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Stochastic H_2/H_∞ Control for Poisson Jump-Diffusion Systems*

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Abstract This paper is concerned with stochastic H_2/H_∞ control problem for Poisson jump-diffusion systems with (x, u, v)-dependent noise, which are driven by Brownian motion and Poisson random jumps. A stochastic bounded real lemma (SBRL for short) for Poisson jump-diffusion systems is firstly established, which stands out on its own as a very interesting theoretical problem. Further, sufficient and necessary conditions for the existence of a state feedback H_2/H_∞ control are given based on four coupled matrix Riccati equations. Finally, a discrete approximation algorithm and an example are presented.

Keywords Poisson jump-diffusion systems, Stochastic H_2/H_∞ control, Stochastic bounded real lemma, Indefinite stochastic Riccati equation
2000 MR Subject Classification 93E20, 60H10, 93B36, 49N10

1 Introduction

In many realistic situations, the noise for a dynamic system may be Wiener- or Poisson-type. Poisson noises cause the random discontinuity of the system dynamic, whereas Wiener noises cause the continuous perturbation of the system dynamic. Poisson processes should be also considered when a dynamic system is exposed to sudden, infrequent, highly localized changes that occur in a short period of time such as earthquakes, large random weather fluctuations, or occasional mass mortalities. Poisson processes quite frequently arise in engineering, manufacturing, economics, and biosystem applications. For example, in financial market, the stock price is classically described by geometric Brownian motion, however, in practice, the price of stocks can be made a sudden shift by the exogenous disturbance such as wars, decisions of large banks or the corporation, national policy, block transaction etc. In order to describe such phenomena, Poisson jumps is usually inserted in and thus the control system is governed by independent Brownian motion and Poisson random jumps, which is called Poisson jump-diffusion systems (see [18, 24]). The optimal control problem with random jumps was first considered by Boel and Varaiya [1]. In their case, the control system is disturbed by random jumps and the optimal solution is a discontinuous stochastic process. From then on, many scholars began to study the jump-diffusion system and its applications, for further reference, we refer to [8, 17, 20] and their references. Those results mostly concentrate on optimal control and its application in financial markets or their corresponding theories.

H_∞ control and H_2/H_∞ control are important robust control design methods in modern control theory. H_∞ control requires a controller to eliminate the external disturbance below a

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given disturbance attenuation level, while H_2/H_∞ control demands one to minimize an average output energy while guaranteeing a prescribed disturbance attenuation level. Before 1998, most works are focused on deterministic H_∞ and H_2/H_∞ control for the systems governed by an ordinary differential equation (see [4, 13–14, 19]). In recent years, quite a lot of research interests have been devoted to stochastic Itô systems. For example, in Hinrichsen and Pritchard [10], H_∞ control for general linear stochastic Itô systems was firstly discussed very extensively. Moreover, a very useful lemma called "stochastic bounded real lemma (SBRL for short)" was given therein using linear matrix inequalities. Then Zhang and Chen [26] investigated H_∞ control of nonlinear stochastic systems with multiplicative noise, where a Hamilton-Jacobi equation (HJE for short) associated with nonlinear stochastic H_∞ control was derived. As for the H_2/H_∞ control, Chen and Zhang [2] studied the mixed H_2/H_∞ control with state-dependent noise, and then a further discussion on the case of nonlinear stochastic Itô systems with state, control input and external disturbance-dependent noise ((x, u, v)-dependent noise for short) was given in [27], both of which extended the deterministic H_2/H_∞ control results of [14] to the stochastic setting. More recently, Wang [23] discussed the H_2/H_∞ control with state-dependent noise and random coefficients, where the sufficient and necessary conditions for the existence of the H_2/H_∞ control are given by a pair of coupled backward stochastic Riccati equations. This result extended the work of Chen and Zhang [2] to the case of random coefficients.

In addition, stochastic H_∞ and H_2/H_∞ control for discrete or continuous time Itô systems with Markovian jumping parameters also have attracted many researchers' attention. For instance, in a recent monograph (see [7]), H_∞ control has been elaborately addressed for discrete-time Markov jump systems with multiplicative noise. The state and output feedback H_∞ control of nonlinear stochastic Markov jump systems with state and disturbance dependent noise has been tackled in [15] based on coupled Hamilton-Jacobi inequalities. Hou et al. [11] studied the infinite horizon H_2/H_∞ control problem for a broad class of discrete-time Markov jump systems with (x, u, v)-dependent noise. For other relevant developments in this regard, interested readers can refer to the monographs as Dragan and Morozan [5–6], Todorov and Fragoso [21–22], etc.

As mentioned above, most works on stochastic H_∞ and H_2/H_∞ control are constrict in Itô stochastic systems or Markovian jump systems, but Poisson noise was not considered in these papers. While in engineering practice, the dynamic systems, such as electric power systems, aircraft flight control systems and manufacturing systems, may experience dramatic changes when suffering sudden external impact as power failure, atmospheric turbulence in extreme weather and machine breakage or repair. In this situation, it is very nature to utilize the Poisson jump-diffusion process to describe such phenomena (see [3, 16]). Lin [16] studied the H_∞ control problem for a class of Poisson jump-diffusion stochastic linear systems with constant coefficients. Chen [3] investigated the H_∞ robust control designs for nonlinear stochastic systems with external disturbance and Poisson noise, where a fuzzy approach to solve the Hamilton-Jacobi inequality (HJI for short) was employed. However, to the best of our knowledge, there are rare literature concerning stochastic H_2/H_∞ control for jump-diffusion systems. The objective of this paper is to develop an H_2/H_∞ control theory for Poisson jump-diffusion systems with (x, u, v)-dependent noise and time-varying coefficients, and sufficient and necessary conditions are derived for the existence of a state feedback H_2/H_∞ control in terms of four coupled matrix-valued Riccati equations. To some extent, the results of this paper can be viewed as an extension of that of [2] to the case of (x, u, v)-dependent noise and random jumps.

The rest of this paper is organized as follows. In Section 2, we present two useful lemmas and describe some basic theories on stochastic H_∞ and H_2/H_∞ control. In Section 3, inspired by the ideas in [10], we develop the stochastic bounded real lemma for Poisson jump-diffusion system, which enables us to obtain necessary and sufficient conditions for the existence of a stabilizing controller which keeps the effect of the perturbation on the to-be-controlled output below a given disturbance attenuation level. Based on this result, we then derive necessary and sufficient conditions for the existence of stochastic H_2/H_∞ control in Section 4. Concluding remarks are presented in Section 5.

2 Problem Formulation and Preliminaries

2.1 Notations

We make use of the following notations in this paper.

M' is the transpose of the vector or matrix M. $|M|$ denotes the square root of the summarized squares of all the components of the vector or matrix M. $\langle M_1, M_2\rangle$ is the inner product of two vectors M_1 and M_2. M^{-1} is the inverse of a nonsingular square matrix M. $\mathbb{R}^m$ stands for the m-dimensional Euclidean space. $C(\tau,T;H)$ is the space of H-valued continuous functions on $[\tau,T]$ endowed with the maximum norm. $\mathcal{S}^2_{\mathcal{F}}(\tau,T;H)$ is the space of H-valued $\mathcal{F}_t$-adapted càdlàg processes f on $[\tau,T]$ satisfying $\|f\| = E\sup_{\tau\leqslant t\leqslant T}|f_t|^2 < \infty$. $\mathcal{L}^2_{\mathcal{F}}(\tau,T;H)$ is the space of H-valued $\mathcal{F}_t$-adapted square-integrable stochastic processes f on $[\tau,T]$ satisfying $\|f\| = \big[E\int_\tau^T|f_t|^2\mathrm{d}t\big]^{\frac{1}{2}}$. $\mathcal{L}^{\nu,2}(\mathscr{E};H)$ is the space of H-valued measurable functions f defined on the measurable space $(\mathscr{E},\mathscr{B}(\mathscr{E}),\nu)$ satisfying $\|f\| = \sqrt{\int_{\mathscr{E}}|f(\theta)|^2\nu(\mathrm{d}\theta)}$. $\mathcal{L}^{\nu,2}_{\mathcal{F}}([0,T]\times\mathscr{E};H)$ is the space of $\mathcal{L}^{\nu,2}(\mathscr{E};H)$-valued and $\mathcal{F}_t$-predictable processes f satisfying $\|f\| = \sqrt{E\iint_{\mathscr{E}\times(0,T]}|f_t(\theta)|^2\nu(\mathrm{d}\theta)\mathrm{d}t}$. $\mathcal{U}^k[\tau,T]$ is the space of R^k-valued and $\mathcal{F}_t$-predictable processes f on $[\tau,T]$ satisfying $E\int_\tau^T|f_t|^2\mathrm{d}t<\infty$.

Sometimes we may write f for a deterministic function f_t, omitting the variable t, whenever no confusion arises. Under this convention, when $f\geqslant(>)\ 0$ means $f_t\geqslant(>)\ 0$ for all $t\in[0,T]$.

2.2 Two useful lemmas

Throughout this paper, let T be a fixed strictly positive real number and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ be a complete filtered probability space on which is defined one-dimensional standard Brownian motion $\{W_t\}_{0\leqslant t\leqslant T}$. Denote by $\mathscr{B}(\Xi)$ the Borel-σ-algebra of any topological space Ξ. Let $(\mathscr{E},\mathscr{B}(\mathscr{E}),\nu)$ be a measurable space with $\nu(\mathscr{E})<\infty$ and $\eta:\Omega\times\mathscr{D}_\eta\to\mathscr{E}$ be an $\mathcal{F}_t$-adapted stationary Poisson point process with characteristic measure ν, where $\mathscr{D}_\eta$ is a countable subset of $(0,\infty)$. Then the counting measure induced by η is

$$\mu((0,t]\times M) = \sharp\{s\in\mathscr{D}_\eta; s\leqslant t, \eta_s\in M\},\quad \forall t\geqslant 0,\ M\in\mathscr{B}(\mathscr{E}).$$

And $\widetilde{\mu}(\mathrm{d}\theta,\mathrm{d}t) := \mu(\mathrm{d}\theta,\mathrm{d}t) - \nu(\mathrm{d}\theta)\mathrm{d}t$ is a compensated Poisson random martingale measure which is assumed to be independent of Brownian motion. Assume that $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is P-completed natural filtration generated by $\{W_t, 0\leqslant t\leqslant T\}$ and $\big\{\iint_{A\times(0,t]}\widetilde{\mu}(\mathrm{d}\theta,\mathrm{d}r), 0\leqslant t\leqslant T, A\in\mathscr{B}(\mathscr{E})\big\}$.

