Chin. Ann. Math.
37B(5), 2016, 719–728
DOI: 10.1007/s11401-016-0978-0

Chinese Annals of Mathematics, Series B

Some Properties of Meromorphic Solutions to Systems of Complex Differential-Difference Equations

Haichou LI[1]

Abstract Applying Nevanlinna theory of the value distribution of meromorphic functions, the author studies some properties of Nevanlinna counting function and proximity function of meromorphic solutions to a type of systems of complex differential-difference equations. Specifically speaking, the estimates about counting function and proximity function of meromorphic solutions to systems of complex differential-difference equations can be given.

Keywords Differential-difference equation, Systems of equation, Meromorphic solutions, Proximity function, Counting function
2000 MR Subject Classification 30D05, 39B32, 30D35

1 Introduction and Main Results

Let $f(z)$ be meromorphic function in the complex plane $\mathbf{C}$. We assume that the reader is familiar with the standard notations of Nevanlinna theory of the value distribution of meromorphic functions, such as the characteristic function $T(r,f)$, the proximity function $m(r,f)$, the counting function $N(r,f)$, as well as the first and second main theorems (see [1–4]). The notation $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set E of r of finite linear measure $\lim\limits_{r\to\infty}\int_{[1,\infty)\cap E}\frac{\mathrm{d}r}{r}<\infty$. A meromorphic function $a(z)$ is called a small function of $f(z)$ if and only if $T(r,a(z))=S(r,f)$.

Many authors have studied the problems of the existence or the growth of meromorphic solutions of systems of complex differential equations, and obtained some results (see [5–9]). In recent years, there has been renewed interests in difference (discrete) equations and difference analogues of Nevanlinna's theory in the complex plane $\mathbf{C}$ (see [10–24]). Chiang and Feng [13], as well as Halburd and Korhonen [21] established a difference analogue of the Logarithmic derivative lemma independently. The foundations of the theory of complex difference equations were laid by Julia, Birkhoff, Batchelder and others in the early twentieth century. Later on, Shimomura [24] and Yanagihara [22–23] considered nonlinear complex difference equations by the method of Nevanlinna's theory.

In 2011, Korhonen [21] investigated the properties of finite-order meoromorphic solutions of the equation

$$H(z,w)P(z,w)=Q(z,w), \tag{1.1}$$

Manuscript received October 29, 2014. Revised April 14, 2015.
[1]Department of Mathematics, South China Agricultural University, Guangzhou 510642, China.
E-mail: lihaichou@126.com hcl2016@scau.edu.cn

where $P(z,w)=P(z,w(z),w(z+c_1),\cdots,w(z+c_n))$, $c_1,\cdots,c_n\in\mathbf{C}$, and obtained the following result.

Theorem A Let $w(z)$ be a finite-order meromorphic solution of (1.1), where $P(z,w)$ is a homogeneous difference polynomial with meromorphic coefficients, and $H(z,w)$ and $Q(z,w)$ are polynomials in $w(z)$ with meoromorphic coefficients having no common factors. If

$$\max\{\deg_w(H),\deg_w(Q)-\deg_w(P)\}>\min\{\deg_w(P),\mathrm{ord}_0(Q)-\mathrm{ord}_0(P)\},$$

then $N(r,w)\neq S(r,w)$, where $\mathrm{ord}_0(P)$ denotes the order of zero of $P(z,x_0,x_1,\cdots,x_n)$ at $x_0=0$ with respect to the variable x_0.

