

Abstract Elliptic Equations with Integral Boundary Conditions

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Abstract This paper focuses on nonlocal integral boundary value problems for elliptic differential-operator equations. Here given conditions guarantee that maximal regularity and Fredholmness in L_p spaces. These results are applied to the Cauchy problem for abstract parabolic equations, its infinite systems and boundary value problems for anisotropic partial differential equations in mixed L_p norm.

Keywords Boundary value problems, Integral boundary conditions, Differential-operator equations, Maximal L_p regularity, Abstract parabolic equation, Operator valued multipliers, Interpolation of Banach spaces, Semigroups of operators

2000 MR Subject Classification: 34G10, 35J25, 35J70

1 Introduction

Boundary value problems (for short BVPs) for differential-operator equations (for short DOEs) have been studied extensively by many researchers (see [1–3, 8, 10–11, 14–23, 25–32] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [16, 28]. The maximal regularity properties for differential-operator equations have been investigated, e.g., in [1, 8, 10–11, 17–22, 26, 28, 30–32].

In recent years, integral boundary conditions (for short IBC) for evolution problems have found many applications in various disciplines such as chemical engineering, thermoplasticity, underground water flow and population dynamics (see [4, 7, 9, 13, 24] and the references therein).

The main purpose aim of the present paper, is to show the separability properties of the integral boundary value problem (for short IBVP) for the following DOE:

$$-u^{(2)}(x) + Au(x) + \lambda u = f(x) \quad (1.1)$$

and maximal regularity of Cauchy problem for the following abstract parabolic equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + Au(x, t) = f(x, t), \quad (1.2)$$

with integral boundary conditions, where A is a linear operator in the Banach space E and λ is a complex parameter.

Manuscript received January 14, 2013. Revised January 6, 2015.

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Unlike the previous results, the boundary conditions here contain nonlocal integral terms. The maximal L_p -regularity and the Fredholmness are obtained. Moreover, it is proven that the corresponding elliptic operator is \mathbb{R} -positive and is a generator of an analytic semigroup. These results are applied to nonlocal BVPs for partial differential equations and it's finite or infinite systems on cylindrical domains.

We shall prove the separability of the problem (1.1), i.e., we show that for each $f \in L_p(0, 1; E)$, there exists a unique strong solution u of the problem (1.1) and the following uniform coercive estimate holds:

$$\sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \|u^{(i)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \leq C \|f\|_{L_p(0,1;E)}.$$

Let E be a Banach space and $L_p(\Omega; E)$ denotes the space of strongly measurable E -valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega; E)} = \left(\int_{\Omega} \|f(x)\|_E^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The Banach space E is called a UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L_p(\mathbb{R}, E)$, $p \in (1, \infty)$ (see [6]), where $\mathbb{R} = (-\infty, \infty)$. UMD spaces include, e.g., L_p , l_p spaces and Lorentz spaces L_{pq} , $p, q \in (1, \infty)$.

Let \mathbb{C} be the set of the complex numbers and

$$S_{\varphi} = \{\lambda; \lambda \in \mathbb{C}, |\arg \lambda| \leq \varphi\} \cup \{0\}, \quad 0 \leq \varphi < \pi.$$

A linear operator A is said to be φ -positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\|(A + \lambda I)^{-1}\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$$

for any $\lambda \in S_{\varphi}$, $0 \leq \varphi < \pi$, where I is the identity operator in E , and $B(E)$ is the space of bounded linear operators in E . Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by A_{λ} . It is known in [27, §1.15.1] that there exist the fractional powers A^{θ} of a positive operator A . Let $E(A^{\theta})$ denote the space $D(A^{\theta})$ with norm

$$\|u\|_{E(A^{\theta})} = (\|u\|^p + \|A^{\theta}u\|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. By $(E_1, E_2)_{\theta, p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, we denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the K -method (see [27, §1.3.2]).

Let $S(\mathbb{R}^n; E)$ denote the Schwartz class, i.e., the space of all E -valued rapidly decreasing smooth functions on \mathbb{R}^n . Let F denote the Fourier transformation. A function $\Psi \in C(\mathbb{R}^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ if the map $u \rightarrow \Lambda u = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$\Lambda : L_p(\mathbb{R}^n; E_1) \rightarrow L_p(\mathbb{R}^n; E_2).$$

The set of all multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ will be denoted by $M_p^p(E_1, E_2)$. For $E_1 = E_2 = E$, it will be denoted by $M_p^p(E)$. The most important facts about Fourier multipliers and some related references, can be found in [11–12, 27, 30].

Let $\Phi_h = \{\Psi_h \in M_p^p(E_1, E_2), h \in Q\}$ be a collection of multipliers. We say that W_h is a uniform collection of multipliers if there exists a positive constant M independent of $h \in Q$ such that

$$\|F^{-1}\Psi_h F u\|_{L_p(\mathbb{R}^n; E_2)} \leq M \|u\|_{L_p(\mathbb{R}^n; E_1)}$$

for all $h \in Q$ and $u \in S(\mathbb{R}^n; E_1)$.

Let \mathbb{N} denote the set of natural numbers. A set $G \subset B(E_1, E_2)$ is called \mathbb{R} -bounded (see [10]) if there is a positive constant C such that for all $T_1, T_2, \dots, T_m \in G$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_{\Omega} \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$ -valued random variables on Ω (see [11]). The smallest C for which the above estimate holds is called an \mathbb{R} -bound of the collection G and denoted by $\mathbb{R}(G)$.

A set $G_h \subset B(E_1, E_2)$ depending on parameter $h \in Q$ is called uniformly \mathbb{R} -bounded with respect to h if there exists a constant C , independent of $h \in Q$, such that for all $T_1(h), T_2(h), \dots, T_m(h) \in G_h$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_{\Omega} \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy.$$

It implies that $\sup_{h \in Q} \mathbb{R}(G_h) \leq C$. Let

$$\begin{aligned} \beta &= (\beta_1, \beta_2, \dots, \beta_n), \quad \xi = (\xi_1, \xi_2, \dots, \xi_n), \quad \xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} \dots \xi_n^{\beta_n}, \quad |\beta| = \sum_{k=1}^n \beta_k, \\ D_j &= \frac{\partial}{\partial \xi_j}, \quad D^\beta = D_1^{\beta_1} D_2^{\beta_2} \dots D_n^{\beta_n}, \quad U = \{\beta; \beta_j \in \{0, 1\}, j = 1, 2, \dots, n\}. \end{aligned}$$

Definition 1.1 A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in C^{(n)}(\mathbb{R}^n; B(E))$, the \mathbb{R} -boundedness of the set $\{|\xi|^{|\beta|} D_\xi^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U\}$ implies that Ψ is a Fourier multiplier, i.e., $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.

The uniform \mathbb{R} -boundedness of the set $|\xi|^{|\beta|} D_\xi^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U$, i.e.,

$$\sup_{h \in Q} \mathbb{R}(|\xi|^{|\beta|} D_\xi^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U) \leq K$$

implies that Ψ_h is a uniform collection of Fourier multipliers.

Remark 1.1 Definition 1.1 is a restriction on the Banach spaces E . If E is a UMD space, then this space satisfies the multiplier condition. All UMD spaces satisfy the multiplier condition (see [12]).

