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Symmetric Periodic Orbits and Uniruled Real Liouville Domains*

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Abstract A real Liouville domain is a Liouville domain with an exact anti-symplectic involution. The authors call a real Liouville domain uniruled if there exists an invariant finite energy plane through every real point. Asymptotically, an invariant finite energy plane converges to a symmetric periodic orbit. In this note, they work out a criterion which guarantees uniruledness for real Liouville domains.

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1 Introduction

For closed symplectic manifolds, Hu, Li and Ruan [8] have defined the notion of symplectical uniruledness by requiring the existence of a non-zero Gromov-Witten invariant of genus zero with a point constraint (see [8]). It was pointed out by Li [9], that it is not meaningful to naively mimic the definition of algebraic geometry, since he showed that for a simply-connected, closed symplectic manifold, there always exists a symplectic surface in a suitable fixed homology class going through a point. On the other hand, for Liouville domains, there are no non-constant holomorphic spheres, and McLean proposed to use holomorphic planes instead (see [14]). Roughly speaking, these holomorphic planes are asymptotic to periodic Reeb orbits, and therefore play an important role in the dynamics on the boundary of the Liouville domains. For instance, Hofer-Viterbo and Lu used a stretching construction involving related ideas to prove versions of the Weinstein conjecture (see [4, 11]).

Many interesting symplectic manifolds come equipped with a symmetry in the form of an anti-symplectic involution, also known as a real structure. In that case, one can investigate holomorphic curves that are invariant under this involution to get more specialized information. In the case of closed symplectic manifolds, this was done by Welschinger [20] in the form of real Gromov-Witten invariants and it is the subject of ongoing research (see for instance [3]). Therefore it is also natural to investigate the notion of uniruledness for Liouville domains that have such a symmetry. An application of this notion consists of existence results for symmetric periodic Reeb orbits.

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We will now state our notions and results more precisely. We define a real Liouville domain (W, λ, ϱ) as a triple consisting of a Liouville domain (W, λ) and an exact anti-symplectic involution $\varrho \in \mathrm{Diff}(W)$, i.e., a map ϱ satisfying

$$\varrho^2 = \mathrm{id}, \quad \varrho^*\lambda = -\lambda.$$

If we restrict ϱ to the boundary ∂W of the Liouville domain W, we get a real contact manifold, meaning a contact manifold together with an involution, under which the contact form is anti-invariant. If R denotes the Reeb vector field on ∂W, then R is anti-invariant under ϱ as well, i.e.,

$$\varrho_* R = T\varrho R = -R.$$

If $T > 0$ and $v \in C^\infty([0,T], \partial W)$ is a T-periodic orbit for R, then $v_\varrho \in C^\infty([0,T], \partial W)$ defined as

$$v_\varrho(t) = \varrho(v(T-t))$$

is a T-periodic orbit as well.

Definition 1.1 *A T-periodic orbit $v \in C^\infty([0,T], \partial W)$ is called symmetric if it satisfies $v = v_\varrho$.*

Symmetric periodic orbits play a prominent role in the restricted three body problem (see [2]) as well as in the Seifert conjecture on brake orbits (see [19]).

The Weinstein conjecture asserts that on every closed contact manifold, the Reeb flow admits a periodic orbit. Affirmative answers to this conjecture can be obtained in various cases by taking advantage of the interplay between holomorphic curves and closed Reeb orbits (see [4, 10–11, 21]). To examine this connection in the real case, we introduce the notion of an uniruled real Liouville domain. Note that for a real Liouville domain (W, λ, ϱ), the Liouville vector field X defined by the equation $\iota_X d\lambda = \lambda$ is invariant under ϱ and therefore ϱ extends to the completion V of W. By abuse of notation, we will also use the symbols λ and ϱ for the extensions to V. If we choose on V an SFT-like almost-complex structure anti-invariant under ϱ, then ϱ induces an involution of finite energy planes on V. Inspired by the paper of McLean [14], we give the following definition.

Definition 1.2 *A real Liouville domain (W, λ, ϱ) is called (real) uniruled if for every anti-invariant SFT-like complex structure J on the completion (V, λ, ϱ), there exists an invariant finite energy plane of SFT-energy less than or equal to 1 through any given point on the Lagrangian submanifold $\mathrm{Fix}(\varrho) \subset V$.*

Note that the above consists of a requirement for every anti-invariant complex structure J, which makes this notion meaningful. In particular, there are real Liouville domains without any finite energy plane for any SFT-like complex structure, such as the disk bundle $D^*\Sigma_g$ associated with a hyperbolic metric on a surface of higher genus. For a simply-connected example, consider the affine part of Fermat-type hypersurfaces,

$$\left\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum_{k=0}^{n} z_j^k = 1\right\} \cap B_R,$$

where B_R is a ball of sufficiently large radius R. If the degree k is larger than $3n-2$, then the Fredholm index of any finite energy plane is negative. This implies that for a generic choice of the SFT-like complex structure, no finite energy planes exist, and hence such an affine hypersurface is in particular not uniruled. Alternatively, one could use the real Gromov-Witten theory, developed in [20], to formulate the above notion.

The asymptotic behavior of finite energy planes as studied in [5–7, 16] immediately implies the following theorem.

Theorem 1.1 *Assume that* (W, λ, ϱ) *is a uniruled real Liouville domain. Then there exists a symmetric periodic orbit of the Reeb vector field* R *on* ∂W *of period less than or equal to* 1.

Remark 1.1 If one requires that the SFT-energy of the invariant finite energy planes in Definition 1.2 is less than or equal to a constant $\kappa > 0$ instead of being less than or equal to 1, the period of the symmetric Reeb orbit in Theorem 1.1 can be estimated from the above by the constant κ. However, we can always scale λ to $\frac{1}{\kappa}\lambda$ so that one does not gain anything by considering this more general notion.

The purpose of this note is to provide a condition which guarantees uniruledness for a real Liouville domain. For this, we embed the real Liouville domain into a closed symplectic manifold and use the Gromov-Witten theory on this ambient manifold. One could use Welschinger's invariants ("real Gromov-Witten theory") as are used for instance in [20], but we will argue indirectly. Let us now explain the properties we require on the ambient manifold.

Assume that (M, ω) is a closed symplectic manifold that satisfies the Bohr-Sommerfeld condition, that is, the cohomology class represented by the symplectic form is integral in the sense that the class $[\omega]$ lies in the image of $H^2(M;\mathbb{Z})$ in $H^2(M;\mathbb{R})$. We suppose in addition that $[\omega]$ is primitive in the sense that for every $k > 1$, the cohomology class $\frac{1}{k}[\omega]$ is not integral.

Definition 1.3 *We say that a symplectic hypersurface* $\Sigma \subset M$ *is primitive if* $[\Sigma]$ *is Poincaré dual to* $[\omega]$.

Remark 1.2 If $H^2(M;\mathbb{Z})$ is torsion-free, this notion is unambiguous. If $H^2(M;\mathbb{Z})$ has torsion, the class $[\omega] \in H^2_{dR}(M)$ does not uniquely determine an integral cohomology class. In this latter case, we mean that $[\Sigma]$ is Poincaré dual to $[\omega]$ when regarded as a real homology class in $H_{2n-2}(M;\mathbb{R})$.

Denote by $h\colon \pi_2(M) \to H_2(M;\mathbb{Z})$ the Hurewicz homomorphism.

