Lebesgue area measure on $\mathbf{D}$, where $-1 < \alpha < \infty$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $L_a^{p,\alpha}$ consists of those functions f analytic on $\mathbf{D}$ such that

$$\|f\|_{L_a^{p,\alpha}} = \Big(\int_{\mathbf{D}} |f(z)|^p \,\mathrm{d}A_\alpha(z)\Big)^{\frac{1}{p}} < \infty.$$

If $\alpha = 0$, we get the Bergman space L_a^p.

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the spaces of Dirichlet type $\mathcal{D}_\alpha^p$ consist of those functions f analytic on $\mathbf{D}$ such that

$$\|f\|_{\mathcal{D}_\alpha^p} = \quad \Big(|f(0)|^p + \int_{\mathbf{D}} |f'(z)|^p \mathrm{d}A_\alpha(z)\Big)^{\frac{1}{p}} < \infty.$$

That is, $f \in \mathcal{D}^p_\alpha$ if and only if $f' \in L^{p,\alpha}_a$. For $\alpha = 0$, the spaces $\mathcal{D}^2_0$ is the classical Dirichlet space. For other values of p and α, the spaces $\mathcal{D}^p_\alpha$ have been extensively studied in number of papers (see [6–7, 15, 19]). The spaces $\mathcal{D}^p_\alpha$ are called Dirichlet spaces if $p \geq \alpha+1$. For $\alpha = 0$, the space $\mathcal{D}^2_0$ is the classical Dirichlet space. If $p < \alpha + 1$, then it is well known that $\mathcal{D}^p_\alpha = L^{p,\alpha-p}_a$ (see [5, Theorem 6]). Also $\mathcal{D}^2_1$ equals to the Hardy spaces H^2. Further, $\mathcal{D}^p_\alpha \subset \mathcal{D}^q_\alpha$, if $1 \leq q < p$. On the other hand, we have

$$\mathcal{D}^p_{p-1} \subset H^p, \quad 0 < p \leq 2 \tag{1.1}$$

and

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \leq p < \infty \tag{1.2}$$

and the inclusions are strict when $p \neq 2$. The inclusion (1.1) for $1 \leq p \leq 2$ can be proved by Riesz-Thorin Theorem and the case $0 < p < 1$ has been proved by Flett in [5]. The inclusion (1.1) follows by a classical result due to Littlewood and Paley [13], see also the proof by Luecking [14]. It is also well known that, for every p, the Hardy space H^p is contained in the Bergman space $A^{2p.}$ This is also true for the spaces $\mathcal{D}^p_{p-1}$, that is, we have $H^p \subset \mathcal{D}^p_{p-1} \subset A^{2p}$ for $2 \leq p < \infty$ and $\mathcal{D}^p_{p-1} \subset H^p \subset A^{2p}$ for $0 < p \leq 2$. If $0 < p \leq 2$ and $f \in \mathcal{D}^p_{p-1}$, then $f \in H^p$ and so f has non-tangential limit a.e. $\mathbb{T}$. Therefore we have that if $0 < p \leq 2$ and $f \in \mathcal{D}^p_{p-1}$, then $|f(r\mathrm{e}^{\mathrm{i}\theta})| = O(1)$ as $r \to 1^-$ for a.e. $\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}$. Zygmund proved in [23] that if f is an analytic function in $\mathbb{D}$, then

$$\int_0^r |f'(\rho \mathrm{e}^{\mathrm{i}t})| \mathrm{d}\rho = o\Big[\Big(\log\frac{1}{1-r}\Big)^{\frac{1}{2}}\Big], \quad \text{as } r \to 1^- \tag{1.3}$$

for almost every point $\mathrm{e}^{\mathrm{i}t}$ in the Fatou set of f, F_f, which consists of those $\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}$ such that f has a finite non-tangential limit at $\mathrm{e}^{\mathrm{i}\theta}$. Then obviously, (1.3) implies

$$|f(r\mathrm{e}^{\mathrm{i}t})| = o\Big[\Big(\log\frac{1}{1-r}\Big)^{\frac{1}{2}}\Big], \quad \text{as } r \to 1^-. \tag{1.4}$$

An analytic function f on $\mathbb{D}$ is said to belong to the Bloch-type space $\mathfrak{B}^\beta(\mathbb{D}) = \mathfrak{B}^\beta$ if

$$B(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)| < \infty.$$

The expression $B(f)$ defines a semi-norm while a natural norm is given by $\|f\|_{\mathfrak{B}^\beta} = |f(0)| + B(f)$. It makes $\mathfrak{B}^\beta$ into a Banach space.

