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New Subclasses of Biholomorphic Mappings and the Modified Roper-Suffridge Operator*

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Abstract The authors propose a new approach to construct subclasses of biholomorphic mappings with special geometric properties in several complex variables. The Roper-Suffridge operator on the unit ball B^n in $\mathbb{C}^n$ is modified. By the analytical characteristics and the growth theorems of subclasses of spirallike mappings, it is proved that the modified Roper-Suffridge operator $[\Phi_{G,\gamma}(f)](z)$ preserves the properties of $S^*_\Omega(A,B)$, as well as strong and almost spirallikeness of type β and order α on B^n. Thus, the mappings in $S^*_\Omega(A,B)$, as well as strong and almost spirallike mappings, can be constructed through the corresponding functions in one complex variable. The conclusions follow some special cases and contain the elementary results.

Keywords Biholomorphic mappings, Spirallike mappings, Starlike mappings, Roper-Suffridge operator
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1 Introduction

In 1995, the Roper-Suffridge operator

$$\phi_n(f)(z) = (f(z_1), \sqrt{f'(z_1)}z_0)'$$

was introduced in [17], where $f(z_1)$ is a univalent holomorphic function on the unit disk D, and $z = (z_1,\ z_0) \in B^n,\ z_1 \in D,\ z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1},\ \sqrt{f'(0)} = 1$. By the Roper-Suffridge operator, we can construct a normalized locally biholomorphic mapping on B^n through a normalized locally biholomorphic function on D. Therefore, the Roper-Suffridge operator provides a powerful tool for constructing biholomorphic mappings with special geometric properties in several complex variables. By far, there have been lots of beautiful results about the generalized Roper-Suffridge operator (see [2, 4–5, 7–12, 18]).

In 2005, Muir and Suffridge [14] extended the Roper-Suffridge operator to be

$$F(z) = (f(z_1) + f'(z_1)P(z_0), \sqrt{f'(z_1)}z_0)', \tag{1.1}$$

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where f is a normalized biholomorphic function on D, $z = (z_1,\ z_0) \in B^n$, $z_1 \in D$, $z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$, $\sqrt{f'(0)} = 1$ and $P : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2. Muir and Suffridge proved that the operator (1.1) preserves starlikeness and convexity on $\|P\| \le \frac{1}{4}$ and $\|P\| \le \frac{1}{2}$, respectively. By (1.1), Muir and Suffridge [15–16] discussed the extremal mapping of convex mapping on B^n.

Wang and Liu [20] modified the operator (1.1) to be

$$F(z) = (f(z_1) + f'(z_1)P(z_0), [f'(z_1)]^{\frac{1}{m}} z_0)', \tag{1.2}$$

where $P : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree m. They showed that the operator (1.2) preserves almost starlikeness of order $\alpha \in [0, 1)$ and starlikeness of order $\alpha \in (0, 1)$ under some conditions for P. In 2011, Feng and Yu [3] showed that the operator (1.2) preserves almost spirallikeness of type β and order α, spirallikeness of type β and order α, and strong spirallikeness of type β and order α on B^n.

In 2008, Muir [13] introduced the following generalized Roper-Suffridge operator on the unit ball in complex Banach spaces:

$$[\Phi_{G,\beta}(f)](z) = (f(z_1) + G([f'(z_1)]^{\beta} \widehat{z}), [f'(z_1)]^{\beta} \widehat{z})',$$

where $z = (z_1, \widehat{z})$, $f(z_1)$ is a normalized univalent holomorphic function on D, and G is a holomorphic function in $\mathbb{C}^{n-1}$ with $G(0) = 0$, $DG(0) = 0$, $\gamma \ge 0$ and $[f'(z_1)]^{\beta}|_{z_1=0} = 1$. Muir discussed the relationship between $[\Phi_{G,\beta}(f)](z)$ and Loewner chains.

Now, we extend the Roper-Suffridge operator to be

$$[\Phi_{G,\gamma}(f)](z) = \Big(f(z_1) + G\Big(\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma} z_0\Big), \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma} z_0\Big)'.$$

In this paper, we mainly seek conditions for G under which the modified operator $[\Phi_{G,\gamma}(f)](z)$ preserves the properties of subclasses of biholomorphic mappings. In Sections 2–3, by the properties of k-fold symmetric mappings and the growth theorems of subclasses of biholomorphic mappings, we study the properties of modified operator $[\Phi_{G,\gamma}(f)](z)$ for strong and almost spirallike mappings of type β and order α on B^n. Thus we obtain that $[\Phi_{G,\gamma}(f)](z)$ preserves starlikeness of order α, strong starlikeness of order α, strong and almost starlikeness of order α, and strong spirallikeness of type β.

In order to get the main results, we need the following definitions and lemma.

Definition 1.1 (see [19]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$, and the Minkowski functional $\rho(z)$ of Ω be C^1 except for some submanifolds of lower dimensions. Let $f(z)$ be a normalized locally biholomorphic mapping on Ω, and let $-1 \le A < B < 1$,*

$$\Big|\frac{2}{\rho(z)}\frac{\partial \rho}{\partial z}(z)[Df(z)]^{-1}f(z) - \frac{1-AB}{1-B^2}\Big| < \frac{B-A}{1-B^2}, \quad z \in \Omega\backslash\{0\}.$$

Then we call $f(z) \in S^*_{\Omega}(A, B)$.

For $\Omega = D$, the condition reduces to

$$\Big|\frac{f(z)}{zf'(z)} - \frac{1-AB}{1-B^2}\Big| < \frac{B-A}{1-B^2}, \quad z \in D.$$

If $A = -1$ and $B = 1 - 2\alpha$, then $f(z)$ is a starlike function of order α.

If $A = -\alpha$ and $B = \alpha$, then $f(z)$ is a strong starlike function of order α.