Definition 1.4 *We say that a class* $A \in \mathrm{im}(h)$ *is decomposable if there exist classes* $B, C \in \mathrm{im}(h)$ *satisfying*

$$A = B + C, \quad \langle [\omega], B\rangle > 0, \quad \langle [\omega], C\rangle > 0.$$

We say that A *is indecomposable if it is not decomposable.*

Definition 1.5 *A decoration* $\mathcal{D} = (\Sigma, A, S)$ *of* (M, ω) *is a triple consisting of a primitive symplectic hypersurface* $\Sigma \subset M$, *an indecomposable homology class* $A \in H_*(M)$ *and a submanifold* $S \subset \Sigma$ *satisfying the following two requirements:*

(i) $A \circ [\Sigma] = 1$.

(ii) *The Gromov-Witten invariant* $GW_A([S], [p])$ *is odd, where* $[p]$ *is the homology class of a point.*

We refer to the triple $(M, \omega, \mathcal{D})$ *as a decorated symplectic manifold.*

Remark 1.3 By Gromov-Witten invariants we mean the variants defined in [13], and for this we insist that S should be a submanifold rather than a general cycle.

Remark 1.4 Note that for a decoration $\mathcal{D} = (\Sigma, A, S)$, we have

$$\langle [\omega], A\rangle = PD([\omega]) \circ A = [\Sigma] \circ A = 1$$

so that each holomorphic sphere contributing to the Gromov-Witten invariant $GW_A([S],[p])$ has a symplectic area equal to 1.

Definition 1.6 *Assume that $(M,\omega,\mathcal{D})$ is a decorated symplectic manifold with decoration $\mathcal{D}=(\Sigma,A,S)$. An anti-decorating involution $\rho\colon M\to M$ is an anti-symplectic involution satisfying the following conditions:*

(i) *Both Σ and S are invariant under ρ.*

(ii) $\rho_* A = -A$.

A decorated real symplectic manifold $(M,\omega,\mathcal{D},\rho)$ is a quadruple consisting of a decorated symplectic manifold $(M,\omega,\mathcal{D})$ together with an anti-decorating involution ρ.

Definition 1.7 *Assume that (W,λ,ϱ) is a real Liouville domain and $(M,\omega,\mathcal{D},\rho)$ is a decorated real symplectic manifold. An embedding of a real Liouville domain into a decorated symplectic manifold*

$$\varepsilon\colon (W,\lambda,\varrho)\to(M,\omega,\mathcal{D},\rho)$$

is an embedding $\varepsilon\colon W\to M\setminus\Sigma$ satisfying

$$\mathrm{d}\lambda=\varepsilon^*\omega,\quad \varrho=\varepsilon^*\rho.$$

A Christmas tree is a quadruple $(W,\lambda,\rho,\varepsilon)$ consisting of a real Liouville domain (W,λ,ρ) and an embedding $\varepsilon\colon(W,\lambda,\rho)\to(M,\omega,\mathcal{D},\rho)$ into a decorated real symplectic manifold.

The main result of this paper is the following theorem.

Theorem 1.2 *Assume that $(W,\lambda,\rho,\varepsilon)$ is a Christmas tree satisfying $b_1(W)=0$. Then (W,λ,ρ) is real uniruled.*

Combining Theorem 1.2 with Theorem 1.1, we obtain the following corollary.

Corollary 1.1 *Assume that $(W,\lambda,\rho,\varepsilon)$ is a Christmas tree satisfying $b_1(W)=0$. Then there exists a symmetric periodic orbit of a period less than or equal to 1 for the Reeb flow on ∂W.*

2 Definitions and Notions of the Symplectic Field Theory (SFT for short)

By a real symplectic manifold we mean a triple (M,ω,ρ) where (M,ω) is a symplectic manifold and $\rho\in\mathrm{Diff}(M)$ is an anti-symplectic involution, so

$$\rho^2=\mathrm{id},\quad \rho^*\omega=-\omega.$$

A Liouville domain is a compact exact symplectic manifold $(W,\omega=\mathrm{d}\lambda)$ with a global Liouville vector field, defined by $\iota_X\omega=\lambda$, such that the boundary is smooth and convex, meaning that the Liouville vector field X points outward at the boundary.

The boundary of a Liouville domain carries a natural cooriented contact structure. Indeed, the Liouville condition implies that $\alpha:=\lambda|_{\partial W}$ is a positive contact form on ∂W, so $\alpha\wedge(\mathrm{d}\alpha)^{n-1}>0$. The hyperplane distribution defined by

$$\xi=\ker\alpha\subset T\partial W$$

is called the contact structure and the vector field R on ∂W defined by the equations

$$\iota_R\alpha=1,\quad \iota_R\mathrm{d}\alpha=0$$

is called the Reeb vector field.

The following procedure can be used to complete a Liouville domain W into a so-called Liouville manifold, which has cylindrical ends instead of convex boundary components. For each boundary component C of ∂W, we attach the positive end of a symplectization, given by the symplectic manifold $([0,\infty[\times C, \mathrm{d}(\mathrm{e}^t\alpha))$, to W along C. The Liouville vector field on the cylindrical end is

$$X = \frac{\partial}{\partial t}.$$

After this process we obtain a complete Liouville manifold, which we will denote by (V,λ).

An almost-complex structure J on a complete Liouville manifold V is called compatible with the symplectic form $\omega = \mathrm{d}\lambda$ if $\omega(\cdot, J\cdot)$ is a Riemannian metric. An ω-compatible almost-complex structure J is called SFT-like if it satisfies the following conditions:

(1) J preserves the hyperplane distribution ξ on $\partial W \subset V$.

(2) On ∂W it rotates the Liouville vector field into the Reeb vector field in the sense that $JX = R$ and $JR = -X$.

(3) On the cylindrical end $\partial W \times [0,\infty[$ the almost-complex structure is invariant under the Liouville flow ϕ_X^t for $t \in [0,\infty)$.

Pick an SFT-like almost-complex structure J on V and assume that $w\colon (\mathbb{C}, i) \to (V, J)$ is a J-holomorphic plane. We now explain how to define the energy of w. This will be a variation of the Hofer energy. Choose a small $\delta > 0$, indicating the size of a collar neighborhood of ∂W, and define

$$\Lambda := \{\phi \in C^\infty(]-\delta,\infty[,[0,1]) : \phi' \geq 0,\ \phi'|_{]-\delta,0]} = 0\}.$$

For $\phi \in \Lambda$, define a 1-form $\lambda_\phi \in \Omega^1(V)$ by

$$\lambda_\phi(y) = \begin{cases} \phi(r)\alpha(x), & \text{if } y = (x,r) \in \partial W \times [0,\infty[, \\ \phi(0)\lambda(y), & \text{if } y \in W \end{cases}$$

and abbreviate $\omega_\phi = \mathrm{d}\lambda_\phi$. The Hofer energy or SFT energy of w is then defined as

$$E(w) = \sup_{\phi\in\Lambda} \int_{\mathbb{C}} w^*\omega_\phi \in [0,\infty].$$

The holomorphic plane w is called a finite energy plane if it satisfies

$$0 < E(w) < \infty.$$

We also have the following non-real version of uniruledness, somewhat different from [14].