2 Order Boundedness

In this section, we characterize the order boundedness of weighted composition operators $W_{h,\varphi}$ acting between Dirichlet type spaces. Before formulating and proving the results, we first give some auxiliary results.

For any $0<p<\infty,\ \alpha>-1$. Let $\delta_z(f)=f(z)$ for $f\in\mathcal{D}_\alpha^p$ and $z\in\mathbb{D}$. The following point evaluation estimate is frequently used in this area: If $0<p<\infty,\ p<\alpha+2,\ \alpha>-1$ and $f\in\mathcal{D}_\alpha^p$, we have

$$|f(z)|\le C_{p,\alpha}\frac{\|f\|_{\mathcal{D}_\alpha^p}}{(1-|z|^2)^{\frac{1}{p}(\alpha+2-p)}},\quad z\in\mathbb{D}. \tag{2.1}$$

We also have the following lemma.

Lemma 2.1 *Suppose $c>-1$ and $d\ge 0$. Let*

$$J_{c,d}(z)=\int_{\mathbb{D}}\frac{(1-|w|^2)^c\mathrm{d}A(w)}{|1-\overline{z}\omega|^{2+c+d}}.$$

Then we have the following asymptotic properties:

(1) *If $d>0$, then*

$$J_{c,d}(z)\asymp\frac{1}{(1-|z|^2)^d}.$$

(2) *If $d=0$, then*

$$J_{c,d}(z)\asymp\log\frac{1}{1-|z|^2}.$$

Now we can prove the following lemma.

Lemma 2.2 *For $0<p<\alpha+2$ and $\alpha>-1$, then there are positive constants C_1 and C_2 depending only on α and p such that*

$$\frac{C_1}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}}\le\|\delta_z\|\le\frac{C_2}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}}.$$

Proof By the equation (2.1), we have

$$|f(z)|\le C_2\frac{\|f\|_{\mathcal{D}_\alpha^p}}{(1-|z|^2)^{\frac{1}{p}(\alpha+2-p)}},$$

where C_2 depends only on α and p and $f\in\mathcal{D}_\alpha^p$. This yields the second inequality. For the remaining inequality, let

$$f_z(w)=\frac{(1-|z|^2)^{\frac{\alpha+2}{p}}}{(1-\bar{z}w)^{\frac{2\alpha+4-p}{p}}},\quad |z|<1.$$

Then by using Lemma 2.1, we can find $C_1>0$ such that $\|f_z\|_{\mathcal{D}_\alpha^p}\le C_1$. Therefore,

$$\|\delta_z\|\ge\frac{|f_z(z)|}{\|f_z\|_{\mathcal{D}_\alpha^p}}\ge C_1\Big(\frac{1}{1-|z|^2}\Big)^{\frac{\alpha+2-p}{p}}.$$

The next theorem is our first equivalent condition of the order boundedness of $W_{h,\varphi}$ acting on $\mathcal{D}_\alpha^p$.

Theorem 2.1 *Suppose $0<p,q<\infty,\ p<\alpha+2$ and $\alpha>-1$. Let $h\in L^q(m)$ and φ be an analytic self-map of the unit disk such that $|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|<1$ a.e. $[m]$. Then the following are equivalent:*

(i) $W_{h,\varphi}:\mathcal{D}^p_\alpha\to L^q(m)$ *is order bounded;*

(ii) $\frac{h}{(1-|\varphi^*|^2)^{\frac{1}{p}(\alpha+2-p)}}\in L^q(m)$.

