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An Initial-Boundary Value Problem for Parabolic Monge-Ampère Equation in Mathematical Finance

Ming LI[1] Changyu REN[2]

Abstract This paper deals with some parabolic Monge-Ampère equation raised from mathematical finance: $V_sV_{yy}+ryV_yV_{yy}-\theta V_y^2=0$ $(V_{yy}<0)$. The existence and uniqueness of smooth solution to its initial-boundary value problem with some requirement is obtained.

Keywords Initial-boundary value problem, Parabolic Monge-Ampère equation, Strong convex monotonic function
2000 MR Subject Classification 17B40, 17B50

1 Introduction

In [1], an optimal investment problem was proposed on the time interval $[0,T]$. They used variables r,b,σ to describe the financial market. The attitude of the investors at the terminal time T for risk and interest can be describe by the utility function $g(y)$. The goal is to search the optimal portfolio, and to achieve the maximum profit for investors. In [1], the following model was deduced to solve the above problem:

$$\begin{cases} V_sV_{yy}+ryV_yV_{yy}-\theta V_y^2=0, & V_{yy}<0,\ (s,y)\in[0,T)\times\mathbb{R},\\ V(T,y)=g(y), \quad g'(y)\geq 0, & y\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $V=V(s,y)$ is the undetermined function and r,b,σ are given constants, satisfying $r\geq 0,\ \sigma>0,\ b-r>0,\ \theta=\frac{b-r}{\sigma}$. Usually, we can assume that $g(y)=1-\mathrm{e}^{-\lambda y}$, where λ is some positive constant. In [2–3], the authors studied problem (1.1) and got the existence and uniqueness of smooth solution.

Note that the variable y in (1.1) means "the initial capital for investors". Then, it is obvious that, for $y<0$, the investors can not invest anything. Meanwhile, "the initial capital" has to be finite. For the sake of using in the real world, we only consider the initial-boundary value problem in the domain $[0,T)\times(0,X)$. In this paper, we discuss the classical solution for the following problem:

$$\begin{cases} -(u_t-rxu_x)u_{xx}=\theta u_x^2, \quad u_{xx}>0, & (x,t)\in(0,X)\times(0,T],\\ u(x,0)=g(x), \quad g'(x)\leq 0, & x\in[0,X],\\ u(0,t)=u_0(t), \quad u(X,t)=u_X(t), & t\in[0,T]. \end{cases} \tag{1.2}$$

Here, we have used transformation $(s,y)=(T-t,x),\ V(s,y)=-u(x,t)$ in (1.1).

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[1]Department of Mathematics, Fudan University, Shanghai 200433, China. E-mail:liming12@fudan.edu.cn
[2]Department of Mathematics, Jilin University, Changchun 130012, China. E-mail: rency@jlu.edu.cn

By the way, from the viewpoint of partial differential equations, these problems are still valuable to discuss. In the famous work of Caffarelli- Nirenberg-Spruck [4] for Monge-Ampère equations, they proved the existence of the strictly convex solution to the following problem:

$$\begin{cases} \det D_x^2 u = \psi(x, u, Du) & \text{for } x \in (\text{bounded smooth convex domain})\, \Omega \subset \mathbb{R}^n, \\ u = \phi(x) & \text{for } x \in \partial\Omega \end{cases} \tag{1.3}$$

with the requirement of strictly convexity and the increasing condition:

$$0 < \psi(x, z, p) \leq C(1 + |p|^2)^{\frac{n}{2}} \quad \text{for } x \in \overline{\Omega},\ z \leq \max\phi. \tag{1.4}$$

As Krylov [5] pointed out that the corresponding parabolic problem matching (1.3) should be

$$\begin{cases} -u_t \det D_x^2 u = \widetilde{\psi}(x, t, u, Du) & \text{for } (x,t) \in Q = \Omega \times (0, T], \\ u(x,t) = \widetilde{\phi}(x,t) & \text{for } (x,t) \in \partial_p Q. \end{cases} \tag{1.5}$$

Since we need to use the method of [4], we also require the condition appearing in (1.4) (also see [7]):

$$0 < \widetilde{\psi}(x, t, z, p) \leq C(1 + |p|^2)^{\frac{n}{2}} \quad \text{for } (x,t) \in \overline{Q},\ z \leq \max\phi. \tag{1.6}$$

It is obvious that, for $r = 0$, (1.2) becomes the type of (1.5), where $n = 1$, $\det D_x^2 u = u_{xx}$. But the requirement (1.6) can not be satisfied for the right-hand side of the problem (1.2) which may even be zero.

