The Cauchy Problem for Coupled Nonlinear Schrödinger Equations with Linear Damping: Local and Global Existence and Blowup of Solutions*

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Abstract   The authors study, by applying and extending the methods developed by Cazenave (2003), Dias and Figueira (2014), Dias et al. (2014), Glassey (1994–1997), Kato (1987), Ohta and Todorova (2009) and Tsutsumi (1984), the Cauchy problem for a damped coupled system of nonlinear Schrödinger equations and they obtain new results on the local and global existence of H^1-strong solutions and on their possible blowup in the supercritical case and in a special situation, in the critical or supercritical cases.

Keywords   Nonlinear Schrödinger equations, Cauchy problem, Blowup of solutions, Dissipation
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1 Introduction

The study of blowup of solutions for a damped nonlinear Schrödinger equation has been developed in the papers by Tsutsumi [14] and by Ohta and Todorova [10]. More recently, the problem was addressed by [4], for the case of inhomogeneous damping. Stimulated by the relevance for physical applications, there was also significant interest in exploring the blowup phenomenon in a system of coupled nonlinear Schrödinger equations with cubic nonlinearity, with the linear coupling [7] and without the linear coupling [12–13]. A rather complete list of the available results can be found in [7]. Two sufficient conditions for the finite-time blowup have also been established for the supercritical case of the coupled nonlinear Schrödinger equations, one of which has gain and another has dissipation, both balanced with each other (see [5]).

In this paper we consider the system:

$$\begin{cases} \mathrm{i}\, u_t = -\Delta u + \mathrm{i}\,\gamma_1\, u + k\, v - (g_1|u|^{p-1} + g|v|^2)\, u, \\ \mathrm{i}\, v_t = -\Delta v + \mathrm{i}\,\gamma_2\, v + k\, u - (g|u|^2 + g_2|v|^{p-1})\, v \end{cases} \tag{1.1}$$

with initial data $u_0, v_0 \in H^1(\mathbb{R}^N)$, where $1 \leq N \leq 3$, and $\frac{4}{N} \leq p - 1$, with $p - 1 < \frac{4}{N-2}$ if $N = 3$, $\gamma_1, \gamma_2 \in \mathbb{R}$ describe gain ($\gamma_{1,2} > 0$) or dissipation ($\gamma_{1,2} < 0$), $k \in \mathbb{R}$ is the linear coupling,

$g_1, g_2 > 0$, $g \in \mathbb{R}$, $u(x,t), v(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^N$, $t \geq 0$. The particular case $p - 1 = 2$, $N \geq 3$ and $\gamma_1 = -\gamma_2$ was considered in [5] for the study of the possible blowup of H^1-strong solutions.

The system (1.1) may appear in various physical contexts. As a few examples, we mention an optical coupler ($N = 1$) with passive and active arms (see [1]) and the self-phase modulation (described by g_1 and g_2) stronger than the Kerr nonlinearity ($p > 3$). Alternatively, the model describes propagation of a pulse in an elliptically polarized medium (see [9]) with dissipation where the two polarizations are linearly coupled. In two- and three-dimensional settings, the model can describe diffraction, focusing and filamentation of a transversely polarized electromagnetic wave (see [3]) where the orthogonally polarized components (they are described by u and v) are linearly coupled (or alternatively, two beams are linearly coupled) and are subject to absorption or gain (described by $\gamma_{1,2}$). In these cases, the evolutional variable t describes distance along the propagation direction of the beam. Further, at $N = 3$, the model describes a collapse of an unstable binary mixture of Bose-Einstein condensates (see [11]) subject to the removal and adding atoms.

In this paper we first study the existence and uniqueness of H^1-strong solutions of the system (1.1) in the sense of Kato (see [2, 8]) by applying some variants of Strichartz's inequalities (see [10]) and some convenient a priori estimates (Theorems 2.1 and 3.1). In the second part of this paper, we extend the main result of [5] in the supercritical case (Theorem 4.1) and give a new result in the critical case (Theorem 4.2).

2 Local Existence in $H^1(\mathbb{R}^N)$

In this section we will study the local existence in $H^1(\mathbb{R}^N)$ to the Cauchy problem for the system (1.1) with initial data $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$. Recall that we have $\frac{4}{N} \leq p - 1 < \frac{4}{N-2}$ ($< +\infty$ if $N = 1, 2$) and $1 \leq N \leq 3$. The case $p - 1 = \frac{4}{N}$ is called the critical case.

To prove the local existence of solutions, we apply Kato's method (see [2, 8]) by adapting the proof of Theorem 4.4.6 in [2].

We start by writing system (1.1) in the form:

$$\begin{cases} i\,\varphi_t = H\varphi + \mathcal{G}(\varphi) + B\varphi, \\ \varphi(0) = \varphi_0, \end{cases} \tag{2.1}$$

where $\varphi = \left(\begin{smallmatrix} u \\ v \end{smallmatrix} \right)$, $\varphi_0 = \left(\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right) \in (H^1(\mathbb{R}^N))^2$,

$$H\varphi = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \varphi,$$

$$\mathcal{G}(\varphi) = \begin{pmatrix} \mathcal{G}_1(\varphi) \\ \mathcal{G}_2(\varphi) \end{pmatrix} = - \begin{pmatrix} g_1 |u|^{p-1} + g|v|^2 & 0 \\ 0 & g|u|^2 + g_2|v|^{p-1} \end{pmatrix} \varphi$$

and

$$B\varphi = \begin{pmatrix} i\,\gamma_1 & k \\ k & i\,\gamma_2 \end{pmatrix} \varphi = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix},$$

i.e.,

$$B_1(u,v) = i\,\gamma_1 u + k\,v, \quad B_2(u,v) = k\,u + i\,\gamma_2 v.$$

We decompose $\mathcal{G}_{1,2}$ as follows:

$$\mathcal{G}_1(\varphi) = g_1 G(u) + \widetilde{G}(v, u), \quad \mathcal{G}_2(\varphi) = g_2 G(v) + \widetilde{G}(u, v)$$

with the functions $G(\cdot)$ and $\widetilde{G}(\cdot, \cdot)$ defined as follows:

$$G(u) = -|u|^{p-1} u, \quad \widetilde{G}(u, v) = -g|u|^2 v.$$

Now we easily derive

$$\left| |v_1|^2 u_1 - |v_2|^2 u_2 \right| \leq c \left(|v_1|^2 + |v_2|^2 + |u_1|^2 + |u_2|^2 \right) \left(|u_1 - u_2| + |v_1 - v_2| \right)$$

and the same estimate for $\left| |u_1|^2 v_1 - |u_2|^2 v_2 \right|$. Moreover,

$$\left| |u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right| \leq c \left(|u_1|^{p-1} + |u_2|^{p-1} \right) |u_1 - u_2|,$$

and we get a similar estimate for $\left| |v_1|^{p-1} v_1 - |v_2|^{p-1} v_2 \right|$.