Definition 1.2 (see [1]) *Let $f(z)$ be a normalized locally biholomorphic mapping on B^n. Let $\alpha\in[0,1)$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$, and*

$$\Big|\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}+\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{1}{\|z\|^2}\overline{z}'[Df(z)]^{-1}f(z)-\frac{1+c^2}{1-c^2}\Big|<\frac{2c}{1-c^2}.$$

Then $f(z)$ is called a strong and almost spirallike mapping of type β and order α on B^n.

For $\Omega=D$, the condition reduces to

$$\Big|\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}+\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z)}{zf'(z)}-\frac{1+c^2}{1-c^2}\Big|<\frac{2c}{1-c^2}.$$

Setting $\alpha=0$, $\beta=0$ and $\alpha=\beta=0$, Definition 1.2 reduces to the definition of strong spirallike mappings of type β, strong and almost starlike mappings of order α, and strong starlike mappings, respectively.

Lemma 1.1 (see [13]) *Let $P(z)$ be a homogeneous polynomial of degree m, and let $DP(z)$ be the Fréchet derivative of P at z. Then*

$$DP(z)z=mP(z).$$

2 The Invariance of $S^*_\Omega(A,B)$

Lemma 2.1 *Let*

$$F(z)=\Big(f(z_1)+G\Big(\Big[\frac{f(z_1)}{z_1}\Big]^\gamma z_0\Big),\Big[\frac{f(z_1)}{z_1}\Big]^\gamma z_0\Big)',$$

where $z=(z_1,z_0)$, $f(z_1)$ is a normalized univalent holomorphic function on D, G is holomorphic in $\mathbb{C}^{n-1}$ with $G(0)=0$, $DG(0)=0$, $\gamma\geq 0$ and $[\frac{f(z_1)}{z_1}]^\gamma|_{z_1=0}=1$. The homogeneous expansion of $G(z)$ is $\sum\limits_{j=0}^{\infty}P_j(z)$, where $P_j(z)$ is a homogeneous polynomial of degree j. Then

$$\begin{aligned}\overline{z}'(DF(z))^{-1}F(z)={}&\overline{z_1}\frac{f(z_1)}{f'(z_1)}+\frac{\overline{z_1}}{f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)+\Big\{1-\gamma\\&+\gamma\frac{f(z_1)}{z_1f'(z_1)}-\gamma\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\Big\}\|z_0\|^2.\end{aligned}$$

Proof Since

$$F(z)=\Big(f(z_1)+G\Big(\Big[\frac{f(z_1)}{z_1}\Big]^\gamma z_0\Big),\Big[\frac{f(z_1)}{z_1}\Big]^\gamma z_0\Big)',$$

we get

$$DF(z)=\begin{pmatrix}f'(z_1)+\sum\limits_{j=2}^{\infty}\gamma j\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)}P_j(z_0) & \sum\limits_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}DP_j(z_0)\\ \gamma\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma}\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)}z_0 & \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma}I_{n-1}\end{pmatrix},$$

where I_{n-1} is the identity operator on $\mathbb{C}^{n-1}$.

Let $(DF(z))^{-1}F(z) = h(z)$. Then $DF(z)h(z) = F(z)$. Let $h(z) = (A, B)'$. From Lemma 1.1, we have

$$\begin{cases} \Big\{f'(z_1) + \sum_{j=2}^{\infty} \gamma j \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j} \frac{z_1 f'(z_1) - f(z_1)}{z_1 f(z_1)} P_j(z_0)\Big\} A z_0 + \sum_{j=2}^{\infty} \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j} j P_j(z_0) B \\ \quad = f(z_1) z_0 + \sum_{j=2}^{\infty} \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j} P_j(z_0) z_0, \\ \gamma \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma} \frac{z_1 f'(z_1) - f(z_1)}{z_1 f(z_1)} z_0 A + \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma} B = \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma} z_0. \end{cases}$$

It follows that

$$\begin{cases} A = \frac{f(z_1)}{f'(z_1)} + \frac{1}{f'(z_1)} \sum_{j=2}^{\infty} \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j} (1-j) P_j(z_0), \\ B = \Big\{1 - \gamma + \gamma \frac{f(z_1)}{z_1 f'(z_1)} - \gamma \frac{z_1 f'(z_1) - f(z_1)}{z_1 f(z_1) f'(z_1)} \sum_{j=2}^{\infty} \Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j} (1-j) P_j(z_0)\Big\} z_0. \end{cases}$$

So we have the desired conclusion.

Lemma 2.2 *Let $f(z_1) \in S^*_{\Omega}(A, B)$ with $f'(z_1) \neq 0$ and $-1 \leq A < B \leq \frac{A+1}{2}$. Let*

$$q(z_1) = \frac{f(z_1)}{z_1 f'(z_1)} - \frac{1 - AB}{1 - B^2}.$$

Then

$$|q(z_1)| < \frac{B - A}{1 - B^2}, \quad \Big|q(z_1) + \frac{B^2 - AB}{1 - B^2}\Big| \leq |z_1|.$$

Proof Since $f(z_1) \in S^*_{\Omega}(A, B)$, by Definition 1.1, we have

$$\Big|\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1 - AB}{1 - B^2}\Big| < \frac{B - A}{1 - B^2}.$$

Thus $|q(z_1)| < \frac{B-A}{1-B^2}$. Let

$$g(z_1) = \begin{cases} \frac{f(z_1)}{z_1 f'(z_1)} - 1, & z_1 \neq 0, \\ 0, & z_1 = 0. \end{cases}$$