To overcome the difficulties, we observe that we can construct a sub-solution if the data of our problem satisfy appropriate conditions. Then, we derive the needed a prior estimates and use the degree theory to obtain the existence of the smooth solutions.

2 Preliminaries

For the equation (1.2) with a given function in the right-hand side, namely, the initial-boundary value problem of the parabolic Monge-Ampère equation is

$$\begin{cases} -(u_t - rxu_x)u_{xx} = f(x,t), & (x,t) \in Q, \\ u(x,0) = g(x), \quad g'(x) \leq 0, & x \in [0, X], \\ u(0,t) = u_0(t), \quad u(X,t) = u_X(t), & t \in [0, T]. \end{cases} \tag{2.1}$$

In this section we use Q to denote $(0, X) \times [0, T]$.

We assume that

(h_1) There exist some constants $\alpha \in (0,1)$, $\mu > 0$, such that

$$f(x,t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q}), \quad f(x,t) > \mu$$

in Q.

(h_2) $g(x) \in C^{4+\alpha}([0, X])$ satisfies $g''(x) \geq \mu$ in $[0, X]$, and $u_0(t), u_X(t) \in C^{2+\alpha}([0, T])$ satisfy $-u_0'(t) \geq \mu$, $-u_X'(t) - ru_0(t) + ru_X(t) \geq \mu$ in $[0, T]$.

(h_3) The data of problem (2.1) satisfy the compatibility conditions up to the second order.

The compatibility condition is necessary for parabolic equations, which is one of the main differences to the elliptic type. The key point is that the initial data of the problem satisfy the smooth conditions at corner points. For instance, the 0th-order compatibility condition is

$$g(0) = u_0(0), \quad g(X) = u_X(0).$$

The 1st order compatibility condition is

$$-u_0'(0)g''(0) = f(0,0), \quad -(u_X'(0) - rXg'(X))g''(X) = f(X,0).$$

Using the method in [6], by some modification, we can get the following result.

Theorem 2.1 *If conditions* (h_1)–(h_3) *hold, then the problem* (2.1) *has a unique solution* $u(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$ *satisfying* $-(u_t - rxu_x) > 0,\ u_{xx} > 0$ *in* $\overline{Q}$.

3 Main Result

We note that (1.2) is invariant with the transformation $x \mapsto Xx$. Hence, without loss of generality, we can assume that $X = 1$, and in the following, we use Q to denote $(0,1) \times (0,T]$. Then, (1.2) becomes

$$\begin{cases} -(u_t - rxu_x)u_{xx} = \theta u_x^2, \quad u_{xx} > 0, & (x,t) \in Q, \\ u(x,0) = g(x), \quad g'(x) \le 0, & x \in [0,1], \\ u(0,t) = u_0(t), \quad u(1,t) = u_1(t), & t \in [0,T]. \end{cases} \tag{3.1}$$

Now, we only consider the non-degenerate case of Equation (3.1). In our case, the solution should be called a strong convex monotonic function. Here, a function $u(x,t)$ is called a strong convex monotonic function, if $u(x,t) \in C^{2,1}(\overline{Q})$ and

$$u_{xx}(x,t) > 0, \quad u_x(x,t) < 0, \quad (u_t(x,t) - rxu_x(x,t)) < 0, \quad \forall (x,t) \in \overline{Q}.$$

Since we will establish the existence of problem (3.1) in $C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$, we find the following necessary conditions:

(H_1) There exist some constants $\alpha \in (0,1),\ \mu > 0,\ u_0(t), u_1(t) \in C^{2+\alpha}([0,T])$, and $g(x) \in C^{4+\alpha}([0,1])$ satisfying $-u_0'(t) \ge \mu,\ -u_1'(t) - ru_0(t) + ru_1(t) \ge \mu$ in $[0,T]$, and $g'(x) < 0,\ g''(x) \ge \mu$ in $[0,X]$.

(H_2) The data of problem (3.1) satisfy compatibility conditions up to the second order.