The φ -positive operator A is said to be \mathbb{R} -positive in a Banach space E if the set

$$L_A = \{\xi(A + \xi)^{-1} : \xi \in S_\varphi\}, \quad 0 \leq \varphi < \pi$$

is \mathbb{R} -bounded.

Note that in Hilbert spaces, all norm-bounded sets are \mathbb{R} -bounded. Therefore, in Hilbert spaces, all positive operators are \mathbb{R} -positive. If A is the generator of a contraction semigroup on L_q , $1 \leq q \leq \infty$ or A has the bounded imaginary powers with $\|A^{it}\|_{B(E)} \leq Ce^{\nu|t|}$, $\nu < \frac{\pi}{2}$ (see [11]) in $E \in \text{UMD}$, then those operators are \mathbb{R} -positive.

Let $D(\Omega; E)$ denote the class of all E -valued infinite differentiable functions on domain Ω with compact supports. For $E = \mathbb{C}$, denotes it by $D(\Omega)$.

Let $\sigma_\infty(E)$ denote the space of all compact operators in E . Let E_0 and E be two Banach spaces and E_0 be continuously and densely embedded into E . Let m be a positive integer. Let us consider the space $W_p^m(\Omega; E_0, E)$, consisting of all functions $u \in L_p(\Omega; E_0)$ that have the generalized derivatives $D_k^m u = \frac{\partial^m u}{\partial x_k^m} \in L_p(\Omega; E)$ with the norm

$$\|u\|_{W_p^m(\Omega; E_0, E)} = \|u\|_{L_p(\Omega; E_0)} + \sum_{k=1}^n \|D_k^m u\|_{L_p(\Omega; E)} < \infty.$$

For $\Omega = (a, b)$, $a, b \in (-\infty, \infty)$, the space $W_p^m(\Omega; E_0, E)$ will be denoted by $W_p^m(a, b; E_0, E)$ and for $E_0 = E$, denotes it by $W_p^m(\Omega; E)$.

By using the techniques of [12, Theorem 3.7] we obtain the following proposition.

Proposition 1.1 *Let E_1 and E_2 be two UMD spaces and*

$$\Psi_h \in C^n(\mathbb{R}^n \setminus \{0\}; B(E_1, E_2)).$$

Suppose that there exists a positive constant K such that

$$\sup_{h \in Q} \mathbb{R}(\{|\xi|^{\|\beta\|} D^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U\}) \leq K.$$

Then Ψ_h is a uniform collection of multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ for $p \in (1, \infty)$.

From [21, Theorem 4] we obtain the following theorem.

Theorem 1.1 *Let the following conditions be satisfied:*

(1) E is a Banach space satisfying the uniform multiplier condition $p \in (1, \infty)$ and $0 < h \leq h_0 < \infty$ are certain parameters;

(2) m is a positive integer and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ are n -tuples of nonnegative integer numbers such that

$$\varkappa = \frac{|\alpha|}{m} \leq 1, \quad 0 \leq \mu \leq 1 - \varkappa;$$

(3) A is an \mathbb{R} -positive operator in E with $0 \leq \varphi < \pi$;

(4) $\Omega \in \mathbb{R}^n$ is a region such that there exists a bounded linear extension operator between from $W_p^m(\Omega; E(A), E)$ to $W_p^m(\mathbb{R}^n; E(A), E)$ and from $L_p(\Omega; E)$ to $L_p(\mathbb{R}^n; E)$.

Then the embedding $D^\alpha W_p^m(\Omega; E(A), E) \subset L_p(\Omega; E(A^{1-\varkappa-\mu}))$ is continuous and there exists a positive constant C_μ such that

$$\prod_{k=1}^n \|D^\alpha u\|_{L_p(\Omega; E(A^{1-\varkappa-\mu}))} \leq C_\mu [h^\mu \|u\|_{W_p^m(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega; E)}] \quad (1.3)$$

for all $u \in W_p^m(\Omega; E(A), E)$ and $0 < h \leq h_0$.

Remark 1.2 If $\Omega \subset \mathbb{R}^n$ is a region satisfying m -horn condition (see [5, §7]), $E = \mathbb{R}$, $A = I$, then for $p \in (1, \infty)$ there exists a bounded linear extension operator from $W_p^m(\Omega) = W_p^m(\Omega; \mathbb{R}, \mathbb{R})$ to $W_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n; \mathbb{R}, \mathbb{R})$.

From [21, Theorem 6] we obtain the following theorem.

Theorem 1.2 Suppose that all conditions of Theorem 1.1 are satisfied. Let $0 < \mu \leq 1 - \varkappa$. Then the embedding

$$D^\alpha W_p^m(\Omega; E(A), E) \subset L_p(\Omega; (E(A), E)_{\varkappa, p})$$

is continuous and there exists a positive constant C_μ such that for all $u \in W_p^m(\Omega; E(A), E)$, the uniform estimate holds, i.e.,

$$\|D^\alpha u\|_{L_p(\Omega; (E(A), E)_{\varkappa+\mu, p})} \leq C_\mu [h^\mu \|u\|_{W_p^m(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega; E)}].$$

Remark 1.3 Note that the constant C_μ in the above estimate depends only on h_0 , μ and T . Since these numbers are bounded and fixed, this dependence does not influence the further results.

Theorem 1.3 (see [22]) Let E be a Banach space and A be a φ -positive operator in E with bound M , $0 \leq \varphi < \pi$. Let m be a positive integer, $1 < p < \infty$ and $\alpha \in (\frac{1+\gamma}{2p}, \frac{1+\gamma}{2p} + m)$, $0 \leq \gamma < pl - 1$. Then for $\lambda \in S_\varphi$, an operator $-A_\lambda^{\frac{1}{2}}$ generates a semigroup $e^{-xA_\lambda^{\frac{1}{2}}}$ which is holomorphic for $x > 0$. Moreover, there exists a positive constant C (depending only on M, φ, m, α and p) such that for every $u \in (E, E(A^m))_{\frac{\alpha}{m} - \frac{1+\gamma}{2mp}, p}$ and $\lambda \in S_\varphi$,

$$\int_0^\infty \|A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}} u\|^p x^\gamma dx \leq C [\|u\|_{(E, E(A^m))_{\frac{\alpha}{m} - \frac{1+\gamma}{2mp}, p}}^p + |\lambda|^{\alpha p - \frac{1+\gamma}{2}} \|u\|_E^p].$$

From [28, §1.7.7, Theorem 2] we obtain the following theorem.

Theorem 1.4 Let m and j be integer numbers, $0 \leq j \leq m - 1$, $\theta_j = \frac{pj+1}{pm}$, $x_0 \in [0, b]$. Then, for $u \in W_p^m(0, b; E_0, E)$, the transformations $u \rightarrow u^{(j)}(x_0)$ are bounded linearly from $W_p^m(0, b; E_0, E)$ onto $(E_0, E)_{\theta_j, p}$ and the following inequality holds:

$$\|u^{(j)}(x_0)\|_{(E_0, E)_{\theta_j, p}} \leq C (\|u^{(m)}\|_{L_p(0, b; E)} + \|u\|_{L_p(0, b; E_0)}).$$

By using the integral representation formula, in a similar way as in [5, §10.1] we have the following theorem.