In 2012, Gao [17] extended the above result of (1.1) to the systems, and obtained some properties of the proximity function and the counting function of meoromorphic solutions to systems of difference equations such as

$$\begin{cases}\Phi_1(z,w_1,w_2)=R_1(z,w_1),\\ \Phi_2(z,w_1,w_2)=R_2(z,w_2),\end{cases}\tag{1.2}$$

where $R_i(z,w_i(z))$ $(i=1,2)$ are rational functions in $w_i(z)$ $(i=1,2)$ with meromorphic coefficients which are small functions of $f_i(z)$ $(i=1,2)$ respectively, and $\Phi_1(z,w_1,w_2)$, $\Phi_2(z,w_1,w_2)$ are difference polynomials which are defined as

$$\begin{aligned}&\Phi_1(z,w_1,w_2)=\Phi_1(z,w_1(z),w_2(z),w_1(z+c_1),w_2(z+c_1),\cdots,w_1(z+c_n),w_2(z+c_n))\\ =&\sum_{(i)}a_{(i)}(z)\prod_{k=1}^{2}w_k^{i_{k0}}w_k^{i_{k1}}(z+c_1)\cdots w_k^{i_{kn}}(z+c_n),\\ &\Phi_2(z,w_1,w_2)=\Phi_2(z,w_1(z),w_2(z),w_1(z+c_1),w_2(z+c_1),\cdots,w_1(z+c_n),w_2(z+c_n))\\ =&\sum_{(j)}b_{(j)}(z)\prod_{k=1}^{2}w_k^{j_{k0}}w_k^{j_{k1}}(z+c_1)\cdots w_k^{j_{kn}}(z+c_n),\end{aligned}$$

with the coefficients $\{a_{(i)}(z)\}$, $\{b_{(j)}(z)\}$ being small functions with respect to both w_1 and w_2, and $c_i\in\mathbf{C}$, for all $i=1,2,\cdots,n$.

So far, the previous researches are only on the complex differential equations (systems) or difference equations (systems), but not on difference-differential equations (systems). Therefore, it is very important and meaningful to study the cases of difference-differential equations (systems), and this paper will mainly investigate some properties of meromorphic solutions of the systems of difference-differential equations. By the way, let me give out the definition of difference-differential equation and the system of difference-differential equations as follows.

Definition 1.1 *We call an equation a difference-differential equation, if this equation contains the difference and the differential of one function at the same time.*

Definition 1.2 *Corresponding to difference-differential equations in Definition* 1.1*, we will call the systems which contain difference-differential equations the systems of difference-differential equations.*

In this paper, inspired by the ideas of Gao, the author will mainly investigate some properties of the meromorphic solutions of the systems of complex differential-difference equations, and extend the results obtained by Gao [17] to the systems of the following form (1.5), which is different from the systems of complex differential equations or systems of complex difference equations. That will be an innovative contribution of this paper.

Let c_l $(l=1,2,\cdots,n)\in\mathbf{C}$, I, J be two finite sets of multi-indexes $(i_0,i_1,\cdots,i_n)$, $(j_0,j_1,\cdots,j_n)$ respectively. $\Omega_1(z,w_1,w_2)$, $\Omega_2(z,w_1,w_2)$ are difference-differential polynomials which are defined as

$$\Omega_1(z,w_1,w_2)=\sum_{(i\in I)} a_{(i)}(z)\prod_{k=1}^{2} w_k^{i_{k0}}(w_k'(z+c_1))^{i_{k1}}(w_k''(z+c_2))^{i_{k2}}\cdots(w_k^{(n)}(z+c_n))^{i_{kn}}, \quad (1.3)$$

$$\Omega_2(z,w_1,w_2)=\sum_{(j\in J)} b_{(j)}(z)\prod_{k=1}^{2} w_k^{j_{k0}}(w_k'(z+c_1))^{j_{k1}}(w_k''(z+c_2))^{j_{k2}}\cdots(w_k^{(n)}(z+c_n))^{j_{kn}}, \quad (1.4)$$

respectively, where the coefficients $\{a_{(i)}(z)\}$, $\{b_{(j)}(z)\}$ are small functions with respect to both w_1 and w_2 in the sense that

$$T(r,a_{(i)})=S(r,w_k),\quad T(r,b_{(i)})=S(r,w_k),\quad k=1,2,$$

as r tends to infinity outside of an exceptional set E of finite logarithmic measure $\int_E \frac{\mathrm{d}x}{x}<\infty$. Denote