(H_3) There exists some constant $\nu > 0$ and two strong convex monotonic functions $u^0(x,t)$, $\underline{u}(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$, satisfying

$$\begin{aligned} &-(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx} > -(u_t^0 - rxu_x^0)u_{xx}^0, && \forall (x,t) \in \overline{Q}, \\ &-(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx} > \theta \underline{u}_x^2, && \forall (x,t) \in \overline{Q}, \\ &\underline{u}_x(x,t) \le -\nu < 0, && \forall (x,t) \in \overline{Q}. \end{aligned}$$

The following two lemmas give the sufficient conditions for (H_3) to hold.

Lemma 3.1 *If* (H_1)–(H_2) *and the following condition holds:*

$$\begin{aligned} &-[(1-x)u_0'(t) + xu_1'(t)]g''(x) + rx[g'(x) - u_0(t) + u_0(0) + u_1(t) - u_1(0)]g''(x) \\ &> \theta\{g'(x) - [u_0(t) - u_0(0)] + [u_1(t) - u_1(0)]\}^2 \end{aligned} \tag{3.2}$$

and for some constant $c_1 > 0$, *we have*

$$g'(x) - [u_0(t) - u_0(0)] + [u_1(t) - u_1(0)] \le -c_1, \quad \forall (x,t) \in \overline{Q}. \tag{3.3}$$

Then, we have some convex monotonic function $\underline{u}(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$ *satisfying*

$$\begin{cases} -(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx} > \theta \underline{u}_x^2 > 0, & \forall (x,t) \in Q, \\ \underline{u}(x,0) = g(x), \quad g'(x) \le 0, & x \in [0,1], \\ \underline{u}(0,t) = u_0(t), \quad \underline{u}(1,t) = u_1(t), & t \in [0,T]. \end{cases} \tag{3.4}$$

Proof We let

$$\underline{u}(x,t) = g(x) + (1-x)[u_0(t) - u_0(0)] + x[u_1(t) - u_1(0)].$$

Hence, we get our result.

Lemma 3.2 *If the condition in the above lemma is satisfied, then there is some strong convex monotonic function* $u^0(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$ *satisfying*

$$\begin{cases} -(u_t^0 - rxu_x^0)u_{xx}^0 < -(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx}, & \forall (x,t) \in Q, \\ u^0(x,0) = g(x), \quad g'(x) \le 0, & x \in [0,1], \\ u^0(0,t) = u_0(t), \quad u^0(1,t) = u_1(t), & t \in [0,T], \end{cases} \tag{3.5}$$

and $|u^0|_{C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})} \le K$. *Here the constant* K *only depends on the data of our problem* (3.1).

Proof Let $A(x,t) = -t^2\eta(t) - \frac{\theta c_1^2}{4}(1-\eta(t))$, and

$$\eta(t) = \begin{cases} 1, & t \in \left[0, \frac{\delta}{2}\right], \\ 0, & t \in [\delta, T], \end{cases} \quad \delta \le \frac{\sqrt{\theta} c_1^2}{2}.$$

Then, it is obvious that $A(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$, $-\dfrac{\theta c_1^2}{2} \le A(x,t) \le 0$, and

$$A(0,0) = A(1,0) = A_t(0,0) = A_t(1,0) = A_x(0,0) = A_x(1,0) = A_{xx}(0,0) = A_{xx}(1,0) = 0.$$

Let $f^0(x,t) = -(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx} + A(x,t)$. We can check $f^0(x,t) > 0$ in $\overline{Q}$ and

$$f^0(x,t) < -(\underline{u}_t - rx\underline{u}_x)\underline{u}_{xx}, \quad \forall (x,t) \in Q = (0,1) \times (0,T].$$

On the other hand, f^0 also satisfies the compatibility conditions up to the second order.

Using Theorem 2.1, we have a unique strong convex monotonic function

$$u^0(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$$

satisfying (3.5) and

$$|u^0|_{C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})} \le K.$$

Since $\underline{u}$ is the sub-solution for u^0, $u_x^0(1,t) \le \underline{u}_x(1,t)$. By $u_{xx}^0 > 0$, we have $u_x^0(x,t) < 0$.