Theorem 1.5 Let m and j be integer numbers, $0 \leq j \leq m - 1$, $\theta_j = \frac{pj+1}{pm}$, $0 < h \leq h_0$, $x_0 \in [0, b]$. Then, for $u \in W_p^m(0, b; E)$, $0 \leq j \leq m - 1$, the transformation $u \rightarrow u^{(j)}(x_0)$ is bounded linearly from $W_p^m(0, b; E)$ onto E and the following inequality holds:

$$\|u^{(j)}(x_0)\|_E \leq C (h^{1-\theta_j} \|u^{(m)}\|_{L_p(0, b; E)} + h^{-\theta_j} \|u\|_{L_p(0, b; E)}).$$

Let A be a positive operator in a Banach space E . Consider the differential-operator equation

$$Lu = u^{(m)}(x) + \sum_{k=1}^m a_k A^{\frac{k}{m}} u^{(m-k)}(x) = 0, \quad x \in (a, b). \quad (1.4)$$

Let $\omega_1, \omega_2, \dots, \omega_m$ be the roots of the equation

$$\omega^m + a_1\omega^{m-1} + \dots + a_m = 0$$

and

$$\begin{aligned}\omega_{\min} &= \min\{\arg \omega_j, j = 1, \dots, q; \arg \omega_j + \pi, j = q+1, \dots, m\}, \\ \omega_{\max} &= \max\{\arg \omega_j, j = 1, \dots, q; \arg \omega_j + \pi, j = q+1, \dots, m\},\end{aligned}$$

where q is some integer number from $(1, m)$.

A system of complex numbers $\omega_1, \omega_2, \dots, \omega_m$ is called q -separated if there exists a straight line P passing through 0 such that no value of the numbers ω_j lies on it, and $\omega_1, \omega_2, \dots, \omega_q$ are on one side of P while $\omega_{q+1}, \dots, \omega_m$ are on the other side.

Lemma 1.1 (see [20]) *Let the following conditions be satisfied:*

- (1) $a_m \neq 0$ and $\omega_j, j = 1, \dots, m$, are q -separated;
- (2) E is a Banach space satisfying the multiplier condition for $p \in (1, \infty)$;
- (3) A is an \mathbb{R} -positive operator in E .

Then for a function $u(x)$ to be a solution of Equation (1.4), which belongs to the space $W_p^m(a, b;$

$E(A), E)$, it is necessary that $u = \sum_{k=1}^q e^{-(x-a)\omega_k A^{\frac{1}{m}}} g_k + \sum_{k=q+1}^m e^{-(b-x)\omega_k A^{\frac{1}{m}}} g_k$, where

$$g_k \in (E(A), E)_{\frac{1}{mp}, p}, \quad k = 1, 2, \dots, m.$$

2 Statement of the Problem

In a Banach space E , consider the integral boundary value problem

$$\begin{aligned}Lu &= -u^{(2)}(x) + Au(x) + \lambda u = f(x), \quad x \in (0, 1), \\ L_k u &= \int_0^1 B_k(x)u(x)dx = f_k,\end{aligned}\tag{2.1}$$

where $f \in L_p(0, 1; E)$, $f_k \in E_k = (E(A), E)_{\alpha_k, p}$, $p \in (1, \infty)$; A and B_k are linear operators in E and λ is a complex parameter.

3 Homogeneous Equations

Let us first consider the following problem:

$$(L + \lambda)u = -u^{(2)}(x) + (A + \lambda)u(x) = 0,\tag{3.1}$$

$$L_k u = \int_0^1 B_k(x)u(x)dx = f_k, \quad f_k \in E_k, \quad k = 1, 2,\tag{3.2}$$

where λ is a complex parameter, A and B_k are linear operators in E , $E_k = (E(A), E)_{\alpha_k, p}$.

Remark 3.1 Let A be a positive operator in E . By definition, the operator $A_\lambda = A + \lambda$ is φ_1 -positive in E for $|\arg \lambda| \leq \varphi$ and $\varphi + \varphi_1 < \pi$. From the beginning of the proof of [28, Lemma 5.4.2/4], for $|\arg \lambda| \leq \varphi$ and $|\arg \mu| \leq \varphi_1$, $\varphi, \varphi_1 \in (0, \pi)$, we have the estimate

$$\|(A_\lambda + \mu)^{-1}\| \leq \frac{M_0}{|\mu|}$$

with M_0 depending on φ only. Then in view of [8, Lemma 2.6], there exist semigroups $U_{1\lambda}(x) = e^{-xA_\lambda^{\frac{1}{2}}}$, $U_{2\lambda}(x) = e^{-(1-x)A_\lambda^{\frac{1}{2}}}$ which are holomorphic for $x > 0$ and strongly continuous for $x \geq 0$.

Let

$$\eta_{ij} = \int_0^1 B_i(x)U_{j\lambda}dx, \quad i, j = 1, 2, \quad \eta = \eta_{11}\eta_{22} - \eta_{12}\eta_{21}.$$

Condition 3.1 Assume that

- (1) E is a Banach space satisfying the multiplier condition and $\eta \neq 0$;
- (2) A is an \mathbb{R} -positive operator in E ;
- (3) $D(B_k) \subset (E(A), E)_{\varkappa+\varepsilon, p}$ for $\varkappa = \alpha_0 - \frac{1}{2p}$, $\frac{1}{2p} < \alpha_0 < 1$, $\alpha_0 = \max_k \{\alpha_k\}$, $\varepsilon \in (0, 1 - \varkappa)$

and

$$\|B_k(x)u\|_E \leq C\|u\|_{(E(A), E)_{\varkappa+\varepsilon, p}}, \quad u \in (E(A), E)_{\varkappa+\varepsilon, p};$$

- (4) the operators $u \rightarrow L_k u$ are bounded from $W_p^2(0, 1; E(A), E)$ into E_k .

Theorem 3.1 Suppose that Condition 3.1 is satisfied. Then, the problem (3.1)–(3.2) for $f_k \in E_k$, $p \in (1, \infty)$ and $\lambda \in S_\varphi$ with sufficiently large $|\lambda|$ has a unique solution $u \in W_p^2(0, 1; E(A), E)$ and the following coercive uniform estimate holds:

$$\begin{aligned} & \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \|u^{(i)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \\ & \leq M \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1-\alpha_k-\frac{1}{2p}} \|f_k\|_E). \end{aligned} \quad (3.3)$$

Proof By virtue of Lemma 1.1 and Remark 1.3, an arbitrary solution of Equation (3.2) belonging to the space $W_p^2(0, 1; E(A), E)$, has the form

$$u(x) = U_{1\lambda}(x)g_1 + U_{2\lambda}(x)g_2, \quad (3.4)$$

where

$$g_k \in (E(A), E)_{\frac{1}{2p}, p}, \quad k = 1, 2.$$

Now taking into account boundary conditions (3.3) we obtain the algebraic linear equations with respect to g_1, g_2 :

$$\begin{aligned} & \int_0^1 B_1(x)U_{1\lambda}(x)dx g_1 + \int_0^1 B_1(x)U_{2\lambda}(x)dx g_2 = f_1, \\ & \int_0^1 B_2(x)U_{1\lambda}(x)dx g_1 + \int_0^1 B_2(x)U_{2\lambda}(x)dx g_2 = f_2. \end{aligned} \quad (3.5)$$