Now we state some basic results on stochastic differential equation (SDE for short) driven by both martingale and Poisson jumps which will be frequently used in this paper. One is the Itô's formula of this type process (see [9]), the other is the solution's existence and uniqueness of stochastic differential equation driven by Brownian motion and Poisson random jumps (see [12]).

Lemma 2.1 *Let M_t be a square integral continuous martingale and A_t be a continuous adapted process with finite variance. $g=(g^1,\cdots,g^n)$, here $g^i(s,x,\theta),\ i=1,\cdots,n$ is $\{\mathcal{F}_t\}_{t\geqslant 0}$ locally square integrable, and x_0 is an $\mathcal{F}_0$-measurable variable. x_t satisfies the following Itô type stochastic process:*

$$x_t = x_0 + M_t + A_t + \iint_{\mathscr{E}\times(0,t]} g(s,x_{s-},\theta)\widetilde{\mu}(\mathrm{d}\theta,\mathrm{d}s).$$

Then for every $F(t,x)\in C^{1,2}(\mathbb{R}_+,\mathbb{R}^n)$, we have

$$\begin{aligned}\mathrm{d}F(t,x_t) &= F_t(t,x_t)\mathrm{d}t + \sum_{i=1}^n F_{x^i}(t,x_t)\mathrm{d}M_t^i + \sum_{i=1}^n F_{x^i}(t,x_t)\mathrm{d}A_t^i + \frac{1}{2}\sum_{i,j=1}^n F_{x^ix^j}\mathrm{d}\langle M^i,M^j\rangle_t \\ &\quad + \int_{\mathscr{E}}\Big[F(t,x_t+g(t,x_t,\theta)) - F(t,x_t) - \sum_{i=1}^n F_{x^i}(t,x_t)g^i(t,x_t,\theta)\Big]\nu(\mathrm{d}\theta) \\ &\quad + \int_{\mathscr{E}}[F(t,x_t+g(t,x_{t-},\theta)) - F(t,x_t)]\widetilde{\mu}(\mathrm{d}\theta,\mathrm{d}t). \end{aligned} \tag{2.1}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the quadratic variation process of the semi-martingale, and $F(t,x)\in C^{1,2}(\mathbb{R}_+,\mathbb{R}^n)$ denotes function $F(t,x)$ being twice continuously differentiable in $x\in\mathbb{R}^n$ and once in $t\in\mathbb{R}_+$. F_x and F_{xx} denote the gradient row vector and Hessian matrix with elements of second partial derivatives of n-dimensional function $F(t,x)$, respectively.

Lemma 2.2 *Let x_0 be an $\mathcal{F}_0$-measurable random variable and $b:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^n,\ \sigma:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^n,\ \pi:[0,T]\times\Omega\times\mathscr{E}\times\mathbb{R}^n\to\mathbb{R}^n$ are given mappings satisfying*

(i) *$b(\cdot,0)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$; $\sigma(\cdot,0)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$; $\pi(\cdot,\cdot,0)\in\mathcal{L}^{\nu,2}_{\mathcal{F}}([0,T]\times\mathscr{E};\mathbb{R}^n)$;*

(ii) *for some positive constant $C>0$, and for all $(t,x,\overline{x})\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, there exists*

$$\begin{aligned}&|b(t,x)-b(t,\overline{x})|^2 + |\sigma(t,x)-\sigma(t,\overline{x})|^2 \\ &+\int_{\mathscr{E}}|\pi(t,\theta,x)-\pi(t,\theta,\overline{x})|^2\nu(\mathrm{d}\theta)\leqslant C|x-\overline{x}|^2.\end{aligned}$$

Then the stochastic differential equation with jumps

$$x_t = x_0 + \int_0^t b(s,x_s)\mathrm{d}s + \int_0^t \sigma(s,x_s)\mathrm{d}W_s + \iint_{\mathscr{E}\times(0,t]}\pi(s,\theta,x_{s-})\widetilde{\mu}(\mathrm{d}\theta,\mathrm{d}s)$$

has a unique solution $x\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$. Moreover, a priori estimate holds:

$$\begin{aligned}E\sup_{0\leqslant t\leqslant T}|x_t|^2 &\leqslant K\Big[E|x_0|^2 + E\int_0^T|b(t,0)|^2\mathrm{d}t + E\int_0^T|\sigma(t,0)|^2\mathrm{d}t \\ &\quad + E\iint_{\mathscr{E}\times(0,T]}|\pi(t,\theta,0)|^2\nu(\mathrm{d}\theta)\mathrm{d}t\Big],\end{aligned} \tag{2.2}$$

where K is a positive constant depending only on Lipschitz constant C and T.

2.3 Problem formulation

Consider the following stochastic time-varying linear system governed by Brownian motion W_t and Poisson random martingale measure $\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t)$:

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t^1 v_t + B_t^2 u_t)\mathrm{d}t + (A_t^0 x_t + B_t^{01} v_t + B_t^{02} u_t)\mathrm{d}W_t \\ \qquad\quad + \displaystyle\int_{\mathscr{E}} [E_t(\theta) x_{t-} + F_t^1(\theta) v_t + F_t^2(\theta) u_t]\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ z_t = \begin{pmatrix} C_t x_t \\ D_t^1 v_t \\ D_t^2 u_t \end{pmatrix}, \quad (D_t^2)' D_t^2 = I \end{cases} \tag{2.3}$$

with $x_0 = x^0$. We assume the coefficient matrices $A, A^0 : [0,T] \to \mathbb{R}^{n\times n}$, $B^1, B^{01} : [0,T] \to \mathbb{R}^{n\times m}$, $B^2, B^{02} : [0,T] \to \mathbb{R}^{n\times s}$, $C : [0,T] \to R^{q\times n}$, $D^1 : [0,T] \to \mathbb{R}^{l\times m}$, $D^2 : [0,T] \to \mathbb{R}^{p\times s}$, $E : [0,T] \to \mathcal{L}^{\nu,2}(\mathscr{E}; \mathbb{R}^{n\times n})$, $F^1 : [0,T] \to \mathcal{L}^{\nu,2}(\mathscr{E}; \mathbb{R}^{n\times m})$, $F^2 : [0,T] \to \mathcal{L}^{\nu,2}(\mathscr{E}; \mathbb{R}^{n\times s})$ are matrix-valued continuous functions. Then from Lemma 2.2 for all $(u, v, x^0) \in \mathcal{U}^s[0,T] \times \mathcal{U}^m[0,T] \times \mathbb{R}^n$, there exists a unique solution $x = x(\cdot, u, v, x^0) \in \mathcal{S}^2_{\mathcal{F}}(0,T; \mathbb{R}^n)$ to the state equation of system (2.3).

We view v as an external disturbance which adversely affects the to-be-controlled output $z \in \mathbb{R}^{q+l+p}$ (whose desired value is represented by 0). The disturbing effect is to be ameliorated via control input $u \in \mathcal{U}^s[0,T]$. The effect of the disturbance on the to-be-controlled output z of the system (2.3) is then described by the perturbation operator $\mathcal{L}_{cl} : \mathcal{U}^m[0,T] \to \mathcal{U}^{q+l+p}[0,T]$ being defined as $\mathcal{L}_{cl}(v) = (C_t x(\cdot, u, v, 0), D_t^1 v_t, D_t^2 u_t)'$, which (for zero initial state) maps finite energy disturbance signals v into the corresponding finite energy output signals z of the closed loop system. The size of this linear operator is measured by the induced norm. The larger this norm is, the larger is the effect of the unknown disturbance v on the to-be-controlled output z in the worst case. Then the H_∞ control problem is to determine whether or not for each $\gamma > 0$ there exists a stabilizing controller u^* achieving $\|\mathcal{L}_{u^*}\| < \gamma$, where $\mathcal{L}_{u^*} : \mathcal{U}^{\Updownarrow}[\prime, \mathcal{T}] \to \mathcal{U}^{\amalg+\updownarrow+\checkmark}[\prime, \mathcal{T}]$ can be defined as $\mathcal{L}_{u^*}(v) = (C_t x(\cdot, u_t^*, v, 0), D_t^1 v_t, D_t^2 u_t^*)'$. Obviously, there may be more than one solution satisfying the required condition. We want the control not only to guarantee robust stability, but also to minimize the output energy when the worst case disturbance is implemented to the system, this is the so-called H_2/H_∞ control problem. That is, we wish to:

(1) Find a feedback control $u^* \in \mathcal{U}^s[0,T]$ such that the norm of the perturbation operator of the system (2.3) is less than some given disturbance attenuation level $\gamma > 0$, i.e., $\|\mathcal{L}_{u^*}\| < \gamma$;

(2) We require the control u^* to minimize the output energy z when the worst case disturbance $v^* \in \mathcal{U}^m[0,T]$ is applied to the system (2.3).

As we will show, this problem may be formulated as an LQ nonzero sum game. The two cost functionals we use are defined as follows:

$$J_1(u, v) = E\int_0^T [\gamma^2 |v_t|^2 - |z_t|^2]\mathrm{d}t, \tag{2.4}$$

$$J_2(u, v) = E\int_0^T |z_t|^2 \mathrm{d}t. \tag{2.5}$$

The first one is associated with an H_∞ robustness, while the second one reflects an H_2 optimality requirement. The aim is to find equilibrium strategies u^* and v^* defined by

$$J_1(u^*, v^*) \leqslant J_1(u^*, v), \quad \forall v \in \mathcal{U}^m[0,T],$$

$$J_2(u^*, v^*) \leqslant J_2(u, v^*), \quad \forall u \in \mathcal{U}^s[0,T].$$

If $J_1(u^*, v^*) \geqslant 0$ with $x^0 = 0$, certainly $\|z\|^2 \leqslant \gamma^2\|v\|^2$ for all $v \in \mathcal{U}^m[0,T]$, which ensures that $\|\mathcal{L}_{u^*}\| \leqslant \gamma$. The second Nash inequality shows that u^* minimizes output energy z when the external disturbance is at its worst and given by v^*. For example, in flight control system, the worst case disturbance means the extreme whether as atmospheric turbulence, while the corresponding control input means the control effort which minimizes the energy loss and sensitivity to the worst case atmospheric disturbances. Clearly, if the Nash equilibria (u^*, v^*) exist, then u^* is our desired H_2/H_∞ controller, and v^* is the corresponding worst case disturbance. Then the H_2/H_∞ control problem can be converted into finding the Nash equilibria (u^*, v^*). We approach this problem as linear quadratic (LQ for short) optimal control problem and obtain the solution by studying the associated stochastic Riccati equation.

