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Constructions of Metric $(n+1)$-Lie Algebras*

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Abstract Metric n-Lie algebras have wide applications in mathematics and mathematical physics. In this paper, the authors introduce two methods to construct metric $(n+1)$-Lie algebras from metric n-Lie algebras for $n \geq 2$. For a given m-dimensional metric n-Lie algebra $(\mathfrak{g}, [, \cdots,], B_g)$, via one and two dimensional extensions $\mathfrak{L} = \mathfrak{g}\dot{+}\mathbb{F}c$ and $\mathfrak{g}_0 = \mathfrak{g}\dot{+}\mathbb{F}x^{-1}\dot{+}\mathbb{F}x^0$ of the vector space $\mathfrak{g}$ and a certain linear function f on $\mathfrak{g}$, we construct $(m+1)$- and $(m+2)$-dimensional $(n+1)$-Lie algebras $(\mathfrak{L}, [, \cdots,]_{cf})$ and $(\mathfrak{g}_0, [, \cdots,]_1)$, respectively. Furthermore, if the center $Z(\mathfrak{g})$ is non-isotropic, then we obtain metric $(n+1)$-Lie algebras $(\mathfrak{L}, [, \cdots,]_{cf}, B)$ and $(\mathfrak{g}_0, [, \cdots,]_1, B)$ which satisfy $B|_{\mathfrak{g}\times\mathfrak{g}} = B_g$. Following this approach the extensions of all $(n+2)$-dimensional metric n-Lie algebras are discussed.

Keywords n-Lie algebra, Metric n-Lie algebra, Extension, Isotropic center
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1 Introduction

Lie algebras have played an extremely important role in physics for a long time. Their generalizations, known as n-Lie algebras (see [1–3]), also arise naturally in physics and have recently been studied in the context of M-branes (see [4–8]). The applications of n-Lie algebras to M-branes, quantization of Nambu mechanics, volume-preserving diffeomorphisms, integrable systems, and related generalizations of Lax equations have been considered in [9].

Due to the multiple multiplication, structures of n-Lie algebras are complex. Ling proved in [10] that there is only one finite dimensional simple n-Lie algebra over the complex numbers under the isomorphism, which is the $(n+1)$-dimensional n-Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}^1 = n+1$. Kasymov in [11] defined nilpotent n-Lie algebras and Cartan subalgebras of n-Lie algebras, and proved that Cartan subalgebras of an n-Lie algebra are conjugate, and the similar Engel's theorem is true in n-Lie algebras. More results on structures of n-Lie algebras can be seen in [12–17]. A metric n-Lie algebra $(L, [, \cdots,], B)$ over a field F is an n-Lie algebra $(L, [, \cdots,])$ endowed with a non-degenerate symmetric bilinear form $B : L \otimes L \to F$ satisfying the ad-invariant Equation (2.2). The structure of metric 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident $M2$-branes (see [4]); the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra (see [5, 18]). Papers [19–20] studied structures of metric n-Lie algebras over the real numbers. Paper [21] discussed the isotropic ideals of metric

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n-Lie algebras over the complex numbers, and obtained the formulation on metric dimensions and numbers of minimal ideals of metric n-Lie algebras.

Realization of n-Lie algebras is very important, but it is indeed difficult. In reference [1], Fillipov constructed n-Lie algebras, called Jacobian algebras, by a commutative associative algebra A and n commutative derivations $D_1, \cdots, D_n$, where the multiplication is defined as: For $x_1, \cdots, x_n \in A$, $[x_1, \cdots, x_n] = \det(D_i(x_j))$. Authors in [17, 22–23] constructed some infinite dimensional simple Jacobian n-Lie algebras over the fields of positive characters. P. Ho, Y. Imamura and Y. Matsuo studied two derivations of the multiple $D2$ action from the multiple $M2$-brane model proposed by Bagger-Lambert and Gustavsson; they constructed metric 3-Lie algebras by arbitrary metric Lie algebras through 2-dimensional extensions (see [24]). Bai and collaborators realized 3-Lie algebras by Lie algebras, pre-Lie algebras, commutative associative algebras and linear functions and derivations (see [25–26]), and constructed $(n+1)$-Lie algebras by arbitrary n-Lie algebras (see [27]). They also constructed two-step nilpotent 3-Lie algebras by four index matrices (see [28]). In this paper we construct metric $(n+1)$-Lie algebras from metric n-Lie algebras $(n \geq 2)$ and linear functions.

Throughout this paper, all n-Lie algebras $(n \geq 2)$ are of finite dimension and over an algebraically closed field $\mathbb{F}$ of characteristic zero. Any bracket which is not listed in the multiplication of an n-Lie algebra is assumed to be zero.

2 Fundamental Notions

First we introduce some basic notions used in the paper (see [1, 21]). An n-Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ endowed with an n-ary multi-linear skew-symmetric operation $[x_1, \cdots, x_n]$ satisfying the n-Jacobi identity

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \tag{2.1}$$

The mapping $\mathrm{ad}(x_1, \cdots, x_{n-1})$: $\mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}(x_1, \cdots, x_{n-1})(x_n) = [x_1, \cdots, x_n]$ for all $x_n \in \mathfrak{g}$, is called a left multiplication defined by elements $x_1, \cdots, x_{n-1} \in \mathfrak{g}$.

A subspace I of $\mathfrak{g}$ is called a subalgebra (an ideal) if $[I, \cdots, I] \subseteq I$ $([I, \mathfrak{g}, \cdots, \mathfrak{g}] \subseteq I)$. The subalgebra generated by the vectors $[x_1, \cdots, x_n]$ for all $x_1, \cdots, x_n \in \mathfrak{g}$ is called the derived algebra of $\mathfrak{g}$, which is denoted by $\mathfrak{g}^1$. If $\mathfrak{g}^1 = 0$, then $\mathfrak{g}$ is called abelian. If $\mathfrak{g}^1 \neq 0$ and $\mathfrak{g}$ has no ideals other than 0 and itself, then $\mathfrak{g}$ is called a simple n-Lie algebra. A minimal (maximal) ideal is a nontrivial ideal I such that if $J \neq 0$ is an ideal of $\mathfrak{g}$ and $J \subseteq I$ $(J \supsetneqq I)$, then $J = I$ $(J = \mathfrak{g})$.