Since the operator determinant $\eta \neq 0$, the system (3.5) has the solution $g_1 = \frac{\gamma_1}{\eta}$, $g_2 = \frac{\gamma_2}{\eta}$, where

$$\gamma_1 = \eta_{22}f_1 - \eta_{12}f_2, \quad \gamma_2 = \eta_{11}f_2 - \eta_{21}f_1.$$

Hence, the problem (3.1)–(3.2) has a solution given bellow

$$u(x) = \frac{1}{\eta} [U_{1\lambda}(x)(\eta_{22}f_1 - \eta_{12}f_2) + U_{2\lambda}(x)(\eta_{11}f_2 - \eta_{21}f_1)]$$

$$= \frac{1}{\eta} \{ [U_{1\lambda}(x)\eta_{22} - U_{2\lambda}(x)\eta_{21}]f_1 + [U_{2\lambda}(x)\eta_{11} - U_{1\lambda}(x)\eta_{12}]f_2 \}. \quad (3.6)$$

By virtue of [8, Lemma 2.6], we have

$$\|e^{-xA\lambda^{\frac{1}{2}}}\| \leq Ce^{-\nu|\lambda|^{\frac{1}{2}}x}, \quad \nu > 0, \quad x \in (0, 1), \quad \lambda \in S(\varphi). \quad (3.7)$$

In view of (3.6), by estimate (3.7), the condition 3.1(2) and in view of the positivity of A , we obtain

$$\begin{aligned} & \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \|u^{(i)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \\ & \leq \frac{1}{\|\eta\|} \left\{ \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \|A_{\lambda}^{-(1-\frac{i}{2})} A_{\lambda} [U_{1\lambda}(x)\eta_{22} - U_{2\lambda}(x)\eta_{21}]f_1\|_{L_p(0,1;E)} \right. \\ & \quad \left. + \|A_{\lambda}^{-(1-\frac{i}{2})} A_{\lambda} [U_{1\lambda}(x)\eta_{11} - U_{2\lambda}(x)\eta_{12}]f_2\|_{L_p(0,1;E)} \right\} \\ & \leq \frac{1}{\|\eta\|} \sum_{k=1}^2 [\|A_{\lambda}^{1-\alpha_k+\frac{1}{2p}} U_{1\lambda} f_k\|_{L_p(0,1;E)} + \|A_{\lambda}^{1-\alpha_k+\frac{1}{2p}} U_{2\lambda} f_k\|_{L_p(0,1;E)}]. \end{aligned}$$

Hence, from Theorem 1.3 we obtain the assertion

$$\begin{aligned} D^{k+1}[V_{k+1}, V_{k+1}] &= \sum_{i=0}^{k+1} C_{k+1}^i [D^i(V_{k+1}), D^{k+1-i}(V_{k+1})] \\ &= [D^{k+1}(V_{k+1}), V_{k+1}] + [V_{k+1}, D^{k+1}(V_{k+1})]. \end{aligned}$$

4 Non-homogenous Equations

Now consider IBVPs for non-homogenous equation

$$(L + \lambda)u = -u^{(2)}(x) + (A + \lambda)u(x) = f(x), \quad x \in (0, 1), \quad (4.1)$$

$$L_k u = \int_0^1 B_k(x)u(x)dx = f_k, \quad f_k \in (E(A), E)_{\alpha_k, p}. \quad (4.2)$$

Theorem 4.1 *Suppose that Condition 3.1 is satisfied. Then the operator $u \rightarrow \{(L + \lambda)u, L_1 u, L_2 u\}$ for $|\arg \lambda| \leq \varphi$, $0 \leq \varphi < \pi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2(0, 1; E(A), E)$ onto $L_p(0, 1; E) \times E_1 \times E_2$. Moreover, for these λ , the following uniform coercive estimate holds:*

$$\begin{aligned} & \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \\ & \leq C \left[\|f\|_{L_p(0,1;E)} + \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1-\alpha_k-\frac{1}{2p}} \|f_k\|_E) \right]. \end{aligned} \quad (4.3)$$

Proof We have proved the uniqueness of solution of the problem (4.1)–(4.2) in Theorem 3.1. By definition of $W_p^2(0, 1; E(A), E)$ and by (4) of Condition 3.1, it is easy to see that the operator $u \rightarrow \{(L + \lambda)u, L_1 u, L_2 u\}$ is bounded from $W_p^2(0, 1; E(A), E)$ into $L_p(0, 1; E) \times E_1 \times E_2$. Hence,

by the Banach theorem, it is sufficient to show that the operator $u \rightarrow \{(L + \lambda)u, L_1u, L_2u\}$ is surjective from $W_p^2(0, 1; E(A), E)$ into $L_p(0, 1; E) \times E_1 \times E_2$. Let us define

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in [0, 1], \\ 0, & \text{if } x \notin [0, 1]. \end{cases}$$

We now show that the problem (4.1)–(4.2) has a solution $u \in W_p^2(0, 1; E(A), E)$ for all $f \in L_p(0, 1; E)$, $f_k \in E_k$ and $u = u_1 + u_2$, where u_1 is the restriction on $[0, 1]$ of the solution of the equation

$$[L + \lambda]u = \bar{f}(x), \quad x \in (-\infty, \infty) \quad (4.4)$$

and u_2 is a solution of the problem

$$[L + \lambda]u = 0, \quad L_k u = f_k - L_k u_1. \quad (4.5)$$

By virtue of Theorem 3.1, the problem (4.4) has a unique solution

$$u_1 \in W_p^2(0, 1; E(A), E).$$

A solution of Equation (4.4) is given by the formula

$$u(x) = F^{-1}L_0^{-1}(\lambda, \xi)F\bar{f} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} L^{-1}(\lambda, \xi)(F\bar{f})(\xi) d\xi,$$

where $L(\lambda, \xi) = A + \xi^2 + \lambda$. It follows from the above expression that

$$\begin{aligned} & \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L_p(\mathbb{R}; E)} + \|Au\|_{L_p(\mathbb{R}; E)} \\ &= \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|F^{-1}\xi^j L^{-1}(\lambda, \xi)F\bar{f}\|_{L_p(\mathbb{R}; E)} + \|F^{-1}AL^{-1}(\lambda, \xi)F\bar{f}\|_{L_p(\mathbb{R}; E)}. \end{aligned} \quad (4.6)$$

Let us show that operator-functions

$$\Psi_\lambda(\xi) = AL^{-1}(\lambda, \xi), \quad \sigma_\lambda(\xi) = \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \xi^j L^{-1}(\lambda, \xi)$$

are Fourier multipliers in $L_p(\mathbb{R}; E)$ uniformly with respect to λ . In fact, due to positivity of A and by virtue of [8, Lemma 2.3] we have the following uniform estimates:

$$\begin{aligned} \|L^{-1}(\lambda, \xi)\| &\leq M(1 + \xi^2 + |\lambda|)^{-1}, \\ \|\Psi_\lambda(\xi)\| &= \|A[A + \lambda + \xi^2]^{-1}\| \leq C_1. \end{aligned} \quad (4.7)$$

It is clear to see that

$$\xi \frac{d}{d\xi} \Psi_\lambda(\xi) = -2\xi^2 AL^{-2}(\lambda, \xi) = [-2\xi^2 L^{-1}(\lambda, \xi)]AL^{-1}(\lambda, \xi).$$