In the following, we will give sufficient and necessary conditions for the existence of linear state feedback pairs (u^*, v^*). To this end, we will make some preliminaries in the next section.

3 Stochastic Bounded Real Lemma for Jump-Diffusion Systems

In this section we shall develop a version of stochastic bounded real lemma (SBRL for short), which states necessary and sufficient conditions for a given stochastic system with jumps to be stable with $\|\mathcal{L}\| < \gamma$. It is of independent interest, because it allows one to determine $\|\mathcal{L}\|$ which measures the influence of the disturbances in the worst case scenario. To this end, we consider the following linear stochastic system:

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t v_t)\mathrm{d}t + (A_t^0 x_t + B_t^0 v_t)\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} (E_t(\theta)x_{t-} + F_t(\theta)v_t)\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_\tau = \xi, \\ z_t = \begin{pmatrix} C_t x_t \\ D_t v_t \end{pmatrix}, \end{cases} \tag{3.1}$$

where $(\tau, \xi) \in [0,T] \times \mathbb{R}^n$ are the initial time and initial state, respectively. We denote $v \in \mathcal{U}^m[\tau, T]$ as the external disturbance and $z \in \mathcal{U}^{q+l}[0,T]$ the controlled output. All the coefficients $A, A^0 : [0,T] \to \mathbb{R}^{n\times n}$, $B, B^0 : [0,T] \to \mathbb{R}^{n\times m}$, $C : [0,T] \to \mathbb{R}^{q\times n}$, $D : [0,T] \to \mathbb{R}^{l\times m}$, $E : [0,T] \to \mathcal{L}^{\nu,2}(\mathscr{E}; \mathbb{R}^{n\times n})$, $F : [0,T] \to \mathcal{L}^{\nu,2}(\mathscr{E}; \mathbb{R}^{n\times m})$ are matrix-valued continuous functions. For all $(v, \xi) \in \mathcal{U}^m[\tau, T] \times \mathbb{R}^n$, there exists a unique solution $x = x(\cdot, v; \tau, \xi) \in \mathcal{S}^2_{\mathcal{F}}(\tau, T; \mathbb{R}^n)$ to the state equation of system (3.1). We use z defined in (3.1) rather than the more natural $z_t = C_t x_t + D_t v_t$ only to avoid the appearance of cross terms when computing $z_t' z_t$. Note that we assume the Brownian motion to be one-dimensional just for simplicity, there is no essential difficulty in the analysis below for the multidimensional cases.

Definition 3.1 *The system* (3.1) *with initial state zero is said to be externally stable or L^2 input-output stable if there exists a constant $\gamma \geqslant 0$ such that*

$$\|z\| \leqslant \gamma\|v\|, \quad v \in \mathcal{U}^m[0,T]. \tag{3.2}$$

Definition 3.2 *Suppose that the system* (3.1) *is externally stable. The operator $\mathcal{L}$: $\mathcal{U}^m[0,T] \to \mathcal{U}^{q+l}[0,T]$ defined by*

$$(\mathcal{L}v)(t) = \begin{pmatrix} C_t x(t, v; 0, 0) \\ D_t v_t \end{pmatrix}, \quad (t, v) \in [0,T] \times \mathcal{U}^m[0,T]$$

is called the perturbation operator of (3.1). *Its norm is defined as the minimal $\gamma \geqslant 0$ such that* (3.2) *is satisfied, i.e.,*

$$\|\mathcal{L}\| = \sup_{\substack{v\in\mathcal{U}^m[0,T]\\ v\neq 0,\ x_0=0}} \frac{\|(\mathcal{L}v)\|}{\|v\|} = \sup_{\substack{v\in\mathcal{U}^m[0,T]\\ v\neq 0,\ x_0=0}} \frac{\left\{E\int_0^T(x_t'C_t'C_tx_t + v_t'D_t'D_tv_t)\mathrm{d}t\right\}^{\frac{1}{2}}}{\left\{E\int_0^T v_t'v_t\mathrm{d}t\right\}^{\frac{1}{2}}}.$$

$\|\mathcal{L}\|$ is a measure of the worst effect that the stochastic disturbance v may have on the to-be-controlled output z of the system. Therefore it is important to find a way of determining the norm $\|\mathcal{L}\|$. The stochastic bounded real lemma which we will derive in this section provides a method for computing $\|\mathcal{L}\|$.

We proceed by associating a finite time quadratic cost functional with the problem parameterized by the initial data $(\tau,\xi)\in[0,T]\times\mathbb{R}^n$, $v\in\mathcal{U}^m[\tau,T]$ and $z\in\mathcal{U}^{q+l}[\tau,T]$:

$$\begin{aligned} J(v;\tau,\xi) &= E\int_\tau^T[\gamma^2|v_t|^2 - |z_t|^2]\mathrm{d}t \\ &= E\int_\tau^T[\langle(\gamma^2I - D_t'D_t)v_t, v_t\rangle - \langle C_t'C_tx_t, x_t\rangle]\mathrm{d}t, \end{aligned} \tag{3.3}$$

where x denotes the solution of (3.1). Note that for a given $\gamma>0$, the cost functional $J(v;0,0)$ is nonnegative for all $v\in\mathcal{U}^m[0,T]$ if and only if $\|\mathcal{L}\|\leqslant\gamma$. Therefore, the problem of minimizing this functional will lead us a method for estimating $\|\mathcal{L}\|$. We will analyze the linear quadratic (LQ for short) stochastic optimal control problem: Minimize the functional $J(v;\tau,\xi)$ over $v\in\mathcal{U}^{\Updownarrow}[\tau,\mathcal{T}]$ subject to (3.1). In our development in this section, we will employ the usual convention in LQ theory and refer to the disturbance v as a "control".

The above LQ problem is indefinite as the control weighting matrix in the cost is positive definite, while the state weighting matrix is negative semi-definite, which leads to the following indefinite stochastic Riccati equation (SRE for short):

$$\begin{cases} \dot{P}_t = -\Big[A_t'P_t + P_tA_t + (A_t^0)'P_tA_t^0 \\ \qquad + \displaystyle\int_{\mathscr{E}} E_t'(\theta)P_tE_t(\theta)\nu(\mathrm{d}\theta) - C_t'C_t - \Phi_t(P_t)'\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)\Big], \\ P_T = 0, \\ \Lambda_t^\gamma(P_t) > 0 \end{cases} \tag{3.4}$$

with

$$\begin{aligned} \Phi(P) &:= PB + (A^0)'PB^0 + \int_{\mathscr{E}} E'(\theta)PF(\theta)\nu(\mathrm{d}\theta), \\ \Lambda^\gamma(P) &:= \gamma^2I - D'D + (B^0)'PB^0 + \int_{\mathscr{E}} F'(\theta)PF(\theta)\nu(\mathrm{d}\theta). \end{aligned} \tag{3.5}$$

The global solvability of SRE (3.4) is hard to prove due to the following reasons: First, it is a highly nonlinear ordinary differential equation (ODE for short), especially in view of the matrix inverse term $(\Lambda^\gamma(P))^{-1}$, the existence and uniqueness theorem of solution of linear ODE is not valid. Second, the indefiniteness of coefficient matrices makes possible the singularity of the term $\Lambda^\gamma(P)$ when one tries to use the typical approximation scheme to construct a solution.

Third, the final positive definiteness constraint in (3.4) is the part of the equation and must be satisfied by any solution. Finally, (3.4) is a matrix equation, thus certain terms do not commute which adds substantial difficulty to the analysis. Hence, a new way to obtain its solution should be found.

In the following, we will prove the so-called stochastic bounded real lemma, which plays an essential role in this paper.

Theorem 3.1 (Stochastic Bounded Real Lemma) *Given* $\gamma > 0$, $\|\mathcal{L}\| < \gamma$ *if and only if the stochastic Riccati equation* (3.4) *parameterized by* γ *has a unique negative semi-definite solution* $P \leqslant 0$.

The next proposition proves the above theorem in one direction, i.e., establishes a relation between $\|\mathcal{L}\| < \gamma$ and the existence of some $P \leqslant 0$ such that the stochastic Riccati equation (3.4) is solvable.

Proposition 3.1 *Suppose that SRE* (3.4) *is solvable for some pair* (γ, P) *with* $\gamma > 0$ *and* P *being negative definite and uniformly bounded, then LQ problem is solvable with the optimal control* $v_t = \Psi_t x_{t-},\ t \in (0, T]$ *and* $\|\mathcal{L}\| < \gamma$, *where* $\Psi_t = -\Lambda_t^\gamma(P_t)^{-1}\Phi(P_t)$.

Proof Suppose that SRE (3.4) has a solution, and let x be the solution of (3.1) with $x_\tau = \xi$. Applying Itô's formula (2.1) to $\langle P_t x_t, x_t\rangle$, together with considering (3.3) and using the completion of squares, we have

$$
\begin{aligned}
J(v;\tau,\xi) &= J(v;\tau,\xi) + E\Big\{\int_\tau^T \mathrm{d}(x_t' P_t x_t) - x_T' P_T x_T + x_\tau' P_\tau x_\tau\Big\} \\
&= \xi' P_\tau \xi + E\Big\{\int_\tau^T \langle \Lambda_t^\gamma(P_t)(v_t - \Psi_t x_t), v_t - \Psi_t x_t\rangle \mathrm{d}t\Big\} \\
&\geqslant \xi' P_\tau \xi, \qquad (3.6)
\end{aligned}
$$

where $\Psi_t = -\Lambda_t^\gamma(P_t)^{-1}\Phi(P_t)$. It follows immediately that the optimal feedback control would be $v_t = \Psi_t x_{t-}$ if the corresponding solution to the system equation exists. In this case, the optimal cost is $\min\limits_{v\in\mathcal{U}^m[\tau,T]} J(v;\tau,\xi) = \xi' P_\tau \xi$. In fact, when $v_t = \Psi_t x_{t-}$, the system (3.1) reduces to

$$
\begin{cases}
\mathrm{d}x_t = (A_t + B_t\Psi_t)x_t\mathrm{d}t + (A_t^0 + B_t^0\Psi_t)x_t\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} (E_t(\theta) + F_t(\theta)\Psi_t)x_{t-}\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\
x_\tau = \xi.
\end{cases} \qquad (3.7)
$$

In view of the third positive definiteness constraint in (3.4), there exists a sufficient small $\epsilon > 0$ such that $\Lambda_t^\gamma(P_t) \geqslant \epsilon I$ for all $t \in [\tau, T]$. Moreover, since P is negative definite and uniformly bounded, all the coefficients of (3.7) are continuous and uniformly bounded. Therefore linear SDE (3.7) indeed has a unique solution $x \in \mathcal{S}_{\mathcal{F}}^2(\tau, T; \mathbb{R}^n)$, thus, $v = \Psi x \in \mathcal{U}^m[\tau, T]$. Here and in the following, $v = \Psi x$ denotes $\{v_t = \Psi_t x_{t-}\}_{t\in(\tau,T]}$, where $\Psi \in C(\tau, T; \mathbb{R}^{m\times n})$.