Definition 2.1 *We call a Liouville domain (W,λ) uniruled if for every SFT-like almost-complex structure J on its completion (V,λ), there exists a finite energy plane through every point of V.*

3 Examples of Christmas Trees

In this section, we will discuss some examples of Christmas trees. An interesting example concerns the canonical contact form and structure on the unit cotangent bundle of a sphere, $(T^*S^n, \lambda_{\mathrm{can}}, \rho)$, which can be embedded as a real Liouville manifold into the projective quadric with various anti-symplectic involutions ρ. We will check that the projective quadric can be decorated by computing a suitable Gromov-Witten invariant. Real Liouville structures on

T^*S^2 include the regularized, planar circular restricted three-body problem (see [1]), which has one anti-symplectic involution, and the Hill's lunar problem, which has two commuting anti-symplectic involutions.

Before we verify the decoration requirements for the quadric, we start by giving the following basic lemma.

Lemma 3.1 *Let $(M, \omega, \mathcal{D} = (\Sigma, A, B))$ be a decorated symplectic manifold with an anti-decorating involution ρ. Then $M - \nu_M(\Sigma)$ carries the structure of a real Liouville domain, where $\nu_M(\Sigma)$ denotes a tubular neighborhood of Σ in M.*

Proof We first show that $W := M - \nu_M(\Sigma)$ is an exact symplectic manifold. For this, consider the long exact sequence of the pair in cohomology,

$$H^2(M, W) \stackrel{j_\Sigma^*}{\to} H^2(M) \stackrel{j_W^*}{\to} H^2(W).$$

By Corollary 11.2 of [15], the cohomology ring $H^*(M, W)$ is canonically isomorphic to the cohomology ring $H^*(\nu_M(\Sigma), \nu_M(\Sigma)_0)$, associated with the normal bundle of Σ. Here $\nu_M(\Sigma)_0$ denotes the normal bundle of Σ with its zero-section removed. Thus the Thom class $u \in H^2(\nu_M(\Sigma), \nu_M(\Sigma)_0)$ corresponds to a class u' in $H^2(M, W)$. As the homology class $[\Sigma]$ is Poincaré dual to $[\omega]$ (over the reals), it follows that $j_\Sigma^* u'$ equals $[\omega]$ by Problem 11-C from [15]. By exactness of the long exact sequence of the pair, we see $j_W^*[\omega] = j_W^* \circ j_\Sigma^* u' = 0$, so there exists a 1-form $\lambda \in \Omega_W^1$ such that $\mathrm{d}\lambda = \Omega := \omega|_W$.

We now show that we can choose a real Liouville form $\widetilde{\lambda}$, i.e., $\rho^* \widetilde{\lambda} = -\widetilde{\lambda}$. Since $\mathrm{d}\lambda = \Omega$ and $\rho^* \Omega = -\Omega$, we see that there exists a closed 1-form μ such that

$$\rho^* \lambda = -\lambda + \mu.$$

Since $\lambda = \rho^* \circ \rho^* \lambda = \lambda - \mu + \rho^* \mu$, we see that $\mu = \rho^* \mu$. Define $\widetilde{\lambda} := \lambda - \frac{1}{2}\mu$. Then $\rho^* \widetilde{\lambda} = \rho^* \lambda - \frac{1}{2}\rho^* \mu = -\lambda + \frac{1}{2}\mu = -\widetilde{\lambda}$. Hence $(W, \widetilde{\lambda}, \rho)$ is the desired real Liouville domain.

3.1 Smooth quadrics in a projective space

We define a quadric in a projective space as the zero-set of a non-zero homogeneous quadratic polynomial. Note that a homogeneous quadratic polynomial can always be written as $p(z) = z^t B z$, where B is a symmetric matrix. By Sylvester's theorem, we can assume that B is diagonal. We then easily see the following result.

Lemma 3.2 *A quadric is smooth if and only if B has the maximal rank.*

We have the following identification of the smooth projective quadric with an oriented Grassmannian.

Lemma 3.3 *The smooth projective quadric given by*

$$Q^n = \Big\{ [z_0, \cdots, z_{n+1}] \in \mathbb{CP}^{n+1} \Big| \sum_j z_j^2 = 0 \Big\}$$

is diffeomorphic to the symmetric space $Gr^+(2, n+2) \cong \frac{\mathrm{SO}(n+2)}{\mathrm{SO}(2)} \times \mathrm{SO}(n)$. Furthermore, $\mathrm{SO}(n+2)$ *acts transitively via biholomorphisms.*

Proof For the first part, we exhibit the diffeomorphism

$$\begin{aligned} Gr^+(2, n+2) &\to Q^n, \\ \mathrm{span}(x, y) &\mapsto x + \mathrm{i}y, \end{aligned}$$

where $x, y \in \mathbb{R}^{n+2}$ form an orthonormal basis of the 2-plane they span. We use that $\sum_j z_j^2 = \|x\|^2 - \|y\|^2 + 2\mathrm{i}\langle x, y\rangle$. To see that $\mathrm{SO}(n+2)$ acts by biholomorphisms, just observe that

$$\begin{aligned}\mathrm{SO}(n+2) \times Q^n &\to Q^n,\\ (A, [x+\mathrm{i}y]) &\mapsto [Ax + iAy] = [A(x+\mathrm{i}y)].\end{aligned}$$

By an affine quadric we mean the zero-set of a non-zero quadratic polynomial in $\mathbb{C}^{n+1}$. Away from possible singular points, an affine quadric inherits a symplectic structure as a complex submanifold of a Kähler manifold. It is well-known (see [12, Exercise 6.20]), that a smooth affine quadric is symplectomorphic to T^*S^n with its canonical symplectic structure.

Lemma 3.4 *There is a symplectomorphism*

$$\Big(V = \Big\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \Big| \sum_j {z_j}^2 = 1\Big\}, \omega_0\Big) \to (T^*S^n, \omega_{\mathrm{can}}) \subset T^*\mathbb{R}^{n+1},$$
$$z = x + \mathrm{i}y \mapsto \Big(\frac{x}{\|x\|}, \|x\|y\Big).$$

The singular affine quadric appearing in the following lemma is also of interest.

Lemma 3.5 *The symplectization of $(ST^*S^n, \lambda_{\mathrm{can}})$ is symplectomorphic to*

$$V_0 = \Big\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \Big| \sum_j {z_j}^2 = 0\Big\} \setminus \{0\}.$$

In addition, the standard complex structure i is an SFT-like complex structure for the symplectization.

3.2 Naive Gromov-Witten invariants of quadrics

We consider a smooth quadric Q^n given as the zero locus of the symmetric bilinear form B. The Lefschetz hyperplane theorem implies that for $n > 2$, we have $H_2(Q^n; \mathbb{Z}) \cong \mathbb{Z}$ (see [12, Example 4.27]). Moreover, this homology group is generated by a line L, by which we mean a map of the form $[\lambda : \mu] \in \mathbb{CP}^1 \mapsto \lambda p + \mu q$, where $p, q \in Q^n \subset \mathbb{CP}^{n+1}$ (so $B(p,p) = B(q,q) = 0$) and $B(p,q) = 0$. The quadric Q^2 in 4-dimensions is diffeomorphic to $S^2 \times S^2$, so $H_2(Q^2; \mathbb{Z}) \cong \mathbb{Z}^2$, and there are two types of lines, distinguished by their homology classes. We will equip Q^n with its natural complex structure J_0.

Let $\mathrm{Hol}(J_0, [L])$ denote the space of J_0-holomorphic maps from $\mathbb{CP}^1$ to Q^n representing the homology class $[L]$. Write $\mathcal{M}(J_0, [L])$ for the moduli space of J_0-holomorphic curves with the homology class $[L]$. We have

$$\mathcal{M}(J_0, [L]) = \frac{\mathrm{Hol}(J_0, [L])}{\mathrm{Aut}(\mathbb{CP}^1)}.$$

We will compute some Gromov-Witten invariants by "naive counting" (see [18]). To show that this works, we need to establish the regularity of J_0.