With $r = (p-1) + 2 = p + 1$, we derive, for r' such that $\frac{1}{r} + \frac{1}{r'} = 1$, $(u_{1,2}, v_{1,2}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$,

$$\|G(u_2) - G(u_1)\|_{L^{r'}} \leq c \left(\|u_1\|_{L^r}^{p-1} + \|u_2\|_{L^r}^{p-1} \right) \|u_2 - u_1\|_{L^r},$$

$$\|\widetilde{G}(u_2, v_2) - \widetilde{G}(u_1, v_1)\|_{L^{\frac{4}{3}}} \leq c \left(\|u_1\|_{L^4}^2 + \|u_2\|_{L^4}^2 + \|v_1\|_{L^4}^2 + \|v_2\|_{L^4}^2 \right)$$
$$\times \left(\|u_1 - u_2\|_{L^4} + \|v_1 - v_2\|_{L^4} \right),$$

and similar estimates for $G(v)$ and $\|\widetilde{G}(v_2, u_2) - \widetilde{G}(v_1, u_1)\|_{L^{\frac{4}{3}}}$;

$$\|\nabla G(u)\|_{L^{r'}} \leq c \|u\|_{L^r}^{p-1} \|\nabla u\|_{L^r},$$

$$\|\nabla \widetilde{G}(u, v)\|_{L^{\frac{4}{3}}} \leq c \left[\|v\|_{L^4}^2 \|\nabla u\|_{L^4} + \|\nabla v\|_{L^4} \|v\|_{L^4} \|u\|_{L^4} \right]$$
$$\leq c \left[\left(\|v\|_{L^4}^2 + \|u\|_{L^4}^2 \right) \left(\|\nabla u\|_{L^4} + \|\nabla v\|_{L^4} \right) \right],$$

and similar estimates for $\|\nabla G(v)\|_{L^{r'}}$ and $\|\nabla \widetilde{G}(v, u)\|_{L^{\frac{4}{3}}}$.

Moreover, we have

$$\|B_1(u_2, v_2) - B_1(u_1, v_1)\|_{L^2} \leq c \left(\|u_2 - u_1\|_{L^2} + \|v_2 - v_1\|_{L^2} \right),$$

$$\|\nabla B_1(u, v)\|_{L^2} \leq c \left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2} \right),$$

and similar estimates for B_2.

Now we fix $M, T > 0$ to be chosen latter as in the proof of Theorem 4.4.6 in [2] and, with $r = \max(2, p + 1, 4) = \max(p + 1, 4)$, we consider the admissible pair (in Strichartz's sense, see [2, Section 2.3])

$$(q, r), \quad \frac{2}{q} = N \left(\frac{1}{2} - \frac{1}{r} \right).$$

We introduce the space

$$X = \{ \varphi \in (L^\infty(-T, T; H^1) \cap L^q(-T, T; W^{1,r}))^2 \}$$

with the distance

$$d(\varphi_1, \varphi_2) = \|u_1 - u_2\|_{L^q(-T,T;L^r)} + \|u_1 - u_2\|_{L^\infty(-T,T,L^2)}$$
$$+ \|v_1 - v_2\|_{L^q(-T,T;L^r)} + \|v_1 - v_2\|_{L^\infty(-T,T;L^2)},$$

where $\varphi_{1,2} = \begin{pmatrix} u_{1,2} \\ v_{1,2} \end{pmatrix}$ and the subset

$$E = \{\varphi \in X \mid \|u\|_{L^q(-T,T;W^{1,r})} + \|u\|_{L^\infty(-T,T;H^1)}$$
$$+ \|v\|_{L^q(-T,T;W^{1,r})} + \|v\|_{L^\infty(-T,T;H^1)} \le M\},$$

which is a complete metric space with distance d.

Now, with $S(t) = e^{i\Delta t}$ and $t \in \mathbb{R}$, denoting the Schrödinger group in L^2, we introduce, for $\varphi \in E$, $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$,

$$\mathcal{H}(\varphi) = \begin{pmatrix} \mathcal{H}_1(\varphi) \\ \mathcal{H}_2(\varphi) \end{pmatrix}, \tag{2.2}$$

with the entries

$$\mathcal{H}_1(\varphi) = S(t)u_0 + i \int_0^t S(t-s)\, \mathcal{G}_1(u(s), v(s))\mathrm{d}s$$
$$+ i \int_0^t S(t-s)\, B_1(u(s), v(s))\mathrm{d}s, \tag{2.3}$$
$$\mathcal{H}_2(\varphi) = S(t)\, v_0 + i \int_0^t S(t-s)\, \mathcal{G}_2(u(s), v(s))\mathrm{d}s$$
$$+ i \int_0^t S(t-s)\, B_2(u(s), v(s))\mathrm{d}s. \tag{2.4}$$

Now, reasoning as in the proof of Theorem 4.4.6 in [2], we can prove, by the previous estimates and applying Strichartz's inequalities, that

$$\mathcal{H}(\varphi) \in C([-T,T]; H^1) \cap L^q(-T,T; W^{1,r}),$$

and for a convenient M and a sufficiently small $T > 0$, $\mathcal{H}(u,v) \in E$ and

$$d(\mathcal{H}(\varphi_1), \mathcal{H}(\varphi_2)) \le \frac{1}{2}\, d(\varphi_1, \varphi_2) \quad \text{for } \varphi_1, \varphi_2 \in E.$$

The uniqueness in $C([-T,T]; H^1)$ and the blowup alternative follow as in Theorem 4.4.6 in [2]. We have the following theorem.

Theorem 2.1 *Let* $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$. *Then, the Cauchy problem for the system* (1.1) *has a unique strong solution* $(u,v) \in C([0, T_{\max}); (H^1)^2(\mathbb{R}^N))$ *with initial data* (u_0, v_0) *defined on a maximal time interval* $[0, T_{\max})$.

3 Global Existence for $k = 0$

In this section we prove the global existence of the particular case when the linear coupling is absent, and the system obeys sufficiently strong dissipation.

Given $\gamma \in \mathbb{R}$, let us consider the semigroup $(S_\gamma(t))_{t \geq 0}$ in $L^2(\mathbb{R}^N)$ defined by

$$S_\gamma(t) = e^{\gamma t} S(t).$$

We need to apply Strichartz's estimates (see [2]). We recall that a pair (q, r) is admissible if

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right)$$

with $2 \leq r \leq \frac{2N}{N-2}$ ($2 \leq r \leq \infty$ if $N = 1$, $2 \leq r < \infty$ if $N = 2$).

Using the same notation as in [2, 10], we define

$$\Phi_f(t) = \int_0^t S(t - s) \, f(s) \mathrm{d}s$$

and

$$\Phi_f^\gamma(t) = \int_0^t S_\gamma(t - s) \, f(s) \mathrm{d}s = e^{\gamma t} \int_0^t S(t - s) \, e^{-\gamma s} \, f(s) \mathrm{d}s.$$

We have the following estimates (see [2, 10]).

For every admissible pair (q, r) and $\forall \varphi \in L^2$, there exists a constant $c > 0$ such that, with $L^p = L^p(\mathbb{R}^N)$ and $T > 0$,

$$\|S(\cdot) f\|_{L^q(0,T;L^r)} \leq c \|f\|_{L^2}, \quad \forall \varphi \in L^2, \tag{3.1}$$

with c independent of T;

$$\|\Phi_f^\gamma\|_{L^q(0,T;L^r)} \leq c \|f\|_{L^{q'}(0,T;L^{r'})}, \tag{3.2}$$

$$\|\Phi_f^\gamma\|_{L^\infty(0,T;L^r)} \leq c \|f\|_{L^{q'}(0,T;L^{r'})}, \tag{3.3}$$

with c independent of T, $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Moreover, if $2 < r < \frac{2N}{N-2}$ (or $2 < r < +\infty$ if $N = 1$) and $\theta, \widetilde{\theta} \in \,]1, +\infty[$ are such that $\frac{1}{\theta} + \frac{1}{\widetilde{\theta}} = N\left(\frac{1}{2} - \frac{1}{r}\right)$, then

$$\|\Phi_f^\gamma\|_{L^\theta(0,T;L^r)} \leq c \|f\|_{L^{\widetilde{\theta}'}(0,T;L^{r'})} \tag{3.4}$$

with c independent of T and $\frac{1}{\theta} + \frac{1}{\theta'} = 1$.