Then $g(z_1) = q(z_1) + \frac{B^2 - AB}{1-B^2}$ for $z_1 \neq 0$. It follows that $q(z_1) = g(z_1) - \frac{B^2-AB}{1-B^2}$. Therefore

$$|q(z_1)| = \Big|g(z_1) - \frac{B^2 - AB}{1 - B^2}\Big| < \frac{B - A}{1 - B^2}.$$

Then

$$|g(z_1)| < \frac{|B|(B - A)}{1 - B^2} + \frac{B - A}{1 - B^2} = \frac{(1 + |B|)(B - A)}{1 - B^2}.$$

For $B \leq 0$, we have

$$\frac{(1 + |B|)(B - A)}{1 - B^2} = \frac{B - A - B^2 + BA}{1 - B^2} = \frac{(B - 1)(1 + A)}{1 - B^2} + 1 \leq 1.$$

For $0 < B \leq \frac{A+1}{2}$, we have

$$\frac{(1 + |B|)(B - A)}{1 - B^2} = \frac{B^2 + B(1 - A) - A}{1 - B^2} = \frac{(B + 1)[2B - (A + 1)]}{1 - B^2} + 1 \leq 1.$$

Hence $|g(z_1)| < 1$ for $-1 \le A < B \le \frac{A+1}{2}$. Observing that $g(z_1)$ is holomorphic on D and $g(0) = 0$, from the Schwarz lemma, we obtain $|g(z_1)| \le |z_1|$, that is

$$\Big|q(z_1) + \frac{B^2 - AB}{1 - B^2}\Big| \le |z_1|.$$

In the following, $f(z)$ is said to be k-fold symmetric (see [10]), where k is a positive integer if

$$f(z) = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{k}} f(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k}} z).$$

Lemma 2.3 (see [19]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain, and the Minkowski functional $\rho(z)$ of Ω be C^1 except for some submanifolds of lower dimensions. Let $f(z) \in S_\Omega^*(A, B)$ be k-fold symmetric. Then*

$$\begin{cases} \dfrac{\rho(z)}{(1 - A\rho(z)^k)^{\frac{A-B}{Ak}}} \le \rho(f(z)) \le \dfrac{\rho(z)}{(1 + A\rho(z)^k)^{\frac{A-B}{Ak}}}, & A \ne 0, \\ \rho(z)\mathrm{e}^{\frac{-B\rho(z)^k}{k}} \le \rho(f(z)) \le \rho(z)\mathrm{e}^{\frac{B\rho(z)^k}{k}}, & A = 0, \end{cases}$$

or equivalently,

$$\begin{cases} \dfrac{|z|}{(1 - A|z|^k)^{\frac{A-B}{Ak}}} \le |f(z)| \le \dfrac{|z|}{(1 + A|z|^k)^{\frac{A-B}{Ak}}}, & A \ne 0, \\ |z|\mathrm{e}^{\frac{-B|z|^k}{k}} \le |f(z)| \le |z|\mathrm{e}^{\frac{B|z|^k}{k}}, & A = 0. \end{cases}$$

The above estimates are all accurate.

The following are our main results.

Theorem 2.1 *Let $f(z_1) \in S_\Omega^*(A, B)$ be a k-fold symmetric function with $f'(z_1) \ne 0$, $-1 \le A < B \le \frac{A+1}{2}$ and k is a positive integer. Let $F(z)$ be the mapping defined in Lemma 2.1 with $\gamma \in [0, 1)$.*

(1) *If $A = 0$, then $F(z) \in S_\Omega^*(A, B)$ provided that*

$$\sum_{j=2}^{\infty} \mathrm{e}^{(1+\gamma j)\frac{B}{k}} (j-1)\|P_j\| \le \frac{B(1-\gamma)}{2(B+1)}.$$

(2) *If $0 < A < 1$, then $F(z) \in S_\Omega^*(A, B)$ provided that $P_j = 0$ $\big(j < 2\big(1 + \frac{B-A}{Ak}\big)\big)$ and*

$$\sum_{j=2}^{\infty} (1+A)^{\frac{B-A}{Ak}\gamma j} 2^{\frac{B-A}{Ak}} (j-1)\|P_j\| \le \frac{(B-A)(1-\gamma)}{2(B+1)}.$$

(3) *If $-1 \le A < 0$, then $F(z) \in S_\Omega^*(A, B)$ provided that $\gamma < \min\big\{\frac{-Ak}{2(B-A)}, 1\big\}$, $P_j = 0$ $\big(j < \frac{2Ak}{Ak + 2(B-A)\gamma}\big)$ and*

$$\sum_{j=2}^{\infty} 2^{\frac{A-B}{Ak}(1+\gamma j)} (j-1)\|P_j\| \le \frac{(B-A)(1-\gamma)}{2(B+1)}.$$

Proof By Definition 1.1, we only need to prove

$$\Big|\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2} - \frac{1 - AB}{1 - B^2}\Big| < \frac{B - A}{1 - B^2}. \tag{2.1}$$

It is obvious that (2.1) holds for $z_0 = 0$, since $F(z)$ is holomorphic for $z \in \overline{B^n}$ $(z_0 \neq 0)$. From the maximum modulus principle of holomorphic functions, we only need to prove that (2.1) holds for $z \in \partial B^n$ $(z_0 \neq 0)$. In the following, let $\|z\|^2 = |z_1|^2 + \|z_0\|^2 = 1$.