By the proof of the above two lemmas, we have that the sufficient condition for (H_3) is that Equation (3.1) has a strong convex monotinic sub-solution or that the following sub-solution

$$\underline{u}(x,t) = g(x) + (1-x)[u_0(t) - u_0(0)] + x[u_1(t) - u_1(0)]$$

satisfies $\underline{u}_x(x,t) \le -c_1$, $\forall (x,t) \in \overline{Q}$. In what follows, we use the degree theory to prove that the problem (3.1) has a strong convex monotonic solution. Let us consider the Banach space

$$C_0^{4,2}(\overline{Q}) = \{w(x,t) \in C^{4,2}(\overline{Q}); w(x,t)|_{\partial_p Q} = 0\}$$

and its open subset,

$$S = \{v(x,t) \in C_0^{4,2}(\overline{Q}) \mid v > 0 \text{ in } Q, v_x|_{x=0} > 0, v_x|_{x=1} < 0, (v + \underline{u}) \text{ is a strong cover solution}\}.$$

For a sufficiently large constant $\mathbb{R}$, denote

$$S_{\mathbb{R}} = S \cap \{v(x,t) \in C_0^{4,2}(\overline{Q}); |v|_{c_0^{4,2}(\overline{Q})} < \mathbb{R}\}.$$

We need the following lemma from [8].

Lemma 3.3 *Denote K_ρ to be an interval in $\mathbb{R}^1$ with radius ρ. Let $w(x,t)$ in $\overline{Q}$ satisfy the Hölder condition for t with an exponential α and a Hölder constant μ_1. Also assume that the derivative w_x exists. It means that for any $t \in [0,T]$, $w(x,t)$ is Hölder continuous with respect to x. More explicitly, we have*

$$\max_{K_\rho, 0\le t\le T} \mathrm{osc}_x\{w_x(x,t), K_\rho \cap (0,X)\} \le \mu_2 \rho^\beta.$$

Then, the derivative w_x in Q satisfies the Hölder condition for t with an exponential $\delta = \frac{\alpha\beta}{(1+\beta)}$. The Hölder constant μ only depends on $\alpha, \beta, \mu_1, \mu_2$.

Lemma 3.4 *If* (H_1)–(H_3) *hold, then, for any $v \in S$, $\tau \in [0,1]$, the following problem*

$$\begin{cases} -(u_t^\tau - rxu_x^\tau)u_{xx}^\tau = f^\tau(x,t) = \tau\theta(\underline{u}+v)_x^2 + (1-\tau)f^0(x,t), & (x,t) \in Q, \\ u^\tau(x,0) = g(x), \quad g'(x) \le 0, & x \in [0,1], \\ u^\tau(0,t) = u_0(t), \quad u(1,t) = u_1(t), & t \in [0,T] \end{cases} \tag{3.6}$$

has a unique solution $u^\tau(x,t) \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$, and satisfies $|u^\tau|_{C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})} \le K_0$, where K_0 only depends on the data of the problem but not on τ.

Proof By Lemma 3.3, we know that the right-hand-side function $f^\tau(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$. Then, it is easy to check that $f^\tau(x,t) > 0$ in $\overline{Q}$, and satisfies compatibility conditions up to the second order. Hence, by Theorem 2.1, we have the conclusion.

Lemma 3.5 *Suppose that* (H_1)–(H_3) *hold. For any $v \in S$, the problem* (3.6) *has a unique solution $u^\tau \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})$. Denote $v^\tau = u^\tau - \underline{u}$. Define some map $T^\tau v = v^\tau$. Then, we get that*

$$T^\tau : v \in S \mapsto v^\tau \in C_0^{4,2}(\overline{Q})$$

is a compact continuous map.

Proof By Lemma 3.4 and the parabolic equation theory (see [10–11]), the image of the map T^τ is $C^{4+\alpha,2+\frac{\alpha}{2}}$. Since $|u^\tau|_{C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})} \le K_0$, not depending on τ, we have that T^τ is a compact map.