Due to \mathbb{R} -positivity of the operator A , the sets

$$\{-2\xi^2[A + \xi^2 + \lambda]^{-1} : \xi \in \mathbb{R} \setminus \{0\}\}, \quad \{A[A + \xi^2 + \lambda]^{-1} : \xi \in \mathbb{R} \setminus \{0\}\}$$

are \mathbb{R} -bounded. Then in view of the Kahane's contraction principle and from the product properties of the collection of \mathbb{R} -bounded operators (see [11, Lemma 3.5, Proposition 3.4]) we obtain

$$\mathbb{R}\left\{\xi \frac{d}{d\xi} \Psi_\lambda(\xi) : \xi \in \mathbb{R} \setminus \{0\}\right\} \leq C.$$

That is to say, the \mathbb{R} -bound of the set

$$\left\{\xi \frac{d}{d\xi} \Psi_\lambda(\xi) : \xi \in \mathbb{R} \setminus \{0\}\right\}$$

is independent of λ . Next, let us consider $\sigma_\lambda(\xi)$. It is clear to see that

$$\|\sigma_\lambda(\xi)\|_{B(E)} \leq C|\lambda| \sum_{j=0}^2 [|\xi||\lambda|^{-\frac{1}{2}}]^j \|L^{-1}(\lambda, \xi)\|_{B(E)}.$$

Then by using the well-known inequality $y^j \leq C(1 + y^m)$, $y \geq 0$, $j \leq m$ for $y = (|\lambda|^{-\frac{1}{2}}|\xi|)^j$ and $m = 2$, we get the uniform estimate

$$\left| \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \xi^j \right| \leq C(|\lambda| + \xi^2). \quad (4.8)$$

From (4.7)–(4.8) we have

$$\|\sigma_\lambda(\xi)\|_{B(E)} \leq C(|\lambda| + \xi^2)(1 + \xi^2 + |\lambda|)^{-1} \leq C.$$

Due to the \mathbb{R} -positivity of the operator A , the set

$$\{(|\lambda| + \xi^2)L^{-1}(\lambda, \xi) : \xi \in \mathbb{R} \setminus \{0\}\}$$

is \mathbb{R} -bounded. Then from the equality

$$\sigma_\lambda(\xi) = \left[\sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \xi^j \right] (|\lambda| + \xi|\lambda| + \xi^2)L^{-1}(\lambda, \xi)$$

and by Kahane's contraction principle we obtain the \mathbb{R} -boundedness of set

$$\{\sigma_\lambda(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}.$$

Now by differentiating $\sigma_\lambda(\xi)$ we have

$$\begin{aligned} \xi \frac{d}{d\xi} \sigma_\lambda(\xi) &= -|\lambda|2\xi^2 L^{-2} + \sum_{j=1}^2 |\lambda|^{1-\frac{j}{2}} [\xi^j L^{-1} - 2\xi^{j+2} L^{-2}] \\ &= (-2\xi^2 L^{-1})(|\lambda| L^{-1}) + \sum_{j=1}^2 [|\lambda|^{1-\frac{j}{2}} \xi^j] (|\lambda| + \xi^2)^{-1} (|\lambda| + \xi^2) L^{-1} \\ &\quad + \left[\left(-2 \sum_{j=1}^2 |\lambda|^{1-\frac{j}{2}} \xi^j \right) (|\lambda| + \xi^2)^{-1} \right] [(|\lambda| + \xi^2) L^{-1}] \xi^2 L^{-1}. \end{aligned}$$

Consider the set

$$\{ \sigma_0(\lambda, \xi) : \xi \in \mathbb{R} \setminus \{0\} \}, \quad \sigma_0(\lambda, \xi) = \xi \frac{d}{d\xi} \sigma_\lambda(\xi).$$

Due to the \mathbb{R} -positivity of the operator A , the sets

$$\begin{aligned} \{ \sigma_1(\lambda, \xi) : \xi \in \mathbb{R} \setminus \{0\} \}, \quad \sigma_1(\lambda, \xi) &= \lambda L^{-1}, \\ \{ \sigma_2(\lambda, \xi) : \xi \in \mathbb{R} \setminus \{0\} \}, \quad \sigma_2(\lambda, \xi) &= (|\lambda| + \xi^2) L^{-1}, \\ \{ \sigma_3(\lambda, \xi) : \xi \in \mathbb{R} \setminus \{0\} \}, \quad \sigma_3(\lambda, \xi) &= \xi^2 L^{-1} \end{aligned}$$

are \mathbb{R} -bounded. Then from the above formula in view of estimate (4.7), by virtue of the Kahane's contraction principle and from the product and additional properties of the collection of \mathbb{R} -bounded operators, for all $\xi_1, \xi_2, \dots, \xi_m \in \mathbb{R}$, $\sigma_k(\xi_1, \lambda), \sigma(\xi_2, \lambda), \dots, \sigma(\xi_m, \lambda)$, $u_1, u_2, \dots, u_m \in E$ and independent symmetric $\{-1, 1\}$ -valued random variables $r_j(y)$, $j = 1, 2, \dots, m$, $m \in \mathbb{N}$, we obtain the uniform estimate

$$\begin{aligned} \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) \sigma_0(\lambda, \xi_j) u_j \right\|_E dy &\leq C \sum_{k=1}^3 \int_{\Omega} \left\| \sum_{j=1}^m \sigma_k(\lambda, \xi_j) r_j(y) u_j \right\|_E dy, \\ &\leq C \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) u_j \right\|_E dy, \end{aligned}$$

i.e.,

$$\sup_{\lambda} \mathbb{R} \left(\left\{ \xi \frac{d}{d\xi} \sigma_\lambda(\xi) : \xi \in \mathbb{R} \setminus \{0\} \right\} \right) \leq C.$$

Then in view of Definition 1.1, it follows that $\Psi_\lambda(\xi)$ and $\sigma_\lambda(\xi)$ are the uniform collection of multipliers in $L_p(\mathbb{R}; E)$. Then, by using the equality (4.6) we obtain that the problem (4.4) has a solution $u \in W_p^2(\mathbb{R}; E(A), E)$ and the uniform estimate holds, i.e.,

$$\sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L_p(\mathbb{R}; E)} + \|Au\|_{L_p(\mathbb{R}; E)} \leq C \|\bar{f}\|_{L_p(\mathbb{R}; E)}. \quad (4.9)$$

Let u_1 be the restriction of u on $(0, 1)$. Then the estimate (4.9) implies that

$$u_1 \in W_p^2(0, 1; E(A), E).$$

By virtue of Theorem 1.5, we get

$$u_1^{(m_k)}(\cdot) \in (E(A); E)_{\theta_k, p}, \quad k = 1, 2.$$

Hence, $L_k u_1 \in E_k$. Thus by virtue of Theorem 4.1, the problem (4.9) has a unique solution $u_2(x)$ that belongs to the space $W_p^2(0, 1; E(A), E)$ and for sufficiently large $|\lambda|$ we have

$$\begin{aligned} &\sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u_2^{(j)}\|_{L_p(0, 1; E)} + \|Au_2\|_{L_p(0, 1; E)} \\ &\leq C \sum_{k=1}^2 [\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E + \|u_1^{(m_k)}\|_{C([0, 1]; E_k)} + |\lambda|^{1-\theta_k} \|u_1\|_{C([0, 1]; E)}]. \end{aligned} \quad (4.10)$$

Moreover, from (4.10), for $|\arg \lambda| \leq \varphi$ we obtain

$$\sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u_1^{(j)}\|_{L_p(0,1;E)} + \|Au_1\|_{L_p(0,1;E)} \leq C \|f\|_{L_p(0,1;E)}. \quad (4.11)$$

By Theorem 4.1, we have

$$\begin{aligned} & \sum_{j=0}^2 |\lambda|^{1-\frac{j}{2}} \|u_2^{(j)}\|_{L_p(0,1;E)} + \|Au_2\|_{L_p(0,1;E)} \\ & \leq C \left(\|f\|_{L_p(0,1;E)} + \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E) \right). \end{aligned} \quad (4.12)$$

Finally, from (4.11)–(4.12) we obtain the estimate (4.3).