Now, by using the Duhamel formula, we write the system (1.1), for the local solution, in the integral form. In the case $k = 0$ for $t \in [0, T_{\max})$ we have

$$\begin{cases} u(t) = S_{\gamma_1}(t) \, u_0 + \mathrm{i} \int_0^t S_{\gamma_1}(t - s) \left(-g_1|u(s)|^{p-1}u(s) - g|v(s)|^2 u(s)\right) \mathrm{d}s, \\[2mm] v(t) = S_{\gamma_2}(t) \, v_0 + \mathrm{i} \int_0^t S_{\gamma_2}(t - s) \left(-g|u(s)|^2 v(s) - g_2|v(s)|^{p-1}v(s)\right) \mathrm{d}s. \end{cases} \tag{3.5}$$

Next we state a global existence result of the Cauchy problem for the system (1.1) with $k = 0$.

Theorem 3.1 *Assume $\gamma_1, \gamma_2 < 0$ and $k = 0$. Then, for any $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$ there exists $\gamma^*(\|u_0\|_{H^1}, \|v_0\|_{H^1}) > 0$ such that, for all $\gamma_1, \gamma_2 < -\gamma^*$, $T_{\max} = +\infty$.*

First we prove the following important result.

Lemma 3.1 *Under the conditions of the Theorem* 3.1, *assume that there exist constants* $\varepsilon > 0$ *and* $\overline{\gamma} < 0$ *such that for* $\gamma_i \leq \overline{\gamma}$, $i = 1, 2$, *we have, with* $w_0^1 = u_0$, $w_0^2 = v_0$,

$$\|S_{\gamma_i}(\cdot)\, w_0^i\|_{L^\theta(0,+\infty;L^{p+1})} \leq \varepsilon, \quad if \ \ p - 1 \geq 2, \tag{3.6}$$

where $\theta = \frac{2(p-1)(p+1)}{4-(N-2)(p-1)}$,

$$\|S_{\gamma_i}(\cdot)\, w_0^i\|_{L^8(0,+\infty;L^4)} \leq \varepsilon, \quad if \ \ p - 1 < 2. \tag{3.7}$$

Then $T_{\max} = +\infty$.

Proof Let $\widetilde{\theta}$ be defined by $\frac{1}{\theta} + \frac{1}{\widetilde{\theta}} = N\left(\frac{1}{2} - \frac{1}{r}\right)$ for $2 < r < \frac{2N}{N-2}$ (or $2 < r < +\infty$ if $N = 1$) (see (3.4)). We have

$$p\,r' = r, \quad p\,\widetilde{\theta}' = \theta, \quad \frac{1}{q'} = \frac{1}{q} + \frac{p-1}{\theta}. \tag{3.8}$$

Applying (3.5) we derive, for $t \in [0, T]$, $T < T_{\max}$,

$$\begin{aligned}
&\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \\
&\leq \|u_0\|_{H^1} + \|v_0\|_{H^1} + \|\Phi_{|u|^{p-1}u}^{\gamma_1}\|_{L^\infty(0,T;H^1)} + \|\Phi_{|v|^{p-1}v}^{\gamma_2}\|_{L^\infty(0,T;H^1)} \\
&\quad + \|\Phi_{(|u|^2+|v|^2)u}^{\gamma_1}\|_{L^\infty(0,T;H^1)} + \|\Phi_{(|u|^2+|v|^2)v}^{\gamma_2}\|_{L^\infty(0,T;H^1)}.
\end{aligned} \tag{3.9}$$

Now we start with the following case.

(I) $p - 1 \geq 2$.

In this case we have

$$|u|^2 \leq 1 + |u|^{p-1}, \quad |v|^2 \leq 1 + |v|^{p-1}.$$

We estimate

$$\begin{aligned}
\|\Phi_u^{\gamma_1}\|_{L^\infty(0,T;H^1)} &\leq \sup_{t\in(0,T)} \mathrm{e}^{\gamma_1 t}\left\|\int_0^t S(t-s)\,\mathrm{e}^{-\gamma_1 s}\,u(s)\mathrm{d}s\right\|_{H^1} \\
&\leq \sup_{t\in(0,T)} \left(\mathrm{e}^{\gamma_1 t}\frac{\mathrm{e}^{-\gamma_1 t}-1}{|\gamma_1|}\right)\|u\|_{L^\infty(0,T;H^1)} \\
&\leq \frac{1}{|\gamma_1|}\|u\|_{L^\infty(0,T;H^1)}, \tag{3.10}
\end{aligned}$$

and similarly,

$$\|\Phi_v^{\gamma_2}\|_{L^\infty(0,T;H^1)} \leq \frac{1}{|\gamma_2|}\|v\|_{L^\infty(0,T;H^1)}. \tag{3.11}$$

Hence, for $|\gamma_i|$ large enough, it follows from (3.3) and (3.9) that

$$\begin{aligned}
\|u\|_{L^\infty(0,T;H^1)} + \|v\|_{L^\infty(0,T;H^1)} &\leq 2\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right) + c\||u|^{p-1}u\|_{L^{q'}(0,T;W^{1,r'})} \\
&\quad + c\||v|^{p-1}v\|_{L^{q'}(0,T;L^{r'})} \\
&\quad + c\||u|^{p-1}(v + |\nabla v|)\|_{L^{q'}(0,T;L^{r'})} \\
&\quad + c\||v|^{p-1}(u + |\nabla u|)\|_{L^{q'}(0,T;L^{r'})} \tag{3.12}
\end{aligned}$$

with c being a constant independent of T.

Now we estimate the typical term $\||u|^p Dv\|_{L^{q'}(0,T;L^{r'})}$ $\left(D = \frac{\partial}{\partial x_j}\right)$ in the right-hand side of (3.12).

From (3.8) we derive

$$\||u|^{p-1}Dv\|_{L^{q'}(0,T;L^{r'})} \le c\,\|u\|_{L^\theta(0,T;L^r)}^{p-1}\,\|Dv\|_{L^q(0,T,W^{1,r})}. \tag{3.13}$$

Next, we fix $0 < t \le T$. By using the estimate (3.4), from the Duhamel formula (3.5), we deduce by (3.6):

$$
\begin{aligned}
\|u\|_{L^\theta(0,t;L^r)} &\le \|S_{\gamma_1}(\cdot)\,u_0\|_{L^\theta(0,t;L^r)} \\
&\quad + \|\Phi_{|u|^{p-1}u}^{\gamma_1}\|_{L^\theta(0,t;L^r)} + \|\Phi_{|u|^2u+|v|^2u}^{\gamma_1}\|_{L^\theta(0,t;L^r)} \\
&\le \varepsilon + c\||u|^{p-1}u\|_{L^{\tilde{\theta}'}(0,t;L^{r'})} \\
&\quad + c\||u|^2 u\|_{L^{\tilde{\theta}'}(0,t;L^{r'})} + c\||v|^2 v\|_{L^{\tilde{\theta}'}(0,t;L^{r'})}.
\end{aligned}
\tag{3.14}
$$

Now, we remark that, for $D_u = \{x \in \mathbb{R}^N \mid |u(x)| \ge 1 \text{ a.e.}\}$, with χ_{D_u} being the characteristic function of D_u and for each t,

$$
\begin{aligned}
\||u|^2 u\|_{L^{r'}} &= \|\chi_{D_u}|u|^2 u + (1 - \chi_{D_u})|u|^2 u\|_{L^{r'}} \\
&\le \||u|^{p-1}u\|_{L^{r'}} + \|(1 - \chi_{D_u})|u|^2 u\|_{L^{r'}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|(1 - \chi_{D_u})|u|^2 u\|_{L^{r'}} &\le \|(1 - \chi_{D_u})u\|_{L^{2\frac{p+1}{p-1}}}^2 \|u\|_{L^r} \\
&\le \|u\|^{2\frac{p-1}{p+1}} \|u\|_{L^r} \\
&\le e^{2\gamma_1 \frac{p-1}{p+1} t} \|u_0\|_{L^2} \|u\|_{L^r}.
\end{aligned}
$$

Since $\widetilde{\theta}' = \frac{\theta}{p} < \theta$, it follows that

$$
\begin{aligned}
\||u|^2 u\|_{L^{\tilde{\theta}'}(0,t;L^{r'})} &\le c\||u|^{p-1}u\|_{L^{\tilde{\theta}'}(0,t;L^{r'})} + \frac{c}{|\gamma_1|}\|u\|_{L^\theta(0,t;L^r)} \\
&\le c\|u\|_{L^\theta(0,t;L^r)}^p + \frac{c}{|\gamma_1|}\|u\|_{L^\theta(0,t;L^r)}
\end{aligned}
$$

with c being a constant independent of t.