Lemma 3.6 *There is a bounded constant $M > 0$, such that all solutions $u(t,x)$ of the following problem*

$$\begin{cases} -(u_t - rxu_x)u_{xx} = \tau\theta u_x^2 + (1-\tau)f^0(x,t), & (x,t) \in Q, \\ u(x,0) = g(x), \quad g'(x) \le 0, & x \in [0,1], \\ u(0,t) = u_0(t), \quad u(1,t) = u_1(t), & t \in [0,T] \end{cases} \tag{3.7}$$

satisfy

$$|u|_{C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q})} \le M.$$

Proof We note that $u \ge \underline{u}$ and

$$u \le \overline{u} = (1-x)u_0(t) + xu_1(t). \tag{3.8}$$

Hence, there is some bounded constant $M_1 > 0$, such that

$$\sup_Q |u| \le M_1, \quad \sup_Q |u_x| \le M_1. \tag{3.9}$$

Let us estimate $-u_t$. Define some linear operator

$$\widetilde{L}_u = [-u_{xx}]\frac{\partial}{\partial t} + [-(u_t - rxu_x)]\frac{\partial^2}{\partial x^2} + [rxu_{xx}]\frac{\partial}{\partial x} - 2\tau\theta u_x \frac{\partial}{\partial x}.$$

Consider the following test function $w(x,t) = -u_t + \dfrac{k}{2}u$. We have

$$\begin{aligned}\widetilde{L}_u(w) &= u_{xx}u_{tt} - \frac{k}{2}u_t u_{xx} + u_t u_{txx} - \frac{k}{2}u_t u_{xx} - rxu_x u_{txx} + \frac{k}{2}rxu_x u_{xx}\\ &\quad - rxu_{xx}u_{tx} + \frac{k}{2}rxu_x u_{xx} + 2\tau\theta u_x u_{tx} - k\tau\theta u_x u_x\\ &= -(1-\tau)f_t^0(x,t) + k(1-\tau)f^0(x,t)\\ &= (1-\tau)f^0(x,t)\{k - [\ln f^0(x,t)]_t\}.\end{aligned}$$

If $k \ge k_3 \equiv \sup\limits_Q |[\ln f^0(x,t)]_t|$, we have$\widetilde{L}_u(w) \ge 0$. By the maximum principle, we have

$$\sup_{\partial_p Q}\Big(-u_t + \frac{k_3}{2}u\Big) \ge \sup_Q\Big(-u_t + \frac{k_3}{2}u\Big). \tag{3.10}$$

Using the same linear operator $\widetilde{L}_u$, we consider another two test functions

$$v_3 = \mathrm{e}^{kt}u_t, \quad v_4 = rxu_x\mathrm{e}^{kt}.$$

Direct calculation shows

$$\begin{aligned}\widetilde{L}_u(v_3) &= -u_{xx}u_{tt}\mathrm{e}^{kt} - ku_t u_{xx}\mathrm{e}^{kt} - u_t u_{txx}\mathrm{e}^{kt} + rxu_x u_{txx}\mathrm{e}^{kt}\\ &\quad + rxu_{xx}u_{xt}\mathrm{e}^{kt} - 2\tau\theta u_x u_{tx}\mathrm{e}^{kt}\\ &= (1-\tau)f_t^0\mathrm{e}^{kt} - ku_t u_{xx}\mathrm{e}^{kt},\\ \widetilde{L}_u(v_4) &= -u_{xx}rxu_{xt}\mathrm{e}^{kt} - rxu_x u_{xx}\mathrm{e}^{kt}k - 2ru_t u_{xx}\mathrm{e}^{kt} - rxu_t u_{xxx}\mathrm{e}^{kt}\\ &\quad + 2r^2xu_x u_{xx}\mathrm{e}^{kt} + r^2x^2u_x u_{xxx}\mathrm{e}^{kt} + r^2xu_x u_{xx}\mathrm{e}^{kt}\\ &\quad + r^2x^2u_{xx}u_{xx}\mathrm{e}^{kt} - 2\tau\theta ru_x u_x\mathrm{e}^{kt} - 2\tau\theta rxu_x u_{xx}\mathrm{e}^{kt}\\ &= rx\mathrm{e}^{kt}(1-\tau)f_x^0 + 2r(1-\tau)f^0\mathrm{e}^{kt} - krxu_x u_{xx}\mathrm{e}^{kt},\\ \widetilde{L}_u(-v_3+v_4) &= -(1-\tau)\mathrm{e}^{kt}f_t^0 + rx\mathrm{e}^{kt}(1-\tau)f_x^0 + 2r(1-\tau)f^0\mathrm{e}^{kt}\\ &\quad - k\mathrm{e}^{kt}(-u_t u_{xx} + rxu_x u_{xx})\\ &= (1-\tau)\mathrm{e}^{kt}(-f_t^0 + rxf_x^0 + 2rf^0) - k\mathrm{e}^{kt}[\tau\theta u_x^2 + (1-\tau)f^0]\\ &= -\mathrm{e}^{kt}\{k[\tau\theta u_x^2 + (1-\tau)f^0] + (1-\tau)(f_t^0 - rxf_x^0 - 2rf^0)\}.\end{aligned}$$