The subset $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y_1, \cdots, y_{n-1}] = 0, \ \forall y_1, \cdots, y_{n-1} \in \mathfrak{g}\}$ is called the center of $\mathfrak{g}$. Obviously, $Z(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$.

Let $(\mathfrak{g}, [, \cdots,])$ be an n-Lie algebra. If a non-degenerate symmetric bilinear form $B_g : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F}$ is ad-invariant, that is, for arbitrary $x_1, \cdots, x_{n-1}, y, z \in \mathfrak{g}$,

$$B_g([x_1, \cdots, x_{n-1}, y], z) = -B_g(y, [x_1, \cdots, x_{n-1}, z]), \tag{2.2}$$

then $(\mathfrak{g}, [, \cdots,], B_g)$ is called a metric n-Lie algebra, and B_g is called a metric on $\mathfrak{g}$. For a subspace W of $\mathfrak{g}$, the orthogonal complement of W is defined by

$$W^{\perp} = \{x \in \mathfrak{g} \mid B_g(w, x) = 0 \text{ for all } w \in W\}.$$

If W is an ideal, so is $W^{\perp}$ and $(W^{\perp})^{\perp} = W$. Notice that W is a minimal ideal if and only if $W^{\perp}$ is maximal. We say that W is isotropic (coisotropic) if $W \subseteq W^{\perp}$ ($W^{\perp} \subseteq W$). It is not difficult to see that $\mathfrak{g}^1 = [\mathfrak{g}, \cdots, \mathfrak{g}] = Z(\mathfrak{g})^{\perp}$.

Lemma 2.1 (see [27]) *Let $(\mathfrak{g}, [,\cdots,], B_g)$ be an m-dimensional metric n-Lie algebra with a basis $\{x_1, \cdots, x_m\}$. Suppose that x^{-1} and x_0 are not contained in $\mathfrak{g}$. Set*

$$\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1} \quad (\textit{a direct sum of vector spaces}).$$

Then $(\mathfrak{g}_0, [,\cdots,]_0, B_0)$ is a metric $(n+1)$-Lie algebra whose bracket and metric are given below. For every $1 \le k \le n+1$,

$$\begin{cases} [x_{i_1}, \ \cdots, \underbrace{x_0}_{k}, \cdots, \ x_{i_n}]_0 = (-1)^{k-1}[x_{i_1}, \cdots, x_{i_n}], \quad 1 \le i_1, \cdots, i_n \le m; \\ [x_{i_1}, \ \cdots, \underbrace{x^{-1}}_{k}, \cdots, \ x_{i_n}]_0 = 0, \quad 0 \le i_1, \cdots, i_n \le m, \\ [x_{i_1}, \ \cdots, \ x_{i_n}, \ x_{i_{n+1}}]_0 = B_g([x_{i_1}, \cdots, x_{i_n}], x_{i_{n+1}})x^{-1}, \quad 1 \le i_1, \cdots, i_{n+1} \le m, \end{cases} \tag{2.3}$$

$$B_0 : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathbb{F} : \begin{cases} B_0(x, y) = B_g(x, y), \\ B_0(x_0, x_0) = 1, \\ B_0(x^{-1}, x_0) = (-1)^{n-1}, \\ B_0(x^{-1}, y) = B_0(x^{-1}, x^{-1}) = B_0(x_0, y) = 0, \quad \forall x, y \in \mathfrak{g}. \end{cases} \tag{2.4}$$

Lemma 2.2 (see [25]) *Let $(\mathfrak{g}, [,\cdots,])$ be an n-Lie algebra and $f \in \mathfrak{g}^*$ the dual space of $\mathfrak{g}$. If $f \ne 0$ and $f(\mathfrak{g}^1) = 0$, then $(\mathfrak{g}, [,\cdots,]_f)$ is an $(n+1)$-Lie algebra, where $[,\cdots,]_f$ is defined as follows: for all $x_1, \cdots, x_{n+1} \in \mathfrak{g}$,*

$$[x_1, \cdots, x_{n+1}]_f = \sum_{i=1}^{n+1} (-1)^{i-1} f(x_i)[x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]. \tag{2.5}$$

3 One Dimensional Extensions of Metric n-Lie Algebras

Let $(\mathfrak{g}, [,\cdots,], B_g)$ be an m-dimensional metric n-Lie algebra. If $f \in \mathfrak{g}^*$ and $f(\mathfrak{g}^1) = 0$, we construct $(m+1)$-dimensional $(n+1)$-Lie algebras.

Theorem 3.1 *Let $(\mathfrak{g}, [,\cdots,], B_g)$ be a non-abelian metric n-Lie algebra. If $0 \ne f \in \mathfrak{g}^*$ and $f(\mathfrak{g}^1) = 0$, then B_g is not ad-invariant on the $(n+1)$-Lie algebra $(\mathfrak{g}, [,\cdots,]_f)$, where $[,\cdots,]_f$ is defined as Equation* (2.5).

Proof If for all $x_1, \cdots, x_n, y, z \in \mathfrak{g}$, B_g satisfies $B_g([x_1, \cdots, x_n, y]_f, z) + B_g(z, [x_1, \cdots, x_n, z]_f) = 0$, then by Equation (2.5)

$$\begin{aligned} & B_g([x_1, \cdots, x_n, y]_f, z) + B_g(y, [x_1, \cdots, x_n, z]_f) \\ = {} & B_g\Big(\sum_{i=1}^{n} (-1)^{i-1} f(x_i)[x_1, \cdots, \widehat{x}_i, \cdots, x_n, y] + (-1)^n f(y)[x_1, \cdots, x_n], z\Big) \\ & + B_g\Big(y, \sum_{i=1}^{n} (-1)^{i-1} f(x_i)[x_1, \cdots, \widehat{x}_i, \cdots, x_n, z] + (-1)^n f(z)[x_1, \cdots, x_n]\Big) \\ = {} & B_g((-1)^n f(y)[x_1, \cdots, x_n], z) + B_g(y, (-1)^n f(z)[x_1, \cdots, x_n]) \\ = {} & (-1)^n B_g([x_1, \cdots, x_n], f(z)y + f(y)z) = 0. \end{aligned}$$