Since $f(z_1) \in S^*_\Omega(A,B)$, by Definition 1.1, we have

$$\Big|\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1-AB}{1-B^2}\Big| < \frac{B-A}{1-B^2}.$$

Let

$$q(z_1) = \frac{f(z_1)}{z_1 f'(z_1)} - \frac{1-AB}{1-B^2}.$$

It follows that

$$\frac{f(z_1)}{z_1 f'(z_1)} = q(z_1) + \frac{1-AB}{1-B^2}.$$

From Lemma 2.1, we have

$$\begin{aligned}
&\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2} - \frac{1-AB}{1-B^2}\\
&= \overline{z_1}\frac{f(z_1)}{f'(z_1)} + \frac{\overline{z_1}}{f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0) - \frac{1-AB}{1-B^2}\\
&\quad + \Big\{1-\gamma+\gamma\frac{f(z_1)}{z_1 f'(z_1)} - \gamma\frac{z_1 f'(z_1) - f(z_1)}{z_1 f(z_1) f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\Big\}\|z_0\|^2\\
&= (|z_1|^2 + \gamma\|z_0\|^2)\frac{f(z_1)}{z_1 f'(z_1)} + \frac{|z_1|^2}{z_1 f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0) - \frac{1-AB}{1-B^2}\\
&\quad + \Big\{1-\gamma-\gamma\Big[\frac{1}{f(z_1)} - \frac{1}{z_1 f'(z_1)}\Big]\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\Big\}\|z_0\|^2\\
&= (|z_1|^2 + \gamma\|z_0\|^2)\frac{f(z_1)}{z_1 f'(z_1)} + (1-\gamma)\|z_0\|^2 - \frac{1-AB}{1-B^2}\\
&\quad + \Big[(|z_1|^2 + \gamma\|z_0\|^2)\frac{f(z_1)}{z_1 f'(z_1)} - \gamma\|z_0\|^2\Big]\frac{1}{f(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\\
&= (|z_1|^2 + \gamma\|z_0\|^2)\Big[q(z_1) + \frac{1-AB}{1-B^2}\Big] + (1-\gamma)\|z_0\|^2 - \frac{1-AB}{1-B^2}\\
&\quad + \Big[(|z_1|^2 + \gamma\|z_0\|^2)\Big(q(z_1) + \frac{1-AB}{1-B^2}\Big) - \gamma\|z_0\|^2\Big]\frac{1}{f(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\\
&= (\gamma + (1-\gamma)|z_1|^2)q(z_1) + (1-\gamma)\|z_0\|^2\frac{AB-B^2}{1-B^2}\\
&\quad + \Big[(\gamma + (1-\gamma)|z_1|^2)\Big(q(z_1) + \frac{B^2-AB}{1-B^2}\Big) + |z_1|^2\Big]\frac{1}{f(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0).
\end{aligned}$$

Since $f(z_1) \in S^*_\Omega(A,B)$ is k-fold symmetric, by Lemma 2.3, we have

$$\begin{aligned}
&\frac{|z_1|}{(1-A|z_1|^k)^{\frac{A-B}{Ak}}} \le |f(z_1)| \le \frac{|z_1|}{(1+A|z_1|^k)^{\frac{A-B}{Ak}}}, \quad A \neq 0,\\
&|z_1|\mathrm{e}^{\frac{-B|z_1|^k}{k}} \le |f(z_1)| \le |z_1|\mathrm{e}^{\frac{B|z_1|^k}{k}}, \quad A = 0.
\end{aligned}$$

We discuss the properties of $F(z)$ from Lemma 2.2 in the following 3 cases.

(1) In the case of $A=0$, we have

$$\begin{aligned}
&\Big|\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2}-\frac{1-AB}{1-B^2}\Big|\\
&\le (\gamma+(1-\gamma)|z_1|^2)|q(z_1)|+(1-\gamma)\|z_0\|^2\frac{B^2-AB}{1-B^2}\\
&\quad+\Big[(\gamma+(1-\gamma)|z_1|^2)\Big|q(z_1)+\frac{B^2-AB}{1-B^2}\Big|+|z_1|^2\Big]\frac{1}{|f(z_1)|}\sum_{j=2}^{\infty}\Big|\frac{f(z_1)}{z_1}\Big|^{\gamma j}(j-1)\|P_j\|\|z_0\|^j\\
&< (\gamma+(1-\gamma)|z_1|^2)\frac{B-A}{1-B^2}+(1-\gamma)\|z_0\|^2\frac{B^2-AB}{1-B^2}\\
&\quad+[(\gamma+(1-\gamma)|z_1|^2)|z_1|+|z_1|^2]\frac{\mathrm{e}^{\frac{B|z_1|^k}{k}}}{|z_1|}\sum_{j=2}^{\infty}\mathrm{e}^{\frac{B|z_1|^k}{k}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}}\\
&=\frac{B}{1-B^2}-\frac{B}{B+1}(1-\gamma)\|z_0\|^2\\
&\quad+[\gamma+(1-\gamma)|z_1|^2+|z_1|]\mathrm{e}^{\frac{B|z_1|^k}{k}}\sum_{j=2}^{\infty}\mathrm{e}^{\frac{B|z_1|^k}{k}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}}\\
&=\frac{B}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B}{B+1}(1-\gamma)\\
&\quad-[\gamma+(1-\gamma)|z_1|^2+|z_1|]\mathrm{e}^{\frac{B|z_1|^k}{k}}\sum_{j=2}^{\infty}\mathrm{e}^{\frac{B|z_1|^k}{k}\gamma j}(1-|z_1|^2)^{\frac{j}{2}-1}(j-1)\|P_j\|\Big\}\\
&\le\frac{B}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B}{B+1}(1-\gamma)-\sum_{j=2}^{\infty}2\mathrm{e}^{(1+\gamma j)\frac{B}{k}}(j-1)\|P_j\|\Big\}\\
&\le\frac{B}{1-B^2},
\end{aligned}$$

where

$$\sum_{j=2}^{\infty}\mathrm{e}^{(1+\gamma j)\frac{B}{k}}(j-1)\|P_j\|\le\frac{B(1-\gamma)}{2(B+1)}.$$

Then $F(z)\in S^*_\Omega(A,B)$ by Definition 1.1.