The same conclusion can be obtained for the term $\||v|^2 v\|_{L^{\tilde{\theta}'}(0,t;L^{r'})}$. Therefore, putting together all the terms, we obtain, for $|\gamma_1|$ and $|\gamma_2|$ big enough,

$$\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)} 4\varepsilon + c_1(\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)})^p \tag{3.15}$$

with c_1 being a constant independent of t.

On the other hand, using again the Duhamel formula and the Strichartz's estimates, we

derive

$$
\begin{aligned}
&\|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})} \\
&\leq \|S^{\gamma_1}(\cdot)\,u_0\|_{L^q(0,t;W^{1,r})} + \|S^{\gamma_2}(\cdot)\,v_0\|_{L^q(0,t;W^{1,r})} \\
&\quad + \|\Phi^{\gamma_1}_{|u|^{p-1}u}\|_{L^q(0,t;W^{1,r})} + \|\Phi^{\gamma_2}_{|v|^{p-1}v}\|_{L^q(0,t;W^{1,r})} \\
&\leq c\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right) + c\||u|^{p-1}u\|_{L^{q'}(0,t;W^{1,r'})} \\
&\quad + c\||v|^{p-1}v\|_{L^{q'}(0,t;W^{1,r'})} + c\|(|u|^2 + |v|^2)\,u\|_{L^{q'}(0,t;W^{1,r'})} \\
&\quad + c\|(|v|^2 + |u|^2)\,v\|_{L^{q'}(0,t;W^{1,r'})}.
\end{aligned} \tag{3.16}
$$

Next we proceed as before to estimate the last two terms on the right-hand side more precisely (with $D_v = \{x \in \mathbb{R}^N \mid |v(x)| \geq 1 \text{ a.e.}\}$):

$$
\begin{aligned}
\||v|^2\,Du\|_{L^{q'}(0,t;L^{r'})} &= \|\chi_{D_v}|v|^2 Du + (1-\chi_{D_v})|v|^2 Du\|_{L^{q'}(0,t;L^{r'})} \\
&\leq \||v|^{p-1}Du\|_{L^{q'}(0,t;L^{r'})} + \|(1-\chi_{D_v})|v|^2 Du\|_{L^{q'}(0,t;L^{r'})} \\
&\leq \|v\|^{p-1}_{L^\theta(0,t;L^r)}\|u\|_{L^q(0,t;W^{1,r})} + \frac{c}{|\gamma_2|}\|u\|_{L^q(0,t;W^{1,r})}.
\end{aligned}
$$

Then, for $|\gamma_1|$ and $|\gamma_2|$ large enough, it follows from (3.16) that

$$
\begin{aligned}
&\|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})} \\
&\leq c_2\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right) + c_2\left(\|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)}\right)^{p-1} \\
&\quad \cdot \left(\|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^q(0,t;W^{1,r})}\right)
\end{aligned} \tag{3.17}
$$

with c_2 being a constant independent of t.

Now, let $c_0 = \max(c_1, c_2)$ and choose ε such that $2^{3p-2}\,c_0\,\varepsilon^{p-1} \leq 1$. By the continuity of the functions $t \to \|u\|_{L^\theta(0,t;L^r)} + \|v\|_{L^\theta(0,t;L^r)}$ and $t \to \|u\|_{L^q(0,t;W^{1,r})} + \|v\|_{L^\theta(0,t;W^{1,r})}$, it follows from (3.15) and (3.17) that

$$
\|u\|_{L^\theta(0,T;L^r)} + \|v\|_{L^\theta(0,T;L^r)} \leq 8\,\varepsilon
$$

and

$$
\|u\|_{L^q(0,T;W^{1,r})} + \|v\|_{L^q(0,T;W^{1,r})} \leq 2\,c_0\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right).
$$

The conclusion follows now from (3.12)–(3.13).

(II) $p - 1 < 2$.

Notice that, since $\frac{4}{N} \leq p-1$, the condition $p-1 < 2$ implies $N > 2$, which means in our case $N = 3$. The proof follows the same steps used in the previous case $p-1 \geq 2$. The first estimate (3.12) remains true with $p-1$ replaced by 2 when the admissible pair $(\overline{q}, \overline{r})$ corresponds now to $\overline{r} = 4$. The estimate (3.13) is now

$$
\||u|^2 Dv\|_{L^{\overline{q}'}(0,T;L^{\frac{4}{3}})} \leq c\|u\|^2_{L^8(0,T;L^4)}\|Dv\|_{L^{\overline{q}}(0,T;W^{1,4})}
$$

and the estimates (3.15) and (3.17) are obtained by following the same scheme. For example, to estimate $\|u\|_{L^8(0,t;L^4)}$, just like in (3.14), we use the assumption $\|S_{\gamma_1}(\cdot)\,u_0\|_{L^8(0,+\infty,L^4)} \leq \varepsilon$ and we must only estimate

$$
\||u|^{p-1}u\|_{L^{\tilde{8}}(0,t;L^{\frac{4}{3}})}, \quad \frac{1}{8} + \frac{1}{8} = 3\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{4},
$$

using the decomposition $|u|^{p-1}u = \chi_{D_u}|u|^{p-1}u + (1-\chi_{D_u})|u|^{p-1}u$. With the corresponding estimates (3.15) and (3.17) we conclude in the same way.

Now we can pass to the following proof.

Proof of Theorem 3.1 Assume first $p - 1 \geq 2$. We will prove that (3.6) holds. Since we have $\|S(t)\,u_0\|_{L^{p+1}} \leq c\,\|u_0\|_{H^1}$, $t \geq 0$, we derive

$$\|S_{\gamma_1}u_0\|^{\theta}_{L^{\theta}(0,+\infty;L^{p+1})} = \int_0^{+\infty} e^{\theta\gamma_1 t}\,\|S(t)\,u_0\|^{\theta}_{L^{p+1}}\,\mathrm{d}t \leq -\frac{c}{\theta\gamma_1}\,\|u_0\|^{\theta}_{H^1}.$$

Therefore,

$$\|S_{\gamma_1}u_0\|_{L^{\theta}(0,+\infty;L^{p+1})} \leq c\,\|u_0\|_{H^1}\left(\frac{1}{|\gamma_1|\,\theta}\right)^{\frac{1}{\theta}} \xrightarrow[|\gamma_1|\to+\infty]{} 0$$

and the same conclusion holds for $S_{\gamma_2}v_0$. Similar estimates prove (3.7) in the case $p - 1 < 2$. Hence, the assumptions in Lemma 3.1 are satisfied and thus Theorem 3.1 is proved.