If $k \ge k_4 \equiv \sup\limits_Q \Big|\dfrac{(1-\tau)(f_t^0 - rxf_x^0 - 2rf^0)}{\tau\theta u_x^2 + (1-\tau)f^0}\Big|$, we get $\widetilde{L}_u(-v_3+v_4) \le 0,\ \forall (x,t) \in \overline{Q}$. Then, we have

$$\inf_{\partial_p Q}(-v_3 + v_4) \le \inf_Q(-v_3 + v_4).$$

Hence, we obtain

$$\inf_Q(-u_t + rxu_x) \ge \mathrm{e}^{-k_4T}\inf_{\partial_p Q}(-u_t + rxu_x). \tag{3.11}$$

Note that on $\{x=0\}\times[0,T]$,

$$-u_t+rxu_x=u_0'(t).$$

On $[0,x]\times\{t=0\}$, we have

$$-u_t+rxu_x=\frac{\tau\theta[g'(x)]^2+(1-\tau)f^0(x,t)}{g''(x)}.$$

On $\{x=1\}\times[0,T]$, we have

$$\begin{aligned}-u_t+rxu_x&=-u_t+ru_x\geq -u_1'(t)+r\overline{u}_x(x,t)|_{x=1}\\&=-u_1'(t)-ru_0(t)+ru_1(t),\end{aligned}$$

where $\overline{u}$ is the sup-solution of u defined by (3.8). Now combining (H_1)–(H_2), (3.9)–(3.11), there are some bounded constants $c_1, M_2>0$, such that

$$|u_t|\leq M_2,\quad \inf_Q(-u_t+rxu_x)\geq c_1>0. \tag{3.12}$$

Then, using Equation (3.7), there are bounded constants $\nu_1>0,\ M_3>0$, such that

$$0<\nu_1\leq u_{xx}\leq M_3,\quad \forall (x,t)\in\overline{Q}. \tag{3.13}$$

Taking the derivative with respect to t in (3.7), we have

$$-[u_{xx}]\frac{\partial}{\partial t}(u_t)+[-u_t+rxu_x]\frac{\partial^2}{\partial x^2}(u_t)+[rxu_{xx}-2\tau\theta u_x]\frac{\partial}{\partial x}(u_t)=(1-\tau)f_t^0.$$

The above equation is a linear equation about u_t. Using Hölder estimates for bounded coefficient linear parabolic equations, we have

$$[u_t]_{c^{\alpha,\frac{\alpha}{2}}(\overline{Q})}\leq M_4,\qquad \text{where } M_4 \text{ only depends on problem data.}$$

Combining (3.12)–(3.13) and using Equation (3.7), we have

$$[u_{xx}]_{c^{\alpha,\frac{\alpha}{2}}(\overline{Q})}\leq M_5.$$

Combining all the results that we have obtained and using Schauder estimates, we have our conclusion.

Lemma 3.7 *There exists some bounded constant $\mathbb{R}>0$, such that the operator $I-T^\tau$ has no zero on $\partial S_{\mathbb{R}}$.*

Proof By the definition of $S_{\mathbb{R}}$, we know that there exists no solution of Equation (3.7) on $|v|_{C^{4,2}}=\mathbb{R}$. In what follows, we will have no solution of Equation (3.7) on $\partial S_{\mathbb{R}}$, either. Suppose that u satisfies (3.7) on $\partial S_{\mathbb{R}}$. Then, we have $u\geq \underline{u}$ and at least one of the following three cases holds: $u=\underline{u}$ holds at some interior point of Q; $(u-\underline{u})_x|_{x=0}=0$; $(u-\underline{u})_x|_{x=1}=0$. By the maximum principle and Hopf lemma (as Theorem 3 in [9]), we have $\tau>0$. But we also have

$$\begin{aligned}&(\underline{u}_t-rx\underline{u}_x)\underline{u}_{xx}-(u_t-rxu_x)u_{xx}-\tau(\theta u_x^2-\theta\underline{u}_x^2)\\&=-\tau[-(\underline{u}_t-rx\underline{u}_x)\underline{u}_{xx}-\theta\underline{u}_x^2]+(1-\tau)[f^0(x,t)+(\underline{u}_t-rx\underline{u}_x)\underline{u}_{xx}]\\&\leq 0.\end{aligned}$$

Since $\tau>0$, the equality will not appear in the above inequality. Again using the maximum principle and Hopf lemma, we have $u\equiv\underline{u}$, which is a contradiction. Now, we can prove the main theorem.