Consider the problem

$$[L + \lambda]u = -u^{(2)}(x) + (A + \lambda)u(x) = f(x), \quad x \in (0, 1), \quad (4.13)$$

$$L_k u = \int_0^1 B_k(x) u(x) dx = 0, \quad k = 1, 2. \quad (4.14)$$

Let B denote the operator in $F = L_p(0, 1; E)$ generated by the problem (4.13)–(4.14), i.e.,

$$D(B) = W_p^2(0, b; E(A), E, L_{2k}), \quad Bu = -u^{(2)}(x) + Au(x).$$

Theorem 4.1 implies the following corollary.

Corollary 4.1 Suppose that Condition 3.1 is satisfied. Then for sufficiently large $k > 0$, there exist positive constants C_1 and C_2 so that

$$C_1 \|u\|_{W_p^2(0,1,E(A),E)} \leq \|(B + k)u\|_{L_p(0,1;E)} \leq C_2 \|u\|_{W_p^2(0,1,E(A),E)}$$

for $u \in W(0, 1, E(A), E)$.

Theorem 4.2 Suppose that Condition 3.1 is satisfied. Then the operator B is uniformly \mathbb{R} -positive in $L_p(0, 1; E)$.

Proof The estimate (4.3) implies that, for $\lambda \in S_\varphi$ and enough large $|\lambda|$, $B + \lambda$ is invertible and the operator B is positive in $L_p(0, 1; E)$. By using a similar technique as in [28, Lemma 5.3.2/1], we obtain that, for $f \in D(0, 1; E(A))$, the solution of the Equation (4.13) is represented as

$$u(x) = U_{1\lambda}(x)g_1 + U_{2\lambda}(x)g_2 + \int_0^1 U_{0\lambda}(x-y)f(y)dy, \quad g_k \in E, \quad (4.15)$$

where

$$U_{0\lambda}(x-y) = \begin{cases} -A_\lambda^{-\frac{1}{2}} e^{-(x-y)A_\lambda^{\frac{1}{2}}}, & x \geq y, \\ A_\lambda^{-\frac{1}{2}} e^{-(y-x)A_\lambda^{\frac{1}{2}}}, & x \leq y \end{cases}$$

and $U_{j\lambda}(x)$, $j = 1, 2$ are analytic semigroups generated by operator $A_\lambda^{\frac{1}{2}}$. By taking into account the boundary conditions (4.14), we obtain the following equation with respect to g_1 and g_2 :

$$L_k(U_{1\lambda})g_1 + L_k(U_{2\lambda})g_2 = L_k(\Phi_\lambda), \quad k = 1, 2, \quad \Phi_\lambda = \int_0^1 U_{0\lambda}(x-y)f(y)dy.$$

By solving the above system and substituting it into (4.15), in a similar way as in Theorem 3.1, we obtain the representation of the solution for the problem (4.13)–(4.14):

$$\begin{aligned} u(x) = & \{e^{-xA_\lambda^{\frac{1}{2}}}[C_{11} + \tilde{d}_{11}(\lambda)] + e^{-(1-x)A_\lambda^{\frac{1}{2}}}[C_{12} + \tilde{d}_{12}(\lambda)]\}A_\lambda^{-\frac{m_1}{2}}L_1(\Phi_{\lambda t}) \\ & + \{e^{-xA_\lambda^{\frac{1}{2}}}[C_{21} + \tilde{d}_{21}(\lambda)] + e^{-(1-x)A_\lambda^{\frac{1}{2}}}[C_{22} + \tilde{d}_{22}(\lambda)]\}A_\lambda^{-\frac{m_2}{2}}L_2(\Phi_\lambda), \end{aligned}$$

where C_{kj} and \tilde{d}_{kj} are the same as in (3.6). By calculating $L_k(\Phi_\lambda)$ we obtain from the above

$$\begin{aligned} u(x) &= [B + \lambda]^{-1}f = \int_0^1 G(\lambda, x, y)f(y)dy, \\ G(\lambda, x, y) &= \sum_{k=1}^2 \sum_{j=1}^2 A_\lambda^{-\frac{1}{2}}B_{kj}(\lambda)U_{j\lambda}(x)\tilde{U}_{kj\lambda}(x-y) + U_{0\lambda}(x-y), \end{aligned}$$

where $B_{kj}(\lambda)$ are like \tilde{d}_{jk} . So are the uniformly bounded operators in E and

$$\tilde{U}_{kj\lambda}(x-y) = \begin{cases} b_{kj}e^{-(x-y)A_\lambda^{\frac{1}{2}}}, & x \geq y, \\ \delta_{kj}e^{-(y-x)A_\lambda^{\frac{1}{2}}}, & x \leq y, \end{cases} \quad b_{kj}, \delta_{kj} \in \mathbf{C}.$$

Let us first show that the set $\{G(\lambda, x, y); \lambda \in S(\varphi)\}$ is uniformly \mathbb{R} -bounded. Really, by using the generalized Minkowski's Young inequalities, by semigroup estimates (3.7), have the uniform estimate

$$\begin{aligned} \|G(\lambda, x, y)f\|_F &\leq C \left\{ \sum_{k=1}^2 \sum_{j=1}^2 \|A_\lambda^{-\frac{1}{2}}\| \|B_{kj}(\lambda)\| \|\tilde{U}_{kj\lambda}(x)f\|_F + \|U_{0\lambda}(x)f\|_F \right\} \\ &\leq C \|f\|_F. \end{aligned}$$