3.3 Moduli space and regularity

Let L be a line on a smooth projective quadric with a primitive homology class $[L] \in H_2(Q^n; \mathbb{Z})$. We linearize the Cauchy-Riemann equations at a parametrization of L given by $u : \mathbb{CP}^1 \to Q^n$.

Lemma 3.6 *The linearized operator at u is surjective. In particular, the space of holomorphic maps* $\mathrm{Hol}(J_0,[L])$ *in* Q^n *is a smooth manifold of dimension* $\dim \mathrm{Hol}(J_0,[L]) = 2n+2n$.

We give two arguments for this statement.

3.3.1 Regularity via sheaves and splitting of the normal bundle

In the language of sheaves, triviality of the cokernel is equivalent to vanishing of the sheaf cohomology group $H^1(L, \mathcal{T}Q^n|_L)$ (see the statement of Riemann-Roch). We have the short exact sequence of sheaves

$$0 \to \mathcal{T}L \to \mathcal{T}Q^n|_L \to \nu_L \to 0,$$

where ν_L is the sheaf of germs of holomorphic sections of the normal bundle of L. A piece of the corresponding long exact sequence in cohomology looks like

$$H^1(L, \mathcal{T}L) \to H^1(L, \mathcal{T}Q^n|_L) \to H^1(L, \nu_L).$$

It is a well-known classical fact that $H^1(\mathbb{CP}^1, \mathcal{O}(k)) = 0$ for $k \geq -1$ (a generalization of this formula is known as the Bott formula (see [17, Chapter 1]), so we see directly that $H^1(L, \mathcal{T}L) = 0$ as $\mathcal{T}L \cong \mathcal{O}(2)$. For the normal bundle, note that a line L in a smooth quadric Q^n is always contained in a tower of smooth quadrics of the form

$$L \subset Q^2 \subset Q^3 \subset \cdots \subset Q^n.$$

The normal bundle $\nu_{Q^k}(Q^{k-1})$ is isomorphic to $\mathcal{O}(1)$, and the normal bundle $\nu_{Q^2}(L)$ is trivial, so ν_L splits as

$$\mathcal{O}(1)^{n-2} \oplus \mathcal{O}.$$

By the earlier mentioned Bott formula $H^1(L, \nu_L) \cong H^1(\mathbb{CP}^1, \mathcal{O}(1))^{\oplus n-2} \oplus H^1(\mathbb{CP}^1, \mathcal{O}) = 0$, we conclude that $H^1(L, \mathcal{T}Q^n|_L) = 0$.

3.3.2 Regularity via holomorphic transitive actions

Lemma 3.3 tells us that we have a holomorphic transitive action on Q^n, so by [13, Proposition 7.4.3], every holomorphic sphere is regular, and the claim of the lemma follows.

3.4 Lines through a point

Now consider the evaluation map

$$\begin{aligned} ev : \mathrm{Hol}(J_0,[L]) \times_{\mathrm{Aut}(\mathbb{CP}^1)} \mathbb{CP}^1 &\to Q^n, \\ [u,z] &\mapsto u(z). \end{aligned}$$

By Sard's theorem we find a regular value p of ev, and in fact, since $SO(n+2)$ acts transitively on Q^n, every value is regular. Define the moduli space of lines through p as $\mathcal{M}_p = ev^{-1}(p)$.

Geometrically, we can describe $\mathcal{M}_p$ as follows. If $L = pq$ is a line through p and q which is completely contained in Q^n, then $B(\lambda p + \mu q, \lambda p + \mu q) = 0$ for all $[\lambda : \mu] \in \mathbb{CP}^1$. This gives a quadratic equation in λ and μ, which should vanish identically, so by looking at the coefficients, we find

$$B(p,p) = 0, \quad B(p,q) = 0, \quad B(q,q) = 0.$$

As p and q lie on Q^n, we automatically have $B(p,p) = 0 = B(q,q)$. The remaining equation defines a hyperplane in $\mathbb{CP}^{n+1}$, namely, the "geometric tangent plane"

$$P := \{ z \in \mathbb{CP}^{n+1} \mid B(p,z) = 0 \}.$$

Since every line through p intersects the quadric at infinity, which is given by $Q_\infty = \{z = [z_0 : \dots : z_n : 0] \mid z \in Q^n\}$, we can identify the moduli space of lines through p with $\mathcal{M}_p = Q_\infty \cap P$.

To obtain a Gromov-Witten invariant, we will consider lines through p going through an additional cycle C. First define

$$\begin{aligned} ev : \mathrm{Hol}(J_0, [L]) \times_{\mathrm{Aut}(\mathbb{CP}^1)} \mathbb{CP}^1 \times \mathbb{CP}^1 &\to Q^n \times Q^n, \\ [u; z_1, z_2] &\mapsto (u(z_1), u(z_2)). \end{aligned}$$

A dimension count tells us that C should be a 2-cycle if we want $ev^{-1}(\{p\} \times C)$ to consist of points. Hence we take C to be a line (which is of course a smooth submanifold) in Q_∞ which transversely intersects $\mathcal{M}_p$, regarded as a subset in Q_∞, in a point q_0. We get a unique element in $\mathcal{M}(J_0, [L]) \times_{\mathrm{Aut}(\mathbb{CP}^1)} \mathbb{CP}^1 \times \mathbb{CP}^1$ which maps to $(p, q) \in Q^n \times Q^n$, and we may represent this element by $(u; [0:1], [1:0])$.

To check that the evaluation map is transverse to $\{p\} \times C$, we observe that C is transverse to the set

$$\mathrm{Cone}(p, \mathcal{M}_p) = \{q \in Q^n \mid q \text{ lies on the line from } p \text{ to some point in } \mathcal{M}_p \subset Q_\infty \subset Q^n\}.$$

First we show that vectors of the form $(v, 0) \in T_p Q^n \times T_{q_0} Q^n$ lie in the image of $T_{[u;[0:1],[1:0]]} ev$. Indeed, put $p_s := \exp_p(sv)$, and follow the above procedure to define $\mathcal{M}_{p_s}$. For a small s, we find a unique intersection point $q_s := \mathcal{M}_{p_s} \cap C$. Therefore we find a variation $(u_s, [0:1], [1:0])$ which maps to (p_s, q_s) under ev. Note here that the curve q_s is tangent to C.

To see that a vector of the form $(0, w)$ also lies in the image of $T_{[u;[0:1],[1:0]]} ev$, we first note that we can assume that w lies in the tangent space to $\mathrm{Cone}(p, \mathcal{M}_p)$ since the normal to $\mathrm{Cone}(p, \mathcal{M}_p)$ is tangent to C. The curve $\widetilde{q}_s := \exp_p(sw)$ lies in $\mathrm{Cone}(p, \mathcal{M}_p)$, so by definition of this cone, we find a line from p to $\widetilde{q}_s$. Hence we find a variation $(u_s, [0:1], [1:z_s])$ which maps to $(p, \widetilde{q}_s)$.

We conclude the following result.

Proposition 3.1 *The 2-point Gromov-Witten invariant $GW^{Q^n}_{[L]}([p], [C])$ equals 1.*

We remind the reader that $H_2(Q^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, and there are two distinct homology classes $[L]$ represented by a line in this case. We collect the above results in the following theorem.