Since $f \neq 0$, there is a basis $\{v_1, \cdots, v_m\}$ of $\mathfrak{g}$ such that $f(v_1) = 1$ and $f(v_i) = 0$ for $2 \leq i \leq m$. It follows that $v_1 \notin \mathfrak{g}^1$ and

$$\begin{aligned} &B_g([v_{i_1}, \cdots, v_{i_n}], f(v_1)v_k + f(v_k)v_1) \\ = &B_g([v_{i_1}, \cdots, v_{i_n}], v_k) = 0, \quad 1 \leq i_1, \cdots, i_n \leq m, \ k \geq 2. \end{aligned}$$

Since $B_g([v_{i_1}, \cdots, v_{i_n}], v_1) = -B_g(v_{i_n}, [v_{i_1}, \cdots, v_{i_{n-1}}, v_1]) = 0$, $B_g(\mathfrak{g}^1, \mathfrak{g}) = 0$. Therefore, $\mathfrak{g}^1 = 0$, a contradiction. The desired result follows.

Theorem 3.2 *Let $(\mathfrak{g}, [,\cdots,], B_g)$ be a metric n-Lie algebra and $\mathfrak{g}^1 \neq \mathfrak{g}$. Suppose that c is an element not in $\mathfrak{g}$, and suppose that $\mathfrak{L} = \mathfrak{g} \dot{+} \mathbb{F}c$ is the one-dimensional extension of the vector space $\mathfrak{g}$. Then there is a nonzero $f \in \mathfrak{g}^*$ such that $f(\mathfrak{g}^1) = 0$ and $(\mathfrak{L}, [,\cdots,]_{cf})$ is an $(n+1)$-Lie algebra, where the multiplication $[,\cdots,]_{cf}$ is defined as follows:*

$$\begin{cases} [x_1, \cdots, x_{n+1}]_{cf} = [x_1, \cdots, x_{n+1}]_f + cB_{\mathfrak{g}}([x_1, \cdots, x_n], x_{n+1}) \\ \qquad\qquad\qquad\quad = \displaystyle\sum_{i=1}^{n+1} (-1)^{i-1} f(x_i)[x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}] + cB_{\mathfrak{g}}([x_1, \cdots, x_n], x_{n+1}), \\ [x_1, \cdots, x_n, c]_{cf} = 0, \quad \forall x_1, \cdots, x_{n+1} \in \mathfrak{g}. \end{cases} \tag{3.1}$$

Furthermore, if the center $Z(\mathfrak{g})$ is non-isotropic, then there is a metric B on $(\mathfrak{L}, [,\cdots,]_{cf})$ such that $B(v, w) = B_g(v, w)$ for all $v, w \in \mathfrak{g}$.

Proof By Equations (2.2) and (2.5), we only need to prove that $[,\cdots,]_{cf}$ satisfies Equation (2.1). For all $x_1, \cdots, x_{2n+1} \in \mathfrak{g}$, by Equation (3.1),

$$\begin{aligned} &[[x_1, \cdots, x_{n+1}]_{cf}, x_{n+2}, \cdots, x_{2n+1}]_{cf} \\ = &[[x_1, \cdots, x_{n+1}]_f, x_{n+2}, \cdots, x_{2n+1}]_{cf} \\ = &[[x_1, \cdots, x_{n+1}]_f, x_{n+2}, \cdots, x_{2n+1}]_f + cB_{\mathfrak{g}}([[x_1, \cdots, x_{n+1}]_f, x_{n+2}, \cdots, x_{2n}], x_{2n+1}) \\ = &[[x_1, \cdots, x_{n+1}]_f, x_{n+2}, \cdots, x_{2n+1}]_f + (-1)^n cB_{\mathfrak{g}}([x_1, \cdots, x_{n+1}]_f, [x_{n+2}, \cdots, x_{2n+1}]), \end{aligned}$$

$$\begin{aligned} &\sum_{i=1}^{n+1} [x_1, \cdots, [x_i, x_{n+2}, \cdots, x_{2n+1}]_{cf}, \cdots, x_{n+1}]_{cf} \\ = &\sum_{i=1}^{n+1} [x_1, \cdots, [x_i, x_{n+2}, \cdots, x_{2n+1}]_f, \cdots, x_{n+1}]_{cf} \\ = &\sum_{i=1}^{n+1} [x_1, \cdots, [x_i, x_{n+2}, \cdots, x_{2n+1}]_f, \cdots, x_{n+1}]_f \\ &+ c\sum_{i=1}^{n+1} (-1)^{n-i+1} B_{\mathfrak{g}}([x_i, x_{n+2}, \cdots, x_{2n+1}]_f, [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]) \\ = &\sum_{i=1}^{n+1} [x_1, \cdots, [x_i, x_{n+2}, \cdots, x_{2n+1}]_f, \cdots, x_{n+1}]_f \\ &+ c\sum_{i=1}^{n+1} (-1)^{n-i+1} f(x_i) B_{\mathfrak{g}}([x_{n+2}, \cdots, x_{2n+1}], [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]) \\ &+ c\sum_{i=1}^{n+1} \sum_{j=2}^{n+1} (-1)^{n+i+j} f(x_{n+j}) B_{\mathfrak{g}}([x_i, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n+1}[x_1, \cdots, [x_i, x_{n+2}, \cdots, x_{2n+1}]_f, \cdots, x_{n+1}]_f \\
&\quad + (-1)^n c B_{\mathfrak{g}}([x_{n+2}, \cdots, x_{2n+1}], [x_1, \cdots, x_{n+1}]_f) \\
&\quad + c\sum_{i=1}^{n+1}\sum_{j=2}^{n+1}(-1)^{n+i+j} f(x_{n+j}) B_{\mathfrak{g}}([x_i, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n+1}\sum_{j=2}^{n+1}(-1)^{n+i+j} f(x_{n+j}) B_{\mathfrak{g}}([x_i, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]) \\
&= \sum_{i=1}^{n}\sum_{j=2}^{n+1}(-1)^{n+i+j} f(x_{n+j}) B_{\mathfrak{g}}([x_i, x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], [x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}]) \\
&\quad + \sum_{j=2}^{n+1}(-1)^{j-1} f(x_{n+j}) B_{\mathfrak{g}}([x_{n+1}, x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], [x_1, \cdots, x_n]) \\
&= \sum_{i=1}^{n}\sum_{j=2}^{n+1}(-1)^{n+i+j-1} f(x_{n+j}) B_{\mathfrak{g}}([x_1, \cdots, \widehat{x}_i, \cdots, x_n, [x_i, x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}]], x_{n+1}) \\
&\quad + \sum_{j=2}^{n+1}(-1)^{j} f(x_{n+j}) B_{\mathfrak{g}}([[x_1, \cdots, x_n], x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], x_{n+1}) \\
&= \sum_{j=2}^{n+1}(-1)^{j+1} f(x_{n+j}) B_{\mathfrak{g}}([[x_1, \cdots, x_n], x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], x_{n+1}) \\
&\quad + \sum_{j=2}^{n+1}(-1)^{j} f(x_{n+j}) B_{\mathfrak{g}}([[x_1, \cdots, x_n], x_{n+2}, \cdots, \widehat{x}_{n+j}, \cdots, x_{2n+1}], x_{n+1}) = 0.
\end{aligned}
$$