$$\lambda_{11}=\max_{(i)}\Big\{\sum_{l=0}^{n} i_{1l}\Big\},\quad \lambda_{12}=\max_{(i)}\Big\{\sum_{l=0}^{n} i_{2l}\Big\},$$
$$\lambda_{21}=\max_{(j)}\Big\{\sum_{l=0}^{n} j_{1l}\Big\},\quad \lambda_{22}=\max_{(j)}\Big\{\sum_{l=0}^{n} j_{2l}\Big\},$$
$$\mu_{11}=\max_{(i)}\Big\{\sum_{l=0}^{n}(l+1)i_{1l}\Big\},\quad \mu_{12}=\max_{(i)}\Big\{\sum_{l=0}^{n}(l+1)i_{2l}\Big\},$$
$$\mu_{21}=\max_{(j)}\Big\{\sum_{l=0}^{n}(l+1)j_{1l}\Big\},\quad \mu_{22}=\max_{(j)}\Big\{\sum_{l=0}^{n}(l+1)j_{2l}\Big\}.$$

Now, we will investigate the following systems of complex difference-differential equations:

$$\begin{cases}\Omega_1(z,w_1,w_2)=R_1(z,w_1),\\ \Omega_2(z,w_1,w_2)=R_2(z,w_2),\end{cases}\quad (1.5)$$

where $\Omega_1(z,w_1,w_2)$, $\Omega_2(z,w_1,w_2)$ are difference-differential polynomials defined as (1.3)–(1.4), respectively, and

$$R_1(z,w_1)=\frac{P_1(z,w_1)}{Q_1(z,w_1)}=\frac{\sum\limits_{i=0}^{p_1} a_{1i}(z)w_1^i}{\sum\limits_{i=0}^{q_1} b_{1i}(z)w_1^i},$$

$$R_2(z,w_2)=\frac{P_2(z,w_2)}{Q_2(z,w_2)}=\frac{\sum\limits_{i=0}^{p_2} a_{2i}(z)w_2^i}{\sum\limits_{i=0}^{q_2} b_{2i}(z)w_2^i},$$

with coefficients $\{a_{(i)}(z)\},\ \{b_{(j)}(z)\},\ \{a_{ki}(z)\},\ \{b_{kj}(z)\},\ k=1,2$ being all meromorphic functions and small functions with respect to both w_1 and w_2, $a_{1p_1}b_{1q_1}\neq 0,\ a_{2p_2}b_{2q_2}\neq 0$.

The difference-differential polynomial $\Omega_k(z,w_1,w_2)$ is said to be homogeneous with respect to $w_k(z)$ $(k=1,\ 2)$ if the degree $d_k=i_{k0}+i_{k1}+\cdots+i_{kn}$ of each term is non-zero and the same for all $i\in I$.

The order of growth of a meromorphic solution (w_1,w_2) is defined as

$$\rho(w_1,w_2)=\max\{\rho(w_1),\ \rho(w_2)\},$$

where

$$\rho(w_k)=\limsup_{r\to\infty}\frac{\log T(r,w_k)}{\log r}.$$

The main results are as follows.

Theorem 1.1 *If (w_1,w_2) is a finite-order meromorphic solution of system (1.5), where $\Omega_1(z,w_1,w_2),\ \Omega_2(z,w_1,w_2)$ are homogeneous difference-differential polynomials in w_1 and w_2 respectively, $R_k(z,w_k),\ k=1,2$ are irreducible rational functions in w_k, and*

$$\max\{q_1,p_1\}>1+2\mu_{11}+\mu_{12},$$
$$\max\{q_2,p_2\}>1+2\mu_{21}+\mu_{22}.$$

Then $N(r,w_1)=S(r,w_1)$ and $N(r,w_2)=S(r,w_2)$ can not hold at the same time, possibly outside of an exceptional set of finite logarithmic measure.