(2) In the case of $0<A<1$, it is obvious that $1-A|z_1|^k\ge 1-|z_1|$. Then

$$(1-A|z_1|^k)^{\frac{A-B}{Ak}}\le(1-|z_1|)^{\frac{A-B}{Ak}}$$

for $A<B$. Therefore

$$\begin{aligned}
&\Big|\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2}-\frac{1-AB}{1-B^2}\Big|\\
&\le (\gamma+(1-\gamma)|z_1|^2)|q(z_1)|+(1-\gamma)\|z_0\|^2\frac{B^2-AB}{1-B^2}\\
&\quad+\Big[(\gamma+(1-\gamma)|z_1|^2)\Big|q(z_1)+\frac{B^2-AB}{1-B^2}\Big|+|z_1|^2\Big]\frac{1}{|f(z_1)|}\sum_{j=2}^{\infty}\Big|\frac{f(z_1)}{z_1}\Big|^{\gamma j}(j-1)\|P_j\|\|z_0\|^j\\
&< (\gamma+(1-\gamma)|z_1|^2)\frac{B-A}{1-B^2}+(1-\gamma)\|z_0\|^2\frac{B^2-AB}{1-B^2}
\end{aligned}$$

$$
\begin{aligned}
&+[(\gamma+(1-\gamma)|z_1|^2)|z_1|+|z_1|^2]\frac{(1-A|z_1|^k)^{\frac{A-B}{Ak}}}{|z_1|}\\
&\cdot\sum_{j=2}^{\infty}(1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}}\\
=&\frac{B-A}{1-B^2}-\frac{B-A}{B+1}(1-\gamma)\|z_0\|^2+[\gamma+(1-\gamma)|z_1|^2+|z_1|]\\
&\cdot(1-A|z_1|^k)^{\frac{A-B}{Ak}}\sum_{j=2}^{\infty}(1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}}\\
=&\frac{B-A}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma)-[\gamma+(1-\gamma)|z_1|^2+|z_1|](1-A|z_1|^k)^{\frac{A-B}{Ak}}\\
&\cdot\sum_{j=2}^{\infty}(1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(1+|z_1|)^{\frac{j}{2}-1}(1-|z_1|)^{\frac{j}{2}-1}(j-1)\|P_j\|\Big\}\\
\le&\frac{B-A}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma)-[\gamma+(1-\gamma)|z_1|^2+|z_1|]\\
&\cdot\sum_{j=2}^{\infty}(1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(1+|z_1|)^{\frac{j}{2}-1}(1-|z_1|)^{\frac{j}{2}-1-\frac{B-A}{Ak}}(j-1)\|P_j\|\Big\}\\
=&\frac{B-A}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma)-[\gamma+(1-\gamma)|z_1|^2+|z_1|]\\
&\cdot\sum_{j=2}^{\infty}(1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(1-|z_1|^2)^{\frac{j}{2}-1-\frac{B-A}{Ak}}(1+|z_1|)^{\frac{B-A}{Ak}}(j-1)\|P_j\|\Big\}\\
\le&\frac{B-A}{1-B^2}-(1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma)-2\sum_{j=2}^{\infty}(1+A)^{\frac{B-A}{Ak}\gamma j}2^{\frac{B-A}{Ak}}(j-1)\|P_j\|\Big\}\\
\le&\frac{B-A}{1-B^2},
\end{aligned}
$$

where $P_j=0$ $\big(j<2\big(1+\frac{B-A}{Ak}\big)\big)$, $\frac{j}{2}-1-\frac{B-A}{Ak}\ge 0$ $\big(j\ge 2\big(1+\frac{B-A}{Ak}\big)\big)$ and

$$\sum_{j=2}^{\infty}(1+A)^{\frac{B-A}{Ak}\gamma j}2^{\frac{B-A}{Ak}}(j-1)\|P_j\|\le\frac{(B-A)(1-\gamma)}{2(B+1)}.$$

Hence, $F(z)\in S^*_{\Omega}(A,B)$ by Definition 1.1.

(3) In the case of $-1\le A<0$, it is obvious that

$$1-A|z_1|^k\le 1+|z_1|,\quad 1+A|z_1|^k\ge 1-|z_1|.$$

Furthermore, $\frac{A-B}{Ak}>0$ for $A<B$. Then

$$
\begin{aligned}
(1-A|z_1|^k)^{\frac{A-B}{Ak}}&\le(1+|z_1|)^{\frac{A-B}{Ak}},\\
(1+A|z_1|^k)^{\frac{B-A}{Ak}}&\le(1-|z_1|)^{\frac{B-A}{Ak}}.
\end{aligned}
$$

Similar to the case of $0<A<1$, we obtain

$$\Big|\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2}-\frac{1-AB}{1-B^2}\Big|$$

$$
\begin{aligned}
&< \frac{B-A}{1-B^2} - \frac{B-A}{B+1}(1-\gamma)\|z_0\|^2 + [\gamma + (1-\gamma)|z_1|^2 + |z_1|] \\
&\quad \cdot (1-A|z_1|^k)^{\frac{A-B}{Ak}} \sum_{j=2}^{\infty} (1+A|z_1|^k)^{\frac{B-A}{Ak}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}} \\
&\le \frac{B-A}{1-B^2} - \frac{B-A}{B+1}(1-\gamma)\|z_0\|^2 + [\gamma + (1-\gamma)|z_1|^2 + |z_1|] \\
&\quad \cdot (1+|z_1|)^{\frac{A-B}{Ak}} \sum_{j=2}^{\infty} (1-|z_1|)^{\frac{B-A}{Ak}\gamma j}(j-1)\|P_j\|(1-|z_1|^2)^{\frac{j}{2}} \\
&= \frac{B-A}{1-B^2} - (1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma) \\
&\quad - [\gamma + (1-\gamma)|z_1|^2 + |z_1|] \sum_{j=2}^{\infty} (1+|z_1|)^{\frac{A-B}{Ak}(1+\gamma j)}(1-|z_1|^2)^{\frac{B-A}{Ak}\gamma j + \frac{j}{2} - 1}(j-1)\|P_j\|\Big\} \\
&\le \frac{B-A}{1-B^2} - (1-|z_1|^2)\Big\{\frac{B-A}{B+1}(1-\gamma) - 2\sum_{j=2}^{\infty} 2^{\frac{A-B}{Ak}(1+\gamma j)}(j-1)\|P_j\|\Big\} \\
&\le \frac{B-A}{1-B^2},
\end{aligned}
$$