From (3.6), we derive that $J(v; 0, x^0) \geqslant (x^0)' P_0 x^0$. In particular, if $x^0 = 0$, then $J(v; 0, 0) \geqslant 0$, which is equivalent to $\|\mathcal{L}\| \leqslant \gamma$. To show $\|\mathcal{L}\| < \gamma$, we define an operator

$$
\begin{aligned}
\Gamma:\ & \mathcal{U}^m[0,T] \mapsto \mathcal{U}^m[0,T], \\
& \Gamma v_t = \widetilde{v}_t
\end{aligned}
$$

with its realization

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t v_t)\mathrm{d}t + (A_t^0 x_t + B_t^0 v_t)\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} (E_t(\theta)x_{t-} + F_t(\theta)v_t)\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_0 = 0 \end{cases}$$

and

$$\widetilde{v}_t = v_t + \Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)x_{t-}.$$

According to the estimate for SDE (2.2), we conclude that the operator Γ is well defined, moreover, it is a bounded linear operator. Then Γ^{-1} exists, which is determined by

$$\begin{cases} \mathrm{d}x_t = \{[A_t - B_t\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)]x_t + B_t\widetilde{v}_t\}\mathrm{d}t + \{[A_t^0 - B_t^0\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)]x_t + B_t^0\widetilde{v}_t\}\mathrm{d}W_t \\ \qquad + \displaystyle\int_{\mathscr{E}} \{[E_t(\theta) - F_t(\theta)\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)]x_{t-} + F_t(\theta)\widetilde{v}_t\}\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_0 = 0 \end{cases}$$

and

$$\widetilde{v}_t = -\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)x_{t-} + \widetilde{v}_t.$$

Based on the inverse operator theorem in functional analysis, $\|\Gamma^{-1}\|$ is bounded. There exists a positive constant $c = \frac{\epsilon}{\|\Gamma^{-1}\|^2}$ such that

$$\begin{aligned} J(v;0,0) &= E\int_0^T (v_t - \Psi_t x_t)'\Lambda_t^\gamma(P_t)(v_t - \Psi_t x_t)\mathrm{d}t \\ &= E\int_0^T (\Gamma v_t)'\Lambda_t^\gamma(P_t)(\Gamma v_t)\mathrm{d}t \geqslant \epsilon\|\Gamma v_t\|^2 \geqslant c\|v_t\|^2 > 0, \end{aligned}$$

which is equivalent to $\|\mathcal{L}\| < \gamma$. This proposition is proved.

The preceding proposition implies that for any given $\gamma > 0$, as long as the SRE (3.4) is solvable, the worst effect of the unknown disturbance v on the to-be-controlled output z can be controlled below γ.

In order to prove the second part of the SBRL, Theorem 3.1, we proceed by parts, first establishing some intermediate results.

Using the notation in (3.5), we set

$$M(P) \triangleq \begin{pmatrix} A'P + PA + (A^0)'PA^0 + \int_{\mathscr{E}} E'(\theta)PE(\theta)\nu(\mathrm{d}\theta) - C'C & \Phi(P) \\ \Phi'(P) & \Lambda^\gamma(P) \end{pmatrix}. \tag{3.8}$$

The next result provides an alternative form of writing up the functional defined in (3.3). This new way of expressing the cost allows us to solve the aforementioned minimization problem in a rather straightforward manner.

Lemma 3.1 *Suppose that $P : [\tau, T] \mapsto \mathbb{R}^n$ is continuously differentiable. Then for every $\xi \in \mathbb{R}^n$, $v \in \mathcal{U}^{\Updownarrow}[\tau, \mathcal{T}]$,*

$$\begin{aligned} J(v;\tau,\xi) &= \langle \xi, P_\tau \xi\rangle - E\langle x_T, P_T x_T\rangle \\ &\quad + \int_\tau^T E\Big(\langle x_t, \dot{P}_t x_t\rangle + \Big\langle \begin{bmatrix} x_t \\ v_t \end{bmatrix}, M_t(P_t)\begin{bmatrix} x_t \\ v_t \end{bmatrix}\Big\rangle\Big)\mathrm{d}t, \end{aligned}$$

where $M(P)$ is defined by (3.8) and x is the solution to the state equation of system (3.1).

Proof Applying Itô's formula (2.1) to $\langle x_t, P_t x_t\rangle$, and noting that

$$J(v;\tau,\xi) = J(v;\tau,\xi) + E\Big\{\int_\tau^T d(x_t' P_t x_t) - x_T' P_T x_T + x_\tau' P_\tau x_\tau\Big\},$$

we can easily obtain the result.

Lemma 3.2 *Suppose that* $\varphi \in C(\tau, T; \mathbb{R}^{m\times n})$ *and* $P^{\gamma,\varphi}$ *satisfy the linear differential matrix equation*

$$\dot X_t + \begin{pmatrix} I \\ \varphi_t \end{pmatrix}' \begin{pmatrix} A_t' X_t + X_t A_t + (A_t^0)' X_t A_t^0 + \int_{\mathscr{E}} E_t'(\theta) X_t E_t(\theta)\nu(\mathrm{d}\theta) - C_t' C_t & \Phi_t(X_t) \\ \Phi_t(X_t)' & \Lambda_t^\gamma(X_t) \end{pmatrix} \begin{pmatrix} I \\ \varphi_t \end{pmatrix} = 0 \tag{3.9}$$

with $P_T^{\gamma,\varphi} = 0$. *Then if* $v \in \mathcal{U}^m[\tau, T]$, *we have*

$$J(v + \varphi x^\varphi; \tau, \xi) = \langle \xi, P_\tau^{\gamma,\varphi}\xi\rangle + \int_\tau^T E[\langle v_t, N_t x_t^\varphi\rangle + \langle N_t x_t^\varphi, v_t\rangle + \langle v_t, \Lambda_t^\gamma(P_t^{\gamma,\varphi}) v_t\rangle]\mathrm{d}t, \tag{3.10}$$

where $x^\varphi = x(\cdot, v + \varphi x^\varphi; \tau, \xi)$ *is the solution of*

$$\begin{cases} \mathrm{d}x_t = [(A_t + B_t\varphi_t)x_t + B_t v_t]\mathrm{d}t + [(A_t^0 + B_t^0\varphi_t)x_t + B_t^0 v_t]\mathrm{d}W_t \\ \qquad\quad + \int_{\mathscr{E}} [(E_t(\theta) + F_t(\theta)\varphi_t)x_{t-} + F_t(\theta)v_t]\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_\tau = \xi \end{cases} \tag{3.11}$$

and $N_t = \Phi_t'(P_t^{\gamma,\varphi}) + \Lambda_t^\gamma(P_t^{\gamma,\varphi})\varphi_t$. *In particular, if* $v = 0$, *then*

$$J(\varphi x^\varphi; \tau, \xi) = \langle \xi, P_\tau^{\gamma,\varphi}\xi\rangle. \tag{3.12}$$

Proof As $P^{\gamma,\varphi}$ satisfies

$$\begin{cases} \dot X_t + \begin{pmatrix} I \\ \varphi_t \end{pmatrix}' M_t(X_t) \begin{pmatrix} I \\ \varphi_t \end{pmatrix} = 0, \\ X_T = 0, \end{cases}$$

applying Lemma 3.1 with $P = P^{\gamma,\varphi}$ and $v + \varphi x^\varphi$ for v, we obtain

$$\begin{aligned} J(v + \varphi x^\varphi; \tau, \xi) &= \langle \xi, P_\tau^{\gamma,\varphi}\xi\rangle + E\int_\tau^T \Big(\langle x_t^\varphi, \dot P_t^{\gamma,\varphi} x_t^\varphi\rangle \\ &\qquad + \Big\langle \begin{bmatrix} x_t^\varphi \\ v_t + \varphi_t x_t^\varphi \end{bmatrix}, M_t(P_t)\begin{bmatrix} x_t^\varphi \\ v_t + \varphi_t x_t^\varphi \end{bmatrix}\Big\rangle\Big)\mathrm{d}t \\ &= \langle \xi, P_\tau^{\gamma,\varphi}\xi\rangle + E\int_\tau^T [\langle v_t, N_t x_t^\varphi\rangle \\ &\qquad + \langle N_t x_t^\varphi, v_t\rangle + \langle v_t, \Lambda_t^\gamma(P_t^{\gamma,\varphi}) v_t\rangle]\mathrm{d}t. \end{aligned}$$

Hence, (3.10) holds. Setting $v = 0$ in (3.10), we obtain (3.12). The lemma is proved.

In the proof of last lemma, we should notice that as all the coefficients of (3.11) are continuous and uniformly bounded, from Lemma 2.2, there exists a unique solution $x^\varphi \in S^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m)$ to the equation. Hence, $\varphi x^\varphi \in \mathcal{L}^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m)$. Therefore, $v + \varphi x^\varphi \in \mathcal{U}^m[\tau, T]$ holds true.

The following lemma establishes a lower bound for the cost functional, which depends only on the norm of the initial state.