Theorem 1.2 *If (w_1,w_2) is a finite-order meromorphic solution of system (1.5), where $\Omega_1(z,w_1,w_2),\ \Omega_2(z,w_1,w_2)$ are homogeneous difference-differential polynomials in w_1 and w_2 respectively, $R_k(z,w_k),\ k=1,2$ are irreducible rational functions in w_k, and*

$$\max\{q_2,p_2\}-\mu_{22}>0,\quad \max\{q_1,p_1\}-\mu_{11}>0,$$
$$(\mu_{11}-\max\{q_1,p_1\})(\mu_{22}-\max\{q_2,p_2\})>\mu_{12}\mu_{21}+\lambda_{12}\mu_{21},$$
$$(\mu_{21}-\max\{q_2,p_2\})(\mu_{11}-\max\{q_1,p_1\})>\mu_{22}\mu_{11}+\lambda_{22}\mu_{11}.$$

Then there are

$$m(r,w_1)=o(T(r,w_1)),\quad m(r,w_2)=o(T(r,w_2))$$

when r tends to infinity outside of an exceptional set of finite logarithmic measure.

2 Main Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 (see [3]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))}=\frac{\sum\limits_{i=0}^{p}a_i(z)f^i}{\sum\limits_{j=0}^{q}b_j(z)f^j}$$

such that the meromorphic coefficients $a_i(z),\ b_j(z)$ *satisfy*

$$T(r,a_i)=S(r,f),\quad i=0,1,2,\cdots,p,$$
$$T(r,b_j)=S(r,f),\quad j=0,1,2,\cdots,q,$$

one has

$$T(r,R(z,f))=\max\{p,q\}T(r,f)+S(r,f).$$

Lemma 2.2 (see [4]) *If* $f(z)$ *is a transcendental meromorphic function, then*

$$N(r,f^{(k)})\le(k+1)N(r,f)+S(r,f),$$
$$T(r,f^{(k)})\le(k+1)T(r,f).$$

Lemma 2.3 (see [13]) *Let* $f(z)$ *be a meromorphic function with order* $\rho=\rho(f),\ \rho<+\infty$, *and let* c *be a fixed nonzero complex number. Then for each* $\varepsilon>0$, *one has*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\rho-1+\varepsilon})+O(\log r).$$

Lemma 2.4 (see [13]) *Let* $f(z)$ *be a nonconstant meromorphic function with order* $\rho=\rho(f),\ \rho<+\infty$, *and* c *be a fixed nonzero complex number. Then for each* $1>\delta>0$, *one has*

$$m\Big(r,\frac{f(z+c)}{f(z)}\Big)=o\Big(\frac{T(r,f(z))}{r^{\delta}}\Big)=S(r,f),$$
$$m(r,f(z+c))=m(r,f)+S(r,f)$$

for all r *outside of a possible exceptional set with finite logarithmic measure.*

Lemma 2.5 *Let* $f(z)$ *be a nonconstant meromorphic function with order* $\rho=\rho(f),\ \rho<+\infty$, *and* c *be a fixed nonzero complex number. Then*

$$m\Big(r,\frac{f^{(k)}(z+c)}{f(z)}\Big)=S(r,f),$$
$$m(r,f^{(k)}(z+c))\le m(r,f)+S(r,f)$$

for all r *outside of a possibly exceptional set with finite logarithmic measure.*

Proof By the logarithmic derivative lemma, there is

$$m\Big(r,\frac{f^{(k)}(z+c)}{f(z+c)}\Big)=S(r,f(z+c))=S(r,f).$$

Then, from Lemma 2.4 and the above, we have

$$\begin{aligned}m\Big(r,\frac{f^{(k)}(z+c)}{f(z)}\Big)&=m\Big(r,\frac{f^{(k)}(z+c)}{f(z+c)}\frac{f(z+c)}{f(z)}\Big)\\&\le m\Big(r,\frac{f^{(k)}(z+c)}{f(z+c)}\Big)+m\Big(r,\frac{f(z+c)}{f(z)}\Big)\\&=S(r,f).\end{aligned}$$