Therefore, $(\mathfrak{L}, [, \cdots,]_{cf})$ is an $(n+1)$-Lie algebra.

If the center $Z(\mathfrak{g})$ is non-isotropic, by [21, Lemma 2.3], $Z(\mathfrak{g}) \not\subseteq Z(\mathfrak{g})^{\perp} = \mathfrak{g}^1$. There is a non-zero vector $v_1 \in Z(\mathfrak{g}) - \mathfrak{g}^1$ such that $\mathfrak{g} = \mathbb{F}v_1 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}^1 \subseteq \mathfrak{g}_1$, and $[v_1, \mathfrak{g}_1, \cdots, \mathfrak{g}_1] = 0$. Then, $B_g|\mathfrak{g}_1 \times \mathfrak{g}_1$ is non-degenerate and $B_g(v_1, \mathfrak{g}_1) = 0$. We can choose vectors $v_2, \cdots, v_m \in \mathfrak{g}_1$ such that $\{v_1, \cdots, v_m\}$ is a basis of $\mathfrak{g}$, and choose $f \in \mathfrak{g}^*$ satisfying

$$
\begin{cases} f(v_1) = 1 \\ f(v_k) = 0, \quad 2 \le k \le m. \end{cases} \tag{3.2}
$$

Then $\{v_1, \cdots, v_m, c\}$ is a basis of $\mathfrak{L}$. Define the symmetric bilinear form $B : \mathfrak{L} \otimes \mathfrak{L} \to \mathbb{F}$ by

$$
\begin{cases} B(v_i, v_j) = B_g(v_i, v_j), \quad 1 \le i, j \le m, \\ B(v_1, c) = -1, \\ B(v_k, c) = 0, \quad 2 \le k \le m, \\ B(c, c) = 0, \end{cases} \tag{3.3}
$$

where B is non-degenerate, and by Equations (3.2)–(3.3), for $1 \le i_1, \cdots, i_{n+2} \le m$,

$$
\begin{aligned}
&B([v_{i_1}, \cdots, v_{i_{n+1}}]_{cf}, v_{i_{n+2}}) + B(v_{i_{n+1}}, [v_{i_1}, \cdots, v_{i_n}, v_{i_{n+2}}]_{cf}) \\
&= \sum_{t=1}^{n+1} f(v_{i_t}) B_g([v_{i_1}, \cdots, \widehat{v}_{i_t}, \cdots, v_{i_{n+1}}], v_{i_{n+2}}) + B_g([v_{i_1}, \cdots, v_{i_n}], v_{i_{n+1}}) B(c, v_{i_{n+2}})
\end{aligned}
$$

$$+\sum_{t=1}^{n} f(v_{i_t})B_g(v_{i_{n+1}},[v_{i_1},\cdots,\widehat{v}_{i_t},\cdots,v_{i_n},v_{i_{n+2}}])$$
$$+f(v_{i_{n+2}})B_g(v_{i_{n+1}},[v_{i_1},\cdots,v_{i_n}])+B_g([v_{i_1},\cdots,v_{i_n}],v_{i_{n+2}})B(c,v_{i_{n+1}})$$
$$=B_g([v_{i_1},\cdots,v_{i_n}],(f(v_{i_{n+1}})+B(c,v_{i_{n+1}}))v_{i_{n+2}}+(f(v_{i_{n+2}})+B(c,v_{i_{n+2}}))v_{i_{n+1}})=0.$$

Note also that for $\lambda\in\mathbb{F}$,

$$B([v_1,v_{i_1},\cdots,v_{i_n}]_{cf},c)+B(v_{i_n},[v_1,v_{i_1},\cdots,v_{i_{n-1}},c]_{cf})$$
$$=(-1)^n\lambda B_g(v_1,[v_{i_1},\cdots,v_{i_n}])=0.$$

Therefore, B is a metric on the $(n+1)$-Lie algebra $(\mathfrak{L},[,\cdots,]_{cf})$ and satisfies $B(v,w)=B_g(v,w)$ for all $v,w\in\mathfrak{g}$.

Let A,C be n-Lie algebras. An n-Lie algebra B is called a central extension of A by C, if the sequence of n-Lie algebras

$$0\longrightarrow C\overset{i}{\longrightarrow}B\overset{\pi}{\longrightarrow}A\longrightarrow 0$$

is exact and C is in the center of B. From Theorem 3.2 and Lemma 2.2, the $(n+1)$-Lie algebra $(\mathfrak{L},[,\cdots,]_{cf})$ is the one-dimensional central extension of the $(n+1)$-Lie algebra $(\mathfrak{g},[,\cdots,]_f)$ by the one-dimensional abelian $(n+1)$-Lie algebra Fc.

Remark 3.1 Let $(\mathfrak{g},[,\cdots,],B_g)$ be a metric n-Lie algebra. If the center $Z(\mathfrak{g})$ is isotropic, then the $(n+1)$-Lie algebra $(\mathfrak{L},[,\cdots,]_f)$ in Theorem 3.2 may not be a metric $(n+1)$-Lie algebra.

