Chin. Ann. Math.
37B(5), 2016, 713–718
DOI: 10.1007/s11401-016-1004-2

Chinese Annals of Mathematics, Series B

A Description of Fixed Subgroups of Free Groups*

Qiang ZHANG[1]

Abstract Let F be a finitely generated free group. Martino and Ventura gave an explicit description for the fixed subgroups of automorphisms of F. The author generalizes their results to injective endomorphisms.

Keywords Fixed subgroups, Injective endomorphisms, Free groups, Ranks
2000 MR Subject Classification 20F05, 20F65

1 Introduction

Throughout this paper, let F be a finitely generated free group.

The rank of F, denoted by $r(F)$, is the cardinality of a basis of F. As usual, $\mathrm{Aut}(F)$ denotes the automorphisms of F, $\mathrm{End}(F)$ denotes the endomorphisms of F, and $\mathrm{Inj}(F)$ denotes the injective endomorphisms of F.

Let $\phi : F \to F$ be an endomorphism of F. We will denote ϕ as acting right of argument, and $x \mapsto (x)\phi$ (the parentheses will be omitted if there is no risk of confusion). A subgroup $H \leq F$ is called ϕ-invariant if $H\phi \leq H$. In this case, the restriction of ϕ to H will be denoted by $\phi|H : H \to H$, which is an endomorphism of H.

Except when $r(F) = 1$, $\mathrm{Inn}(F)$, the subgroup of inner automorphisms, is isomorphic to F. For any $y \in F$, we will write γ_y to denote the inner automorphism of right conjugation by y (denoted by exponential notation). Thus $\gamma_y : F \to F,\ x \mapsto x\gamma_y = y^{-1}xy = x^y$. Similarly, for any subgroup $H \leq F$, we denote by $H^y = y^{-1}Hy$ its right conjugation by y.

The fixed subgroup of an endomorphism ϕ of F, denoted by $\mathrm{Fix}\,\phi$, is the subgroup of elements in F fixed by ϕ:

$$\mathrm{Fix}\,\phi = \{x \in F : x\phi = x\}.$$

Following [8], a subgroup $H \leq F$ is called 1-auto-fixed (resp. 1-endo-fixed and 1-inj-fixed), when there exists an automorphism (resp. endomorphism and injective endomorphism) ϕ of F such that $H = \mathrm{Fix}\,\phi$.

In [9], Stallings raised a question: What subgroups S of F can be of the form $\mathrm{Fix}\,\beta$? Here, β refers to an automorphism of the free group F.

Manuscript received November 9, 2014. Revised June 22, 2015.
[1]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China.
E-mail: zhangq.math@xjtu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11201364).

It is easy to see that the trivial subgroup is 1-auto-fixed. Also a cyclic subgroup $H = \langle x \rangle$ of F is 1-auto-fixed if and only if it is pure, i.e., $x^r \in H$ implies $x \in H$, and in this case, $H = \mathrm{Fix}\,\gamma_x$ (for more details, see [7]). So, the interesting cases begin with subgroups of rank 2.

The maximal-rank case was completely settled by Collins and Turner. In [3], they gave a complete description of the 1-auto-fixed subgroups $H \leq F$ with $r(H) = r(F)$.

The goal of this paper is to generalize Martino-Ventura's results (Theorem 1.1 below) to injective endomorphisms (Theorem 1.2 below). In [7], Martino and Ventura generalized Collins and Turner's results, finding a similar description which applies to all 1-auto-fixed subgroups without restriction. For later use, we give the Martino-Ventura result below.

Theorem 1.1 (see [7, Theorem 1.4]) *Let F be a nontrivial finitely generated free group and $\phi \in \mathrm{Aut}(F)$ such that $\mathrm{Fix}\,\phi \neq 1$. Then, there exist integers $r, s \geq 0$, ϕ-invariant non-trivial subgroups $K_1, \cdots, K_r \leq F$, primitive elements $y_1, \cdots, y_s \in F$, a subgroup $L \leq F$, and elements $1 \neq h'_j \in H_j = K_1 * \cdots * K_r * \langle y_1, \cdots, y_j \rangle$, $j = 0, \cdots, s-1$, such that*

$$F = K_1 * \cdots * K_r * \langle y_1, \cdots, y_s \rangle * L$$

and $y_j\phi = h'_{j-1}y_j$ for $j = 1, \cdots, s$; moreover,

$$\mathrm{Fix}\,\phi = \langle \omega_1, \cdots, \omega_r, y_1^{-1}h_0y_1, \cdots, y_s^{-1}h_{s-1}y_s \rangle$$

for some non-proper powers $1 \neq \omega_i \in K_i$ and some $1 \neq h_j \in H_j$ such that $h_j\phi = h'_jh_jh'^{-1}_j$, $i = 1, \cdots, r$, $j = 0, \cdots, s-1$.