4 Blowup Results

In this section we will study the possible blowup of the local in-time H^1-strong solutions (u, v) of the Cauchy problem for the system (1.1) with initial data $(u_0, v_0) \in (H^1(\mathbb{R}^N))^2$ such that

$$|x|u_0,\ |x|v_0 \in L^2(\mathbb{R}^N).$$

In the following we perform formal calculations which can be justified by suitable regularisations that allow us to prove that

$$|x|\,u(x,\cdot),\ |x|\,v(\cdot,\cdot) \in C([0, T_{\max}); L^2).$$

The main ideas are based on the seminal work of Glassey [6], in [7, 10] and in the previous paper [5] where the case $p - 1 = 2$ is studied.

We start by proving some preliminar estimates to the local solution $(u, v) \in C([0, T_{\max});$ $(H^1)^2)$.

It is easy to derive, for $t \in [0, T_{\max})$ and with $\int \cdot\, \mathrm{d}x = \int_{\mathbb{R}^N} \cdot\, \mathrm{d}x$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x = \gamma_1\int|u|^2\mathrm{d}x + k\,\mathrm{Im}\int v\,\overline{u}\mathrm{d}x,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|v|^2\mathrm{d}x = \gamma_2\int|v|^2\mathrm{d}x + k\,\mathrm{Im}\int u\,\overline{v}\mathrm{d}x.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(|u|^2 + |v|^2)\mathrm{d}x = 2\,\gamma_1\int|u|^2\mathrm{d}x + 2\,\gamma_2\int|v|^2\mathrm{d}x$$

$$\leq 2\,\gamma\int(|u|^2 + |v|^2)\mathrm{d}x \tag{4.1}$$

with $\gamma = \max(|\gamma_1|, |\gamma_2|)$.

Then we obtain

$$Q(t) = \|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \le \mathrm{e}^{\gamma t}(\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2). \tag{4.2}$$

We define the energy

$$\begin{aligned}
E(t) = \frac{1}{2} \int |\nabla u(t)|^2 \mathrm{d}x &+ \frac{1}{2} \int |\nabla v(t)|^2 \mathrm{d}x + k \,\mathrm{Re} \int u(t)\,\overline{u}(t) \mathrm{d}x \\
&- \frac{g_1}{p+1} \int |u(t)|^{p+1} \mathrm{d}x - \frac{g_2}{p+1} \int |v(t)|^{p+1} \mathrm{d}x \\
&- \frac{g}{2} \int |u(t)|^2 \, |v(t)|^2 \mathrm{d}x.
\end{aligned} \tag{4.3}$$

From the system (1.1) we deduce

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} = \gamma_1 \int |\nabla u|^2 \mathrm{d}x &+ \gamma_2 \int |\nabla v|^2 \mathrm{d}x - \gamma_1 g_1 \int |u|^{p+1} \mathrm{d}x - \gamma_2 \, g_2 \int |v|^{p+1} \mathrm{d}x \\
&- (\gamma_1 + \gamma_2)\, g \int |u|^2 \, |v|^2 \mathrm{d}x + k\,(\gamma_1 + \gamma_2)\,\mathrm{Re} \int u\,\overline{v} \mathrm{d}x.
\end{aligned} \tag{4.4}$$

We need the following result.

Lemma 4.1 *Assume* $p - 1 > \frac{4}{N}$. *Then the solution* $(u,v) \in C([0, T_{\max}); H^1)$ *of the system* (1.1) *with initial data* $(u_0, v_0) \in (H^1)^2$ *verifies the inequality*

$$E(t) \le E(0) + \widetilde{c}_3\, \gamma \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}) \mathrm{d}\tau + \widetilde{\widetilde{c}}_3\, Q(0)\, \mathrm{e}^{2\gamma t}$$

with

$$\widetilde{c}_3 = \begin{cases} \max\{1, g_1, g_2\}, & \text{if } (\gamma_1 + \gamma_2)\, g \ge 0, \\ \max\{1, |g| + g_1, |g| + g_2\}, & \text{if } (\gamma_1 + \gamma_2)\, g < 0, \quad N = 1, 2, \\ & \text{or if } (\gamma_1 + \gamma_2)\, g < 0, \quad p - 1 \ge 2, \ N = 3 \end{cases}$$

and

$$\widetilde{\widetilde{c}}_3 = \begin{cases} \dfrac{|k|}{2}, & \text{if } (\gamma_1 + \gamma_2)\, g \ge 0, \\ \dfrac{|k| + |g|}{2}, & \text{if } (\gamma_1 + \gamma_2)\, g < 0, \quad N = 1, 2, \\ & \text{or if } (\gamma_1 + \gamma_2)\, g < 0, \quad p - 1 \ge 2, \ N = 3. \end{cases}$$

Proof If $(\gamma_1 + \gamma_2)\, g \ge 0$, it follows from (4.4) that

$$E(t) \le E(0) + \widetilde{c}_3\, \gamma \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}) \mathrm{d}\tau + |k|\, \gamma \int_0^t Q(\tau) \mathrm{d}\tau,$$

where $\widetilde{c}_3 = \max\{1, g_1, g_2\}$, and since $Q(\tau) \le \mathrm{e}^{2\gamma\tau} Q(0)$, we obtain the result with $\widetilde{\widetilde{c}}_3 = \frac{|k|}{2}$. If $(\gamma_1 + \gamma_2)\, g < 0$, we remark that

$$\begin{aligned}
\int |u|^2 \, |v|^2 \mathrm{d}x &\le \frac{1}{2} \int |u|^4 \mathrm{d}x + \frac{1}{2} \int |v|^4 \mathrm{d}x \\
&\le \frac{1}{2} \int |u|^{p+1} \mathrm{d}x + \frac{1}{2} \int |v|^{p+1} \mathrm{d}x + \frac{1}{2} \int (|u|^2 + |v|^2) \mathrm{d}x
\end{aligned}$$

for $N = 1, 2$ with $p - 1 \geq \frac{4}{N}$ and for $N = 3$ with $p - 1 \geq 2$. The result now follows from (4.4) as before.

Now we define the variance

$$Y(t) = Y_1(t) + Y_2(t)$$

with

$$Y_1(t) = \int |x|^2 \, |u|^2 \mathrm{d}x, \quad Y_2(t) = \int |x|^2 \, |v|^2 \mathrm{d}x,$$

and let

$$V_1(t) = 4 \operatorname{Im} \int (\nabla u \cdot x) \, \overline{u} \mathrm{d}x, \quad V_2(t) = 4 \operatorname{Im} \int (\nabla v \cdot x) \, \overline{v} \mathrm{d}x.$$

We derive from (1.1) that

$$Y_1'(t) = V_1(t) + 2\,\gamma_1 Y_1(t) + 2k \operatorname{Im} \int |x|^2 \, v \, \overline{u} \mathrm{d}x,$$

$$Y_2'(t) = V_2(t) + 2\,\gamma_2 Y_2(t) + 2k \operatorname{Im} \int |x|^2 \, u \, \overline{v} \mathrm{d}x.$$

