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On the Number of Integral Ideals in Two Different Quadratic Number Fields*

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Abstract Let K be an algebraic number field of finite degree over the rational field $\mathbb{Q}$, and $a_K(n)$ the number of integral ideals in K with norm n. When K is a Galois extension over $\mathbb{Q}$, many authors contribute to the integral power sums of $a_K(n)$,

$$\sum_{n\le x} a_K(n)^l, \quad l = 1, 2, 3, \cdots .$$

This paper is interested in the distribution of integral ideals concerning different number fields. The author is able to establish asymptotic formulae for the convolution sum

$$\sum_{n\le x} a_{K_1}(n^j)^l a_{K_2}(n^j)^l, \quad j = 1, 2,\ l = 2, 3, \cdots ,$$

where K_1 and K_2 are two different quadratic fields.

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1 Introduction

Dirichlet series plays an important role in number theory. Given two Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s}, \quad \sum_{n=1}^{\infty} b_n n^{-s},$$

the convolution

$$\sum_{n=1}^{\infty} a_n b_n n^{-s}$$

of these two series is a classical object studied by many authors, especially in the theory of automorphic forms. In connection with the multidimensional arithmetic of Hecke E. (see [7, 11]), Linnik Yu. V. in [13] suggested to consider the scalar product of Hecke's L-function associated with Größencharakters and asked whether this function can be analytically continued to the whole complex plane. This is the well-known Linnik problem. In this paper, we will consider the generalizations of the special cases of the problem connected to the Linnik problem.

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Let K be an algebraic number field of finite degree d over the rational field $\mathbb{Q}$. Denote the number of integral ideals in K with norm n by $a_K(n)$. Then the Dedekind zeta-function $\zeta_K(s)$ is defined by, for $\sigma > 1$,

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} = \sum_{n=1}^{\infty} a_K(n) n^{-s}, \quad s = \sigma + \mathrm{i}t,$$

where $\mathfrak{a}$ varies over the integral ideals of K, and $\mathfrak{N}(\mathfrak{a})$ denotes its norm. Obviously, the Dedekind zeta function can be seen as the convolution of Riemann zeta function $\zeta(s)$ and itself.

Chandraseknaran and Good [1] showed that $a_K(n)$ is a multiplicative function, and satisfies

$$a_K(n) \leq \tau(n)^d, \tag{1.1}$$

where $\tau(k)$ is the divisor function, and $d = [K : \mathbb{Q}]$.

The number of integral ideals appeals to many authors. It was already known to Weber [21] that

$$\sum_{n \leq x} a_K(n) = c_K x + O(x^{1-\frac{1}{d}}), \tag{1.2}$$

where c_K is the residue of $\zeta_K(s)$ at its simple pole $s = 1$. The estimate on the error term in (1.1) was improved by Landau [12] to

$$\sum_{n \leq x} a_K(n) = c_K x + O(x^{1-\frac{2}{d+1}+\varepsilon}).$$

For quadratic fields, Huxley and Watt [8] established that

$$\sum_{n \leq x} a_K(n) = c_K x + O(x^{\frac{23}{73}} (\log x)^{\frac{315}{146}}).$$

For cubic fields, Müller [17] proved that

$$\sum_{n \leq x} a_K(n) = c_K x + O(x^{\frac{43}{96}+\varepsilon}).$$

For any algebraic number field of degree $d \geq 3$, Nowak [18] made important contributions, and showed that

$$\sum_{n \leq x} a_K(n) = c_K x + \begin{cases} O(x^{1-\frac{2}{d}+\frac{8}{d(5d+2)}} (\log x)^{\frac{10}{5d+2}}) & \text{for } 3 \leq d \leq 6, \\ O(x^{1-\frac{2}{d}+\frac{3}{2d^2}} (\log x)^{\frac{2}{d}}) & \text{for } d \geq 7. \end{cases}$$

The second moment of $a_K(n)$ was first considered in [2], where it was shown that if K is a Galois extension of $\mathbb{Q}$ of degree d, then

$$\sum_{n \leq x} a_K(n)^2 \sim c'_K x (\log x)^{d-1}, \quad \text{as } x \to \infty, \tag{1.3}$$

for a suitable constant c'_K.