Example 3.1 Let $(\mathfrak{g},[,],B_g)$ be a metric Lie algebra with a basis $\{x_1,x_2,x_3,x_4,x_5\}$, and

$$\begin{cases}[x_1,x_2]=x_3,\\ [x_2,x_3]=x_5,\\ [x_3,x_1]=x_4;\end{cases}\qquad \begin{cases}B_g(x_1,x_5)=1,\\ B_g(x_4,x_2)=1,\\ B_g(x_3,x_3)=1.\end{cases}$$

Then the center $Z(\mathfrak{g})=\mathbb{F}x_4+\mathbb{F}x_5$ is a maximal isotropic ideal. For $f\in V^*$ with $f(x_1)=1$ and $f(x_i)=0$, $2\le i\le 5$, by Theorem 3.2, we obtain a six-dimensional 3-Lie algebra $(\mathfrak{L},[,,]_{cf})$, with the multiplication $[x_1,x_2,x_3]_{cf}=x_5+c$. By a direct computation, $(\mathfrak{L},[,,]_{cf},B)$ is a metric 3-Lie algebra, and $Z(\mathfrak{L})=\mathbb{F}x_4+\mathbb{F}x_5+\mathbb{F}c=Z(\mathfrak{g})+\mathbb{F}c$ is coisotropic.

Example 3.2 Let $(\mathfrak{g},[,,],B_g)$ be a metric 3-Lie algebra with a basis $\{x_1,x_2,x_3,x_4,x_5\}$, and

$$\begin{cases}[x_2,x_3,x_4]=x_1,\\ [x_5,x_2,x_3]=x_4,\\ [x_5,x_3,x_4]=x_2,\\ [x_5,x_2,x_4]=x_3.\end{cases}\qquad \begin{cases}B(x_1,x_5)=-1,\\ B(x_2,x_2)=1,\\ B(x_3,x_3)=-1,\\ B(x_4,x_4)=1,\\ B(x_5,x_5)=1.\end{cases}$$

Then $Z(\mathfrak{g})=\mathbb{F}x_1$ is isotropic. For $f\in\mathfrak{g}^*$, $f(x_5)=1$ and $f(x_i)=0$, $1\le i\le 4$, by Theorem 3.2, we obtain a six-dimensional 4-Lie algebra $(\mathfrak{L},[,,]_{cf})$ with the multiplication $[x_5,x_2,x_3,x_4]_{cf}=x_1-c$. By a direct computation, there does not exist a metric on $(\mathfrak{L},[,,]_{cf})$.

4 Two-Dimensional Extension of Metric n-Lie Algebras

In this section we discuss the two-dimensional extensions of metric n-Lie algebras.

Lemma 4.1 (see [27]) *Let $(\mathfrak{g}, [,\cdots,], B_g)$ be an m-dimensional metric n-Lie algebra with a basis $\{x_1, \cdots, x_m\}$ and let x_0, x_{-1} be not contained in $\mathfrak{g}$. Then the vector space $\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1}$ is an $(n+1)$-Lie algebra under the following multiplication $[,\cdots,]_0$, for all natural numbers k, $1 \le k \le n+1$,*

$$\begin{cases} [x_{i_1}, \ \cdots, \underbrace{x_0}_{k}, \cdots, \ x_{i_n}]_0 = (-1)^{k-1}[x_{i_1}, \cdots, x_{i_n}], \quad 1 \le i_1, \cdots, i_n \le m; \\ [x_{i_1}, \ \cdots, \underbrace{x^{-1}}_{k}, \cdots, \ x_{i_n}]_0 = 0, \quad 0 \le i_1, \cdots, i_n \le m; \\ [x_{i_1}, \ \cdots, \ x_{i_n}, \ x_{i_{n+1}}]_0 = B([x_{i_1}, \cdots, x_{i_n}], x_{i_{n+1}})x^{-1}, \quad 1 \le i_1, \cdots, i_{n+1} \le m. \end{cases} \tag{4.1}$$

Lemma 4.2 *Let $(\mathfrak{g}, [,\cdots,], B_g)$ be an m-dimensional metric n-Lie algebra. Then for all $x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n-1} \in \mathfrak{g}$, we have*

$$\sum_{j=1}^{n+1}(-1)^j B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]) = 0. \tag{4.2}$$

Proof By Equations (2.1)–(2.2), for all $x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n-1} \in \mathfrak{g}$,

$$\begin{aligned} &\sum_{j=1}^{n+1}(-1)^j B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]) \\ =& \sum_{j=1}^{n}(-1)^j B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]) \\ &+ (-1)^{n+1} B_g([x_{n+1}, y_1, \cdots, y_{n-1}], [x_1, \cdots, x_n]), \\ &\sum_{j=1}^{n}(-1)^j B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]) \\ =& \sum_{j=1}^{n}(-1)^{j+1} B_g(x_j, [[x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}], y_1, \cdots, y_{n-1}]) \\ =& \sum_{j=1}^{n}(-1)^{j+1} B_g(x_j, \sum_{k\neq j, k=1}^{n+1}[x_1, \cdots, \widehat{x}_j, \cdots, [x_k, y_1, \cdots, y_{n-1}], \cdots, x_{n+1}]) \\ =& \sum_{j=1}^{n}(-1)^{j+1} B_g(x_j, \sum_{k\neq j, k=1}^{n}[x_1, \cdots, \widehat{x}_j, \cdots, [x_k, y_1, \cdots, y_{n-1}], \cdots, x_{n+1}]) \\ &+ \sum_{j=1}^{n}(-1)^{j+1} B_g(x_j, [x_1, \cdots, \widehat{x}_j, \cdots, x_n, [x_{n+1}, y_1, \cdots, y_{n-1}]) \\ =& \sum_{j=1}^{n}(-1)^{n} \sum_{k\neq j, k=1}^{n} B_g([x_1, \cdots, x_j, \cdots, [x_k, y_1, \cdots, y_{n-1}], \cdots, x_n], x_{n+1}) \\ &+ \sum_{j=1}^{n}(-1)^{n} B_g([x_1, \cdots, x_j, \cdots, x_n], [x_{n+1}, y_1, \cdots, y_{n-1}]) \\ =& (n-1)(-1)^n B_g([[x_1, \cdots, x_n], y_1, \cdots, y_{n-1}]], x_{n+1}) \end{aligned}$$

$$+ n(-1)^n B_g([x_1, \cdots, x_n], [x_{n+1}, y_1, \cdots, y_{n-1}])$$
$$= (-1)^n B_g([x_1, \cdots, x_n], [x_{n+1}, y_1, \cdots, y_{n-1}]).$$

The result follows.