So

$$Y'(t) = V_1(t) + V_2(t) + 2\,\gamma_1 Y_1(t) + 2\,\gamma_2 Y_2(t). \tag{4.5}$$

To compute the second derivative we take the derivative of $V_i(t), \; i = 1, 2$. First,

$$V_1'(t) = 4 \operatorname{Im} \int (\nabla u_t \cdot x) \, \overline{u} \mathrm{d}x + 4 \operatorname{Im} \int (x \cdot \nabla u) \, \overline{u}_t \mathrm{d}x$$

$$= -4N \operatorname{Im} \int u_t \, \overline{u} \mathrm{d}x - 8 \operatorname{Im} \int (x \cdot \nabla \overline{u}) \, u_t \mathrm{d}x$$

and

$$-4N \operatorname{Im} \int u_t \, \overline{u} \mathrm{d}x = 4N \Big[\int |\nabla u|^2 \mathrm{d}x - g_1 \int |u|^{p+1} \mathrm{d}x$$

$$- g \int |u|^2 \, |v|^2 \mathrm{d}x + k \operatorname{Re} \int v \, \overline{u} \mathrm{d}x \Big],$$

$$-8 \operatorname{Im} \int (x \cdot \nabla \overline{u}) \, u_t \mathrm{d}x = 8 \operatorname{Re} \int (x \cdot \nabla \overline{u}) \, [-\Delta u - g_1 |u|^{p-1} u - g|v|^2 \, u] \mathrm{d}x$$

$$+ 8k \operatorname{Re} \int (x \cdot \nabla \overline{u}) \, v \mathrm{d}x - 8\,\gamma_1 \operatorname{Im} \int (x \cdot \nabla \overline{u}) \, u \mathrm{d}x$$

$$= 8 \int |\nabla u|^2 \mathrm{d}x - 4N \int |\nabla u|^2 \mathrm{d}x$$

$$+ \frac{8N}{p+1} g_1 \int |u|^{p+1} \mathrm{d}x - 4 g \int (x \cdot \nabla |u|^2) \, |v|^2 \mathrm{d}x$$

$$+ 8k \operatorname{Re} \int (x \cdot \nabla \overline{u}) \mathrm{d}x + 8\,\gamma_1 \operatorname{Im} \int (x \cdot \nabla u) \, \overline{u} \mathrm{d}x.$$

Then

$$V_1'(t) = 8 \int |\nabla u|^2 \mathrm{d}x + 4N g_1 \frac{1-p}{p+1} \int |u|^{p+1} \mathrm{d}x - 4N g \int |u|^2 \, |v|^2 \mathrm{d}x$$

$$- 4g \int (x \cdot \nabla |u|^2) \, |v|^2 \mathrm{d}x + 4k \, N \operatorname{Re} \int v \, \overline{u} \mathrm{d}x$$

$$+ 8k \operatorname{Re} \int (x \cdot \nabla \overline{u}) \, v \mathrm{d}x + 8\,\gamma_1 \operatorname{Im} \int (x \cdot \nabla u) \, \overline{u} \mathrm{d}x. \tag{4.6}$$

Similarly,

$$V_2'(t) = 8 \int |\nabla v|^2 \mathrm{d}x + 4Ng_2 \frac{1-p}{p+1} \int |v|^{p+1} \mathrm{d}x - 4Ng \int |u|^2 |v|^2 \mathrm{d}x$$

$$- 4g \int (x \cdot \nabla |v|^2) |u|^2 \mathrm{d}x + 4k\, N \operatorname{Re} \int u\,\overline{v}\mathrm{d}x$$

$$+ 8k \operatorname{Re} \int (x \cdot \nabla\overline{v})\, u\mathrm{d}x + 8\,\gamma_2 \operatorname{Im} \int (x \cdot \nabla v)\, \overline{v}\mathrm{d}x. \tag{4.7}$$

By (4.5) we derive

$$Y''(t) = V_1'(t) + V_2'(t) + 2\,\gamma_1 Y_1'(t) + 2\,\gamma_2\, Y_2'(t)$$

$$= V_1'(t) + V_2'(t) + 2\,\gamma_1 V_1(t) + 2\,\gamma_2\, V_2(t)$$

$$+ k(\gamma_1 - \gamma_2) \operatorname{Im} \int |x|^2\, v\,\overline{u}\mathrm{d}x + 4\,\gamma_1^2\, Y_1(t) + 4\,\gamma_2^2\, Y_2(t).$$

So we obtain, from (4.6)–(4.7),

$$Y''(t) = 8 \int |\nabla u|^2 \mathrm{d}x + 8 \int |\nabla v|^2 \mathrm{d}x + 4Ng_1 \frac{1-p}{p+1} \int |u|^{p+1}\mathrm{d}x$$

$$+ 4Ng_2 \frac{1-p}{p+1} \int |v|^{p+1}\mathrm{d}x - 4Ng \int |u|^2 |v|^2 \mathrm{d}x$$

$$+ 16\,\gamma_1 \operatorname{Im} \int (x \cdot \nabla u)\, \overline{u}\mathrm{d}x + 16\,\gamma_2 \operatorname{Im} \int (x \cdot \nabla v)\, \overline{v}\mathrm{d}x$$

$$+ 4k\,(\gamma_1 - \gamma_2) \operatorname{Im} \int |x|^2\, v\,\overline{u}\mathrm{d}x + 4\,\gamma_1^2\, Y_1(t) + 4\,\gamma_2^2\, Y_2(t). \tag{4.8}$$

Now, we will assume $p - 1 > \frac{4}{N}$.

Since $p - 1 > \frac{4}{N}$, we can choose δ such that

$$0 < \delta < N, \quad p - 1 > \frac{4}{\delta}.$$

Rearranging the terms on the right-hand side of (4.8), we derive by (4.3),

$$Y''(t) = \frac{16}{\delta} NE(t) + \frac{8}{\delta} \int (\delta - N)\,[|\nabla u|^2 + |\nabla v|^2]\mathrm{d}x$$

$$+ \frac{4N}{p+1} \left(\frac{4}{\delta} + 1 - p\right) \left[g_1 \int |u|^{p+1}\mathrm{d}x + g_2 \int |v|^{p+1}\mathrm{d}x\right]$$

$$+ \left(\frac{8}{\delta} - 4\right) Ng \int |u|^2 |v|^2 \mathrm{d}x + 16\,\gamma_1 \operatorname{Im} \int (x \cdot \nabla u)\, \overline{u}\mathrm{d}x$$

$$+ 16\,\gamma_2 \operatorname{Im} \int (x \cdot \nabla v)\, \overline{v}\mathrm{d}x + 4k\,(\gamma_1 - \gamma_2) \operatorname{Im} \int |x|^2\, v\,\overline{u}\mathrm{d}x$$

$$- 8Nk \operatorname{Re} \int u\,\overline{v}\mathrm{d}x + 4\,\gamma_1^2\, Y_1(t) + 4\,\gamma_2^2\, Y_2(t). \tag{4.9}$$

First we assume $N = 3$. If $g > 0$, since $p - 1 > \frac{4}{N}$ (we keep the notation with N by technical reasons), we choose δ such that

$$2 < \delta < N, \quad p - 1 > \frac{4}{\delta}.$$

If $g < 0$, we must assume $p - 1 \geq 2$. In the case $p - 1 > 2$, we choose $\delta = 2$, so the term $(\frac{8}{\delta} - 4)\, Ng \int |u|^2\, |v|^2 \mathrm{d}x$ in (4.9) can be canceled. If $p - 1 = 2$, we choose $\delta = \frac{N+2}{2}$. Then we easily check that $\delta < N$, $p - 1 = 2 > \frac{4}{\delta}$, and we have

$$\frac{4N}{p+1}\left(\frac{4}{\delta} + 1 - p\right)\left[g_1 \int |u|^{p+1}\mathrm{d}x + g_2 \int |v|^{p+1}\mathrm{d}x\right] + \left(\frac{8}{\delta} - 4\right)Ng \int |u|^2|v|^2\mathrm{d}x$$

$$= 2\,N\left(\frac{N-2}{N+2}\right)\left[g_1 \int |u|^4\mathrm{d}x + g_2 \int |v|^4\mathrm{d}x\right] + 4N\left(\frac{N-2}{N+2}\right)g \int |u|^2\,|v|^2\mathrm{d}x$$

and

$$\left|2g \int |u|^2\,|v|^2\mathrm{d}x\right| \leq 2\,|g|\, \|u\|_{L^4}^2\, \|v\|_{L^4}^2 \leq \frac{|g|}{\sqrt{g_1 g_2}}\,(g_1\|u\|_{L^4}^4 + g_2\|v\|_{L^4}^4).$$