Remark 1.1 For $1 \neq h_0 \in H_0 = K_1 * \cdots * K_r$, we can know that $H_0 = K_1 * \cdots * K_r \neq 1$. So, in fact, $r \geq 1$ in Theorem 1.1.

A subgroup $H \leq F$ is called a free factor of F, if it admits a basis which can be extended to a basis of F. Thus, if H is a free factor of F, then there exists a subgroup $L \leq F$, such that $F = H * L$. For any free factor $H \leq F$, we have $r(H) \leq r(F)$ with equality if and only if $H = F$.

An element $\omega \in F$ is called an F-primitive element when there exist words $\omega_2, \omega_3, \cdots, \omega_n$ such that $\{\omega, \omega_2, \cdots, \omega_n\}$ is a basis of F.

In this paper, we show that the Martino-Ventura result (Theorem 1.1) also holds for inj-fixed subgroups in free groups, that is the following theorem.

Theorem 1.2 *Let F be a nontrivial finitely generated free group and $\phi \in \mathrm{Inj}(F)$ such that $\mathrm{Fix}\,\phi \neq 1$. Then, there exist integers $r \geq 1$, $s \geq 0$, ϕ-invariant non-trivial subgroups $K_1, \cdots, K_r \leq F$, primitive elements $y_1, \cdots, y_s \in F$, a subgroup $L \leq F$ ($L \neq 1$ if $\phi \notin \mathrm{Aut}(F)$), and elements $1 \neq h'_j \in H_j = K_1 * \cdots * K_r * \langle y_1, \cdots, y_j \rangle$, $j = 0, \cdots, s-1$, such that*

$$F = K_1 * \cdots * K_r * \langle y_1, \cdots, y_s \rangle * L$$

and $y_j\phi = h'_{j-1}y_j$ for $j = 1, \cdots, s$; moreover,

$$\mathrm{Fix}\,\phi = \langle \omega_1, \cdots, \omega_r, y_1^{-1}h_0y_1, \cdots, y_s^{-1}h_{s-1}y_s \rangle$$

for some non-proper powers $1 \neq \omega_i \in K_i$ *and some* $1 \neq h_j \in H_j$ *such that* $h_j\phi = h'_j h_j h'^{-1}_j,\ i = 1, \cdots, r,\ j = 0, \cdots, s-1$.

This paper is organized as follows. In Section 2, we will give the proof of Theorem 1.2, and in Section 3, we will give some corollaries and examples.

2 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Definition 2.1 *Suppose* $\phi \in \mathrm{End}(F)$. *Then we denote by* $F\phi^\infty$ *the stable image of* ϕ, *i.e.,*

$$F\phi^\infty = \bigcap_{i=0}^{\infty} F\phi^i$$

and

$$\phi_\infty = \phi | F\phi^\infty : F\phi^\infty \to F\phi^\infty.$$

It is shown in [5] *that* $r(F\phi^\infty) \leq r(F)$, ϕ_∞ *is an automorphism, and clearly* $\mathrm{Fix}\,\phi = \mathrm{Fix}\,\phi_\infty \leq F\phi^\infty$.

Lemma 2.1 *Let* $\phi \in \mathrm{Inj}(F)$ *be an injective endomorphism of the finitely generated free group* F. *Then* $F\phi^\infty$ *is a free factor of* F, *i.e.,*

$$F = F\phi^\infty * L$$

for a subgroup $L \leq F$; *moreover,* $L = 1$ *if and only if* $\phi \in \mathrm{Aut}(F)$.

Proof It follows from [6, Problem 33 on p. 118] that $F\phi^\infty$ is a free factor of $F\phi^n$ for almost all n. So there exists an integer n and a subgroup $L' \leq F$ such that

$$F\phi^n = F\phi^\infty * L'.$$

Since $\phi \in \mathrm{Inj}(F)$, $\phi^n : F \to F\phi^n = F\phi^\infty * L'$ is an isomorphism. Let $L = L'(\phi^n)^{-1}$. Then

$$F = (F\phi^\infty * L')(\phi^n)^{-1} = (F\phi^\infty)(\phi^n)^{-1} * L'(\phi^n)^{-1} = F\phi^\infty * L.$$

Clearly, $L = 1$ if and only if $\phi \in \mathrm{Aut}(F)$.

Proof of Theorem 1.2 If ϕ is surjective, then $\phi \in \mathrm{Aut}(F)$, by Theorem 1.1, we have done.