Theorem 4.1 *Let $(\mathfrak{g}, [, \cdots,], B_g)$ be an m-dimensional metric n-Lie algebra with a basis $\{x_1, \cdots, x_m\}$. Let $0 \neq f \in \mathfrak{g}^*$ and $f(\mathfrak{g}^1) = 0$. Suppose $x^0, x^{-1} \notin \mathfrak{g}$. Then the vector space $\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1}$ is an $(m+2)$-dimensional $(n+1)$-Lie algebra in the following multiplication, for all $x_1, \cdots, x_{n+1} \in \mathfrak{g}$, $1 \leq k \leq n+1$,*

$$\begin{cases} [x_1, \ \cdots, \ x_{n+1}]_1 = [x_1, \ \cdots, \ x_{n+1}]_f + B_g([x_1, \cdots, x_n], x_{n+1})x^{-1} \\ \qquad = \displaystyle\sum_{j=1}^{n+1} (-1)^{j-1} f(x_j)[x_1, \ \cdots, \ \widehat{x}_j, \cdots, \ x_{n+1}] \\ \qquad\quad + B_g([x_1, \cdots, x_n], x_{n+1})x^{-1}, \\ [x_1, \ \cdots, \underbrace{x^0}_{k}, \cdots, \ x_n]_1 = (-1)^{k-1}[x_1, \cdots, x_n], \\ [x_1, \ \cdots, \underbrace{x^{-1}}_{k}, \cdots, \ x_n]_1 = 0. \end{cases} \tag{4.3}$$

Moreover, if the center $Z(\mathfrak{g})$ is non-isotropic, then there is a metric B on the $(n+1)$-Lie algebra $(\mathfrak{g}_0, [, \cdots,]_1)$ such that $B(v, w) = B_g(v, w)$ for all $v, w \in \mathfrak{g}$.

Proof For all $x_1, \cdots, x_{n+1} \in \mathfrak{g}$, by Equations (3.1) and (4.3), we have

$[x_1, \cdots, x_{n+1}]_1 = [x_1, \cdots, x_{n+1}]_{x^{-1}f}$, where $[x_1, \cdots, x_{n+1}]_{x^{-1}f}$ is defined as in Equation (3.1).

Then for all $y_1, \cdots, y_n \in \mathfrak{g}$,

$$\begin{aligned} & [[x_1, \cdots, x_{n+1}]_1, y_1, \cdots, y_n]_1 \\ = & [[x_1, \cdots, x_{n+1}]_{x^{-1}f}, y_1, \cdots, y_n]_{x^{-1}f} \\ = & \sum_{j=1}^{n+1} [x_1, \cdots, [x_j, y_1, \cdots, y_n]_{x^{-1}f}, \cdots, x_{n+1}]_{x^{-1}f} \\ = & \sum_{j=1}^{n+1} [x_1, \cdots, [x_j, y_1, \cdots, y_n]_1, \cdots, x_{n+1}]_1, \\ & [[x_1, \cdots, x_n, x^0]_1, y_1, \cdots, y_n]_1 \\ = & [[x_1, \cdots, x_n], y_1, \cdots, y_n]_1 \\ = & \sum_{j=1}^{n} (-1)^j f(y_j)[[x_1, \cdots, x_n], y_1, \cdots, \widehat{y}_j, \cdots, y_n] \\ & + x^{-1}(-1)^n B_g([x_1, \cdots, x_n], [y_1, \cdots, y_n]) \\ = & [[x_1, \cdots, x_n], y_1, \cdots, y_n]_f \\ & + x^{-1}(-1)^n B_g([x_1, \cdots, x_n], [y_1, \cdots, y_n]), \\ & \sum_{j=1}^{n} [x_1, \cdots, [x_j, y_1, \cdots, y_n]_1, \cdots, x_n, x^0]_1 \end{aligned}$$

$$
\begin{aligned}
&\quad + [x_1, \cdots, x_n, [x^0, y_1, \cdots, y_n]_1]_1 \\
&= \sum_{j=1}^{n} [x_1, \cdots, [x_j, y_1, \cdots, y_n]_f, \cdots, x_n] \\
&\quad + (-1)^n [x_1, \cdots, x_n, [y_1, \cdots, y_n]]_1 \\
&= \sum_{j=1}^{n} f(x_j)[x_1, \cdots, \widehat{x_j}, [y_1, \cdots, y_n], \cdots, x_n] \\
&\quad + (-1)^n [x_1, \cdots, x_n, [y_1, \cdots, y_n]]_f \\
&\quad + \sum_{j,k=1}^{n} (-1)^k f(y_k)[x_1, \cdots, [x_j, y_1, \cdots, \widehat{y}_k, \cdots, y_n], \cdots, x_n] \\
&\quad + (-1)^n B_g([x_1, \cdots, x_n], [y_1, \cdots, y_n]) \\
&= \sum_{k=1}^{n} f(y_k)[[x_1, \cdots, x_n], y_1, \cdots, \widehat{y}_k, \cdots, y_n] \\
&\quad + (-1)^n B_g([x_1, \cdots, x_n], [y_1, \cdots, y_n]) \\
&= [[x_1, \cdots, x_n], y_1, \cdots, y_n]_f + (-1)^n B_g([x_1, \cdots, x_n], [y_1, \cdots, y_n]) \\
&= [[x_1, \cdots, x_n, x^0]_1, y_1, \cdots, y_n]_1.
\end{aligned}
$$