Collecting all these cases and taking into account that

$$16\gamma_1\,\mathrm{Im} \int (x \cdot \nabla u)\,\overline{u}\mathrm{d}x + 16\,\gamma_2\,\mathrm{Im} \int (x \cdot \nabla v)\,\overline{v}\mathrm{d}x$$

$$\leq \frac{4}{\delta}(N - \delta)\,(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_2^2) + \gamma^2\delta\,\frac{16}{N-\delta}\,(\|x\,u\|_{L^2}^2 + \|x\,v\|_{L^2}^2)$$

and

$$4k\,(\gamma_1 - \gamma_2)\,\mathrm{Im} \int |x|^2\,v\,\overline{u}\mathrm{d}x \leq 4|k|\,\gamma\,Y(t),$$

it follows from (4.9) that

$$Y''(t) + c_2(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1})$$

$$\leq c_1 Y(t) + \frac{16}{\delta}\,NE(t) + 4N|k|\,Q(t) \tag{4.10}$$

with

$$c_1 = c_1(\gamma, k, N) = 4\,\gamma^2 + 4\,\gamma|k| + \gamma^2\delta\,\frac{16}{N-\delta} \tag{4.11}$$

and

$$c_2 = c_2(\gamma, k, N)$$

$$= \begin{cases} \min\left\{\frac{4}{\delta}(N - \delta), \dfrac{4}{p+1}\,g_1\left(p - 1 - \dfrac{4}{\delta}\right), \dfrac{4}{p+1}\,g_2\left(p - 1 - \dfrac{4}{\delta}\right)\right\}, & \text{if } g > 0, \\[2mm] \min\left\{2(N-2), \dfrac{4}{p+1}\,g_1(p-3), \dfrac{4}{p+1}\,g_2(p-3)\right\}, & \text{if } g < 0 \text{ and } p - 1 > 2, \\[2mm] \dfrac{2}{5}\,\min\left\{1, g_1 + \dfrac{g\sqrt{g_1}}{\sqrt{g_2}}, g_2 + \dfrac{g\sqrt{g_2}}{\sqrt{g_1}}\right\}, & \text{if } -\sqrt{g_1 g_2} < g < 0 \text{ and } p - 1 = 2. \end{cases} \tag{4.12}$$

By applying Lemma 4.1, from (4.10), we derive the following inequality:

$$Y''(t) + c_2(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1})$$

$$\leq c_1 Y(t) + \frac{16}{\delta}\,NE(0) + c_4\,\mathrm{e}^{2\gamma t}$$

$$+ c_3 \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1})\mathrm{d}\tau \tag{4.13}$$

with

$$c_3 = \frac{16}{\delta} N\gamma \widetilde{c}_3, \quad c_4 = 4N\left(|k| + \frac{4}{\delta}\widetilde{c}_3\right)Q(0).$$

Next, we show that the inequality (4.13), which holds for $N = 3$, can also be verified for $N = 1, 2$, up to a few changes in the constants.

The critical point is to dominate the term $\left(\frac{8}{\delta} - 4\right) Ng \int |u|^2 |v|^2 \mathrm{d}x$ in (4.9).

Assume now $N = 2$. Since $p - 1 > \frac{4}{N} = 2$, we have

$$\int |u|^2 |v|^2 \mathrm{d}x \leq \frac{1}{2}\int |u|^{p+1}\mathrm{d}x + \frac{1}{2}\int |v|^{p+1}\mathrm{d}x + \frac{1}{2}\int (|u|^2 + |v|^2)\mathrm{d}x$$

and we choose $\delta = \delta_2 < 2$ such that

$$p - 1 > \frac{4}{\delta_2}$$

and

$$\left(\frac{8}{\sqrt{2}} - 4\right)|g| < \min\left\{\frac{4}{p+1}\, g_1\left(p - 1 - \frac{4}{\delta_2}\right), \frac{4}{p+1}\, g_2\left(p - 1 - \frac{4}{\delta_2}\right)\right\}.$$

It follows that (4.13) holds with constants c_2 and c_4 given by

$$c_2(\gamma, k, 2) = \min\left\{\frac{4}{\delta_2}\, (2 - \delta_2), \frac{4}{p+1}\, g_1\left(p - 1 - \frac{4}{\delta_2}\right), \frac{4}{p+1}\, g_2\left(p - 1 - \frac{4}{\delta_2}\right)\right\}$$

and

$$c_4(\gamma, k, 2) = \left[8\left(|k| + \frac{4}{\delta_2}\widetilde{\widetilde{c}}(2)\right) + 2\left(\frac{4}{\sqrt{\delta_2}} - 2\right)|g|\right]Q(0).$$

Finally, assume $N = 1$. We have $p - 1 > \frac{4}{N} = 4$ and we choose $\delta = \delta_1 < 1$ such that $p - 1 - \frac{4}{\delta_1} > 0$. By the Gagliardo–Nirenberg inequality, we derive

$$\|u\|_{L^4}^4 \leq c\, \|\nabla u\|_{L^2}\, \|u\|_{L^2}^3 \leq \varepsilon\, \|\nabla u\|_{L^2}^2 + c(\varepsilon)\, \|u\|_{L^2}^6$$

with $\varepsilon > 0$ to be chosen.

Thus

$$\left(\frac{8}{\delta_1} - 4\right)|g| \int |u|^2 |v|^2 \mathrm{d}x$$

$$\leq \left(\frac{4}{\delta_1} - 2\right)|g|\,\varepsilon\,(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) + c(\varepsilon)\,(\|u\|_{L^2}^6 + \|v\|_{L^2}^6)$$

and we choose ε such that

$$\left(\frac{4}{\delta_1} - 2\right)|g|\,\varepsilon < \frac{2}{\delta_1}\,(1 - \delta_1).$$

Once again we obtain (4.13) with the constants

$$c_2(\gamma, k, 1) = \min\left\{\frac{2}{\delta_1}\, (2 - \delta_1), \frac{4}{p+1}\, g_1\left(p - 1 - \frac{4}{\delta_1}\right), \frac{4}{p+1}\, g_2\left(p - 1 - \frac{4}{\delta_1}\right)\right\},$$

and that the term $c_4\, \mathrm{e}^{2\gamma t}$ is now replaced by $c_4(\gamma, k, 1)\, \mathrm{e}^{6\gamma t}$ with

$$c_4(\gamma, k, 1) = 4\left(|k| + \frac{4}{\delta_1}\widetilde{\widetilde{c}}_3(1)\right)Q(0) + c(\varepsilon)\, Q^3(0).$$

Now, let

$$\rho(t) = \int_0^t \int_0^s (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} + \|v\|_{L^{p+1}}^{p+1}) \mathrm{d}\tau \mathrm{d}s.$$

From (4.13) we obtain

$$Y(t) + c_2\, \rho(t) \leq F(t) + c_1 \int_0^t \int_0^s Y(\tau) \mathrm{d}\tau \mathrm{d}s + c_3 \int_0^t \rho(\tau) \mathrm{d}\tau \qquad (4.14)$$

with

$$F(t) = \begin{cases} Y(0) + Y'(0)\, t + \dfrac{8}{\delta}\, N E(0)\, t^2 + \dfrac{c_4(N)}{4\,\gamma^2}\, (\mathrm{e}^{2\gamma t} - 2\,\gamma t - 1), & \text{if } N > 1, \\[3mm] Y(0) + Y'(0)\, t + \dfrac{8}{\delta}\, E(0)\, t^2 + \dfrac{c_4(1)}{36\,\gamma^2}\, (\mathrm{e}^{6\gamma t} - 6\,\gamma t - 1), & \text{if } N = 1. \end{cases}$$

Next, we introduce the functions

$$M(t) = \sup_{\tau \in [0,t]} F(\tau) + 1, \quad t \geq 0, \qquad (4.15)$$

$$G(t) = M(t)\Big[c_1 \frac{t^2}{2} + \exp\Big(\frac{c_3}{c_2} t\Big) - 1\Big]. \qquad (4.16)$$

We can now state a blowup theorem for the supercritical case.