Moreover,

$$\begin{aligned} m(r, f^{(k)}(z+c)) &= m\Big(r, \frac{f^{(k)}(z+c)}{f(z)} f(z)\Big) \\ &\le m\Big(r, \frac{f^{(k)}(z+c)}{f(z)}) + m(r, f\Big) \\ &= m(r, f) + S(r, f). \end{aligned}$$

3 Proofs of the Theorems

Proof of Theorem 1.1 Suppose that (w_1, w_2) is a finite-order meromorphic solution of the system (1.5). By the Lemma 2.1 and the system (1.5), there are

$$T(r, \Omega_1(r, w_1, w_2)) = T\Big(r, \frac{P_1(z, w_1)}{Q_1(z, w_1)}\Big) = \max\{q_1, p_1\} T(r, w_1) + S(r, w_1), \tag{3.1}$$

$$T(r, \Omega_2(r, w_1, w_2)) = T\Big(r, \frac{P_2(z, w_2)}{Q_2(z, w_2)}\Big) = \max\{q_2, p_2\} T(r, w_2) + S(r, w_2) \tag{3.2}$$

for all r outside of an exceptional set of finite logarithmic measure.

By Lemmas 2.4–2.5, we can get that

$$m(r, \Omega_1(r, w_1, w_2)) \le \lambda_{11} m(r, w_1) + \lambda_{12} m(r, w_2) + S(r, w_1) + S(r, w_2), \tag{3.3}$$

$$m(r, \Omega_2(r, w_1, w_2)) \le \lambda_{21} m(r, w_1) + \lambda_{22} m(r, w_2) + S(r, w_1) + S(r, w_2) \tag{3.4}$$

for all r outside of an exceptional set of finite logarithmic measure.

As

$$\lambda_{11} \le \mu_{11}, \quad \lambda_{12} \le \mu_{12}, \quad \lambda_{21} \le \mu_{21}, \quad \lambda_{22} \le \mu_{22},$$

$$m(r, \Omega_1(r, w_1, w_2)) \le \mu_{11} m(r, w_1) + \mu_{12} m(r, w_2) + S(r, w_1) + S(r, w_2), \tag{3.5}$$

$$m(r, \Omega_2(r, w_1, w_2)) \le \mu_{21} m(r, w_1) + \mu_{22} m(r, w_2) + S(r, w_1) + S(r, w_2) \tag{3.6}$$

for all r outside of an exceptional set of finite logarithmic measure.

Thus, from the assumptions of Theorem 1.1, combining (3.1) and (3.5), (3.2) and (3.6), respectively, we have

$$N(r, \Omega_1(r, w_1, w_2)) \ge (1 + \mu_{11} + \mu_{12}) T(r, w_1) - \mu_{12} m(r, w_2) + S(r, w_1) + S(r, w_2), \tag{3.7}$$

$$N(r, \Omega_2(r, w_1, w_2)) \ge (1 + \mu_{21} + \mu_{22}) T(r, w_2) - \mu_{21} m(r, w_1) + S(r, w_1) + S(r, w_2). \tag{3.8}$$

However, on the other hand, it follows that

$$N(r, \Omega_1(r, w_1, w_2)) \le \mu_{11} N(r, w_1) + \mu_{12} N(r, w_2) + S(r, w_1) + S(r, w_2), \tag{3.9}$$

$$N(r, \Omega_2(r, w_1, w_2)) \le \mu_{21} N(r, w_1) + \mu_{22} N(r, w_2) + S(r, w_1) + S(r, w_2). \tag{3.10}$$