By Lemma 4.2,

$$
\begin{aligned}
&\sum_{j=1}^{n+1} [x_1, \cdots, [x_j, y_1, \cdots, y_{n-1}, x^0]_1, \cdots, x_{n+1}]_1 \\
&= \sum_{j=1}^{n+1} [x_1, \cdots, [x_j, y_1, \cdots, y_{n-1}], \cdots, x_{n+1}]_1 \\
&= \sum_{j=1}^{n+1} \sum_{k \neq j, k=1}^{n+1} (-1)^{k-1} f(x_k)[x_1, \cdots, \widehat{x}_k, \cdots, [x_j, y_1, \cdots, y_{n-1}], \cdots, x_{n+1}] \\
&\quad + x^{-1}\Big(\sum_{j=1}^{n+1} (-1)^{n+1-j} B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]\Big) \\
&= [[x_1, \cdots, x_{n+1}]_f, y_1, \cdots, y_{n-1}] \\
&\quad + x^{-1}\Big(\sum_{j=1}^{n+1} (-1)^{n+1-j} B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]\Big) \\
&= [[x_1, \cdots, x_{n+1}]_1, y_1, \cdots, y_{n-1}, x^0]_1 \\
&\quad + x^{-1}\Big(\sum_{j=1}^{n+1} (-1)^{n+1-j} B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]\Big), \\
&= [[x_1, \cdots, x_{n+1}]_1, y_1, \cdots, y_{n-1}, x^0]_1 \\
&\quad + x^{-1}(-1)^{n+1}\Big(\sum_{j=1}^{n+1} (-1)^{j} B_g([x_j, y_1, \cdots, y_{n-1}], [x_1, \cdots, \widehat{x}_j, \cdots, x_{n+1}]\Big), \\
&= [[x_1, \cdots, x_{n+1}]_1, y_1, \cdots, y_{n-1}, x^0]_1.
\end{aligned}
$$

Thanks to Equation (4.3),

$$\begin{aligned}
&[[x_1,\cdots,x_n,x^0]_1,y_1,\cdots,y_{n-1},x^0]_1\\
=&[[x_1,\cdots,x_n],y_1,\cdots,y_{n-1}]\\
=&\sum_{j=1}^{n}[[x_1,\cdots,[x_j,y_1,\cdots,y_{n-1}],\cdots,x_n]\\
=&\sum_{j=1}^{n}[[x_1,\cdots,[x_j,y_1,\cdots,y_{n-1},x^0],\cdots,x_n,x^0]+[[x_1,\cdots,x_n,[x^0,y_1,\cdots,y_{n-1},x^0].
\end{aligned}$$

Therefore, $(\mathfrak{g}_0=\mathfrak{g}\oplus\mathbb{F}x^0\oplus\mathbb{F}x^{-1},\ [,\cdots,]_1)$ is an $(n+1)$-Lie algebra in the multiplication defined by Equation (4.3).

If the center $Z(\mathfrak{g})$ of the metric n-Lie algebra $(\mathfrak{g},[,\cdots,],B_{\mathfrak{g}})$ is non-isotropic, we denote $\mathfrak{g}_1=\mathfrak{g}\oplus\mathbb{F}x^{-1}$. Then by Equations (4.3) and (3.1), $(\mathfrak{g}_1,[,\cdots,]_1)$ is a subalgebra of the $(n+1)$-Lie algebra $(\mathfrak{g}_0,[,\cdots,]_1)$. Define a linear isomorphism

$$\sigma:\mathfrak{g}_1=\mathfrak{g}+\oplus\,\mathbb{F}x^{-1}\to\mathfrak{L}=\mathfrak{g}\oplus\ \mathbb{F}c:\begin{cases}\sigma(x)=x, & x\in\mathfrak{g},\\ \sigma(x^{-1})=c.\end{cases}$$

By Equations (3.1) and (4.3), σ is an algebraic isomorphism from $(\mathfrak{g}_1,[,\cdots,]_1)$ to $(\mathfrak{L},[,\cdots,]_{cf})$. By Theorem 3.2 and Equation (3.3), there is a metric $B_1:\mathfrak{g}_1\times\mathfrak{g}_1\to\mathbb{F}$ such that $B_1|_{g\times g}=B_g$. Now define $B:\mathfrak{g}_0\times\mathfrak{g}_0\to\mathbb{F}$, for all $x_1,x_2\in\mathfrak{g}$,

$$\begin{cases}B(x_1,x_2)=B_1(x_1,x_2),\\ B(x_1,x^{-1})=B_1(x_1,x^{-1}),\\ B(x^{-1},x^{-1})=B_1(x^{-1},x^{-1}),\\ B(x^0,x^{-1})=-1,\\ B(x^0,x^0)=B(x^0,\mathfrak{g})=0.\end{cases}$$

Then B is non-degenerate on $\mathfrak{g}_0$. According to Equations (4.3)–(4.4) and a direct computation, we get that for all $x_1,\cdots,x_{n+2}\in\mathfrak{g}$,

$$\begin{aligned}
&B([x_1,\cdots,x_{n+1}]_1,x^0)=B_g([x_1,\cdots,x_n],x_{n+1})B(x^{-1},x^0)\\
=&-B_g([x_1,\cdots,x_n],x_{n+1})=-B([x_1,\cdots,x_n,x^0]_1,x_{n+1}),\\
&B([x_1,\cdots,x_{n+1}]_1,x_{n+2})=B_1([x_1,\cdots,,x_{n+1}]_f,x_{n+2})\\
=&-B_1([x_1,\cdots,x_n,x_{n+2}]_1,x_{n+1})=-B([x_1,\cdots,x_n,x_{n+2}]_1,x_{n+1}).
\end{aligned}$$

Therefore, $(\mathfrak{g},[,\cdots,]_1,B)$ is a metric $(n+1)$-Lie algebra and satisfies $B|_{\mathfrak{g}\otimes\mathfrak{g}}=B_g$.

Remark 4.1 Let $(\mathfrak{g},[,\cdots,],B_g)$ be a metric n-Lie algebra. If the center $Z(\mathfrak{g})$ is isotropic, then the $(n+1)$-Lie algebra in Theorem 4.3 may not be a metric $(n+1)$-Lie algebra. The counterexample can be seen in Section 5.