Later, Chandraseknaran and Good [1] showed that if K is a Galois extension of $\mathbb{Q}$ of degree d, then for any $\varepsilon > 0$ and any integer $l \geq 2$, we have

$$\sum_{n \leq x} a_K(n)^l = xP_K(\log x) + O(x^{1-\frac{2}{d^l}+\varepsilon}), \tag{1.4}$$

where P_K denotes a suitable polynomial of degree $d^{l-1} - 1$.

In 2010, Lü and Wang [14] improved their result. If K is a Galois extension of $\mathbb{Q}$ of degree d, then for any $\varepsilon > 0$ and any integer $l \geq 2$, we have

$$\sum_{n \leq x} a_K(n)^l = xP_K(\log x) + O(x^{1-\frac{3}{d^l+6}+\varepsilon}),$$

where P_K denotes a suitable polynomial of degree $d^{l-1} - 1$. Furthermore, for Abelian extensions K, some stronger results have also been established.

Recently, Lü and Yang [15] studied the average behavior of the coefficients of Dedekind zeta function over square numbers. For example, it was proved that for Galois fields of degree d which is odd, we have

$$\sum_{n \leq x} a_K(n^2)^l = xP_m(\log x) + O(x^{1-\frac{3}{md+3}+\varepsilon}),$$

where $m = \left(\frac{d+1}{2}\right)^l d^{l-1}$, and $P_m(t)$ is a polynomial in t of degree $m - 1$.

In this paper, we will discuss the special cases of the Linnik problem in different quadratic fields. Let K_1 and K_2 be two different quadratic fields. We are interested in convolution sum

$$\sum_{n \leq x} a_{K_1}(n^j)^l a_{K_2}(n^j)^l, \quad j = 1, 2;\ l = 1, 2, 3, \cdots.$$

In this direction, Fomenko [3] proved that

$$\sum_{n \leq x} a_{K_1}(n) a_{K_2}(n) = c_{K_1,K_2} x + O(x^{\frac{1}{2}+\varepsilon}),$$

where c_{K_1,K_2} is a suitable constant.

We are able to prove the following results.

Theorem 1.1 *Let*

$$K_i = \mathbb{Q}(\sqrt{d_i})\ (i = 1, 2)$$

be the quadratic field of discriminant d_i. Assume that $(d_1, d_2) = 1$. Then for any $\varepsilon > 0$ and any integer $l \geq 2$, we have

$$\sum_{n \leq x} a_{K_1}(n)^l a_{K_2}(n)^l = xP_{K_1,K_2}(\log x) + O(x^{1-\frac{3}{4^l}+\varepsilon}),$$

where P_{K_1,K_2} denotes a suitable polynomial of degree $4^{l-1} - 1$.

Theorem 1.2 *Let*

$$K_i = \mathbb{Q}(\sqrt{d_i}) \ (i = 1, 2)$$

be the quadratic field of discriminant d_i. Assume that $(d_1, d_2) = 1$. Then for any $\varepsilon > 0$ and any integer $l \geq 1$, we have

$$\sum_{n \leq x} a_{K_1}(n^2)^l a_{K_2}(n^2)^l = x\overline{P}_{K_1,K_2}(\log x) + \begin{cases} O(x^{1-\frac{3}{10}+\varepsilon}) & \text{for } l = 1, \\ O(x^{1-\frac{3}{9^l}+\varepsilon}) & \text{for } l \geq 2, \end{cases}$$

where $\overline{P}_{K_1,K_2}$ denotes a suitable polynomial of degree $M^2 + 2M$, and $M = \frac{3^l-1}{2}$.

Remark 1.1 (1) When $l = 1$, by using our method, the result in Theorem 1.1 coincides with the result of Feomenko in [3];

(2) Under the GRH (Generalized Riemann hypothesis), we can improve the error term in the two theorems above as $O(x^{\frac{1}{2}+\varepsilon})$.

As an application, we can get the distribution of the integral ideals in a quadratic field with norm of the sum of two squares.