Proof Suppose that $\frac{h}{(1-|\varphi^*|^2)^{\frac{\alpha+2-p}{p}}}\in L^q(m)$. Since $|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|<1$ a.e. $[m]$, then by Lemma 2.2, there is a constant C depending only on p and α such that $|f(\varphi^*(\mathrm{e}^{\mathrm{i}\theta}))|\le C(1-|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|^2)^{\frac{-(\alpha+2-p)}{p}}$ a.e. $[m]$ for all f with $\|f\|_{\mathcal{D}^p_\alpha}\le 1$. Let

$$t(\mathrm{e}^{\mathrm{i}\theta})=C|h(\mathrm{e}^{\mathrm{i}\theta})|(1-|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|^2)^{\frac{-(\alpha+2-p)}{p}}.$$

Then clearly $t\in L^q(m)$ by hypothesis and previous inequality implies that $|h(\mathrm{e}^{\mathrm{i}\theta})\|f(\varphi^*(\mathrm{e}^{\mathrm{i}\theta}))|\le t(\mathrm{e}^{\mathrm{i}\theta})$ a.e. $[m]$. Thus $W_{h,\varphi}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded.

Next suppose that $W_{h,\varphi}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded. Then there exists $t\in L^q(m),\ t\ge 0$ with $t(\mathrm{e}^{\mathrm{i}\theta})\ge|h(\mathrm{e}^{\mathrm{i}\theta})\|f(\varphi^*(\mathrm{e}^{\mathrm{i}\theta}))|$ a.e. $[m]$ for all f with $\|f\|_{\mathcal{D}^p_\alpha}\le 1$. Thus by Lemma 2.2, the inequality

$$\begin{aligned}t(\mathrm{e}^{\mathrm{i}\theta})&\ge|h(\mathrm{e}^{\mathrm{i}\theta})|\sup\{|\delta_{\varphi^*(\mathrm{e}^{\mathrm{i}\theta})}(f)|:\|f\|_{\mathcal{D}^p_\alpha}\le 1\}\\&=|h(\mathrm{e}^{\mathrm{i}\theta})|\cdot\|\delta_{\varphi^*(\mathrm{e}^{\mathrm{i}\theta})}\|\\&\ge C|h(\mathrm{e}^{\mathrm{i}\theta})|(1-|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|^2)^{\frac{-(\alpha+2-p)}{p}}\quad\text{holds a.e. }[m].\end{aligned}$$

It follows that $\frac{h}{(1-|\varphi^*|^2)^{\frac{\alpha+2-p}{p}}}\in L^q(m)$.

Remark 2.1 It is easy to check that under the condition of Theorem 2.1,

$$\frac{h}{(1-|\varphi^*|^2)^{\frac{\alpha+2-p}{p}}}\in L^q(m)$$

if and only if

$$\frac{h}{(1-|\varphi^*|)^{\frac{\alpha+2-p}{p}}}\in L^q(m).$$

Even though the requirements in Theorem 2.1 seem strong, the result is still useful in many nontrivial cases.

Example 2.1 Let $\varphi(z)=\frac{1}{2}(1+z)$ and $h(\mathrm{e}^{\mathrm{i}\theta})=1-\cos\frac{\theta}{2}$. Then

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{1-|\varphi^*|}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{1-\cos\frac{\theta}{2}}=\infty.$$

Thus we can know that the composition operator C_φ is not order bounded from $\mathcal{D}^2_2$ into $L^1(m)$. However, the behavior of h near the point 1 guarantees that

$$\int_0^{2\pi}\frac{h}{1-|\varphi^*|}\mathrm{d}\theta=2\pi<\infty,$$

so $W_{h,\varphi}$ is order bounded from $\mathcal{D}^2_2$ into $L^1(m)$, with the help of the weight h.

On the other hand, if we take $\widehat{\varphi}(z)=\frac{1}{2}(1-z)$, then even though with the same weight h, $W_{h,\widehat{\varphi}}$ is no longer order bounded from $\mathcal{D}^2_2$ into $L^1(m)$.

In fact, this example shows that the order boundedness of $W_{h,\varphi}$ is determined by h and φ together in the way we show in Theorem 2.1.