Theorem 3.1 *The projective quadric Q^n admits a decoration by $\mathcal{D} = (Q^{n-1}, [L], C)$, where $[L]$ is the homology class of a line and C is the submanifold described above.*

Remark 3.1 It is clear that the projective quadric has many anti-symplectic involutions. For instance, we can compose conjugation with swapping coordinates.

4 Existence of Invariant Curves

Complex conjugation on $\mathbb{CP}^1$ defines an anti-symplectic involution $R_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$, namely

$$\rho_0[z_0 : z_1] = [\overline{z}_0 : \overline{z}_1].$$

Now pick an ω-compatible almost-complex structure J on TM which is anti-invariant under ρ, so

$$\rho^* J = -J.$$

Denote the space of parametrized J-holomorphic maps from $\mathbb{CP}^1$ to M by $\mathrm{Hol}(J)$. We define an involution on this space:

$$I\colon \mathrm{Hol}(J) \to \mathrm{Hol}(J),$$
$$u \mapsto \rho \circ u \circ \rho_0.$$

Now we will write the fixed-point locus of this involution as

$$\mathrm{Hol}(J)^\rho = \{u \in \mathrm{Hol}(J) : I(u) = u\}.$$

Take a point $p \in M$, a submanifold $S \subset M$ and a spherical homology class $A \in \mathrm{im}(h)$, where $h\colon \pi_2(M) \to H_2(M;\mathbb{Z})$ is the Hurewicz homomorphism, and define

$$\mathrm{Hol}(J;(S,p;A)) = \{u \in \mathrm{Hol}(J) : u(\nu) \in S, u(\sigma) = p, [u] = A\},$$

where $\nu = [1:0] \in \mathbb{CP}^1$ is the "north-pole" and $\sigma = [0:1] \in \mathbb{CP}^1$ is the "south-pole". Note that both the north-pole and the south-pole lie on the real part $\mathbb{RP}^1 = \mathrm{Fix}(\rho_0) \subset \mathbb{CP}^1$. The parametrization has not yet been fully determined by just two marked points, so we still have a $\mathbb{C}^*$-action on this space. Later, we will mod out by this action.

Suppose now that S is invariant under ρ, that the point p lies in the Lagrangian $L = \mathrm{Fix}(\rho)$, and that the homology class A is anti-invariant, so $\rho_* A = -A$. Then the space $\mathrm{Hol}(J,(S,p;A))$ is invariant under the involution I and we set

$$\mathrm{Hol}^\rho(J,(S,p;A)) = \mathrm{Hol}(J,(S,p;A)) \cap \mathrm{Hol}(J)^\rho.$$

If $\Sigma \subset M$ is a symplectic submanifold, we will write $\mathscr{J}(\Sigma,\rho)$ for the space of all ω-compatible almost-complex structures on M, which are anti-invariant under the anti-symplectic involution ρ and are restricted on Σ to an $\omega|_\Sigma$-compatible almost-complex structure such that Σ becomes a J-holomorphic submanifold of M. We will denote the complement of Σ in M by Σ^c. The main result of this section is the following theorem.

Theorem 4.1 *Assume that $(M,\omega,\mathcal{D},\rho)$ is a decorated real symplectic manifold with decoration $\mathcal{D} = (\Sigma, A, S)$. Then for every point $p \in L \cap \Sigma^c$ and every almost-complex structure $J \in \mathscr{J}(\Sigma,\rho)$, the moduli space $\mathcal{M}_J^\rho(S,p;A) = \mathrm{Hol}^\rho(J,(S,p;A))/\mathbb{C}^*$ is nonempty.*

The proof of Theorem 4.1 needs some preparation. We first recall from [13, Section 2.5] that a holomorphic curve $u\colon \mathbb{CP}^1 \to M$ is called multiply covered if there exists a holomorphic curve $v\colon \mathbb{CP}^1 \to M$ and a holomorphic map $\phi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfying

$$u = v \circ \phi, \quad \deg(\phi) > 1.$$

If a curve is not multiply covered, it is called simple.

Lemma 4.1 *A holomorphic curve $u \in \mathrm{Hol}(J)$ is simple if and only if $I(u)$ is simple.*

Proof First suppose that u is simple and that $v \in \mathrm{Hol}(J)$ and $\phi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a holomorphic map such that

$$I(u) = v \circ \phi.$$

By using that I is an involution, we compute

$$u = I^2(u) = I(v\phi) = \rho v \phi \rho_0 = \rho v \rho_0 \rho_0 \phi \rho_0 = I(v) \circ (\rho_0 \phi \rho_0).$$

Since u is simple, by assumption, we conclude that

$$\deg(\phi) = \deg(\rho_0 \phi \rho_0) = 1$$

and therefore $I(u)$ is simple as well. This proves the "only if" part and the "if" part follows again from the fact that $I^2(u) = u$.

We now need the fact that $\mathrm{Aut}(\mathbb{CP}^1) = PSL_2(\mathbb{C})$.

Definition 4.1 *A simple holomorphic curve $u \in \mathrm{Hol}(J)$ is called a pseudo-fixed point if there exists $\phi \in PSL_2(\mathbb{C})$ such that $I(u) = u \circ \phi$. It is called a fixed point if ϕ is the identity, i.e., $I(u) = u$.*

Remark 4.1 It follows from [13, Proposition 2.5.1] that a simple holomorphic curve has no nontrivial automorphisms. Therefore the map ϕ for a pseudo-fixed point is uniquely determined.

Lemma 4.2 *Assume that $u \in \mathrm{Hol}(J)$ is a pseudo-fixed point, so that $I(u) = u\phi$ for some $\phi \in PSL_2(\mathbb{C})$. Then $\phi\rho_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is an anti-holomorphic involution.*

Proof It is clear that $\phi\rho_0$ is anti-holomorphic. To check that it is an involution, we compute

$$u = I^2(u) = I(u\phi) = \rho u \phi \rho_0 = \rho u \rho_0 \rho_0 \phi \rho_0 = I(u) \rho_0 \phi \rho_0 = u \phi \rho_0 \phi \rho_0.$$

Since u is simple by assumption, it follows from [13, Proposition 2.5.1] that u has no nontrivial automorphisms so that

$$(\phi\rho_0)^2 = \mathrm{id}.$$

This finishes the proof of the lemma.

We abbreviate by $\mathcal{I} \subset \mathrm{Diff}(\mathbb{CP}^1)$ the space of anti-holomorphic involutions of $\mathbb{CP}^1$.

Proposition 4.1 *The space $\mathcal{I}$ has two connected components.*

Proof We first show that $\mathcal{I}$ is diffeomorphic to the space

$$\mathcal{J} = \{[A] \in PSL_2(\mathbb{C}) : [\overline{A}] = [A^{-1}]\},$$

where for $A \in SL_2(\mathbb{C})$ we denote by $[A]$ its equivalence class in the projectivization $PSL_2(\mathbb{C})$ and by $\overline{A}$ the complex conjugate of the matrix A. We define a map

$$\Phi \colon \mathcal{I} \to \mathcal{J}, \quad \psi \mapsto \psi\rho_0.$$

To check that this map is well defined, we first note that $\psi\rho_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a biholomorphism so that $\psi\rho_0 = [A] \in PSL_2(\mathbb{C})$. Now we compute by using the fact that ρ_0 as well as ψ are involutions

$$[\overline{A}] = \rho_0(\psi\rho_0)\rho_0 = \rho_0\psi = \rho_0^{-1}\psi^{-1} = (\psi\rho_0)^{-1} = [A^{-1}].$$

This proves that Φ is well defined. To show that it is a diffeomorphism we construct its inverse as follows

$$\Psi \colon \mathcal{J} \to \mathcal{I}, \quad \phi \mapsto \phi\rho_0.$$

That Ψ is inverse to Φ is an immediate consequence of the fact that ρ_0 is an involution. Therefore it just remains to check that Ψ is well defined, i.e., $\phi\rho_0$ is actually an involution. This follows from the following computation:

$$(\phi\rho_0)^2 = \phi(\rho_0\phi\rho_0) = \phi\phi^{-1} = \mathrm{id}.$$

This proves that $\mathcal{I}$ and $\mathcal{J}$ are diffeomorphic.