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with the discriminant d. $a_K(n)$ is the number of integral ideals with norm n in K. The distribution of integral ideals is important in algebraic number theory, and we are interested in the distribution of integral ideals with norm of the sum of two squares, i.e., we consider the average sum

$$\sum_{n^2+m^2 \leq x} a_K(n^2 + m^2).$$

We define the function $r(u)$ to be the number of solutions to $n^2 + m^2 = u$ in integers n, m. Then the generating Dirichlet series for $r(u)$ is equal to $4\zeta_{K'}(s)$, where $\zeta_{K'}(s)$ is the zeta function of the imaginary quadratic field $K' = \mathbb{Q}(\sqrt{-1})$.

On the other hand, we have

$$\sum_{n^2+m^2 \leq x} a_K(n^2 + m^2) = \sum_{u \leq x} a_K(u) \sum_{u=n^2+m^2} 1 = \sum_{u \leq x} a_K(u) r(u).$$

Now, let $K_1 = K,\ K_2 = K'$. According to Theorem 1.1, when $l = 1$, we have the following corollary.

Corollary 1.1 *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with discriminant d, and $a_K(n)$ the number of integral ideals with norm n. Then we have*

$$\sum_{n^2+m^2 \leq x} a_K(n^2 + m^2) = cx + O(x^{\frac{1}{2}+\varepsilon}), \tag{1.5}$$

where c is a suitable constant.

2 Proof of Theorem 1.1

For $\Re s \gg 1$, define

$$L_{K_1,K_2}(s) = \sum_{n=1}^{\infty} \frac{a_{K_1}(n)^l a_{K_2}(n)^l}{n^s}. \tag{2.1}$$

In fact, it absolutely converges in the half-plane $\Re s > 1$ on noting (1.1). Since $a_{K_1}(n)^l a_{K_2}(n)^l$ is a multiplicative arithmetic function, we have

$$L_{K_1,K_2}(s) = \prod_p \Big(1 + \frac{a_{K_1}(p)^l a_{K_2}(p)^l}{p^s} + \frac{a_{K_1}(p^2)^l a_{K_2}(p^2)^l}{p^{2s}} + \cdots\Big).$$

Obviously, the term $\frac{a_{K_1}(p)^l a_{K_2}(p)^l}{p^s}$ determines the analytic properties of $L_{K_1,K_2}(s)$ in the half-plane $\Re s > \frac{1}{2}$.

Let $K_1K_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be the composite field. K_1 and K_2 are two intermediate fields of the composite field K_1K_2. Let K_3 be another intermediate field of K_1K_2. Then it is well-known that

$$\zeta(s)^2 \zeta_{K_1K_2}(s) = \zeta_{K_1}(s)\zeta_{K_2}(s)\zeta_{K_3}(s).$$

(see the formula 45 on page 64 in Swinnerton-Dyer [20]). On the other hand, we have

$$\zeta_{K_i}(s) = \zeta(s)L(s,\chi_i), \quad i = 1, 2; \quad \zeta_{K_3}(s) = \zeta(s)L(s,\chi_3), \tag{2.2}$$

where χ_1 and χ_2 are two Dirichlet characters, and $\chi_3 = \chi_1\chi_2$. Hence we have

$$\zeta_{K_1K_2}(s) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_3). \tag{2.3}$$

By comparing the Euler products over prime numbers of both sides in (2.2)–(2.3), we have

$$a_{K_i}(p) = 1 + \chi_i(p) \ (i = 1, 2), \quad a_{K_1K_2}(p) = 1 + \chi_1(p) + \chi_2(p) + \chi_3(p),$$

where p is prime. Hence for any prime number p, we have

$$a_{K_1K_2}(p) = a_{K_1}(p)a_{K_2}(p).$$

The composite field K_1K_2 is a Galois extension of degree 4 over $\mathbb{Q}$. For a Galois extension K over $\mathbb{Q}$ of degree d, Chandraseknaran and Good [1] proved, by the well-known decomposition law of prime ideals, that except for finitely many prime numbers

$$a_K(p)^l = d^{l-1} a_K(p),$$

where l is any positive integer. In particular, we have

$$a_{K_1}(p)^l a_{K_2}(p)^l = a_{K_1K_2}(p)^l = 4^{l-1} a_{K_1K_2}(p).$$

By directly checking the Euler products of $L_{K_1,K_2}(s)$ and $\zeta_{K_1K_2}(s)^{4^{l-1}}$ in the region $\Re s > 1$, we have

$$L_{K_1,K_2}(s) = \zeta_{K_1K_2}(s)^{4^{l-1}} U(s), \tag{2.4}$$

where $U(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$.