Otherwise, following Lemma 2.1, we have $F = F\phi^\infty * L''$, $L'' \neq 1$, and $1 \leq r(F\phi^\infty) < r(F)$, so $F\phi^\infty$ is finitely generated. Applying Theorem 1.1 to $F\phi^\infty$ and $\phi_\infty \in \mathrm{Aut}(F\phi^\infty)$, there exist integers $r \geq 1,\ s \geq 0$, ϕ_∞-invariant non-trivial subgroups $K_1, \cdots, K_r \leq F\phi^\infty$, primitive elements $y_1, \cdots, y_s \in F\phi^\infty$, a subgroup $L' \leq F\phi^\infty$, and $1 \neq h'_j \in H_j = K_1 * \cdots * K_r * \langle y_1, \cdots, y_j \rangle$, $j = 0, \cdots, s-1$, such that

$$F\phi^\infty = K_1 * \cdots * K_r * \langle y_1, \cdots, y_s \rangle * L'$$

and $y_j\phi_\infty = h'_{j-1}y_j$ for $j = 1, \cdots, s$; moreover,

$$\operatorname{Fix}\phi_\infty = \langle \omega_1, \cdots, \omega_r, y_1^{-1}h_0y_1, \cdots, y_s^{-1}h_{s-1}y_s\rangle$$

for some non-proper powers $1 \neq \omega_i \in K_i$ and some $1 \neq h_j \in H_j$ such that $h_j\phi_\infty = h'_jh_jh'^{-1}_j,\ i = 1, \cdots, r,\ j = 0, \cdots, s-1$.

Let $L = L' * L''$. Then $L \neq 1$. Since $\phi_\infty = \phi|F\phi^\infty$ and $\operatorname{Fix}\phi = \operatorname{Fix}\phi_\infty$, we have

$$F = F\phi^\infty * L'' = K_1 * \cdots * K_r * \langle y_1, \cdots, y_s\rangle * L$$

and

$$\operatorname{Fix}\phi_\infty = \langle \omega_1, \cdots, \omega_r, y_1^{-1}h_0y_1, \cdots, y_s^{-1}h_{s-1}y_s\rangle.$$

Thus Theorem 1.2 holds.

3 Some Corollaries and Examples

In this section, we will give some corollaries and examples of Theorem 1.2.

From Theorem 1.2, we immediately have the following corollary.

Corollary 3.1 (see [1, 10]) *Let F_n be the free group of rank n and $\phi : F_n \to F_n$ be an injective endomorphism. If $\phi \in \operatorname{Aut}(F_n)$, then $r(\operatorname{Fix}\phi) \leq n$; if $\phi \notin \operatorname{Aut}(F_n)$, then $r(\operatorname{Fix}\phi) \leq n-1$.*

In fact, it is also shown in [10] that if $\phi : F_n \to F_n$ is an endomorphism which is not an automorphism, then $r(\operatorname{Fix}\phi) \leq n-1$. So, we have the following definition.

Definition 3.1 *A subgroup $H \leq F_n$ $(n \geq 2)$ is called maximum-rank 1-endo-fixed (resp. 1-inj-fixed) if there exists an endomorphism (resp. injective endomorphism) ϕ of F such that $H = \operatorname{Fix}\phi$ and $r(H) = \begin{cases} n, & \phi\in\operatorname{Aut}(F_n), \\ n-1, & \phi\notin\operatorname{Aut}(F_n). \end{cases}$*

Corollary 3.2 *If $\phi \in \operatorname{Inj}(F)$ with maximum-rank 1-inj-fixed subgroup, then there exist integers $r \geq 1$, $s \geq 0$, and primitive elements $\omega_1, \cdots, \omega_r, y_1, \cdots, y_s, z \in F$ ($z = 1$ if and only if $\phi \in \operatorname{Aut}(F)$), such that*

$$F = \langle\omega_1\rangle * \cdots * \langle\omega_r\rangle * \langle y_1\rangle * \cdots * \langle y_s\rangle * \langle z\rangle,$$

and $\omega_i\phi = \omega_i,\ y_j\phi = h'_{j-1}y_j,\ 1 \neq h'_j \in H_j = \langle\omega_1\rangle * \cdots * \langle\omega_r\rangle * \langle y_1\rangle * \cdots * \langle y_j\rangle,\ j = 1, \cdots, s$; moreover,

$$\operatorname{Fix}\phi = \langle \omega_1, \cdots, \omega_r, y_1^{-1}h_0y_1, \cdots, y_s^{-1}h_{s-1}y_s\rangle,$$

for some $1 \neq h_j \in H_j$ such that $h_j\phi = h'_jh_jh'^{-1}_j,\ j = 0, \cdots, s-1$.