Lemma 3.3 *Suppose $\|\mathcal{L}\| < \gamma$. Then there exists $\mu > 0$, such that for any $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ and any $v \in \mathcal{U}^m[\tau, T]$, we have $J(v; \tau, \xi) \geqslant -\mu|\xi|^2$.*

Proof Denote by X the solution of (3.9) with $\varphi \equiv 0$ and final value $X_T = 0$, i.e., X solves

$$\dot{X}_t + A_t' X_t + X_t A_t + (A_t^0)' X_t A_t^0 + \int_{\mathscr{E}} E_t'(\theta) X_t E_t(\theta) \nu(\mathrm{d}\theta) - C_t' C_t = 0. \tag{3.13}$$

By linearity, the solution $x(t, v; \tau, \xi)$ of (3.1) satisfies

$$x(t, v; \tau, \xi) = x(t, v; \tau, 0) + x(t, 0; \tau, \xi).$$

Applying (3.10) with $\varphi = 0$, we get

$$J(v; \tau, \xi) - J(v; \tau, 0) = \xi' X_\tau \xi + E \int_\tau^T [\langle v_t, N_t x(t, 0; \tau, \xi)\rangle + \langle x(t, 0; \tau, \xi), N_t v_t\rangle] \, \mathrm{d}t,$$

where $N_t = \Phi_t'(X_t)$. Let $0 < \epsilon^2 \leqslant \gamma^2 - \|\mathcal{L}\|^2$, then

$$J(v; 0, 0) = \gamma^2 \|v\|^2 - \|\mathcal{L} v\|^2 \geqslant (\gamma^2 - \|\mathcal{L}\|^2)\|v\|^2 \geqslant \epsilon^2 \|v\|^2, \quad \forall v \in \mathcal{U}^m[0, T].$$

We can easily deduce that $J(v; \tau, 0) \geqslant \epsilon^2 \|v\|^2$ for all $v \in \mathcal{U}^m[\tau, T]$. Hence

$$\begin{aligned} J(v; \tau, \xi) &\geqslant \xi' X_\tau \xi + E \int_\tau^T [\epsilon^2 \langle v_t, v_t\rangle + \langle v_t, N_t x(t, 0; \tau, \xi)\rangle + \langle N_t x(t, 0; \tau, \xi), v_t\rangle] \mathrm{d}t \\ &= \xi' X_\tau \xi + E \int_\tau^T [|\epsilon v_t + \epsilon^{-1} N_t x(t, 0; \tau, \xi)|^2 - |\epsilon^{-1} N_t x(t, 0; \tau, \xi)|^2] \mathrm{d}t \\ &\geqslant \xi' X_\tau \xi - E \int_\tau^T |\epsilon^{-1} N_t x(t, 0; \tau, \xi)|^2 \mathrm{d}t. \end{aligned} \tag{3.14}$$

According to the estimate for SDE (2.2), there exists $c_0 > 0$ such that

$$E \int_\tau^T |x(t, 0; \tau, \xi)|^2 \mathrm{d}t \leqslant c_0 |\xi|^2.$$

Hence, by (3.12) there exist constants $c_1, c_2 > 0$ such that

$$0 \geqslant \langle \xi, X_\tau \xi\rangle = J(0; \tau, \xi) = -E \int_\tau^T \langle C_t' C_t x(t, 0; \tau, \xi), x(t, 0; \tau, \xi)\rangle \mathrm{d}t \geqslant -c_1 |\xi|^2$$

and

$$\|N_t\| = \left\| X_t B + (A_t^0)' X_t B_t^0 + \int_{\mathscr{E}} E_t'(\theta) X_t F_t(\theta) \nu(\mathrm{d}\theta) \right\| \leqslant c_2, \quad t \in [0, T].$$

Thus, by (3.14),

$$J(v; \tau, \xi) \geqslant -c_1 |\xi|^2 - c_2^2 \epsilon^{-2} c_0 |\xi|^2.$$

This lemma is proved.

Lemma 3.4 *Suppose that* $\|\mathcal{L}\| < \gamma$, $\varphi \in C(\tau, T; \mathbb{R}^{m\times n})$, *and* $P^{\gamma,\varphi}$ *satisfies* (3.9) *with* $P_T^{\gamma,\varphi} = 0$. *Then*

$$\Lambda_t^\gamma(P_t^{\gamma,\varphi}) \geqslant (\gamma^2 - \|\mathcal{L}\|^2)I, \quad t \in [\tau, T]. \tag{3.15}$$

Proof We will first prove that $\Lambda_t^\gamma(P_t^{\gamma,\varphi}) \geqslant 0$. Suppose that this is false and there exists $\widehat{t} \in [\tau, T]$, $\widetilde{v} \in \mathbb{R}^m$, $\|\widetilde{v}\| = 1$ such that $\langle \widetilde{v}, \Lambda_t^\gamma(P_t^{\gamma,\varphi})\widetilde{v}\rangle \leqslant -\eta$ for some $\eta > 0$. Assume $\widehat{t} < T$. Then, for $\delta > 0$ sufficiently small,

$$\langle \widetilde{v}, \Lambda_t^\gamma(P_t^{\gamma,\varphi})\widetilde{v}\rangle \leqslant -\frac{1}{2}\eta, \quad t \in [\widetilde{t}, \widehat{t} + \delta] \subset [\tau, , T].$$

Define

$$v_t = \begin{cases} 0, & t \in [\tau, \widehat{t}) \cup (\widehat{t} + \delta, T], \\ \widetilde{v}, & t \in [\widehat{t}, \widehat{t} + \delta]. \end{cases}$$

Let $\tau = 0$, $\xi = 0$. Then by a prior estimate (2.2), we have

$$E \sup_{0\leqslant t\leqslant \widehat{t}} |x_t^\varphi|^2 \leqslant K\Big[E\int_0^{\widehat{t}} |B_t v_t|^2 \mathrm{d}t + E\int_0^{\widehat{t}} |B_t^0 v_t|^2 \mathrm{d}t + E\iint_{\mathscr{E}\times(0,\widehat{t}]} |F_t(\theta)v_t|^2 \nu(\mathrm{d}\theta)\mathrm{d}t\Big] = 0,$$

i.e., $E \sup_{0\leqslant t\leqslant \widehat{t}} |x_t^\varphi|^2 = 0$. Particularly, $E|x_{\widehat{t}}^\varphi|^2 = 0$, then $Ex_{\widehat{t}}^\varphi = 0$. Now applying Lemma 3.2 to the aforementioned v, we have

$$\begin{aligned} J(v + \varphi x^\varphi; 0, 0) &= \int_0^T E[\langle v_t, N_t x_t^\varphi\rangle + \langle N_t x_t^\varphi, v_t\rangle + \langle v_t, \Lambda_t^\gamma(P_t^{\gamma,\varphi})v_t\rangle]\mathrm{d}t \\ &\leqslant \int_{\widetilde{t}}^{\widetilde{t}+\delta} \Big(2|N_t'\widetilde{v}||Ex_t^\varphi| - \frac{1}{2}\eta\Big)\mathrm{d}t. \end{aligned}$$

Choosing $\delta > 0$ sufficiently small, the integrand becomes negative, since $Ex_{\widetilde{t}}^\varphi$ is right continuous and $Ex_{\widetilde{t}}^\varphi = 0$. While by Lemma 3.3, we have

$$J(v + \varphi x^\varphi; 0, 0) \geqslant 0.$$

This yields a contradiction whence $\Lambda_t^\gamma(P_t^{\gamma,\varphi}) \geqslant 0$. If $\widehat{t} = T$, a similar proof applies, replacing the interval $[\widehat{t}, \widehat{t} + \delta]$ by $[T - \delta, T]$.

Now let ϵ be any positive number such that $\|\mathcal{L}\|^2 < \gamma^2 - \epsilon^2$. Applying the previous step with $\widetilde{\gamma} = (\gamma^2 - \epsilon^2)^{\frac{1}{2}}$ instead of γ we obtain, for the corresponding solution $P_t^{\widetilde{\gamma},\varphi}$ of (3.9) (with $\widetilde{\gamma}$ instead of γ), $\Lambda_t^{\widetilde{\gamma}}(P_t^{\widetilde{\gamma},\varphi}) \geqslant 0$. By (3.12), we obtain, for any $\tau \in [0, T)$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned} \langle \xi, P_\tau^{\gamma,\varphi}\xi\rangle &= J(\varphi x^\varphi; \tau, \xi) \\ &= E\int_\tau^T [\langle (\gamma^2 I - D_t'D_t)\varphi_t x_t^\varphi, \varphi_t x_t^\varphi\rangle - \langle C_t'C_t x_t^\varphi, x_t^\varphi\rangle]\mathrm{d}t \\ &\geqslant E\int_\tau^T [\langle (\widetilde{\gamma}^2 I - D_t'D_t)\varphi_t x_t^\varphi, \varphi_t x_t^\varphi\rangle - \langle C_t'C_t x_t^\varphi, x_t^\varphi\rangle]\mathrm{d}t \\ &= \langle \xi, P_\tau^{\widetilde{\gamma},\varphi}\xi\rangle. \end{aligned}$$

It follows that if $\gamma > \widetilde{\gamma}$, then $P_\tau^{\gamma,\varphi} \geqslant P_\tau^{\widetilde{\gamma},\varphi}$. Therefore

$$\begin{aligned}\Lambda_t^\gamma(P_t^{\gamma,\varphi}) &= \gamma^2 I - D_t'D_t + (B_t^0)'P_t^{\gamma,\varphi}B_t^0 + \int_{\mathscr{E}} F_t'(\theta)P_t^{\gamma,\varphi}F_t(\theta)\nu(\mathrm{d}\theta)\\ &> \widetilde{\gamma}^2 I - D_t'D_t + (B_t^0)'P_t^{\widetilde{\gamma},\varphi}B_t^0 + \int_{\mathscr{E}} F_t'(\theta)P_t^{\widetilde{\gamma},\varphi}F_t(\theta)\nu(\mathrm{d}\theta)\\ &= \Lambda_t^{\widetilde{\gamma}}(P_t^{\widetilde{\gamma},\varphi}) \geqslant 0,\end{aligned}$$

i.e., $\Lambda_t^\gamma(P_t^{\gamma,\varphi}) \geqslant \epsilon^2 I$ for all $t \in [0,T]$. Since this holds for arbitrary $\epsilon^2 < \gamma^2 - \|\mathcal{L}\|^2$, (3.15) follows. The proof is complete.

We will now study the matrix differential equation (3.4). The function

$$f(P) = A_t'P + PA_t + (A_t^0)'PA_t^0 + \int_{\mathscr{E}} E_t'(\theta)PE_t(\theta)\nu(\mathrm{d}\theta) - C_t'C_t - \Phi_t(P)'\Lambda_t^\gamma(P)^{-1}\Phi_t(P)$$

is continuously differentiable on its domain of definition $D_f = \{P : \det(\Lambda^\gamma(P)) \neq 0\}$. From the existence and uniqueness theorem of local solution to ordinary differential equation (ODE for short), there exists a unique solution to (3.4) on $[T-\delta, T]$ for sufficiently small $\delta > 0$.

We are now in a position to prove the second part of Theorem 3.1, that is, if $\|\mathcal{L}\| < \gamma$, SRE (3.4) has a global solution on $[0,T]$.