Now we begin to complete the proof of Theorem 1.1. By (2.4) we learn that $L_{K_1,K_2}(s)$ can be analytically continued to the half-plane $\Re s > \frac{1}{2}$, where $s = 1$ is the only pole of order 4^{l-1}. Then by (2.1) and Perron's formula (see Proposition 5.54 in [10]), we have

$$\sum_{n\le x} a_{K_1}(n)^l a_{K_2}(n)^l = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} L_{K_1,K_2}(s) \frac{x^s}{s} \mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{2.5}$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. Here we have used (1.1). Next we move the integration to the parallel segment with $\Re s = \frac{1}{2} + \varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}
&\sum_{n\le x} a_{K_1}(n)^l a_{K_2}(n)^l \\
&= \frac{1}{2\pi \mathrm{i}} \Big\{ \int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} + \int_{\frac{1}{2}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{1}{2}+\varepsilon-\mathrm{i}T} \Big\} L_{K_1,K_2}(s) \frac{x^s}{s} \mathrm{d}s \\
&\quad + \operatorname*{Res}_{s=1} L_{K_1,K_2}(s) x + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\
&:= \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + x P_{K_1,K_2}(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).
\end{aligned} \tag{2.6}$$

For J_1, by (2.4) we have (noting that $l \ge 2$)

$$\begin{aligned}
\mathrm{J}_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \Big| L_{K_1,K_2}\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big)\Big| t^{-1} \mathrm{d}t \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \Big| \zeta_{K_1K_2}^{4^{l-1}}\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big)\Big| t^{-1} \mathrm{d}t \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \Big| \zeta_{K_1K_2}^{4^{l-1}-3}\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big)\Big| \\
&\quad \times \Big| \zeta^3\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big) \prod_{j=1}^{3} L\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t, \chi_j\Big)^3 \Big| t^{-1} \mathrm{d}t. \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \Big| \zeta^3\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big) \prod_{j=1}^{3} L\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t, \chi_j\Big)^3 \Big| t^{\frac{4^l-12}{6}-1} \mathrm{d}t,
\end{aligned}$$

where we have

$$\zeta\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t\Big) \ll (1 + |t|)^{\frac{1}{6}+\varepsilon} \tag{2.7}$$

and

$$\Big| L\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t, \chi\Big)\Big| \ll (1 + |t|)^{\frac{1}{6}+\varepsilon}. \tag{2.8}$$

These results can be derived from the two following results:

$$\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big) \ll (1+|t|)^{\frac{1}{6}}\log(|t|+1)$$

and

$$L\Big(\frac{1}{2}+\mathrm{i}t,\chi\Big) \ll (1+|t|)^{\frac{1}{6}}\log(|t|+1)$$

(see Theorems 24.1.1 and 24.2.1 in Pan and Pan [19]) by the Phragmen-Lindelöf principle for a strip (see Theorem 5.53 in Iwaniec and Kowalski [10]).

Then we have

$$\begin{aligned}
\mathrm{J}_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{\frac{4^l-12}{6}-1}\Big(\int_{\frac{T_1}{2}}^{T_1}\Big|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^{12}\mathrm{d}t\Big)^{\frac{1}{4}} \\
&\quad\times\prod_{j=1}^{3}\Big(\int_{\frac{T_1}{2}}^{T_1}\Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi_j\Big)\Big|^{12}\mathrm{d}t\Big)^{\frac{1}{4}}\Big\} \\
&\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{4^l}{6}-1+\varepsilon} + x^{\frac{1}{2}+\varepsilon},
\end{aligned} \tag{2.9}$$

where we have used

$$\int_{\frac{T_1}{2}}^{T_1}\Big|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^{12}\mathrm{d}t \ll T_1^{2+\varepsilon}$$

and

$$\int_{\frac{T_1}{2}}^{T_1}\Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi\Big)\Big|^{12}\mathrm{d}t \ll T_1^{2+\varepsilon}.$$