Theorem 4.1 *Assume $p - 1 > \frac{4}{N}$ and the Cauchy problem (1.1) with initial data $(u_0, v_0) \in (H^1)^2$. Let $(u, v) \in C([0, T_{\max}); (H^1)^2)$ be the corresponding local solution. Assume that $(|x|u_0, |x|v_0) \in (L^2)^2$ and if $N = 3$, we have*

$$\begin{cases} p - 1 \geq 2, & \text{if } (\gamma_1 + \gamma_2)\, g < 0, \\ p - 1 > 2, & \text{if } g < 0, \\ p - 1 \geq 2, & \text{if } -\sqrt{g_1 g_2} < g < 0. \end{cases}$$

Assume also that there exists $T_0 > 0$ such that

$$F(T_0) + 1 < 0, \qquad (4.17)$$

$$G(T_0) < 1. \qquad (4.18)$$

Then the solution (u, v) blows up in finite time with $T_{\max} \leq T_0$.

Proof Let us define

$$T_1 = \sup\{t \in [0, T_0] \mid Y(\tau) \leq M(T_0),\ \tau \in [0, t]\}.$$

It follows from (4.14)–(4.15) and (4.17) that, for $t \in [0, T_1]$,

$$\begin{aligned} Y(t) + c_2\, \rho(t) &\leq F(t) + c_1 M(T_0)\frac{T_0^2}{2} + c_3 \int_0^t \rho(\tau) \mathrm{d}\tau \\ &\leq M(T_0) - 1 + c_1 M(T_0)\frac{T_0^2}{2} + c_3 \int_0^t \rho(\tau) \mathrm{d}\tau \\ &< M(T_0) + c_3 \int_0^t \rho(\tau) \mathrm{d}\tau. \end{aligned} \qquad (4.19)$$

Applying Gronwall's inequality, we obtain

$$\rho(t) \leq \frac{M(T_0)}{c_2} \exp\left(\frac{c_3}{c_2} t\right).$$

Using this estimate, back to the right-hand side of (4.15), we derive

$$Y(t) \leq F(t) + G(T_0), \quad t \in [0, T_1],$$

and by (4.18), $Y(T_1) < M(T_0)$. Then $T_1 = T_0$. Hence,

$$Y(T_0) < F(T_0) + 1 < 0,$$

which is absurd since $Y \geq 0$.

Remark 4.1 As can be seen by an adaptation of the proofs in Lemma 1 in [5], where the particular case $p - 1 = 2$ is considered, the blowup assumptions (4.17)–(4.18) are satisfied in that case, for a certain $T_0 > 0$, if the initial energy

$$E(0) < -\frac{(N+2)\widetilde{M}(T_{0,\max})}{8NT_{0,\min}^2},$$

where

$$\widetilde{M}(t) = 1 + Y(0) + C_0(e^{2\gamma t} - 1)$$

with $\gamma = \gamma_1 = -\gamma_2 > 0$ and $T_{0,\max}$, $T_{0,\min}$ defined in Lemma 1 in [5].

Remark 4.2 The blowup result of Theorem 4.1 can be extended to higher dimensions if $p - 1 > \frac{4}{N}$ with the same proof as in the case $N = 3$.

Now, we consider a special case of the Cauchy problem for the system (1.1), which includes the critical case, although it requires the absence of the linear coupling.

Theorem 4.2 *Assuming $k = 0$, $\gamma_1 = \gamma_2 = \sigma > 0$, $g > 0$ and $p - 1 \geq \frac{4}{N}$ (critical and supercritical cases). Let $(u, v) \in ([0, T_{\max}); (H^1)^2)$ be the local solution to the Cauchy problem for the system (1.1) with initial data $(u_0, v_0) \in (H^1)^2$, $(x\, u_0, x\, v_0) \in (L^2)^2$. Then, if $E(0) < 0$, the solution blows up in finite time, that is, $T_{\max} < +\infty$.*

Proof We have in this case, by (4.4),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma\left[\int|\nabla u|^2\mathrm{d}x + \int|\nabla v|^2\mathrm{d}x - g_1\int|u|^{p+1}\mathrm{d}x - g_2\int|v|^{p+1}\mathrm{d}x - 2\,\sigma g\int|u|^2\,|v|^2\mathrm{d}x\right]$$

$$= \sigma\left[\int|\nabla u|^2\mathrm{d}x + \int|\nabla v|^2\mathrm{d}x - \frac{2\,g_1}{p+1}\int|u|^{p+1}\mathrm{d}x - \frac{2\,g_2}{p+1}\int|u|^{p+1}\mathrm{d}x\right]$$

$$- 2\,\sigma g\int|u|^2\,|v|^2\mathrm{d}x + \sigma\left(\frac{2\,g_1}{p+1} - g_1\right)\int|u|^{p+1}\mathrm{d}x + \sigma\left(\frac{2\,g_2}{p+1} - g_2\right)\int|v|^{p+1}\mathrm{d}x.$$

Since

$$\frac{2}{p+1} - 1 \leq 0,$$

we derive, with

$$\gamma = |\gamma_1| = |\gamma_2|, \quad \frac{\mathrm{d}E}{\mathrm{d}t} \leq 2\,\gamma\,E,$$

so

$$\mathrm{e}^{-2\gamma t}E(t) \leq E(0) < 0.$$

Now, from (4.6) and (4.7), we deduce

$$(V_1(t) + V_2(t))' - 2\,\sigma(V_1(t) + V_2(t)) =$$

$$= 8\int |\nabla u|^2 \mathrm{d}x + 8\int |\nabla v|^2 \mathrm{d}x + 4Ng_1\frac{1-p}{1+p}\int |u|^{p+1}\mathrm{d}x$$

$$+ 4Ng_2\frac{1-p}{1+p}\int |v|^{p+1}\mathrm{d}x - 4Ng\int |u|^2\,|v|^2\mathrm{d}x$$

$$= 16\,E(t) + \frac{4\,g_1}{1+p}\,(N(1-p)+4)\int |u|^{p+1}\mathrm{d}x$$

$$+ \frac{4\,g_2}{1+p}\,(N(1-p)+4)\int |v|^{p+1}\mathrm{d}x - 4Ng\int |u|^2\,|v|^2\mathrm{d}x.$$

Since $g > 0$ and $p - 1 \geq \frac{4}{N}$, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-2\sigma t}(V_1 + V_2)] \leq \mathrm{e}^{-2\sigma t}E(t).$$

Finally, from (4.5) we have

$$Y_1'(t) - 2\,\sigma\,Y_1(t) = V_1(t), \quad Y_2'(t) - 2\,\sigma\,Y_2(t) = V_2(t).$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-2\sigma t}\,Y(t)) = \mathrm{e}^{-2\sigma t}(V_1 + V_2).$$

Therefore

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathrm{e}^{-2\sigma t}\,Y(t)) = \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-2\sigma t}(V_1 + V_2)] \leq \mathrm{e}^{-2\sigma t}E(t) \leq E_0 < 0$$

and the conclusion follows.

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