In view of the diffeomorphism established above, we are left to show that $\mathcal{J}$ has two connected components. We rewrite $\mathcal{J}$ first as the quotient

$$\mathcal{J} = \frac{\widetilde{\mathcal{J}}}{\mathbb{Z}_2},$$

where

$$\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_+ \cup \widetilde{\mathcal{J}}_-$$

with

$$\widetilde{\mathcal{J}}_\pm = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix} \right\}$$

and the $\mathbb{Z}_2$-action identifies A with $-A$. Note that both $\widetilde{\mathcal{J}}_+$ and $\widetilde{\mathcal{J}}_-$ are invariant under the $\mathbb{Z}_2$-action. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{\mathcal{J}}_+$, then this is equivalent to that

$$a = \overline{d}, \quad b, c \in \mathrm{i}\mathbb{R}, \quad |a|^2 - bc = 1.$$

Hence we can identify $\widetilde{\mathcal{J}}_+$ with the hyperboloid of one sheet

$$\mathcal{H}_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1\}$$

via the map

$$\mathcal{H}_1 \to \widetilde{\mathcal{J}}_+, \quad (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & \mathrm{i}(x_3 + x_4) \\ \mathrm{i}(x_3 - x_4) & x_1 - \mathrm{i}x_2 \end{pmatrix}.$$

The hyperboloid of one sheet $\mathcal{H}_1$ is connected and therefore we conclude that $\widetilde{\mathcal{J}}_+$ and $\frac{\widetilde{\mathcal{J}}_+}{\mathbb{Z}_2}$ are connected as well.

It remains to show that $\frac{\widetilde{\mathcal{J}}_-}{\mathbb{Z}_2}$ is connected as well. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{\mathcal{J}}_-$, then this is equivalent to that

$$a = -\overline{d}, \quad b, c \in \mathbb{R}, \quad |a|^2 - bc = 1.$$

Hence we can identify $\widetilde{\mathcal{J}}_+$ with the hyperboloid of two sheets

$$\mathcal{H}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : -x_1^2 - x_2^2 - x_3^2 + x_4^2 = 1\}$$

via the map

$$\mathcal{H}_2 \to \widetilde{\mathcal{J}}_-, \quad (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + \mathrm{i}x_2 & x_3 + x_4 \\ x_3 - x_4 & -x_1 + \mathrm{i}x_2 \end{pmatrix}.$$

The pullback of the involution on $\widetilde{\mathcal{J}}_-$ to $\mathcal{H}_2$ is given by $x \mapsto -x$. This involution interchanges the two sheets of $\mathcal{H}_2$ and therefore $\frac{\widetilde{\mathcal{J}}_-}{\mathbb{Z}_2}$ is connected. This finishes the proof of the proposition.

Keeping the notation from the proof of Proposition 4.1, we abbreviate the two connected components of the space $\mathcal{I}$ by

$$\mathcal{I}_\pm := \Psi(\mathcal{J}_\pm), \quad \mathcal{J}_\pm := \frac{\widetilde{\mathcal{J}}_\pm}{\mathbb{Z}_2}.$$

An example of a holomorphic involution in $\mathcal{I}_+$ is the involution $\rho_0 \colon [z_0 : z_1] \mapsto [\overline{z}_0 : \overline{z}_1]$ and an example of an anti-holomorphic involution in $\mathcal{I}_-$ is the antipodal map $\sigma_0 \colon [z_0 : z_1] \mapsto [\overline{z}_1 : -\overline{z}_0]$. Note that the fixed-point set of ρ_0 is topologically a circle, while σ_0 has no fixed points. Since the topological type of the fixed-point set only depends on the connected component of $\mathcal{I}$, we conclude with the following lemma.

Lemma 4.3 *Each anti-holomorphic involution in $\mathcal{I}_-$ acts freely, while the fixed-point set of each involution in $\mathcal{I}_+$ is topologically a circle.*

Definition 4.2 *A pseudo-fixed-point $u \in \mathrm{Hol}(J)$ satisfying $I(u) = u\phi$ is said to be of type I if $\phi \in \mathcal{J}_+$. Otherwise u is said to be of type II, meaning that $\phi \in \mathcal{J}_-$.*

Proposition 4.2 *Assume that $u \in \mathrm{Hol}(J)$ is a pseudo-fixed-point of type* I. *Then there exists $\psi \in PSL_2(\mathbb{C})$ such that $u \circ \psi$ is a fixed point.*

Proof Since u is a pseudo-fixed-point, we have $I(u) = u\phi$ for $\phi \in PSL_2(\mathbb{C})$ and because u is of type I, we have $\phi\rho_0 \in \mathcal{I}_+$. By Lemma 4.3 we know that the fixed-point set of $\phi\rho_0$ is topologically a circle. Identify $\mathbb{CP}^1$ with the 2-dimensional sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ via stereographic projection. We first claim that the fixed-point set $\mathrm{Fix}(\phi\rho_0)$ is actually a small circle, namely, the intersection of S^2 with an affine plane in $\mathbb{R}^3$. To see this, pick three points on $\mathrm{Fix}(\phi\rho_0)$. These three points uniquely determine a small circle. Since ϕ and ρ_0 map small circles to small circles, we conclude that this small circle is fixed under $\phi\rho_0$ and hence has to agree with $\mathrm{Fix}(\phi\rho_0)$. This shows that $\mathrm{Fix}(\phi\rho_0)$ is a small circle.

Since the group $PSL_2(\mathbb{C})$ acts transitively on small circles, we conclude that there exists $\psi \in PSL_2(\mathbb{C})$ satisfying

$$\psi(\mathrm{Fix}(\rho_0)) = \mathrm{Fix}(\phi\rho_0).$$

This implies that

$$\mathrm{Fix}(\phi\rho_0) = \mathrm{Fix}(\psi\rho_0\psi^{-1}).$$

By analyticity, we conclude that

$$\phi\rho_0 = \psi\rho_0\psi^{-1}.$$

Using this equality, we compute

$$I(u\psi) = \rho u\psi\rho_0 = \rho u\rho_0\rho_0\psi\rho_0 = u\phi\rho_0\psi\rho_0 = u\psi\rho_0\psi^{-1}\psi\rho_0 = u\psi.$$

Hence $u\psi$ is a fixed point. This finishes the proof of the proposition.

Proposition 4.3 *Assume that $\Sigma \subset M$ is a complex ρ-invariant hypersurface and $u \in \mathrm{Hol}(J)$ is a pseudo-fixed-point satisfying $[u] \circ [\Sigma] = 1$ and $\mathrm{im}(u) \not\subset \Sigma$. Then u is of type* I.