These results can be established by Gabriel's convexity theorem (see Lemma 8.3 in Ivić [9]), and the results of Heath-Brown [5] and Meurman [16] respectively, which state that

$$\int_{\frac{T_1}{2}}^{T_1}\Big|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big|^{12}\mathrm{d}t \ll T_1^{2}(\log T_1)^{17}$$

and

$$\int_{\frac{T_1}{2}}^{T_1}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,\chi\Big)\Big|^{12}\mathrm{d}t \ll T_1^{2+\varepsilon}.$$

By (2.7)–(2.8), for the integrals over the horizontal segments we have

$$\begin{aligned}
\mathrm{J}_2+\mathrm{J}_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma}|\zeta_{K_1K_2}^{4^{l-1}}(\sigma+\mathrm{i}T)|T^{-1}\mathrm{d}\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^{\sigma}T^{\frac{4^l}{3}(1-\sigma)+\varepsilon}T^{-1} \\
&= \max_{\frac{1}{2}+\varepsilon\le\sigma\le b}\Big(\frac{x}{T^{\frac{4^l}{3}}}\Big)^{\sigma}T^{\frac{4^l}{3}-1+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{4^l}{6}-1+\varepsilon}.
\end{aligned} \tag{2.10}$$

From (2.6), (2.9)–(2.10), we have

$$\sum_{n\le x} a_{K_1}(n)^l a_{K_2}(n)^l = xP_{K_1,K_2}(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{4^l}{6}-1+\varepsilon}). \tag{2.11}$$

On taking $T = x^{\frac{3}{4^l}}$ in (2.11), we have

$$\sum_{n\le x} a_{K_1}(n)^l a_{K_2}(n)^l = xP_{K_1,K_2}(\log x) + O(x^{1-\frac{3}{4^l}+\varepsilon}).$$

3 Proof of Theorem 1.2

We firstly recall some useful results. Let K be an algebraic number field of degree n, and then

$$\zeta_K\Big(\frac{1}{2}+\mathrm{i}t\Big) \ll t^{\frac{n}{6}+\varepsilon}$$

(see [6]).

By using the Phragmen-Lindelöf principle for a strip, we have that for $\frac{1}{2}\le\sigma\le 1+\varepsilon$,

$$\zeta_K(\sigma+\mathrm{i}t) \ll (1+|t|)^{\frac{n}{3}(1-\sigma)+\varepsilon}.$$

From (1.1), we can easily get

$$a_{K_1}(n^2)^l \le n^{\varepsilon}, \quad a_{K_2}(n^2)^l \le n^{\varepsilon}.$$

Define the series

$$L_{2,l}(s) = \sum_{n=1}^{\infty} \frac{a_{K_1}(n^2)^l a_{K_2}(n^2)^l}{n^s}, \tag{3.1}$$

and then it is absolutely convergent in the half plane $\Re s > 1$. Since $a_{K_i}(n)$ $(i = 1, 2)$ are multiplicative, so are $a_{K_1}(n^2)^l a_{K_2}(n^2)^l$. We can rewrite $L_{2,l}(s)$ as

$$L_{2,l}(s) = \prod_p \Big(1 + \frac{a_{K_1}(p^2)^l a_{K_2}(p^2)^l}{p^s} + \cdots\Big), \tag{3.2}$$

where the product runs over all primes.