where $P_j = 0$ $\big(j < \frac{2Ak}{Ak+2(B-A)\gamma}\big)$, $\frac{B-A}{Ak}\gamma j + \frac{j}{2} - 1 \ge 0$ $\big(j \ge \frac{2Ak}{Ak+2(B-A)\gamma}, \gamma < \min\big\{\frac{-Ak}{2(B-A)}, 1\big\}\big)$ and

$$
\sum_{j=2}^{\infty} 2^{\frac{A-B}{Ak}(1+\gamma j)}(j-1)\|P_j\| \le \frac{(B-A)(1-\gamma)}{2(B+1)}.
$$

Hence, $F(z) \in S^*_\Omega(A,B)$ by Definition 1.1.

Setting $A = -(B+2\alpha) = -1$ and $A = -B = -\alpha$ in Theorem 2.1, respectively, we can get the following results.

Corollary 2.1 *Let $f(z_1)$ be a starlike function of order α on D with $f'(z_1) \ne 0$, $\alpha \in \big[\frac{1}{2}, 1\big)$, and let $f(z_1)$ be k-fold symmetric. Let $F(z)$ be the mapping defined in Lemma* 2.1 *with $\gamma \in [0,1)$, and let $P_j = 0$ for $j < \frac{2k}{k-4(1-\alpha)\gamma}$, and*

$$
\sum_{j=2}^{\infty} 2^{\frac{2}{k}(1-\alpha)(1+\gamma j)}(j-1)\|P_j\| \le \frac{1-\gamma}{2}.
$$

Then $F(z)$ is a starlike mapping of order α on B^n.

Corollary 2.2 *Let $f(z_1)$ be a strong starlike function of order α on D with $f'(z_1) \ne 0$, $\alpha \in \big(0, \frac{1}{3}\big]$, and let $f(z_1)$ be k-fold symmetric. Let $F(z)$ be the mapping defined in Lemma* 2.1 *with $\gamma \in \big[0, \min\big\{\frac{k}{4}, 1\big\}\big)$, and let $P_j = 0$ for $j < \frac{2k}{k-4\gamma}$, and*

$$
\sum_{j=2}^{\infty} 2^{\frac{2}{k}(1+\gamma j)}(j-1)\|P_j\| \le \frac{\alpha(1-\gamma)}{1+\alpha}.
$$

Then $F(z)$ is a strong starlike mapping of order α on B^n.

Setting $\gamma = \frac{1}{m}$, $j = m$, we can get the following result.

Corollary 2.3 *Let $f(z_1) \in S^*_\Omega(A,B)$ be a k-fold symmetric function with $f'(z_1) \neq 0$, $-1 \leq A < B \leq \frac{A+1}{2}$, and k is a positive integer. Let*

$$F(z) = \Big(f(z_1) + \frac{f(z_1)}{z_1}P(z_0), \Big[\frac{f(z_1)}{z_1}\Big]^{\frac{1}{m}} z_0\Big)',$$

where $z = (z_1, z_0) \in B^n$, $P(z_0)$ is a homogeneous polynomial of degree m in $\mathbb{C}^{n-1}$ with $m \in \mathbb{Z}^+$, $m \geq 2$ and $\big[\frac{f(z_1)}{z_1}\big]^{\frac{1}{m}}|_{z_1=0} = 1$.

(1) *If $A = 0$, then $F(z) \in S^*_\Omega(A,B)$ provided that*

$$\|P\| \leq \frac{\mathrm{e}^{-\frac{2B}{k}} B}{2m(B+1)}.$$

(2) *If $0 < A < 1$, then $F(z) \in S^*_\Omega(A,B)$ provided that $m \geq 2(1 + \frac{B-A}{Ak})$ and*

$$\|P\| \leq \frac{[2(1+A)]^{\frac{A-B}{Ak}}(B-A)}{2m(B+1)}.$$

(3) *If $-1 \leq A < 0$, then $F(z) \in S^*_\Omega(A,B)$ provided that $m \geq 2 + \frac{2(A-B)}{Ak}$ and*

$$\|P\| \leq \frac{2^{\frac{2(B-A)}{Ak}}(B-A)}{2m(B+1)}.$$

Remark 2.1 Setting $k = 1$ in Theorem 2.1 and Corollaries 2.1–2.3, we have the corresponding simplified results.

3 The Invariance of Strong and Almost Spirallike Mappings of Type β and Order α

We begin with some helpful lemmas.