Proof Since $[u] \circ [\Sigma] = 1$, the image of u is not contained in Σ and Σ is complex, we deduce from positivity of intersections that $\#u^{-1}(\Sigma) = 1$, i.e., there exists $w_0 \in \mathbb{CP}^1$ such that

$$u^{-1}(\Sigma) = \{w_0\}. \tag{4.1}$$

Since u is a pseudo-fixed-point, there exists $\phi \in PSL_2(\mathbb{C})$ such that $I(u) = u\phi$. We compute by using the ρ-invariance of Σ,

$$u\phi\rho_0(w_0) = \rho u\rho_0\rho_0(w_0) = \rho u(w_0) \in \rho\Sigma = \Sigma.$$

We deduce from (4.1) that

$$\phi\rho_0(w_0) = w_0.$$

In particular, the fixed-point set of the anti-holomorphic involution $\phi\rho_0$ is not empty. We conclude by Lemma 4.3 that $\phi\rho_0 \in \mathcal{I}_+$ or equivalently that $\phi \in \mathcal{J}_+$, and therefore u is a pseudo-fixed-point of type I. This proves the proposition.

Definition 4.3 *Assume $u \in \mathrm{Hol}(J)$. A point $w \in \mathbb{CP}^1$ is called a ρ-injective point of u if*

$$\mathrm{d}u(w) \neq 0, \quad u^{-1}\{u(w),\ \rho u(w)\} = \{w\}.$$

Lemma 4.4 *Assume that $u \in \mathrm{Hol}(J)$ is a simple holomorphic map which is not a pseudo-fixed-point. Then the complement of the set of ρ-injective points of u is finite.*

Proof Denote by $Z_\rho \subset \mathbb{CP}^1$ the complement of the set of ρ-injective points. Abbreviate further

$$Z = \{w \in \mathbb{CP}^1 : \mathrm{d}u(w) = 0 \text{ or } \#u^{-1}(u(w)) > 1\}$$

the set of non-injective points of u and

$$\mathcal{T} = \{(w_0, w_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : u(w_0) = \rho u(w_1),\ w_0 \neq w_1\}.$$

Consider the map

$$\pi \colon \mathcal{T} \to \mathbb{CP}^1, \quad \pi(w_0, w_1) = w_0.$$

Note that

$$Z_\rho = Z \cup \mathrm{im}(\pi).$$

Since u is simple, the set Z is finite by positivity of intersection (see [13, Theorem E.1.2]). It therefore suffices to show that the set $\mathcal{T}$ is finite as well. To see that, first note that by Lemma 4.1, $I(u)$ is simple as well. Therefore it follows from [13, Corollary 2.5.3] that

$$\mathrm{im}(u) \neq \mathrm{im}(I(u)).$$

Hence by positivity of intersection

$$\#\{(w_0, w_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : u(w_0) = I(u)(w_1)\} < \infty.$$

Note that

$$\begin{aligned} &\#\{(w_0, w_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : u(w_0) = I(u)(w_1)\} \\ = &\#\{(w_0, w_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : u(w_0) = \rho u(w_1)\}. \end{aligned}$$

We deduce that

$$\#\mathcal{T} < \infty.$$

This finishes the proof of the lemma.

We are now ready to prove the main result of this section.

Proof of Theorem 4.1 We argue by contradiction and assume that there exists $J \in \mathscr{J}(\Sigma, \rho)$ such that the moduli space $\mathcal{M}^\rho_J(S, p; A) = \mathrm{Hol}^\rho \frac{J', (S,p;A)}{\mathbb{C}^*}$ is empty. Since A is indecomposable, there is no bubbling and therefore it follows from compactness of holomorphic curves that there exists an open neighborhood $\mathscr{J}_0 \subset \mathscr{J}(\Sigma, \rho)$ of J such that $\mathcal{M}^\rho_{J'}(S, p; A) = \emptyset$ for every $J' \in \mathscr{J}_0$. In view of Proposition 4.2, there is therefore no pseudo-fixed-point of type I in the space of holomorphic maps $\mathrm{Hol}^\rho(J', (S, p; A))$ for every $J' \in \mathscr{J}_0$. Together with Proposition 4.3, the assumptions of the theorem show that there does not exist a pseudo-fixed-point of type II either and therefore there are no pseudo-fixed-points at all in $\mathrm{Hol}(J', (S, p; A))$ for every $J' \in \mathscr{J}_0$.

Furthermore, A is indecomposable, so each holomorphic curve u representing A is simple and we conclude by Lemma 4.4 that for every $J' \in \mathscr{J}_0$, every holomorphic map $u \in \mathrm{Hol}(J', (S, p; A))$

has ρ-injective points. Transversality arguments (see [13, Section 6.2, Section 6.3]), then show that there exists an open and dense subset $\mathscr{J}_0^{\mathrm{reg}} \subset \mathscr{J}_0$ such that for every $J' \in \mathscr{J}_0^{\mathrm{reg}}$, the Gromov-Witten invariant $GW_A([S],[p])$ can be obtained as the signed count of points in the moduli space $\mathcal{M}(J',(S,p;A)) = \mathrm{Hol}(J',(S,p;A))/\mathbb{C}^*$. Since this Gromov-Witten invariant is odd by assumption, we conclude that

$$1 = GW_A\big([S],[p]\big) \bmod 2 = \#\mathcal{M}_{J'}(S,p;A) \bmod 2.$$

However, the moduli space $\mathcal{M}_{J'}(S,p;A)$ is invariant under the involution I which has no fixed points by construction. Therefore the cardinality of the moduli space $\mathcal{M}_{J'}(S,p;A)$ has to be even. This contradiction finishes the proof of the theorem.

5 The Proof

The basic idea to prove Theorem 1.2 is to embed a real Liouville domain into a decorated symplectic manifold, making it into a Christmas tree. By hanging up some Christmas balls, or in other words, taking holomorphic spheres through Σ and a given real point p, and applying a stretching construction, we obtain an invariant finite energy plane through a given point in the real locus.

We need some lemmas to prepare the Christmas tree for the Christmas balls.

Lemma 5.1 *Let $(M,\omega,\mathcal{D}=(\Sigma,A,S))$ be a decorated symplectic manifold with an anti-symplectic involution ρ, and assume that $(W_0,\lambda_0,\rho|_{W_0})$ is a real Liouville domain that embeds into the interior of $M-\nu(\Sigma)$ for some ρ-invariant neighborhood $\nu(\Sigma)$ of Σ. Suppose in addition that $b_1(W_0)=0$.*

Then $W_1 := M-\nu(\Sigma)$ carries the structure of a real Liouville domain $(W_1,\lambda_1,\rho|_{W_1})$ such that $(W_0,\lambda_0,\rho|_{W_0})$ is a real Liouville subdomain in the sense that $\lambda_1|_{W_0}=\lambda_0$.

Proof Since we will need a cut-off function, we first extend λ_0 to a neighborhood of W_0. By Lemma 3.1, $W_1 := M-\nu(\Sigma)$ is a real Liouville domain $(W_1,\widetilde{\lambda}_1,\rho)$. As $\omega = \mathrm{d}\widetilde{\lambda}_1 = \mathrm{d}\lambda_0$ on a neighborhood of W_0, we see that $\widetilde{\lambda}_1-\lambda_0$ is closed, and as $b_1(W)=0$, we find a function f on a neighborhood of W_0 such that $\lambda_0=\widetilde{\lambda}_1-\mathrm{d}f$. It follows directly that $\rho^*\mathrm{d}f=-\mathrm{d}f$. If $\rho^*f\neq -f$, then we replace f by $\frac{1}{2}(f-\rho^*f)$.