Due to \mathbb{R} -positivity of A , in view of properties of holomorphic semigroups $U_{j\lambda}(x)$ and the uniform boundedness of operators $B_{kj}(\lambda)$ and by using the Kahane's contraction principle, we get that the sets

$$\begin{aligned} &\{b_{kj}(\lambda, x, y) : \lambda \in S_\varphi\}, \\ b_{kj}(\lambda, x, y) &= B_{kj}(\lambda)A_\lambda^{-\frac{1}{2}}U_{j\lambda}(x)[U_{1\lambda}(1-y) + U_{2\lambda}(y)], \\ &\{b_0(\lambda, x, y) : \lambda \in S_\varphi\}, \quad b_0(\lambda, x, y) = U_{0\lambda}(x-y) \end{aligned}$$

are uniformly \mathbb{R} -bounded. Then by using the Kahane's contraction principle, product and additional properties of the collection of \mathbb{R} -bounded operators and the \mathbb{R} -boundedness of the sets b_{kj}, d_0 for all $u_1, u_2, \dots, u_\mu \in F$, $\lambda_1, \lambda_2, \dots, \lambda_\mu \in S(\varphi)$, for independent symmetric $\{-1, 1\}$ -valued random variables $r_i(y)$, $i = 1, 2, \dots, \mu$, $\mu \in N$, we have the estimate

$$\begin{aligned} &\int_\Omega \left\| \sum_{i=1}^\mu r_i(y)G(\lambda_i, x, y)u_i \right\|_F d\tau \\ &\leq C \left\{ \sum_{k,j=1}^2 \int_\Omega \left\| \sum_{i=1}^\mu r_i(y)b_{kj}(\lambda_i, x, y)u_i \right\|_F d\tau \right. \\ &\quad \left. + \int_\Omega \left\| \sum_{i=1}^\mu r_i(y)b_0(\lambda_i, x, y)u_i \right\|_F d\tau \right\} \end{aligned}$$

$$\leq C e^{\beta|\lambda|^{\frac{1}{2}}|x-y|} \int_{\Omega} \left\| \sum_{i=1}^{\mu} r_i(y) u_i \right\|_F d\tau, \quad \beta < 0$$

uniformly in x and y . This implies that

$$\mathbb{R}\{G(\lambda, x, y) : \lambda \in S_{\varphi}\} \leq C e^{\beta|\lambda|^{\frac{1}{2}}|x-y|}, \quad \beta < 0, \quad x, y \in (0, 1).$$

In view of \mathbb{R} -bondedness property of kernel operators (see [11, Proposition 4.12]) and due to the density of $D(0, 1; E(A))$ in $L_p(0, 1; E)$ (see [23]), we obtain the assertion.

5 Cauchy Problem for Abstract Parabolic Equation

Consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au(x, t) &= f(x, t), \quad x \in (0, 1), \quad t \in (0, \infty), \\ L_k u &= \int_0^1 B_k(x) u(x, y) dx = 0, \quad k = 1, 2, \quad u(x, 0) = 0. \end{aligned} \quad (5.1)$$

In this section we obtain the well-posedness of problem (4.14) in mixed $L_{\mathbf{p}}$ space.

If $G_+ = (0, \infty) \times (0, 1)$, $\mathbf{p} = (p, p_1)$, $L_{\mathbf{p}}(G_+; E)$ will denote the space of all \mathbf{p} -summable scalar-valued functions with mixed norm (see, [5, §1] for $E = \mathbb{C}$), i.e., the space of all measurable functions f defined on G , for which

$$\|f\|_{L_{\mathbf{p}}(G_+)} = \left(\int_{\mathbb{R}_+} \left(\int_0^1 \|f(x, t)\|_E^p dt \right)^{\frac{p_1}{p}} dx \right)^{\frac{1}{p_1}} < \infty.$$

Analogously, $W_{\mathbf{p}}^2(G_+; E)$ denotes the E -valued Sobolev space with corresponding mixed norm (see, [5, §10] for $E = \mathbb{C}$).

Theorem 5.1 *Suppose that Condition 3.1 is satisfied for $\varphi \in (\frac{\pi}{2}, \pi)$. Then for $f \in L_{\mathbf{p}}(G_+; E)$ and sufficiently large $a > 0$, the problem (5.1) has a unique solution belonging to $W_{\mathbf{p}}^{1,2}(G_+; E(A), E)$ and the uniform estimate holds, i.e.,*

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}(G_+; E)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_{\mathbf{p}}(G_+; E)} + \|Au\|_{L_{\mathbf{p}}(G_+; E)} \leq C \|f\|_{L_{\mathbf{p}}(G_+; E)}.$$

Proof The problem (4.14) can be express as the following Cauchy problem:

$$\frac{du}{dt} + (B + a)u(t) = f(t), \quad u(0) = 0. \quad (5.2)$$

Theorem 4.2 implies that the operator O is uniformly \mathbb{R} -positive and is a generator of analytic semigroups in $F = L_p(0, 1; E)$. Then by virtue of [30, Theorem 4.2], we obtain that for all $f \in L^{p_1}(\mathbb{R}_+; F)$, the problem (4.15) has a unique solution belonging to $W_{p_1}^1(\mathbb{R}_+; D(B), F)$ and the following estimate holds:

$$\left\| \frac{du}{dt} \right\|_{L_{p_1}(\mathbb{R}_+; F)} + \|(B + a)u\|_{L_{p_1}(\mathbb{R}_+; F)} \leq C \|f\|_{L_{p_1}(\mathbb{R}_+; F)}.$$

Since $L_{p_1}(\mathbb{R}_+; F) = L_{\mathbf{p}}(G_+; E)$, by Theorem 4.1 we have

$$\|(B + a)u\|_{L_{p_1}(\mathbb{R}_+; F)} = \|u\|_{W_{\mathbf{p}}^2(G_+; E(A), E)}.$$

These relations and the above estimate imply the assertion.

6 Nonlocal Boundary Value Problems for Elliptic Equations

The Fredholm property of BVPs for elliptic equations with parameters in smooth domains was studied in [1]. In this section, the coercive estimate on the solution of integral boundary conditions for elliptic equations will be established in mixed $L_{\mathbf{p}}$ spaces.

Let $G \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain with an $(m-1)$ -dimensional boundary $\partial G \in C^\infty$ which locally admits rectification. Consider the following nonlocal BVP for the following anisotropic elliptic equation:

$$(L + \lambda)u = -D_x^2 u(x, y) + \sum_{|\beta| \leq 2m} a_\beta(y) D_y^\beta u(x, y) + \lambda u(x, y) = f(x, y), \quad (6.1)$$

$$L_k u = \int_0^1 B_k(x) u(x, y) dx = f_k(y), \quad k = 1, 2, \quad y \in G, \quad (6.2)$$

$$B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D_y^\beta u(x, y)|_{y \in \partial\Omega} = 0, \quad x \in (0, 1), \quad j = 1, 2, \dots, m, \quad (6.3)$$

where $B_k(x)$ are bounded operator from $B_{p_1, p}^{2m(1-\kappa-\varepsilon)}(G)$ to $L_{p_1}(G)$ for $\kappa = \theta_0 - \frac{1}{2p}$, $\frac{1}{2p} < \theta_0 < 1$, $\theta_0 = \max_k \{\theta_k\}$, $\varepsilon \in (0, 1 - \kappa)$, $\frac{1}{2p} - 1 < \theta_k < \frac{1}{2p}$, and $B_{p_1, p}^s(G) = B_{\mathbf{p}}^s(G)$ denotes the Besov space [6, §18].

$D_x = \frac{\partial}{\partial x}$, $D_j = -i \frac{\partial}{\partial y_j}$, $D_y = (D_1, \dots, D_m)$, $m_k \in \{0, 1\}$, α_k, β_k are complex numbers, $r = 0$ or $r = 1$, $y = (y_1, \dots, y_m)$. Let Q denote the operator generated by problem (6.1)–(6.3) for $\lambda = 0$.