So combing (3.9) and (3.7), (3.10) and (3.8), respectively, we can obtain

$$\begin{aligned}&(1+\mu_{11}+\mu_{12})T(r,w_1)\\ \le\ &\mu_{11}T(r,w_1)+\mu_{12}N(r,w_2)+\mu_{12}T(r,w_2)+S(r,w_1)+S(r,w_2),\end{aligned}\tag{3.11}$$

$$\begin{aligned}&(1+\mu_{21}+\mu_{22})T(r,w_2)\\ \le\ &\mu_{21}N(r,w_1)+\mu_{22}T(r,w_2)+\mu_{21}T(r,w_1)+S(r,w_1)+S(r,w_2).\end{aligned}\tag{3.12}$$

By the suppositions that $N(r,w_1)=S(r,w_1)$ and $N(r,w_2)=S(r,w_2)$, we can get by the last inequality that

$$(1+\mu_{12})T(r,w_1)\le\mu_{12}T(r,w_2)+S(r,w_1)+S(r,w_2),$$
$$(1+\mu_{21})T(r,w_2)\le\mu_{21}T(r,w_1)+S(r,w_1)+S(r,w_2).$$

That is,

$$(1+\mu_{12}+o(1))T(r,w_1)\le(\mu_{12}+o(1))T(r,w_2),$$
$$(1+\mu_{21}+o(1))T(r,w_2)\le(\mu_{21}+o(1))T(r,w_1).$$

By the last inequation, we can get that

$$(1+\mu_{12})(1+\mu_{21})<\mu_{12}\mu_{21}.$$

Thus, from the last inequality, we can get a contradiction. Therefore, the proof of Theorem 1.1 is complete.

Proof of Theorem 1.2 Suppose that (w_1,w_2) is a finite-order meromorphic solution of the system (1.5). By Lemmas 2.4–2.5, we can get that

$$m(r,\Omega_1(r,w_1,w_2))\le\lambda_{11}m(r,w_1)+\lambda_{12}m(r,w_2)+S(r,w_1)+S(r,w_2),\tag{3.13}$$
$$m(r,\Omega_2(r,w_1,w_2))\le\lambda_{21}m(r,w_1)+\lambda_{22}m(r,w_2)+S(r,w_1)+S(r,w_2)\tag{3.14}$$

for all r outside of an exceptional set of finite logarithmic measure.

By Lemma 2.1 and the system (1.5), there are

$$T(r,\Omega_1(r,w_1,w_2))=T\Big(r,\frac{P_1(z,w_1)}{Q_1(z,w_1)}\Big)=\max\{q_1,p_1\}T(r,w_1)+S(r,w_1),\tag{3.15}$$
$$T(r,\Omega_2(r,w_1,w_2))=T\Big(r,\frac{P_2(z,w_2)}{Q_2(z,w_2)}\Big)=\max\{q_2,p_2\}T(r,w_2)+S(r,w_2)\tag{3.16}$$

for all r outside of an exceptional set of finite logarithmic measure.

So, combining (3.13) and (3.15), (3.14) and (3.16), respectively, we can have

$$\begin{aligned}&N(r,\Omega_1(r,w_1,w_2))+\lambda_{11}m(r,w_1)+\lambda_{12}m(r,w_2)+S(r,w_1)+S(r,w_2)\\ \ge\ &m(r,\Omega_1(r,w_1,w_2))+N(r,\Omega_1(r,w_1,w_2))=T(r,\Omega_1(r,w_1,w_2))\\ =\ &\max\{q_1,p_1\}T(r,w_1)+S(r,w_1),\end{aligned}\tag{3.17}$$

$$\begin{aligned}&N(r,\Omega_2(r,w_1,w_2))+\lambda_{21}m(r,w_1)+\lambda_{22}m(r,w_2)+S(r,w_1)+S(r,w_2)\\ \ge\ &m(r,\Omega_2(r,w_1,w_2))+N(r,\Omega_2(r,w_1,w_2))=T(r,\Omega_2(r,w_1,w_2))\\ =\ &\max\{q_2,p_2\}T(r,w_2)+S(r,w_1)\end{aligned}\tag{3.18}$$

for all r outside of an exceptional set of finite logarithmic measure.