Lemma 3.1 *Let $f(z_1)$ be a strong and almost spirallike function of type β and order α on D with $f'(z_1) \neq 0$, $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $c \in (0, \frac{1}{3}]$. Let*

$$q(z_1) = \frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1+c^2}{1-c^2}.$$

Then

$$|q(z_1)| < \frac{2c}{1-c^2}, \quad \Big|q(z_1) + \frac{2c^2}{1-c^2}\Big| \leq |z_1|.$$

Proof Since $f(z_1)$ is a strong and almost spirallike function of type β and order α on D, by Definition 1.2, we have

$$\Big|\frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1+c^2}{1-c^2}\Big| < \frac{2c}{1-c^2}.$$

That is $|q(z_1)| < \frac{2c}{1-c^2}$. Let

$$g(z_1) = \begin{cases} \dfrac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big[1 - \dfrac{f(z_1)}{z_1 f'(z_1)}\Big], & z_1 \neq 0, \\ 0, & z_1 = 0. \end{cases}$$

Then $g(z_1) = \frac{-2c^2}{1-c^2} - q(z_1)$ for $z_1 \neq 0$. It follows that $q(z_1) = -g(z_1) - \frac{2c^2}{1-c^2}$. Therefore

$$|q(z_1)| = \Big|g(z_1) + \frac{2c^2}{1-c^2}\Big| < \frac{2c}{1-c^2}.$$

Then

$$|g(z_1)| < \frac{2c^2+2c}{1-c^2} = \frac{2c}{1-c} < 1$$

for $c \in (0, \frac{1}{3}]$. Observing that $g(z_1)$ is holomorphic on D and $g(0) = 0$, we have $|g(z_1)| \leq |z_1|$ by the Schwarz lemma. Thus

$$\Big|q(z_1) + \frac{2c^2}{1-c^2}\Big| \leq |z_1|.$$

Lemma 3.2 (see [6]) *Let $f(z)$ be a normalized univalent holomorphic function on D. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

The following are our main results.

Theorem 3.1 *Let $f(z_1)$ be a strong and almost spirallike function of type β and order α on D with $f'(z_1) \neq 0$, $\alpha \in [0,1)$, $\beta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ and $c \in \big(0, \frac{1}{3}\big]$. Let $F(z)$ be the mapping defined in Lemma 2.1 with $\gamma < \frac{1}{4}$, $P_j = 0$ $\big(j < \frac{2}{1-4\gamma}\big)$ and*

$$\sum_{j=2}^{\infty} 2^{2\gamma j+1}(j-1)\|P_j\| \leq \frac{c(1-\gamma)(1-\alpha)\cos\beta}{(1+c)(1+(1-\alpha)\cos\beta)}.$$

Then $F(z)$ is a strong and almost spirallike mapping of type β and order α on B^n.

Proof By Definition 1.2, we only need to prove

$$\Big|\frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2} - \frac{1+c^2}{1-c^2}\Big| < \frac{2c}{1-c^2}. \tag{3.1}$$

It is obvious that (3.1) holds for $z_0 = 0$, since $F(z)$ is holomorphic for $z \in \overline{B^n}$ $(z_0 \neq 0)$. From the maximum modulus principle of holomorphic functions, we only need to prove that (3.1) holds for $z \in \partial B^n$ $(z_0 \neq 0)$. In the following, let $\|z\|^2 = |z_1|^2 + \|z_0\|^2 = 1$.

Since $f(z_1)$ is a strong and almost spirallike function of type β and order α on D, by Definition 1.2, we have

$$\Big|\frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1+c^2}{1-c^2}\Big| < \frac{2c}{1-c^2}.$$

Let

$$q(z_1) = \frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1 f'(z_1)} - \frac{1+c^2}{1-c^2}.$$

It follows that

$$\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1 f'(z_1)} = q(z_1) - \frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1+c^2}{1-c^2}.$$

By Lemma 2.1, we obtain

$$\frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2} - \frac{1+c^2}{1-c^2}$$

$$= \frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big\{\overline{z_1}\frac{f(z_1)}{f'(z_1)} + \frac{\overline{z_1}}{f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)$$

$$+ \Big[1-\gamma+\gamma\frac{f(z_1)}{z_1 f'(z_1)} - \gamma\frac{z_1 f'(z_1) - f(z_1)}{z_1 f(z_1) f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\Big]\|z_0\|^2\Big\} - \frac{1+c^2}{1-c^2}$$

$$= \frac{-\alpha + \mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big\{(|z_1|^2+\gamma\|z_0\|^2)\frac{f(z_1)}{z_1 f'(z_1)} + \frac{|z_1|^2}{z_1 f'(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)$$

$$+ (1-\gamma)\|z_0\|^2 + \Big(\frac{\gamma\|z_0\|^2}{z_1 f'(z_1)} - \frac{\gamma\|z_0\|^2}{f(z_1)}\Big)\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)\Big\} - \frac{1+c^2}{1-c^2}$$

$$= (|z_1|^2+\gamma\|z_0\|^2)\Big[q(z_1) - \frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha} + \frac{1+c^2}{1-c^2}\Big] + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}(1-\gamma)\|z_0\|^2 + \frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}$$

$$- \frac{1+c^2}{1-c^2} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big[\frac{|z_1|^2+\gamma\|z_0\|^2}{z_1 f'(z_1)} - \frac{\gamma\|z_0\|^2}{f(z_1)}\Big]\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)$$

$$= (|z_1|^2+\gamma\|z_0\|^2)q(z_1) - \frac{2c^2}{1-c^2}(1-\gamma)\|z_0\|^2$$

$$+ \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big[(|z_1|^2+\gamma\|z_0\|^2)\Big(\frac{f(z_1)}{z_1 f'(z_1)} - 1\Big) + |z_1|^2\Big]\frac{1}{f(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0)$$

$$= (|z_1|^2+\gamma\|z_0\|^2)q(z_1) - \frac{2c^2}{1-c^2}(1-\gamma)\|z_0\|^2$$

$$+ \Big[(|z_1|^2+\gamma\|z_0\|^2)\Big(q(z_1) + \frac{2c^2}{1-c^2}\Big) + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}|z_1|^2\Big]\frac{1}{f(z_1)}\sum_{j=2}^{\infty}\Big[\frac{f(z_1)}{z_1}\Big]^{\gamma j}(1-j)P_j(z_0).$$