Proof of Theorem 3.1 It only remains to prove the converse of Proposition 3.2. Assume $\|\mathcal{L}\| < \gamma$, then $\gamma^2 I - D_t'D_t \geqslant \epsilon^2 I$ holds for every $t \in [0,T]$. And as $P_T = 0$, it follows that for sufficiently small $\delta > 0$,

$$\Lambda_t^\gamma(P_t) = \gamma^2 I - D_t'D_t + (B_t^0)'P_tB_t^0 + \int_{\mathscr{E}} F_t'(\theta)P_tF_t(\theta)\nu(\mathrm{d}\theta) \geqslant \epsilon_\delta^2, \quad t \in [T-\delta, T],$$

where ϵ_δ is a positive constant depending on δ. Then P is continuously differentiable on $[T-\delta, T]$ and the Riccati equation (3.4) has a unique solution P on $[T-\delta, T]$. Setting φ replaced by $\Psi = -\Lambda^\gamma(P)^{-1}\Phi'(P) \in C(T-\delta, T; \mathbb{R}^{n\times m})$ on the left hand side of (3.9), we obtain

$$\begin{aligned}&\dot{P}_t + \begin{pmatrix} I \\ \Psi_t \end{pmatrix}' M_t(P_t) \begin{pmatrix} I \\ \Psi_t \end{pmatrix}\\ &= \dot{P}_t + A_t'P_t + P_tA_t + (A_t^0)'P_tA_t^0 + \int_{\mathscr{E}} E_t'(\theta)P_tE_t(\theta)\nu(\mathrm{d}\theta) - C_t'C_t - \Phi_t(P_t)'\Lambda_t^\gamma(P_t)^{-1}\Phi_t(P_t)\\ &= 0, \quad t \in [T-\delta, T].\end{aligned}$$

The last equality holds because of the local solvability of (3.4). Hence P satisfies (3.9) on $[T-\delta, T]$ with $\Psi = \varphi$, i.e., $P_t^{\gamma,\varphi} = P_t$, $t \in [T-\delta, T]$. Moreover, with this choice of Ψ_t,

$$N_t = \Phi_t'(P_t) + \Lambda_t^\gamma(P_t)\Psi_t = 0, \quad t \in [T-\delta, T],$$

and so Lemma 3.2 implies that,

$$J(v + \Psi x; \tau, \xi) = \langle \xi, P_\tau \xi \rangle + \int_\tau^T E[\langle v_t, \Lambda_t^\gamma(P_t)v_t \rangle]\mathrm{d}t, \quad \tau \in [T-\delta, T].$$

But by Lemma 3.4,

$$\Lambda_t^\gamma(P_t) = \Lambda_t^\gamma(P_t^{\gamma,\Psi}) \geqslant (\gamma^2 - \|\mathcal{L}\|^2)I > 0, \quad t \in [T-\delta, T].$$

Hence, for all $t \in [T-\delta, T]$, the optimal feedback control is

$$v_t = \Psi_t x^{\Psi}_{t-}, \quad \Psi_t = -\Lambda^{\gamma}_t(P_t)^{-1}\Phi'_t(P_t)$$

with x^{Ψ} satisfying

$$\begin{cases} \mathrm{d}x_t = (A_t + B_t\Psi_t)x_t\mathrm{d}t + (A^0_t + B^0_t\Psi_t)x_t\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}}(E_t(\theta) + F_t(\theta)\Psi_t)x_{t-}\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_\tau = \xi, \end{cases}$$

and the optimal cost is

$$\min_{v \in \mathcal{U}^m[\tau,T]} J(v;\tau,\xi) = \langle \xi, P_\tau \xi\rangle. \tag{3.16}$$

As a consequence, we obtain

$$\langle \xi, P_\tau\xi\rangle = J(\Psi x;\tau,\xi) \leqslant J(0;\tau,\xi) = E\int_\tau^T \langle -C'_tC_tx_t, x_t\rangle \mathrm{d}t \leqslant 0, \quad \tau \in [T-\delta, T].$$

On the other hand, by Lemma 3.3,

$$\langle \xi, P_\tau\xi\rangle = J(\Psi x;\tau,\xi) \geqslant -\mu|\xi|^2, \quad \tau \in [T-\delta, T].$$

Hence,

$$-\mu I \leqslant P_\tau \leqslant 0, \quad \tau \in [T-\delta, T].$$

Now, suppose that there exists a solution of (3.4) backwards in time on a maximal interval $(\sigma, T] \subset [0, T]$, and as $t \downarrow \sigma$, P_t becomes unbounded, i.e., (3.4) exhibits the phenomenon of a finite escape time. We shall show that the existence of a finite escape time will lead to a contraction. In fact, by the discussion above, the following estimates hold in the interval $[\sigma + \sigma_\epsilon, T]$ with $\sigma_\epsilon > 0$ sufficiently small:

$$-\mu I \leqslant P_\tau \leqslant 0, \quad \Lambda^{\gamma}_t(P_t) \geqslant (\gamma^2 - \|\mathcal{L}\|^2)I. \tag{3.17}$$

As the constant μ is independent of the left interval endpoint $\sigma + \sigma_\epsilon$, letting $\sigma_\epsilon \downarrow 0$, we obtain $-\mu I \leqslant P_\tau \leqslant 0$ on $(\sigma, T]$. Hence, the solution P_t of (3.4) cannot escape to ∞ as $t \downarrow \sigma$. It follows that there exists a boundary point P^0, $\det(\Lambda^\gamma(P^0)) = 0$ of the domain D_f which is a limit point of P_t as $t \downarrow \sigma$. But this contradicts the fact that by (3.17), $\Lambda^{\gamma}_t(P_t) \geqslant (\gamma^2 - \|\mathcal{L}\|^2)I$ as $t \downarrow \sigma$. Thus, the maximal solution interval is $[0, T]$. The uniqueness of the solution follows from (3.16).

The proof of Theorem 3.1 is completed.

So far, we have shown the stochastic bounded real lemma for Poisson jump-diffusion system, that is, $\|\mathcal{L}\| < \gamma$ is equivalent to that the SRE (3.4) has a unique negative semi-definite solution. Theoretically, by virtue of this theorem, the infimum of all these given disturbance attenuation levels $\gamma > 0$ such that the corresponding SRE (3.4) has a unique solution, can be used as an estimate of $\|\mathcal{L}\|$.

4 Stochastic H_2/H_∞ Control for Jump-Diffusion Systems

In this section, we shall give necessary and sufficient conditions for the solvability of the stochastic H_2/H_∞ control problem in terms of four cross-coupled Riccati equations.

Consider the stochastic linear system (2.3), the finite horizon stochastic H_2/H_∞ control problem can be stated as follows.

Definition 4.1 *Given a disturbance attenuation level $\gamma > 0$, to find, if possible, a state feedback control $u^* \in \mathcal{U}^s[0,T]$, such that with the constraint* (2.3)*, we have that*

(1)

$$\begin{aligned}\|\mathcal{L}_{u^*}\| &:= \sup_{\substack{v\in\mathcal{U}^m[0,T]\\ v\neq 0,\ x_0=0}} \frac{\|z\|}{\|v\|}\\ &= \sup_{\substack{v\in\mathcal{U}^m[0,T]\\ v\neq 0,\ x_0=0}} \frac{\left\{E\int_0^T (x_t'C_t'C_tx_t + u_t^{*\prime}u_t^* + v_t'(D_t^1)'D_t^1v_t)\mathrm{d}t\right\}^{\frac{1}{2}}}{\{E\int_0^T v_t'v_t\mathrm{d}t\}^{\frac{1}{2}}} < \gamma;\end{aligned}$$

(2) *when the worst case disturbance $v^* \in \mathcal{U}^m[0,T]$, if it exists, is applied to the system* (2.3)*, u^* minimizes the output energy*

$$J_2(u,v^*) = E\int_0^T (x_t'C_t'C_tx_t + u_t'u_t + (v_t^*)'(D_t^1)'D_t^1v_t^*)\mathrm{d}t.$$

Here, the so-called worst case disturbance v^* means that for any $v \in \mathcal{U}^m[0,T]$ and any $x^0 \in \mathbb{R}^n$,

$$v^* = \arg\min_v J_1(u^*,v) = \arg\min_v E\int_0^T (\gamma^2 v_t'v_t - z_t'z_t)\mathrm{d}t.$$

If the previous (u^*,v^*) exists, then the finite horizon H_2/H_∞ control has a pair of solutions (u^*,v^*).

The following lemma, which is necessary in the derivation that follows, is given without proof and its proof can be analogous to that of Theorem 7.2 of [25].

Lemma 4.1 *Riccati equation*

$$\begin{cases}\dot{X}_t + A_t'X_t + X_tA_t + (A_t^0)'X_tA_t^0 + C_t'C_t + \int_{\mathscr{E}} E_t'(\theta)X_tE_t(\theta)\nu(\mathrm{d}\theta)\\ \quad -\Delta_t(X_t)'\Theta_t(X_t)^{-1}\Delta_t(X_t) = 0,\\ \Theta_t(X_t) > 0,\\ X_T = 0\end{cases} \tag{4.1}$$

with

$$\begin{aligned}\Delta(X) &\triangleq XB + (A^0)'XB^{02} + \int_{\mathscr{E}} E'(\theta)XF^2(\theta)\nu(\mathrm{d}\theta),\\ \Theta(X) &\triangleq I + (B^{02})'XB^{02} + \int_{\mathscr{E}} (F^2(\theta))'XF^2(\theta)\nu(\mathrm{d}\theta)\end{aligned}$$

admits a unique solution X. Moreover, X is semi-positive definite and uniformly bounded.

Remark 4.1 In view of the relationship between the Riccati equation (4.1) and the linear quadratic (LQ for short) optimal control problem, where the dynamic system is driven by:

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t^2 u_t)\mathrm{d}t + (A_t^0 x_t + B_t^{02} u_t)\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} [E_t(\theta)x_{t-} + F_t^2(\theta)u_t]\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t) \\ x_\tau = \xi, \end{cases}$$

and the corresponding cost functional is

$$J_2(u;\tau,\xi) = E\int_\tau^T (|Cx|^2 + |u|^2)\mathrm{d}t = E\int_\tau^T (x_t' C_t' C_t x_t + u_t' u_t)\mathrm{d}t,$$

we immediately obtain that the optimal feedback control is

$$u_t = \Sigma_t x_{t-}, \quad \Sigma_t = -\Theta_t(X_t)^{-1}\Delta_t(X_t), \quad t \in [\tau, T]$$

with x_t satisfying

$$\begin{cases} \mathrm{d}x_t = (A_t + B_t^2\Sigma_t)x_t\mathrm{d}t + (A_t^0 + B_t^{02}\Sigma_t)x_t\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} [E_t(\theta) + F_t^2(\theta)\Sigma_t]x_{t-}\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_\tau = \xi, \end{cases}$$

and the optimal cost is $\min\limits_{u\in\mathcal{U}^s[\tau,T]} J_2(u;\tau,\xi) = \xi' X_\tau \xi$, where X is the solution to (4.1).