Corollary 2.1 *Suppose $0<p<\alpha+2$ and $\alpha>-1$. Let $h\in L^q(m)$ and φ be an analytic self-map of the unit disk such that $|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|<1$ a.e. $[m]$. Then the following are equivalent:*
(i) *$W_{h,\varphi}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded;*
(ii) *$W_{h,\varphi^n}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded for some $n\in\mathbb{N}^*$;*
(iii) *$W_{h,\varphi^n}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded for each $n\in\mathbb{N}^*$;*

Proof By using Theorem 2.1 and the inequality that

$$1-|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|\le 1-|(\varphi^*)^n(\mathrm{e}^{\mathrm{i}\theta})|\le n(1-|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|),$$

we get the desired result.

Remark 2.2 Suppose that $h\in L^\infty$, φ is an analytic self-map of the unit disk such that $|\varphi^*(\mathrm{e}^{\mathrm{i}\theta})|<1$ a.e. $[m]$ and $W_{h,\varphi}:\mathcal{D}^p_\alpha\to L^q(m)$ is order bounded for some $\alpha>-1$. By Theorem 2.1, $\frac{h}{(1-|\varphi^*|)^{\frac{1}{p}(\alpha+2-p)}}\in L^q(m)$. If we can even require $\alpha+1>p$, then it follows that

$$\int_0^{2\pi}\Big(\frac{|h|}{(1-|\varphi^*|)^{\frac{1}{p}}}\Big)^{(\alpha+2-p)q}\mathrm{d}m<\|h\|_\infty^{(\alpha+1-p)q}\int_0^{2\pi}\frac{|h|^q}{(1-|\varphi^*|)^{\frac{(\alpha+2-p)q}{p}}}\mathrm{d}m<\infty.$$

Thus $\frac{|h|}{(1-|\varphi^*|)^{\frac{1}{p}}}\in L^{(\alpha+2-p)q}$.

Now if $\alpha=p-2$, then $\mathcal{D}^p_\alpha=B_p$, the Besov space. Whenever $f\in\mathcal{D}^p_\alpha$, for $\alpha=p-2,\ \alpha>-1$ and $1<p<\infty$, we have the following inequality:

$$|f(z)|\le C\Big(\log\frac{2}{1-|z|}\Big)^{1-\frac{1}{p}}. \tag{2.2}$$

Moreover, we have the following lemma.

Lemma 2.3 *Let $0<p<\infty,\ \alpha=p-2$ and $\alpha>-1$. Then there are positive constants C_1 and C_2 depending only on p such that*

$$C_1\Big(\log\frac{2}{1-|z|}\Big)^{1-\frac{1}{p}}\le\|\delta_z\|\le C_2\Big(\log\frac{2}{1-|z|}\Big)^{1-\frac{1}{p}}.$$

Proof The existence of C_2 follows directly from (2.2). Now take

$$f_z(w)=\frac{\log\frac{2}{1-\overline{z}w}}{\big(\log\frac{2}{1-|z|^2}\big)^{\frac{1}{p}}},\quad |z|<1.$$

Then by using Lemma 2.1, we can find $C_1>0$ such that $\|f_z\|_{\mathcal{D}^p_\alpha}<C_1$. Therefore, we obtain

$$\|\delta_z\|\ge\frac{|f_z(z)|}{\|f_z\|_{\mathcal{D}^p_\alpha}}>C_1\Big(\log\frac{2}{1-|z|}\Big)^{1-\frac{1}{p}}.$$

The proof of the following theorem is almost similar to Theorem 2.1, so we omit the details.

Theorem 2.2 *Suppose* $0 < p, q < \infty$, $\alpha = p - 2$ *and* $\alpha > -1$. *Let* $h \in L^q(m)$ *and* φ *be an analytic self-map of the unit disk such that* $|\varphi^*(e^{i\theta})| < 1$ *a.e.* $[m]$. *Then the following are equivalent:*

(i) $W_{h,\varphi} : \mathcal{D}^p_\alpha \to L^q(m)$ *is order bounded;*

(ii) $\frac{h}{(\log \frac{2}{1-|z|})^{\frac{1}{p}-1}} \in L^q(m)$.

Now let's turn to the Hardy and Bergman spaces. We also begin with the following well known lemma.