Moreover, by Lemma 2.2, it follows that

$$N(r, \Omega_1(r, w_1, w_2)) \le \mu_{11} N(r, w_1) + \mu_{12} N(r, w_2) + S(r, w_1) + S(r, w_2), \tag{3.19}$$
$$N(r, \Omega_2(r, w_1, w_2)) \le \mu_{21} N(r, w_1) + \mu_{22} N(r, w_2) + S(r, w_1) + S(r, w_2) \tag{3.20}$$

for all r outside of an exceptional set of finite logarithmic measure. Therefore, from (3.17) and (3.19), there is

$$\begin{aligned} &\max\{q_1, p_1\} T(r, w_1) \\ &\le \mu_{11} N(r, w_1) + \mu_{12} N(r, w_2) + \lambda_{11} m(r, w_1) + \lambda_{12} m(r, w_2) + S(r, w_1) + S(r, w_2) \\ &\le (\lambda_{11} - \mu_{11}) m(r, w_1) + \mu_{11} T(r, w_1) + (\mu_{12} + \lambda_{12}) T(r, w_2) + S(r, w_1) + S(r, w_2). \end{aligned}$$

That is

$$\begin{aligned} &(\mu_{11} - \lambda_{11}) m(r, w_1) \\ &\le (\mu_{11} - \max\{q_1, p_1\} + o(1)) T(r, w_1) + (\mu_{12} + \lambda_{12} + o(1)) T(r, w_2) \end{aligned} \tag{3.21}$$

for all r outside of an exceptional set of finite logarithmic measure. By (3.18) and (3.20), there is

$$\begin{aligned} &\max\{q_2, p_2\} T(r, w_2) \\ &\le \mu_{21} N(r, w_1) + \mu_{22} N(r, w_2) + \lambda_{21} m(r, w_1) + \lambda_{22} m(r, w_2) + S(r, w_1) + S(r, w_2) \\ &\le \mu_{21} T(r, w_1) + \mu_{22} T(r, w_2) + S(r, w_1) + S(r, w_2). \end{aligned}$$

Thus

$$(\max\{q_2, p_2\} - \mu_{22} + o(1)) T(r, w_2) \le (\mu_{21} + o(1)) T(r, w_1). \tag{3.22}$$

Therefore, by the supposition that

$$\max\{q_2, p_2\} - \mu_{22} > 0,$$

combining (3.21)–(3.22), we can obtain

$$\begin{aligned} (\mu_{11} - \lambda_{11}) m(r, w_1) \le{}& (\mu_{11} - \max\{q_1, p_1\} + o(1)) T(r, w_1) \\ &+ \frac{(\mu_{12} + \lambda_{12} + o(1))(\mu_{21} + o(1))}{\max\{q_2, p_2\} - \mu_{22}} T(r, w_1). \end{aligned}$$

Therefore,

$$(\mu_{11} - \lambda_{11}) m(r, w_1) \le \left(\mu_{11} - \max\{q_1, p_1\} + \frac{\mu_{12}\mu_{21} + \lambda_{12}\mu_{21} + o(1)}{\max\{q_2, p_2\} - \mu_{22}}\right) T(r, w_1).$$

By the supposition that

$$(\mu_{11} - \max\{q_1, p_1\})(\mu_{22} - \max\{q_2, p_2\}) > \mu_{12}\mu_{21} + \lambda_{12}\mu_{21}$$

and the last inequality, we have

$$m(r, w_1) = o(T(r, w_1))$$

for all r outside of a set of finite logarithmic measure.

Similarly, we can also obtain that

$$m(r, w_2) = o(T(r, w_2))$$

for all r outside of a set of finite logarithmic measure.

Therefore, we have completed the proof of Theorem 1.2.

References

[1] Hayman, W. K., Meromorphic Functions, Oxford Mathematical Monongraphs, Clarendon Press, Oxford, 1964.

[2] Cherry, W. and Ye, Z., Nenvanlinna's Theory of Value Distribution, Springer Monographs in Mathematics, Springer-Verlag, Berlin, Germany, 2001.