Find a ρ-invariant cut-off function g such that $g\equiv 1$ on W_0, and $g\equiv 0$ on the complement of a neighborhood of W_0. Then $\lambda_1=\widetilde{\lambda}_1-\mathrm{d}(gf)$ has the desired properties.

Lemma 5.2 *Let $(M,\omega,\mathcal{D}=(\Sigma,A,S))$ be a decorated symplectic manifold, and assume that $(W_0,\lambda_0,\rho|_{W_0})$ is a real Liouville domain that embeds into the interior of $M-\nu(\Sigma)$ for some ρ-invariant neighborhood $\nu(\Sigma)$ of Σ. Suppose in addition that $b_1(W_0)=0$. Let J be an almost-complex structure on M that is compatible with ω and SFT-like near ∂W_0.*

Assume that $u:\mathbb{CP}^1\to M$ is a J-holomorphic sphere through a point $p\in W_0$ such that $[u]\circ[\Sigma]=1$. Then the component C of $u^{-1}(W_0)$ containing z_0 with $u(z_0)=p$ satisfies the following:

(1) *C is diffeomorphic to a disk.*

(2) *$\int_C u|_C^*\omega = \int_C u|_C^*\mathrm{d}\lambda_0 \le 1$. In particular, the SFT energy of $u|_C$ is bounded from above by 1.*

Proof After possibly shifting the boundary ∂W_0 a little bit, we can assume that $u^{-1}(\partial W_0)$ consists of finitely many circles. Let C denote the component of $u^{-1}(W_0)$ containing z_0. We claim that C has only one boundary component. To see why, note that if C has more than one boundary component, then there is a connected component of $\widetilde{C} := \mathbb{CP}^1 - \mathrm{int}(C)$ with the properties

(1) $\widetilde{C}$ shares a boundary component with C.

(2) $u(\widetilde{C})$ does not intersect $\nu(\Sigma)$, and is contained in $M - \mathrm{int}(W_0)$.

To see that, the latter condition can be imposed. We observe that u intersects Σ only once, and we also use that $\mathbb{CP}^1$ has genus 0.

Now apply the previous lemma to see that $M - \nu(\Sigma)$ carries the structure of a real Liouville domain (W_1, λ_1) with a real Liouville subdomain (W, λ). This allows us to compute the energy of $\widetilde{C}$ via Stokes' theorem,

$$E(u|_{\widetilde{C}}) = \int_{\widetilde{C}} u^*_{\widetilde{C}}\omega = \int_{\widetilde{C}} \mathrm{d}u^*_{\widetilde{C}}\lambda_1 = \int_{\partial\widetilde{C}} u^*_{\widetilde{C}}\lambda_1 < 0.$$

The last inequality holds, because the orientation induced by the outward pointing normal is minus the one induced by the Reeb vector field; one can see this by using that J is SFT-like near ∂W_0. Since the energy of the holomorphic curve $u|_{\widetilde{C}}$ is positive, this is a contradiction, so we conclude that C has one boundary component. It follows directly that C is diffeomorphic to a disk. The claimed energy estimate is now also clear, since $\int_{\mathbb{CP}^1} u^*\omega = 1$.

We will now apply a stretching argument to obtain an invariant finite energy plane. This is illustrated in Figure 1. Let X denote the Liouville vector field on $M - \Sigma$. Take a point $p \in W_0$,

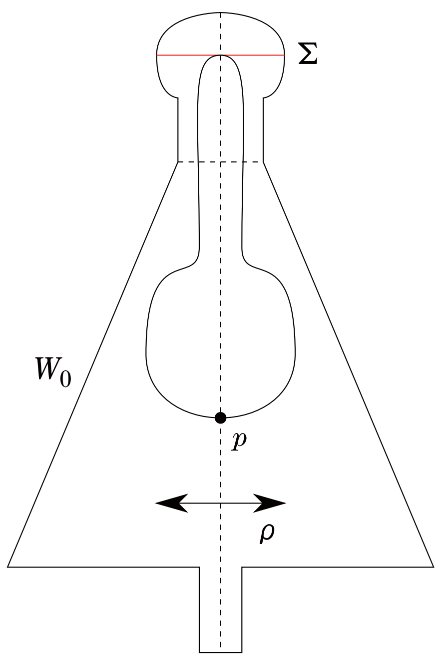

Figure 1 Hanging up Christmas balls (holomorphic spheres) in a Christmas tree

and for $\tau \in \mathbb{R}_{\geq 0}$ define p_τ, by following the Liouville flow backwards, as $p_\tau = Fl^X_{-\tau}(p)$. Define the stretched Liouville domain W_0^τ by

$$W_0^\tau := (W_0, \omega = \mathrm{d}\lambda_0) \cup_\partial ([0,\tau] \times \partial W_0, \mathrm{d}(\mathrm{e}^t \lambda_0|_{\partial W_0})).$$

Choose a compatible complex structure J_τ on W_0^τ, which is SFT-like on $[0,\tau] \times \partial W_0$. We choose this sequence J_τ such that it is a constant sequence of complex structures when restricted to

W_0. Since the map $x \mapsto Fl^X_\tau(x)$ provides a symplectic deformation from W_0 to W_0^τ, we can pull back J_τ to a complex structure on W_0, which is SFT-like near the boundary. Extend this J_τ to a compatible complex structure $\widetilde{J}_\tau$ for (M,ω).

With Lemma 5.2 applied to an invariant holomorphic sphere obtained with Theorem 4.1, we find a $\widetilde{J}_\tau$-holomorphic disk

$$\widetilde{u}_\tau : \widetilde{C}_\tau \subset \mathbb{CP}^1 \to W_0$$

going through p_τ, with its boundary on ∂W_0. We now stretch the Liouville domain W_0 to a Liouville domain W_0^τ by using the above deformation. This deformation also gives us a J_τ-holomorphic curve

$$u_\tau : C_\tau \to W_0^\tau$$

going through p. As the Hofer energy of $\widetilde{u}_\tau$ is bounded by 1, so is the Hofer energy of u_τ.

Denote the norm induced by $\omega_\tau(\cdot, J_\tau \cdot)$ by $\|\cdot\|_\tau$. By rescaling the domain, we can ensure that $\max_{z\in C_\tau} \|du_\tau\|_\tau = 1$; we need to rescale the disk C_τ for this, but we will continue to write C_τ for this rescaled disk. Since p lies in W_0 and the boundary of the disk, $u_\tau(\partial C_\tau)$, lies on $\{\tau\} \times \partial W_0$, we see directly that the radius for the disk C_τ has to be at least τ by a very crude estimate, using $\max_{z\in C_\tau} \|du_\tau\|_\tau = 1$. Taking the limit $\tau \to \infty$, we find a convergent subspace, and obtain a map

$$u_\infty : \mathbb{C} \to W_0^\infty,$$

where W_0^∞ is the completion of W_0. As the Hofer energy of u_∞ is bounded from above by 1, we conclude that u_∞ is the desired finite energy plane through $p \in W_0$.

This stretching construction also implies the well-known corollary (see also [11]).

Corollary 5.1 *Let (W,λ) be a Liouville domain admitting an embedding into a decorated symplectic manifold $(M,\omega,\mathcal{D})$. Suppose that $b_1(W) = 0$. Then W is uniruled. Furthermore, there exists a periodic orbit of a period less than or equal to* 1 *for the Reeb flow on ∂W.*

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