For a quadratic field K, Lü and Yang [15] proved the relation

$$a_K(p^2)^l = 1 + M\cdot a_K(p)$$

holds true for all but finitely many primes, where $M = \frac{3^l-1}{2}$, and l is any positive integer. We can immediately deduce that except for finitely many primes,

$$\begin{aligned} a_{K_1}(p^2)^l a_{K_2}(p^2)^l &= 1 + M\cdot a_{K_1}(p) + M\cdot a_{K_2}(p) + M^2\cdot a_{K_1}(p)a_{K_2}(p) \\ &= 1 + M\cdot a_{K_1}(p) + M\cdot a_{K_2}(p) + M^2\cdot a_{K_1K_2}(p). \end{aligned}$$

By checking the Euler products of $L_{2,l}(s)$ and $\zeta(s)\zeta_{K_1}^M(s)\zeta_{K_2}^M(s)\zeta_{K_1K_2}^{M^2}(s)$, we have

$$L_{2,l}(s) = \zeta(s)\zeta_{K_1}^M(s)\zeta_{K_2}^M(s)\zeta_{K_1K_2}^{M^2}(s)U_1(s), \tag{3.3}$$

where $U_1(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$.

From (3.3), $L_{2,l}(s)$ admits a meromorphic continuation to the half-plane $\Re s > \frac{1}{2}$, and only has a pole at $s = 1$ of order $(M+1)^2$ in this region.

By applying Perron's formula, we have

$$\sum_{n\le x} a_{K_1}(n^2)^l a_{K_2}(n^2)^l = \frac{1}{2\pi \mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} L_{2,l}(s)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{3.4}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later.

We shift the path of integration to the vertical line

$$\Re s = \frac{1}{2} + \varepsilon.$$

By Cauchy's residue theorem, we get

$$\begin{aligned}
&\sum_{n\le x} a_{K_1}(n^2)^l a_{K_2}(n^2)^l \\
&= \frac{1}{2\pi \mathrm{i}}\Big\{\int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} + \int_{\frac{1}{2}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{1}{2}+\varepsilon-\mathrm{i}T}\Big\} L_{2,l}(s)\frac{x^s}{s}\mathrm{d}s \\
&\quad + \operatorname*{Res}_{s=1}\Big(L_{2,l}(s)\frac{x^s}{s}\Big) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\
&:= x\overline{P}_{K_1,K_2}(\log x) + \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),
\end{aligned} \tag{3.5}$$

where $\overline{P}_{K_1,K_2}(t)$ denotes a suitable polynomial in t of degree $M^2 + 2M$.

Case $l = 1$

The horizontal segments contribute

$$\begin{aligned}
\mathrm{J}_2 + \mathrm{J}_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \frac{x^\sigma|\zeta(\sigma+\mathrm{i}T)\zeta_{K_1}(\sigma+\mathrm{i}T)\zeta_{K_2}(\sigma+\mathrm{i}T)\zeta_{K_1K_2}(\sigma+\mathrm{i}T)|}{T}\mathrm{d}\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\le 1+\varepsilon}\{x^\sigma \cdot T^{\frac{1+2+2+4}{3}(1-\sigma)-1+\varepsilon}\} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{1}{2}+\varepsilon},
\end{aligned} \tag{3.6}$$

where we have used that $U_1(s)$ is absolutely convergent in the region $\Re s \ge \frac{1}{2}$ and behaves as $O(1)$ there.

For J_1, we have

$$\begin{aligned}
\mathrm{J}_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \Big|L_{2,l}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|t^{-1}\mathrm{d}t \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \Big|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\zeta_{K_1}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|
\end{aligned}$$

$$\times \left|\zeta_{K_2}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\zeta_{K_1K_2}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\right|t^{-1}\mathrm{d}t$$
$$= x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T \left|\zeta^4\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\cdot L^2\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_1\Big)\right|$$
$$\times \left|L^2\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_2\Big)\cdot L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_3\Big)\right|t^{-1}\mathrm{d}t$$
$$\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\log T\max_{T_1\le T}\Big\{T_1^{\frac{1}{6}-1}\Big(\int_{\frac{T_1}{2}}^{T_1}\zeta^{12}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\mathrm{d}t\Big)^{\frac{1}{3}}$$
$$\times\prod_{i=1}^{2}\Big(\int_{\frac{T_1}{2}}^{T_1}L^6\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_i\Big)\mathrm{d}t\Big)^{\frac{1}{3}}\Big\}$$
$$\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}T^{\frac{2}{3}+\frac{2}{3}\times\frac{5}{4}+\frac{1}{6}-1+\varepsilon}$$
$$= x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}T^{\frac{2}{3}+\varepsilon}. \tag{3.7}$$