Lemma 2.4 *Suppose* $0 < p < \infty$. *Let* δ_z^1 *and* δ_z^2 *denote the point evaluation functional in* H^p *and* A^p *respectively. Then we have*

$$\|\delta_z^1\| \asymp \frac{1}{(1-|z|^2)^{\frac{1}{p}}}$$

and

$$\|\delta_z^2\| \asymp \frac{1}{(1-|z|^2)^{\frac{2}{p}}}.$$

Again, by imitating the proof of Theorem 2.1, we get the following equivalent conditions.

Theorem 2.3 *Suppose* $0 < p, q < \infty$. *Let* $h \in L^q(m)$ *and* φ *be an analytic self-map of the unit disk such that* $|\varphi^*(e^{i\theta})| < 1$ *a.e.* $[m]$. *Then the following are equivalent:*

(i) $W_{h,\varphi} : H^p \to L^q(m)$ *is order bounded;*

(ii) $W_{h,\varphi} : A^{2p} \to L^q(m)$ *is order bounded;*

(iii) $\frac{h}{(1-|\varphi^*|)^{\frac{1}{p}}} \in L^q(m)$.

Finally in this section, we give the following results as corollaries of Theorems 2.1–2.3.

Corollary 2.2 *Suppose* $0 < p, q < \infty$. *Let* $h \in L^q(m)$ *and* φ *be an analytic self-map of the unit disk such that* $|\varphi^*(e^{i\theta})| < 1$ *a.e.* $[m]$. *Then composition operator* $W_{h,\varphi} : H^p \to L^q(m)$ *is order bounded if and only if* $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to L^q(m)$ *is order bounded.*

Corollary 2.3 *Fix* $0 < p < \infty$ *and* $h \in L^q(m)$. *Let* φ *be an analytic self-map of the unit disk such that* $|\varphi^*(e^{i\theta})| < 1$ *a.e.* $[m]$. *Then the following are equivalent:*

(i) $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to L^q(m)$ *is order bounded;*

(ii) $\frac{h}{(1-|\varphi^*|)^{\frac{1}{p}}} \in L^q(m)$.

3 Order Boundedness of $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$

In this section, we characterize the order boundedness of weighted composition operators $W_{h,\varphi}$ acting between Dirichlet type spaces. Recall that in this case, $W_{h,\varphi}$ is order bounded if and only if we can find $g \in L^q(A_{q-1})$, $g \geq 0$ such that for all $f \in \mathcal{D}^q_{q-1}$, $\|f\| \leq 1$, we have

$$|W_{h,\varphi}(f)'(z)| \leq g(z), \quad \text{a.e. } [A_{q-1}].$$

The proof of the following lemma follows similar lines as the proof of Lemma 3.2 of [8]. So we omit the proof.

Lemma 3.1 *Let $0 < p < \infty$ and $z \in \mathbb{D}$. Then the following hold:*
(i) $\sup\{|f(z)| : f \in \mathcal{D}^p_{p-1}, \|f\|_{\mathcal{D}^p_{p-1}} \le 1\} = \frac{1}{(1-|z|^2)^{\frac{1}{p}}}$;
(ii) $\sup\{|f'(z)| : f \in \mathcal{D}^p_{p-1}, \|f\|_{\mathcal{D}^p_{p-1}} \le 1\} = \frac{1}{(1-|z|^2)^{\frac{1+p}{p}}}$.

Theorem 3.1 *Let $0 < p, q < \infty$. If h and φ satisfy the condition*

$$\int_{\mathbb{D}} \frac{|h'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q}{p}}} \mathrm{d}A_{q-1}(z) + \int_{\mathbb{D}} \frac{|h(z)|^q|\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q(1+p)}{p}}} \mathrm{d}A_{q-1}(z) < \infty,$$

then $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is bounded.

Proof Let $f \in \mathcal{D}^p_{p-1}$ and $z \in \mathbb{D}$. Then by Lemma 3.1, we have

$$\begin{aligned} |(W_{h,\varphi} f(z))'| &= |(h(z)f(\varphi(z)))'| \\ &\le \frac{|h'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} \|f\|_{\mathcal{D}^p_{p-1}} + \frac{|h(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{(1+p)}{p}}} \|f\|_{\mathcal{D}^p_{p-1}}. \end{aligned}$$

Therefore,

$$\|(h(z)f(\varphi(z)))\|^q_{\mathcal{D}^q_{q-1}} \le \int_{\mathbb{D}} \frac{|h'(z)|^q \|f\|^q_{\mathcal{D}^p_{p-1}}}{(1-|\varphi(z)|^2)^{\frac{q}{p}}} \mathrm{d}A_{q-1}(z) + \int_{\mathbb{D}} \frac{|h(z)|^q|\varphi'(z)|^q \|f\|^q_{\mathcal{D}^p_{p-1}}}{(1-|\varphi(z)|^2)^{\frac{q(1+p)}{p}}} \mathrm{d}A_{q-1}(z).$$

Combining this with the assumption, we get the desired result.