In the following, we shall give sufficient and necessary conditions for the existence of the linear state feedback pair (u^*, v^*), which generalize the result of Chen and Zhang [2] to the case of stochastic systems with Poisson random jumps and (x,u,v)-dependent noise.

For convenience, we introduce the following notation:

$$\begin{aligned} &\Upsilon_1(P_1) =: \gamma^2 I - (D^1)'D^1 + (B^{01})'P_1 B^{01} + \int_{\mathscr{E}} (F^1(\theta))'P_1 F^1(\theta)\nu(\mathrm{d}\theta), \\ &\Upsilon_2(P_2) =: I + (B^{02})'P_2 B^{02} + \int_{\mathscr{E}} (F^2(\theta))'P_2 F^2(\theta)\nu(\mathrm{d}\theta), \\ &\Pi_1(P_1, K_2) =: P_1 B^1 + (A^0 + B^{02}K_2)'P_1 B^{01} + \int_{\mathscr{E}} (E(\theta) + F^2(\theta)K_2)'P_1 F^1(\theta)\nu(\mathrm{d}\theta), \\ &\Pi_2(P_2, K_1) =: P_2 B^2 + (A^0 + B^{01}K_1)'P_2 B^{02} + \int_{\mathscr{E}} (E(\theta) + F^1(\theta)K_1)'P_2 F^2(\theta)\nu(\mathrm{d}\theta). \end{aligned}$$

Theorem 4.1 *For stochastic system* (2.3)*, if the following four coupled matrix Riccati equations*

$$\begin{cases} \dot{P}_1 + (A + B^2K_2)'P_1 + P_1(A + B^2K_2) + (A^0 + B^{02}K_2)'P_1(A^0 + B^{02}K_2) \\ \quad -C'C - K_2'K_2 + \displaystyle\int_{\mathscr{E}} (E(\theta) + F^2(\theta)K_2)'P_1(E(\theta) + F^2(\theta)K_2)\nu(\mathrm{d}\theta) \\ \quad -\Pi_1(P_1, K_2)'\Upsilon_1(P_1)^{-1}\Pi_1(P_1, K_2) = 0, \\ \Upsilon_1(P_1) > 0, \\ P_{1,T} = 0, \end{cases} \tag{4.2}$$

$$K_1 = -\Upsilon_1(P_1)^{-1}\Pi_1(P_1, K_2), \tag{4.3}$$

$$\begin{cases}\dot{P}_2+(A+B^1K_1)'P_2+P_2(A+B^1K_1)+(A^0+B^{01}K_1)'P_2(A^0+B^{01}K_1)\\ \quad +C'C+K_1'(D^1)'D^1K_1+\int_{\mathscr{E}}(E(\theta)+F^1(\theta)K_1)'P_2(E(\theta)+F^1(\theta)K_1)\nu(\mathrm{d}\theta)\\ \quad -\Pi_2(P_2,K_1)\Upsilon_2(P_2)^{-1}\Pi_2(P_2,K_1)=0,\\ \Upsilon_2(P_2)>0,\\ P_{2,T}=0,\end{cases} \tag{4.4}$$

$$K_2=-\Upsilon_2(P_2)^{-1}\Pi_2(P_2,K_1) \tag{4.5}$$

have solutions (P_1,P_2,K_1,K_2) with $P_1\leqslant 0$ and $P_2\geqslant 0$, then the H_2/H_∞ control problem admits a pair of solutions (u^*,v^*) satisfying

$$u_t^*=K_{2,t}x_{t-},\quad v_t^*=K_{1,t}x_{t-},\quad t\in(0,T]. \tag{4.6}$$

Moreover, setting $x^0=0$ and $u=u^*$, $\|\mathcal{L}_{u^*}\|<\gamma$ for any $v\in\mathcal{U}^m[0,T]$.

Proof Suppose that the coupled matrix Riccati equations (4.2)–(4.5) are solvable. Let us consider the cost functional $J_1(u,v)$ first. Applying Itô's formula (2.1) and completion of squares, we have

$$\begin{aligned}J_1(u,v)&=E\int_0^T[(\gamma^2v_t'v_t-z_t'z_t)\mathrm{d}t+\mathrm{d}x_t'P_{1,t}x_t]+(x^0)'P_{1,0}x^0-x_T'P_{1,T}x_T\\ &=(x^0)'P_{1,0}x^0+E\int_0^T\Big\{\langle\Upsilon_1(P_1)(v-v^*),v-v^*\rangle-u'u+x'P_1B^2(u-K_2x)\\ &\quad+(u-K_2x)'B^2P_1x+(u^*)'u^*+(A^0x+B^{02}u)'P_1(A^0x+B^{02}u)\\ &\quad+\int_{\mathscr{E}}(E(\theta)x+F^2(\theta)u)'P_1(E(\theta)x+F^2(\theta)u)\nu(\mathrm{d}\theta)\\ &\quad-x'(A^0+B^{02}K_2)'P_1(A^0+B^{02}K_2)x\\ &\quad-x'\int_{\mathscr{E}}(E(\theta)+F^2(\theta)K_2)'P_1(E(\theta)+F^2(\theta)K_2)\nu(\mathrm{d}\theta)x\\ &\quad+v'\Big[(B^{01})'P_1B^{02}+\int_{\mathscr{E}}F^1(\theta)P_1F^2(\theta)\nu(\mathrm{d}\theta)\Big](u-K_2x)\\ &\quad+(u-K_2x)'\Big[(B^{01})'P_1B^{02}+\int_{\mathscr{E}}F^1(\theta)P_1F^2(\theta)\nu(\mathrm{d}\theta)\Big]v\Big\}\mathrm{d}t,\end{aligned}$$

where u^* and v^* are determined by (4.6). Setting $u=u^*=K_2x$, we obtain

$$J_1(u^*,v)=(x^0)'P_{1,0}x^0+E\int_0^T\langle\Upsilon_1(P_1)(v-v^*),v-v^*\rangle\mathrm{d}t.$$

Therefore, $J_1(u^*,v^*)\leqslant J_1(u^*,v)$ and $J_1(u^*,v^*)=(x^0)'P_{1,0}x^0$, which implies that v^* is the worst case disturbance corresponding to u^*. A similar method as Proposition 3.1 yields $\|\mathcal{L}_{u^*}\|<\gamma$. Similarly,

$$\begin{aligned}J_2(u,v)&=(x^0)'P_{2,0}x^0+E\int_0^T\Big\{\langle\Upsilon_2(P_2)(u-u^*),u-u^*\rangle-x'K_1'(D^1)'D^1K_1x+v'(D^1)'D^1v\\ &\quad+x'P_2B^1(v-K_1x)+(v-K_1x)'(B^1)'P_2x+(A^0x+B^{01}v)'P_2(A^0x+B^{01}v)\\ &\quad+\int_{\mathscr{E}}(E(\theta)x+F^1(\theta)v)'P_2(E(\theta)x+F^1(\theta)v)\nu(\mathrm{d}\theta)-x'(A^0+B^{01}K_1)P_2(A^0\end{aligned}$$

$$+ B^{01}K_1)x - x'\int_{\mathscr{E}}(E(\theta)+F^1(\theta)K_1)'P_2(E(\theta)+F^1(\theta)K_1)\nu(\mathrm{d}\theta)x + u'\Big[(B^{01})'P_2B^{02} + \int_{\mathscr{E}}F^1(\theta)P_2F^2(\theta)\nu(\mathrm{d}\theta)\Big](v-K_1x) + (v-K_1x)'\Big[(B^{01})'P_2B^{02} + \int_{\mathscr{E}}F^1(\theta)P_2F^2(\theta)\nu(\mathrm{d}\theta)\Big]u\Big\}\mathrm{d}t, \tag{4.7}$$

and setting $v = v^*$ results in $J_2(u^*, v^*) \leqslant J_2(u, v^*)$ and $J_2(u^*, v^*) = (x^0)'P_{2,0}x^0$. The above information implies that the finite horizon H_2/H_∞ control has a pair of solutions (u^*, v^*) with u^* and v^* defined in (4.6).

We establish the signs of $P_{1,t}$ and $P_{2,t}$ as follows.

(i) A completion of squares argument similar to that which led to (4.7), together with setting $u = u^*$ and $v = v^*$, we finally obtain

$$x_t'P_{2,t}x_t = E\int_t^T [x_s'C_s'C_sx_s + (u_s^*)'u_s^* + (v_s^*)'(D_s^1)'D_s^1v_s^*]\mathrm{d}s \geqslant 0$$

for arbitrary x_t. Consequently, $P_{2,t} \geqslant 0$ for all $t \in [0, T]$.

(ii) A similar calculation using $J_1(\cdot,\cdot)$ with $u = u^*$ and $v = 0$ gives

$$x_t'P_{1,t}x_t = -E\int_t^T [x_s'C_s'C_sx_s + (u_s^*)'u_s^* + \langle \Upsilon_1(P_1)v_s^*, v_s^*\rangle]\mathrm{d}s \leqslant 0.$$

Thus $P_{1,t} \leqslant 0$ for all $t \in [0, T]$. The proof is complete.

Theorem 4.2 *Assume that the finite horizon H_2/H_∞ control problem admits a pair of linear state feedback solutions (u_t^*, v_t^*) with $v_t^* = G_t^1x_{t-}$ and $u_t^* = G_t^2x_{t-}$, where G_t^1 and G_t^2 are continuous matrix-valued functions on $[0, T]$. Then the coupled matrix Riccati equations* (4.2)–(4.5) *have a unique quaternion solution (P_1, P_2, G^1, G^2) with $P_1 \leqslant 0$ and $P_2 \geqslant 0$.*

Proof If (u^*, v^*) is the solution of the considered H_2/H_∞ control problem, then we will prove that the matrix-valued equations (4.2)–(4.5) are solvable.