Proof Following from Theorem 1.2, we have

$$r(\operatorname{Fix}\phi) = r + s = r(K_1 * \cdots * K_r * \langle y_1, \cdots, y_s\rangle).$$

Since $r(K_i) \geq 1$, we have $r(K_i) = 1$. Thus $K_i \cong \mathbb{Z}$; moreover, since $K_i\phi \leq K_i$ and $\operatorname{Fix}\phi \cap K_i = \langle\omega_i\rangle$, we have $K_i = \langle\omega_i\rangle,\ i = 1, \cdots, r$.

Remark 3.1 When $\phi \in \mathrm{Aut}(F)$ and $r(\mathrm{Fix}\,\phi) = r(F)$, Corollary 3.2 is the main result of [3].

From the example below, we can easily know that Corollary 3.2 does not hold for $\phi \in \mathrm{End}(F)$ with maximum-rank 1-endo-fixed subgroup.

Example 3.1 Let $F = \langle a,\ b\rangle$ be a free group of rank 2 freely generated by $\{a, b\}$, and let $\phi \in \mathrm{End}(F)$ be given by

$$\phi : F \to F, \quad a \mapsto a^2b^{-1}a^{-1}b, \quad b \mapsto 1.$$

Then it is easy to know that

$$\mathrm{Fix}\,\phi = \langle a^2b^{-1}a^{-1}b\rangle.$$

So, $\langle a^2b^{-1}a^{-1}b\rangle$ is a maximum-rank 1-endo-fixed subgroup of F. However, following from [2], $a^2b^{-1}a^{-1}b$ is not a primitive element of F. So, Corollary 3.2 does not hold for $\phi \in \mathrm{End}(F)$.

Corollary 3.3 *Every injective endomorphism of F_n $(n \geq 2)$ with maximum-rank 1-inj-fixed subgroup fixes a primitive element of F_n.*

Proof It follows immediately from Corollary 3.2.

Remark 3.2 When $\phi \in \mathrm{Aut}(F)$ and $r(\mathrm{Fix}\,\phi) = r(F)$, Corollary 3.3 is the main theorem of [4]: Every automorphism of F_n with a fixed subgroup of rank n fixes a primitive element of F_n.

If the injective endomorphism ϕ of F_n is not with the maximum-rank 1-inj-fixed subgroup, then Corollary 3.3 does not hold.

Example 3.2 Let $F = \langle a,\ b\rangle$ be a free group of rank 2 freely generated by $\{a, b\}$, and let $\phi \in \mathrm{Aut}(F)$ be given by

$$\phi : F \to F, \quad a \mapsto aba, \quad b \mapsto ab.$$

Then it is easy to know that

$$\mathrm{Fix}\,\phi = \langle b^{-1}a^{-1}ba\rangle.$$

Clearly, $b^{-1}a^{-1}ba$ is not a primitive element of F. So, $\mathrm{Fix}\,\phi$ contains no primitive elements.

Acknowledgements The author would like to thank Prof. Boju Jiang for valuable communications and suggestions. The author also thanks the referee for his or her comments.

References

[1] Bestvina, M. and Handel, M., Train tracks and automorphisms of free groups, *Ann. of Math.*, **135**, 1992, 1–51.

[2] Cohen, M., Metzler, W. and Zimmermann, A., What does a basis of $F[a, b]$ look like? *Math. Ann.*, **257**, 1981, 435–445.

[3] Collins, D. and Turner, E., All automorphisms of free groups with maximal rank fixed subgroups, *Math. Proc. Cambridge Philos. Soc.*, **119**, 1996, 615–630.

[4] Collins, D. and Turner, E., An automorphism of a free group of finite rank with maximal rank fixed point subgroup fixes a primitive element, *J. Pure Appl. Algebra*, **88**, 1993, 43–49.

[5] Imrich, W. and Turner, E., Endomorphisms of free groups and their fixed points, *Math. Proc. Cambridge Philos. Soc.*, **105**, 1989, 421–422.

[6] Magnus, W., Karrass, A. and Solitar, D., Combinatorial Group Theory, Wiley, New York, 1966.

[7] Martino, A. and Ventura, E., A description of auto-fixed subgroups in a free group, *Topology*, **43**, 2004, 1133–1164.

[8] Martino, A. and Ventura, E., On automorphism fixed subgroups of a free group, *J. Algebra*, **230**, 2000, 596–607.

[9] Stallings, J., Graphical theory of automorphisms of free groups, Combinatorial Group Theory and Topology, Ann. of Math. Stud., Vol. 111, Princeton University Press, Princeton, NJ, 1987.

[10] Turner, E., Test words for automorphisms of free groups, *Bull. London Math. Soc.*, **28**, 1996, 255–263.