Theorem 3.1 *If conditions* (H_1)–(H_3) *hold, then the problem* (3.1) *only has a unique strong convex monotonic solution* $u = u(x,t) \in C^{4+\alpha, 2+\frac{\alpha}{2}}(\overline{Q})$.

Proof The maximum principle implies the uniqueness. We only need to discuss the existence. Since $S_{\mathbb{R}}$ is a bounded subset in the Banach space $C_0^{4,2}(\overline{Q})$, the map

$$T^\tau : S_{\mathbb{R}} \mapsto C_0^{4,2}(\overline{Q})$$

is a continuous compact operator. There is no zero for the operator $I - T^\tau$ in $\partial S_{\mathbb{R}}$. Thus, for $\tau \in [0,1]$, we can define the Leray-Schauder degree

$$\deg(I - T^\tau, S_{\mathbb{R}}, 0).$$

By the homotopy invariant for the Leray-Schauder degree, we have

$$\deg(I - T^1, S_{\mathbb{R}}, 0) = \deg(I - T^0, S_{\mathbb{R}}, 0).$$

By the uniqueness, we have $T^0 \equiv v^0 = u^0 - \underline{u}$, $\forall v \in S_{\mathbb{R}}$. Hence, we have

$$(I - T^0)(v) \equiv I - v^0, \quad \forall v \in S_{\mathbb{R}},$$

which implies $\deg(I - T^0, S_{\mathbb{R}}, 0) = \deg(I - v^0, S_{\mathbb{R}}, 0)$. Using the translation invariant and normalization of the Leray-Schauder degree, we have $\deg(I - v^0, S_{\mathbb{R}}, 0) = \deg(I, S_{\mathbb{R}}, v^0) = 1$. Thus, we obtain $\deg(I - T^1, S_{\mathbb{R}}, 0) = 1$. We have completed the proof.

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References

[1] Yong, J. M., Introduction to mathematical finance, in "Mathematical Finance —Theory and Applications", Jiongmin Yong, Rama Cont, eds., Beijing, High Education Press, 2000, 19–137.

[2] Wang, G. L. and Lian, S. Z., An initial value problem for parabolic Monge-Ampère equation from investment theory, *J. Partial Differential Equations*, **16**(4), 2003, 381–383.

[3] Lian, S. Z., Existence of solutions to initial value problem for a parabolic Monge-Ampère equation and application, *Nonlinear Anal.*, **65**(1), 2006, 59–78.

[4] Caffarelli, L., Nirenberg, L. and Spruck, J., The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-Ampère equation, Communications on Pure and Applied Mathematics, **37**, 1984, 369–402.

[5] Krylov, N. V., Boundedly inhomogenuous elliptic and parabolic equations, *Izv. Akad. Nauk SSSR Ser. Mat.*, **46**, 1982, 485–523; English transl. in *Math. USSR Izv.*, **20**, 1983.

[6] Wang, G. L., The first boundary value problem for parabolic Monge-Ampère equation, *Northeast. Math. J.*, **3**, 1987, 463–478.

[7] Wang, G. L. and Wang, W., The first boundary value problem for general parabolic Monge-Ampère equation, *J. Partial Differential Equations*, **3**(2), 1990, 1–15.

[8] Ladyzhenskaja, O. A., Solonnikov, V. A. and Ural'ceva, N. N., Linear and quasilinear equations of parabolic type, American Mathematical Society, Providence, Ruode Island, 1968.

[9] Murray, H. P. and Hans, F. W., Maximum Principles in Differential Equations, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1984.

[10] Lieberman, G. M., Second Order Parabolic Differential Equations, World Scientific Publishing, Singapore, 1996.

[11] Gilbarg, D. and Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, 2nd edition, Grundlehren der Mathematischen Wissenschaften, Fundamental Principles of Mathematical Sciences, 224, Springer-Verlag, Berlin, 1983.