(I) Implementing $u_t^* = G_t^2x_{t-}$ in (2.3) with G^2 to be determined, we obtain

$$\begin{cases} \mathrm{d}x_t = [(A_t + B_t^2G_t^2)x_t + B_t^1v_t]\mathrm{d}t + [(A_t^0 + B_t^{02}G_t^2)x_t + B_t^{01}v_t]\mathrm{d}W_t \\ \qquad + \int_{\mathscr{E}}[(E_t(\theta) + F_t^2(\theta)G_t^2)x_{t-} + F_t^1(\theta)v_t]\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ z_t = \begin{pmatrix} \begin{pmatrix} C_t \\ D_t^2G_t^2 \end{pmatrix} x_t \\ D_t^1v_t \end{pmatrix}, \quad (D_t^2)'D_t^2 = I \end{cases}$$

with $x_\tau = \xi$. Since the finite horizon H_2/H_∞ control is solvable, by definition 4.1, $\|\mathcal{L}_{u^*}\| < \gamma$. Hence, according to Theorem 3.1, Riccati equation

$$\begin{cases} \dot{P}_1 + (A + B^2G^2)'P_1 + P_1(A + B^2G^2) + (A^0 + B^{02}G^2)'P_1(A^0 + B^{02}G^2) \\ \quad -C'C - (G^2)'G^2 + \int_{\mathscr{E}}(E + F^2G^2)'P_1(E + F^2G^2)\nu(\mathrm{d}\theta) \\ \quad -\Pi_1(P_1, G^2)'\Upsilon_1(P_1)^{-1}\Pi_1(P_1, G^2) = 0, \\ \Upsilon_1(P_1) > 0, \\ P_{1,T} = 0 \end{cases} \tag{4.8}$$

has a unique solution $P_1 \leqslant 0$. Moreover, the optimal control problem

$$\min_{v \in \mathcal{U}^m[0,T]} J_1(u^*, v)$$

has a unique solution

$$v_t^* = -\Upsilon_1(P_1)^{-1}\Pi_1(P_1, G^2)x_{t-}, \quad t \in (0,T].$$

Hence

$$G^1 = -\Upsilon_1(P_1)^{-1}\Pi_1(P_1, G^2). \tag{4.9}$$

(II) Substituting $v_t = v_t^* = G_t^1 x_{t-}$ into (2.3) with G^1 being defined as (4.9), we obtain

$$\begin{cases} \mathrm{d}x_t = [(A_t + B_t^1 G_t^1)x_t + B_t^2 u_t]\mathrm{d}t + [(A_t^0 + B_t^{01} G_t^1)x_t + B_t^{02} u_t]\mathrm{d}W_t \\ \qquad + \int_{\mathscr{E}} [(E_t(\theta) + F_t^1(\theta)G_t^1)x_{t-} + F_t^2(\theta)u_t]\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ z_t = \begin{pmatrix} \begin{pmatrix} C_t \\ D_t^1 G_t^1 \end{pmatrix} x_t \\ D_t^2 u_t \end{pmatrix}, \quad (D_t^2)' D_t^2 = I \end{cases} \tag{4.10}$$

with $x_\tau = \xi$. Since

$$\min_{u \in \mathcal{U}^s[0,T]} J_2(u, v^*)$$

is a standard stochastic linear quadratic optimal control problem, according to Remark 4.1, there exists a unique optimal control

$$u_t^* = -\,\Upsilon_2(P_2)^{-1}\Pi_2(P_2, G^1)x_{t-}, \quad t \in (0,T],$$

where $P_2 \geqslant 0$ solves

$$\begin{cases} \dot{P}_2 + (A + B^1G^1)'P_2 + P_2(A + B^1G^1) + (A^0 + B^{01}G^1)'P_2(A^0 + B^{01}G^1) \\ \quad +(G^1)'(D^1)'D^1G^1 + C'C + \int_{\mathscr{E}} (E + F^1G^1)'P_2(E + F^1G^1)\nu(\mathrm{d}\theta) \\ \quad -\Pi_2(P_2, G^1)'\Upsilon_2(P_2)^{-1}\Pi_2(P_2, G^1) = 0, \\ \Upsilon_2(P_2) > 0, \\ P_{2,T} = 0. \end{cases} \tag{4.11}$$

Hence

$$G^2 = -\Upsilon_2(P_2)^{-1}\Pi_2(P_2, G^1). \tag{4.12}$$

Therefore, the coupled matrix-valued equations (4.2)–(4.5) have a unique quaternion solution (P_1, P_2, G^1, G^2) with $P_1 \leqslant 0$ and $P_2 \geqslant 0$. The proof is complete.

The last two theorems imply that for the stochastic linear system (2.3), the existence of a state feedback stochastic H_2/H_∞ control is equivalent to the solvability of the four coupled Riccati equations (4.2)–(4.5). As it is generally difficult to solve the aforementioned four coupled equations, we will present a discretization technique. Set $h = \frac{T}{n}$ for a natural number $n > 0$, and denote $t_i = ih$ with $i = 0, 1, 2, \cdots, n$. When n is sufficiently large, or equivalently, when h is sufficiently small, we may replace $\dot{P}_{1,t_{i+1}}$ and $\dot{P}_{2,t_{i+1}}$ with $\frac{P_{1,t_i} - P_{1,t_{i+1}}}{-h}$ and $\frac{P_{2,t_i} - P_{2,t_{i+1}}}{-h}$ in (4.2) and (4.4), respectively. Then a backward recursive algorithm can be given as follows:

(i) By solving (4.3) and (4.5), it follows $K_{1,T} = 0, K_{2,T} = 0$ from given terminal condition $P_{1,T} = 0, P_{2,T} = 0$.

(ii) Solving (4.2) and (4.4) yields $P_{1,t_{n-1}} = P_{1,T-h} = -hC'_T C_T \leqslant 0$ and $P_{2,t_{n-1}} = P_{2,T-h} = hC'_T C_T \geqslant 0$.

(iii) Repeating above steps (i)–(ii), P_{1,t_i}, P_{2,t_i} may be computed if $P_{1,t_{i+1}} \leqslant 0$ and $P_{2,t_{i+1}} \geqslant 0$ are available with

$$\gamma^2 I - (D^1_{t_{i+1}})' D^1_{t_{i+1}} + (B^{01}_{t_{i+1}})' P_{1,t_{i+1}} B^{01}_{t_{i+1}} + \int_{\mathscr{E}} (F^1_{t_{i+1}}(\theta))' P_{1,t_{i+1}} F^1_{t_{i+1}}(\theta) \nu(\mathrm{d}\theta) > 0,$$
$$I + (B^{02}_{t_{i+1}})' P_{2,t_{i+1}} B^{02}_{t_{i+1}} + \int_{\mathscr{E}} (F^2_{t_{i+1}}(\theta))' P_{2,t_{i+1}} F^2_{t_{i+1}}(\theta) \nu(\mathrm{d}\theta) > 0, \quad i = n, n-1, \cdots, 0.$$

The above recursions may proceed for even if $P_{1,t_i} \leqslant 0, P_{2,t_i} \geqslant 0$ and

$$\gamma^2 I - (D^1_{t_i})' D^1_{t_i} + (B^{01}_{t_i})' P_{1,t_i} B^{01}_{t_i} + \int_{\mathscr{E}} (F^1_{t_i}(\theta))' P_{1,t_i} F^1_{t_i}(\theta) \nu(\mathrm{d}\theta) > 0,$$
$$I + (B^{02}_{t_i})' P_{2,t_i} B^{02}_{t_i} + \int_{\mathscr{E}} (F^2_{t_i}(\theta))' P_{2,t_i} F^2_{t_i}(\theta) \nu(\mathrm{d}\theta) > 0, \quad i = 1, 2, \cdots, n.$$

Because if the coupled Riccati equations (4.2)–(4.5) admit a quaternion solution $(\overline{P}_1 \leqslant 0, \overline{P}_2 \geqslant 0, \overline{K}_1, \overline{K}_2)$, then $\overline{P}_{1,t}$ and $\overline{P}_{2,t}$ must be uniformly continuous on $[0, T]$. Therefore, we have

$$\lim_{h \to 0} \max_{t_i \leqslant t \leqslant t_{i+1}, j=1,2} \{ |\overline{P}_{j,t} - P_{j,t_{i+1}}|, |\overline{P}_{j,t} - P_{j,t_i}| \} = 0.$$

In particular, for some special systems, we may solve (4.2)–(4.5) analytically, see the following example.

Example 4.1 Consider the one-dimensional linear stochastic system with jumps as follows:

$$\begin{cases} \mathrm{d}x_t = (-4x_t + v_t - 4u_t)\mathrm{d}t + (2x_t + 5u_t)\mathrm{d}W_t + \int_{\mathscr{E}} [2x_{t-} - 3u_t] \widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), & x_0 = x^0, \\ z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}, & t \in [0, 1]. \end{cases}$$

If we take $\gamma = 1$ and $\nu(\mathscr{E}) = 1$, then the coupled Riccati equations (4.2)–(4.5) specialize to

$$\begin{cases} \dot{P}_1 - P_1^2 - 1 = 0, \quad P_1 \leqslant 0, \ P_{1,1} = 0, \\ K_1 = -P_1 \end{cases} \tag{4.13}$$

and

$$\begin{cases} \dot{P}_2 - 2P_1 P_2 + 1 = 0, \\ 1 + 34P_2 > 0, \quad P_2 \geqslant 0, \ P_{2,1} = 0, \\ K_2 = 0. \end{cases} \tag{4.14}$$

Solving in turn (4.13)–(4.14), yields

$$P_{1,t} = \tan(t-1), \quad P_{2,t} = -\frac{\tan(t-1)}{2} - \frac{(t-1)}{2\cos^2(t-1)}, \quad t \in [0, 1].$$

Therefore, our desired H_2/H_∞ controller and worst case disturbance are $u_t^* = 0$ and $v_t^* = -\tan(t-1)x_{t-}$ respectively, where x satisfies

$$\begin{cases} \mathrm{d}x_t = (-4 - \tan(t-1))x_t\mathrm{d}t + 2x_t\mathrm{d}W_t + \displaystyle\int_{\mathscr{E}} 2x_{t-}\widetilde{\mu}(\mathrm{d}\theta, \mathrm{d}t), \\ x_0 = x^0. \end{cases}$$

5 Concluding Remarks

This paper has discussed the finite horizon H_2/H_∞ control problem for Poisson jump-diffusion systems with (x, u, v)-dependent noise. Necessary and sufficient conditions for the existence of a state feedback H_2/H_∞ control have been respectively given in terms of the solutions of the four coupled matrix Riccati equations. A discretization algorithm for solving the coupled matrix-valued equations is also presented. It is noteworthy that the stochastic bounded real lemma is of independent interest and plays a central role in the analysis of the H_∞ control problem (in fact a disturbance attenuation problem) and estimation. Its further applications will appear in our forthcoming paper. There remain many interesting topics deserving further explorations. For example, for Poisson jump-diffusion systems with control dependent noise and random coefficients, the corresponding Riccati equation associated with H_∞ robustness becomes a backward stochastic integral partial differential equation with highly nonlinearity and possible singularity, whose solvability is a challenging problem and deserves for further study.

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