If $\Omega = (0, 1) \times G$, $\mathbf{p} = (p_1, p)$, $L_{\mathbf{p}}(\Omega)$ will denote the space of all \mathbf{p} -summable scalar-valued functions with a mixed norm (see, [5, §1]), i.e., the space of all measurable functions f defined on Ω , for which

$$\|f\|_{L_{\mathbf{p}}(\Omega)} = \left(\int_0^1 \left(\int_G |f(x, y)|^{p_1} dx \right)^{\frac{p}{p_1}} dy \right)^{\frac{1}{p}} < \infty.$$

Analogously, $W_{\mathbf{p}}^l(\Omega)$ denotes the Sobolev space with corresponding mixed norm (see [6, §10]).

We have the result as following theorem.

Theorem 6.1 *Let the following conditions be satisfied:*

- (1) B_k are bounded operators from $B_{p_1, p}^{2m(1-\kappa-\varepsilon)}(G)$ to $L_{p_1}(G)$;
- (2) $a_\alpha \in C(\overline{\Omega})$ for each $|\alpha| = 2m$ and $a_\alpha \in [L_\infty + L_{r_k}](\Omega)$ for each $|\alpha| = k < 2m$ with $r_k \geq p_1$, $p_1 \in (1, \infty)$ and $2m - k > \frac{1}{r_k}$, $\nu_\alpha \in L_\infty$;
- (3) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ for each j, β , $m_j < 2m$, $p \in (1, \infty)$;
- (4) for $y \in \overline{\Omega}$, $\xi \in \mathbb{R}^\mu$, $\eta \in S(\varphi_1)$, $\varphi_1 \in [0, \frac{\pi}{2})$, $|\xi| + |\eta| \neq 0$, let

$$\eta + \sum_{|\alpha|=2m} a_\alpha(y) \xi^\alpha \neq 0;$$

- (5) for each $y_0 \in \partial\Omega$, the local BVP's in local coordinates corresponding to y_0

$$\begin{aligned} \eta + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) &= 0, \\ B_{j0} \vartheta &= \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta \vartheta(y) = h_j, \quad j = 1, 2, \dots, m \end{aligned}$$

has a unique solution $\vartheta \in C_0(\mathbb{R}_+)$ for all $h = (h_1, h_2, \dots, h_m) \in \mathbb{R}^m$ and for $\xi' \in \mathbb{R}^{\mu-1}$ with $|\xi'| + |\eta| \neq 0$.

Then

(a) for all $f \in L_{\mathbf{p}}(\Omega)$, $\mathbf{p} = (p_1, p)$, $f_k \in B_{\mathbf{p}}^{2m(1-\theta_k)}(G)$, $|\arg \lambda| \leq \varphi$, $0 \leq \varphi < \pi$ and sufficiently large $|\lambda|$, the problem (6.1)–(6.3) has a unique solution $u \in W_{\mathbf{p}}^2(\Omega)$ and the following coercive uniform estimate holds:

$$\|u\|_{W_{\mathbf{p}}^2(\Omega)} \leq C \left[\|(L + \lambda)u\|_{L_{\mathbf{p}}(\Omega)} + \sum_{k=1}^2 \|L_k u\|_{B_{\mathbf{p}}^{2m(1-\theta_k)}(G)} + \|u\|_{L_{\mathbf{p}}(\Omega)} \right];$$

(b) the operator $u \rightarrow Qu = \{Lu, L_1 u, L_2 u\}$ is Fredholm from $W_{\mathbf{p}}^{2,2m}(\Omega)$ into $L_{\mathbf{p}}(\Omega) \times B_{\mathbf{p}}^{2m(1-\theta_1)}(G) \times B_{\mathbf{p}}^{2m(1-\theta_2)}(G)$.

Proof Let $E = L_{p_1}(G)$. Consider the operator A defined by

$$D(A) = W_{p_1}^{2m}(G; L_0), \quad (Au)(x) = \sum_{|\beta| \leq 2m} a_{\beta}(y) D_y^{\beta} u(x, y).$$

Then the problem (6.1)–(6.3) can be rewritten in the form of (2.1), (3.1), i.e.,

$$Lu = f, \quad L_k u = f_k. \quad (6.4)$$

Let us apply Theorem 4.1 to problem (6.4). It is known that $L_{p_1}(G) \in UMD$ for $p_1 \in (1, \infty)$ (see [3]). Then in view of the multiplier theorems in E -valued L_p spaces (see [30]), the space $L_{p_1}(G)$ satisfies the multiplier condition. By virtue of [11, Theorem 8.2], the operator A is \mathbb{R} -positive in $L_{p_1}(G)$ and has the fractional powers, i.e., all conditions of the Theorem 4.1 hold and we obtain the assertion.

7 Infinite Systems of Parabolic Equations

Consider the mixed problem for an infinite system of parabolic equations

$$\begin{aligned} \frac{\partial u_m}{\partial t} - \frac{\partial^2 u_m}{\partial x^2} + \sum_{j=1}^{\infty} (2^{sj} + a) u_j &= f_m(x, t), \\ L_k u &= \int_0^1 B_k(x) u_m(x, t) dx = 0, \quad k = 1, 2, \quad t \in \mathbb{R}_+, \\ u_m(x, 0) &= 0, \quad x \in (0, 1), \quad s > 0, \quad m = 1, 2, \dots, \infty, \end{aligned} \quad (7.1)$$

where $u_m = u_m(x, t)$ and $B_k(x)$ are linear operators from $l_q^{s(1-\kappa-\varepsilon)}$ to l_q (see [27, §1.18.2] for the definition of l_q^s), and here $\kappa = \theta_0 - \frac{1}{2p}$, $\frac{1}{2p} < \theta_0 < 1$, $\theta_0 = \max_k \{\theta_k\}$, $\varepsilon \in (0, 1 - \kappa)$, $\frac{1}{2p} - 1 < \theta_k < \frac{1}{2p}$, $\theta_k \in (\frac{1}{2p}, 1)$. Let $G = (0, 1) \times \mathbb{R}_+$.

Theorem 7.1 Suppose that B_k are bounded operators from $l_q^{s(1-\kappa-\varepsilon)}$ to l_q for $\theta_k \in (\frac{1}{2p}, 1)$ and $\varepsilon \in (0, 1 + \frac{1}{2p} - \theta_0)$. Then, for $f(x, t) = \{f_m(x, t)\}_1^{\infty} \in L_p(G; l_q)$, $p, q \in (1, \infty)$ and for sufficiently large $a > 0$, the problem (7.1) has a unique solution $u = \{u_m(x, t)\}_1^{\infty}$ that belongs to $W_{\mathbf{p}}^{1,2}(G, l_q(D), l_q)$ and the following coercive uniform estimate holds:

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}(G; l_q)} + \sum_{k=1}^n \left\| \frac{\partial^2 u}{\partial x_k^2} \right\|_{L_{\mathbf{p}}(G; l_q)} + \|Au\|_{L_{\mathbf{p}}(G; l_q)} \leq C \|f\|_{L_{\mathbf{p}}(G; l_q)}.$$

Proof Really, let $E = l_q$, A be infinite matrices, defined by

$$A = [2^{sm}\delta_{jm}], \quad m, j = 1, 2, \dots, \infty.$$

It is easy to see that the operator A is \mathbb{R} -positive in l_q . Therefore, all conditions of Theorem 5.1 hold and we obtain the assertion.

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