From Lemma 3.1, we have

$$|q(z_1)| < \frac{2c}{1-c^2}, \quad \Big|q(z_1) + \frac{2c^2}{1-c^2}\Big| \le |z_1|.$$

Furthermore, $P_j = 0$ $\big(j < \frac{2}{1-4\gamma}\big)$ and $\frac{j}{2} - 2\gamma j - 1 \ge 0$ $\big(j \ge \frac{2}{1-4\gamma}\big)$. From the condition

$$\sum_{j=2}^{\infty} 2^{2\gamma j+1}(j-1)\|P_j\| \le \frac{c(1-\gamma)(1-\alpha)\cos\beta}{(1+c)(1+(1-\alpha)\cos\beta)}$$

and Lemma 3.2, we get

$$\Big|\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha} + \frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{\overline{z}'(DF(z))^{-1}F(z)}{\|z\|^2} - \frac{1+c^2}{1-c^2}\Big|$$

$$\le (|z_1|^2+\gamma\|z_0\|^2)|q(z_1)| + \frac{2c^2}{1-c^2}(1-\gamma)\|z_0\|^2$$

$$+ \Big[(|z_1|^2+\gamma\|z_0\|^2)\Big|q(z_1) + \frac{2c^2}{1-c^2}\Big| + \frac{|1-\mathrm{i}\tan\beta|}{1-\alpha}|z_1|^2\Big]\frac{1}{|f(z_1)|}$$

$$\cdot\sum_{j=2}^{\infty}\Big|\frac{f(z_1)}{z_1}\Big|^{\gamma j}(j-1)\|P_j\|\|z_0\|^j$$

$$< (\gamma + (1-\gamma)|z_1|^2)\frac{2c}{1-c^2} + \frac{2c^2}{1-c^2}(1-\gamma)(1-|z_1|^2)$$

$$
\begin{aligned}
&\quad + \Big[\gamma + (1-\gamma)|z_1|^2 + \frac{|z_1|}{(1-\alpha)\cos\beta}\Big] \sum_{j=2}^{\infty} (1+|z_1|)^{2+\frac{j}{2}} (1-|z_1|)^{\frac{j}{2}-2\gamma j} (j-1)\|P_j\| \\
&= \frac{2c}{1-c^2} + (\gamma - 1 + (1-\gamma)|z_1|^2)\frac{2c}{1-c^2} + \frac{2c^2}{1-c^2}(1-\gamma)(1-|z_1|^2) \\
&\quad + \Big[\gamma + (1-\gamma)|z_1|^2 + \frac{|z_1|}{(1-\alpha)\cos\beta}\Big] \sum_{j=2}^{\infty} (1+|z_1|)^{2+\frac{j}{2}} (1-|z_1|)^{\frac{j}{2}-2\gamma j} (j-1)\|P_j\| \\
&= \frac{2c}{1-c^2} - \frac{2c}{1+c}(1-\gamma)(1-|z_1|^2) \\
&\quad + \Big[\gamma + (1-\gamma)|z_1|^2 + \frac{|z_1|}{(1-\alpha)\cos\beta}\Big] \sum_{j=2}^{\infty} (1+|z_1|)^{2+\frac{j}{2}} (1-|z_1|)^{\frac{j}{2}-2\gamma j} (j-1)\|P_j\| \\
&= \frac{2c}{1-c^2} - (1-|z_1|^2)\Big\{\frac{2c}{1+c}(1-\gamma) \\
&\quad - \Big[\gamma + (1-\gamma)|z_1|^2 + \frac{|z_1|}{(1-\alpha)\cos\beta}\Big] \sum_{j=2}^{\infty} (1+|z_1|)^{2\gamma j+2} (1-|z_1|^2)^{\frac{j}{2}-2\gamma j-1} (j-1)\|P_j\|\Big\} \\
&\le \frac{2c}{1-c^2} - (1-|z_1|^2)\Big\{\frac{2c}{1+c}(1-\gamma) - \Big[1 + \frac{1}{(1-\alpha)\cos\beta}\Big] \sum_{j=2}^{\infty} 2^{2(1+\gamma j)} (j-1)\|P_j\|\Big\} \\
&\le \frac{2c}{1-c^2}.
\end{aligned}
$$

Hence $F(z)$ is a strong and almost spirallike mapping of type β and order α on B^n.

Setting $\gamma = \frac{1}{m}$ and $j = m$ in Theorem 3.1, we get the following result.

Corollary 3.1 *Let $f(z_1)$ be a strong and almost spirallike function of type β and order α on D with $f'(z_1) \neq 0$, $\alpha \in [0,1)$, $\beta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ and $c \in \big(0, \frac{1}{3}\big]$, and let $P(z_0)$ be a homogeneous polynomial of degree m in $\mathbb{C}^{n-1}$, where $m \in \mathbb{Z}^+$, $m \ge 6$. Let*

$$F(z) = \Big(f(z_1) + \frac{f(z_1)}{z_1}P(z_0), \Big[\frac{f(z_1)}{z_1}\Big]^{\frac{1}{m}} z_0\Big)',$$

where $z = (z_1, z_0) \in B^n$, and the branch of the power function is chosen such that $\big[\frac{f(z_1)}{z_1}\big]^{\frac{1}{m}}|_{z_1=0} = 1$. If

$$\|P\| \le \frac{c(1-\alpha)\cos\beta}{8m(1+c)(1+(1-\alpha)\cos\beta)},$$

then $F(z)$ is a strong and almost spirallike mapping of type β and order α on B^n.

Remark 3.1 Setting $\alpha = 0$, $\beta = 0$ in Theorem 3.1 or Corollary 3.1 respectively, we get the corresponding results for strong spirallike mappings of type β, and strong and almost starlike mappings of order α on B^n.

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