[3] Laine, I., Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[4] Yi, H. X. and Yang, C. C., Theory of the Uniqueness of Meromorphic Functions, Science Press, Beijing, 1995 (in Chinese).

[5] Gao, L. Y., The growth of systems of complex nonlinear algebraic differential equations, *Acta Mathematica Scientia Series B*, **30**(5), 2010, 1507–1513.

[6] Gao, L. Y., The growth of solutions of systems of complex nonlinear algebraic differential equations, *Acta Mathematica Scientia Series B*, **30**(3), 2010, 932–938.

[7] Tu, Z. H. and Xiao, X. Z., On the meromorphic solutionas of systems of higher-order algebraic differential equations, *Complex Variables*, **15**, 1990, 197–209.

[8] Song, S. G., Meromorphic solutions of differential equations in the complex domain, *Acta Math. Sinica*, **34**, 1991, 779–784.

[9] Li, K. S. and Chan, W. L., Meromorphic solutions of higher order system of algebraic differential equations, *Math. Scand.*, **71**, 1992, 105–121.

[10] Ablowitz, M. J., Halburd, R. and Herbst, B., On the extension of the PainlevLe property to difference equations, *Nonlinearity*, **13**(3), 2000, 889–905.

[11] Heittokangas, J., Korhonen, R., Laine, I., et al., Complex difference equations of Malmquist type, *Computational Methods and Function Theory*, **1**(1), 2001, 27–39.

[12] Laine, I., Rieppo, J. and Silvennoinen, H., Remarks on complex difference equations, *Computational Methods and Function Theory*, **5**(1), 2005, 77–88.

[13] Chiang, Y. M. and Feng, S. J., On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane, *Ramanujan Journal*, **16**(1), 2008, 105–129.

[14] Li, H. C. and Gao, L. Y., Meromorphic solutions of a type of system of complex differential-difference equations, *Acta Mathematica Scientia Series B*, **35**(1), 2015, 195–206.

[15] Gao, L. Y., Systems of complex difference equations of Malmquist type, *Acta Mathematica Scientia Series B*, **55**(2), 2012, 293–300.

[16] Chen, Z. X., Fixed points of meromorphic functions and of their differences and shifts, *Ann. Polon. Math.*, **109**(2), 2013, 153–163.

[17] Gao, L. Y., Estimates of N-function and m-function of meromorphic solutions of systems of complex difference equations, *Acta Mathematica Scientia Series B*, **32**(4), 2012, 1495–1502.

[18] Li, H. C. and Gao, L. Y., Meromorphic solutions of system of functional equations, *J. Math.*, **32**(4), 2012, 593–597.

[19] Wissenborn, G., On the theorem of Tumura an Clunie, *London Math. Soc.*, **18**, 1986, 371–373.

[20] Goldstein, R., Some results on factorisation of meromorphic functions, *Journal of the London Mathematical Society*, **4**(2), 1971, 357–364.

[21] Korhonen, R., A new clunie type theorem for difference polynomials, *J. Differ. Equ. Appl.*, **17**(3), 2011, 387–400.

[22] Yanagihara, N., Meromorphic solutionas of some difference equations, *Funkcialaj Ekvacioj*, **23**, 1980, 309–326.

[23] Yanagihara, N., Meromorphic solutionas of some difference equations of the nth order, *Arch Ration. Mech. Anal.*, **91** 1983, 19–192.

[24] Shimomura, S., Entire solutins of a polynomial difference equation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **28**, 1981, 253–266.

[25] Gao, L. Y., Zhang, Y. and Li, H. C., Growth of solutions of complex non-linear algebraic differential equations, *J. Math.*, **31**(5), 2011, 785–790.

[26] Li, H. C. and Gao, L. Y., Transcendental meromorphic solutions of second-order algebraic differential equations, *J. Math. Res. Exposition*, **31**(3), 2011, 497–502.