Here we have used the following estimation:

$$\int_1^T L^6\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi\Big)\mathrm{d}t\ll T^{\frac{5}{4}}.$$

This is derived form the formulae

$$\int_1^T L^4\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi\Big)\mathrm{d}t\ll T^{1+\varepsilon},\quad \int_1^T L^{12}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi\Big)\mathrm{d}t\ll T^{2+\varepsilon}.$$

According to (3.6)–(3.7), we obtain

$$\sum_{n\le x}a_{K_1}(n^2)a_{K_2}(n^2)=x\overline{P}_{K_1,K_2}(\log x)+O\Big(\frac{x^{1+\varepsilon}}{T}\Big)+O(x^{\frac{1}{2}+\varepsilon}T^{\frac{2}{3}+\varepsilon}), \tag{3.8}$$

where $\overline{P}_{K_1,K_2}(t)$ is a polynomial in t of degree 3.

Taking $T=x^{\frac{3}{10}}$ to formula (3.8), then

$$\sum_{n\le x}a_{K_1}(n^2)a_{K_2}(n^2)=x\overline{P}_{K_1,K_2}(\log x)+O(x^{1-\frac{3}{10}+\varepsilon}).$$

Case $l\ge 2$

For J_1, by (3.3) we have

$$\mathrm{J}_1\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T\left|L_{2,l}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\right|t^{-1}\mathrm{d}t$$
$$\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T\left|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\zeta_{K_1}^M\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\right.$$
$$\left.\times\,\zeta_{K_1}^M\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\zeta_{K_1K_2}^{M^2}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\right|t^{-1}\mathrm{d}t$$
$$\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T\left|\zeta^{(M+1)^2}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)L^{M^2+M}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_1\Big)\right.$$
$$\left.\times\,L^{M^2+M}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_2\Big)L^{M^2}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\ \chi_3\Big)\right|t^{-1}\mathrm{d}t$$

$$\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \Big|\zeta^3\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\prod_{j=1}^{3} L^3\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi_j\Big)\Big|$$
$$\times t^{\frac{(2M+1)^2-12}{6}-1}\mathrm{d}t$$
$$\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{\frac{(2M+1)^2}{6}-3}\Big(\int_{\frac{T_1}{2}}^{T_1}\zeta^{12}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big)^{\frac{1}{4}}$$
$$\times\prod_{j=1}^{3}\Big(L^{12}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi_j\Big)\Big)^{\frac{1}{4}}\Big\}$$
$$\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{(2M+1)^2}{6}-1+\varepsilon}. \tag{3.9}$$

For the integration over horizontal segments, we have

$$\mathrm{J}_2+\mathrm{J}_3 \ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}\frac{x^\sigma|\zeta(\sigma+\mathrm{i}T)\zeta_{K_1}^M(\sigma+\mathrm{i}T)\zeta_{K_2}^M(\sigma+\mathrm{i}T)\zeta_{K_1K_2}^{M^2}(\sigma+\mathrm{i}T)|}{T}\mathrm{d}\sigma$$
$$\ll \max_{\frac{1}{2}+\varepsilon\le 1+\varepsilon}\{x^\sigma T^{\frac{1+2M+2M+4M^2}{3}(1-\sigma)-1+\varepsilon}\}$$
$$\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{(2M+1)^2}{6}-1+\varepsilon}. \tag{3.10}$$

By (3.9) and (3.10), we get

$$\sum_{n\le x} a_{K_1}(n^2)^l a_{K_2}(n^2)^l = x\overline{P}_{K_1,K_2}(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{(2M+1)^2}{6}-1+\varepsilon}), \tag{3.11}$$

where $\overline{P}_{K_1,K_2}(t)$ is a polynomial in t with degree M^2+2M.

Taking $T = x^{\frac{3}{(2M+1)^2}}$ in the formula (3.11), we obtain

$$\sum_{n\le x} a_{K_1}(n^2)^l a_{K_2}(n^2)^l = x\overline{P}_{K_1,K_2}(\log x) + O(x^{1-\frac{3}{9^l}+\varepsilon}). \tag{3.12}$$

We complete the proof.

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