Corollary 3.1 *Let $0 < p, q < \infty$. The weighted composition operator $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded if*

$$\int_{\mathbb{D}} \frac{|h'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q}{p}}} \mathrm{d}A_{q-1}(z) + \int_{\mathbb{D}} \frac{|h(z)|^q|\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q(1+p)}{p}}} \mathrm{d}A_{q-1}(z) < \infty. \tag{3.1}$$

Proof Suppose that condition (3.1) is true and take a function $f \in \mathcal{D}^p_{p-1}$ with $\|f\|_{\mathcal{D}^p_{p-1}} \le 1$. It follows from Theorem 3.1 that $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is bounded. Moreover, as in the proof of Theorem 3.1, we see that Lemma 3.1 gives

$$\begin{aligned} |(W_{h,\varphi} f(z))'| &= |(h(z)f(\varphi(z)))'| \\ &\le \frac{|h'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} \|f\|_{\mathcal{D}^p_{p-1}} + \frac{|h(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{(1+p)}{p}}} \|f\|_{\mathcal{D}^p_{p-1}}. \end{aligned}$$

Take $g = |h'|(1-|\varphi|^2)^{\frac{-1}{p}} + |h||\varphi'|(1-|\varphi|^2)^{\frac{-(1+p)}{p}}$. The condition (3.1) implies that $g \in L^q(\mathrm{d}A_{q-1})$ and $g \ge 0$. That is, the weighted composition operator $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded.

Theorem 3.2 *Let $0 < p, q < \infty$. If the weighted composition operator $W_{h,\varphi} : \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded, then*

$$\int_{\mathbb{D}} \Big| \frac{h'(z)}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} + \frac{h(z)\varphi(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{\frac{1+p}{p}}} \Big|^q \mathrm{d}A_{q-1}(z) < \infty. \tag{3.2}$$

Proof Since $W_{h,\varphi}: \mathcal{D}^p_{p-1} \to \mathcal{D}^q_{q-1}$ is order bounded, there exists a non-negative function $g \in L^q$ such that $|(W_{h,\varphi}f(z))'| \le g(z)$ for all $z \in \mathbb{D}$ and for all $f \in \mathcal{D}^p_{p-1}$ with $\|f\|^p_{\mathcal{D}^p_{p-1}} \le 1$. Fix a.e. $z \in \mathbb{D}$. Let

$$F(w) = \Big\{ \frac{1-|\varphi(z)|^2}{(1-w\overline{\varphi(z)})^2} \Big\}^{\frac{1}{p}}, \quad w \in \mathbb{D}.$$

Then

$$(W_{h,\varphi}f(w))' = h'(w) \frac{(1-|\varphi(z)|^2)^{\frac{1}{p}}}{(1-\varphi(w)\overline{\varphi(z)})^{\frac{2}{p}}} + h(w) \frac{\varphi(z)\varphi'(w)(1-|\varphi(z)|^2)^{\frac{1}{p}}}{(1-\varphi(w)\overline{\varphi(z)})^{\frac{2+p}{p}}}.$$

So by taking $w = z$, we can get

$$\Big| \frac{h'(z)}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} + \frac{h(z)\varphi(z)\varphi'(z)}{(1-|\varphi(z)|^2)^{\frac{1+p}{p}}} \Big| = |(W_{h,\varphi}f)'(z)| \le g(w).$$

Then the result follows directly since $g \in L^q$.

Acknowledgements The authors would like to thank the referee for useful comments and suggestions which helped to improve the presentation of this paper. The second author is extremely thankful to Professor Daniel Girela for the kind suggestions.

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