Let $(\mathfrak{g},[,],B_g)$ be the five-dimensional metric Lie algebra in Example 3.1, and $f\in\mathfrak{g}^*$, $f(x_1)=1$ and $f(x_i)=0$ for $2\le i\le 5$. Then the center $Z(\mathfrak{g})=\mathbb{F}x_4+\mathbb{F}x_5$ is a maximal isotropic ideal. Suppose that x^0,x^{-1} are not contained in $\mathfrak{g}$, and $\mathfrak{g}_0=\mathfrak{g}\oplus\mathbb{F}x^0\oplus\mathbb{F}x^{-1}$. By

Theorem 4.3, $(\mathfrak{g}_0, [, \cdots,]_1)$ is an $(n+1)$-Lie algebra with the multiplication

$$\begin{cases} [x^0, x_1, x_2]_1 = x_3, \\ [x^0, x_2, x_3]_1 = x_5, \\ [x^0, x_3, x_1]_1 = x_4, \\ [x_1, x_2, x_3]_1 = x_5 + x^{-1}. \end{cases}$$

Define $B : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathbb{F}$,

$$\begin{cases} B(x_1, x_5) = 1, \\ B(x_2, x_4) = 1, \\ B(x_3, x_3) = 1, \\ B(x_1, x^{-1}) = -1, \\ B(x^0, x^{-1}) = -1, \\ B(x^0, x_1) = B(x^0, x_i) = B(x^{-1}, x_i) = 0, \quad 2 \leq i \leq 5. \end{cases}$$

By the direct computation, $(\mathfrak{g}_0, [, ,]_1, B)$ is a metric 3-Lie algebra, and $Z(\mathfrak{g}_0) = \mathbb{F}x_4 + \mathbb{F}x_5 + \mathbb{F}x^{-1} = Z(\mathfrak{g}) + \mathbb{F}x^{-1}$ is isotropic.

5 The Extensions of $(n+2)$-Dimensional Metric n-Lie Algebras

In this section, we give applications of Section 3 and Section 4. We provide the extensions of all $(n+2)$-dimensional metric n-Lie algebras.

By the paper [21] and Theorem 3.2 in [29], if $(\mathfrak{g}, [, \cdots,], B_g)$ is an $(n+2)$-dimensional metric n-Lie algebra, then $(\mathfrak{g}, [, \cdots,], B_g)$ is isomorphic to one and only one of the following:

$$(r^1) \quad \begin{cases} [x_2, \cdots, x_{n+1}] = x_1, \\ [x_{n+2}, x_2, \cdots, \widehat{x_i}, \cdots, x_{n+1}] = x_i, \quad 2 \leq i \leq n+1. \\ B_g(x_i, x_j) = (-1)^{n-i+1}\lambda\delta_{ij}, \quad 2 \leq i, j \leq n+1, \\ B_g(x_1, x_{n+2}) = (-1)^n\lambda, \quad \lambda \in \mathbb{F},\ \lambda \neq 0, \\ B(x_{n+2}, x_{n+2}) = \mu, \quad \mu \in \mathbb{F},\ \mu \neq 0, \\ B_g(x_1, x_i) = 0, \quad 1 \leq i \leq n+1, \\ B_g(x_{n+2}, x_i) = 0, \quad 2 \leq i \leq n+1, \end{cases} \tag{5.1}$$

$$(r^2) \quad \begin{cases} [x_1, \cdots, \widehat{x_i}, \cdots, x_{n+1}] = x_i, \quad 1 \leq i \leq n+1. \\ B(x_i, x_j) = (-1)^{n-i+1}\lambda\delta_{ij}, \quad \lambda \in \mathbb{F},\ \lambda \neq 0,\ 1 \leq i, j \leq n+1, \\ B_g(x_{n+2}, x_{n+2}) = \mu, \quad \mu \in \mathbb{F},\ \mu \neq 0, \\ B_g(x_{n+2}, x_i) = 0, \quad 1 \leq i \leq n+1, \end{cases} \tag{5.2}$$

where $\{x_1, \cdots, x_{n+1}, x_{n+2}\}$ is a basis of $\mathfrak{g}$ and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Now we discuss the extensions of the cases (r^1) and (r^2), respectively.

(a) Let $(\mathfrak{g}, [, \cdots,], B_g)$ be isomorphic to the case (r^1). Then by Equation (5.1), $Z(\mathfrak{g}) = \mathbb{F}x_1$ is isotropic.

($\mathrm{i_a}$) Let c not be contained in $\mathfrak{g}$. Set

$$\mathfrak{L} = \mathfrak{g} \oplus \mathbb{F}c$$

and

$$f \in \mathfrak{g}^*, \quad f(x_{n+2}) = 1$$

and

$$f(x_k) = 0, \quad 1 \le k \le n+1.$$

Then by Theorem 3.2 and Equations (3.1) and (5.1), we obtain an $(n+3)$-dimensional $(n+1)$-Lie algebra $(\mathfrak{L}, [, \cdots,]_{cf})$ with the multiplication

$$\begin{cases} [x_{n+2}, x_2, \cdots, x_{n+1}]_{cf} = x_1 + \lambda c, \\ [x_{n+2}, x_1, x_2, \cdots, \widehat{x}_i, \cdots, x_{n+1}]_{cf} = 0, \quad 2 \le i \le n+2, \\ [c, x_1, \cdots, \widehat{x}_i, \cdots, \widehat{x}_j, \cdots, x_{n+2}]_{cf} = 0, \quad 1 \le i, j \le n+2. \end{cases}$$

The center $Z(\mathfrak{L}) = \mathbb{F}x_1 \oplus \mathbb{F}c$.

If a symmetric bilinear form $B : \mathfrak{L} \times \mathfrak{L} \to \mathbb{F}$ satisfies Equation (2.2), then by a direct computation we get $B(x_1 + \lambda c, \mathfrak{L}) = 0$, that is, B is degenerate. Therefore, there does not exist a metric on $(\mathfrak{L}, [, \cdots,]_{cf})$.

($\mathrm{ii_a}$) Let x^0, x^{-1} be not contained in $\mathfrak{g}$. Set

$$\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1}$$

and

$$f \in \mathfrak{g}^*, \quad f(x_{n+2}) = 1$$

and

$$f(x_k) = 0, \quad 1 \le k \le n+1.$$

Then by Theorem 4.3 and Equations (4.3) and (5.1), the multiplication of the $(n+4)$-dimensional $(n+1)$-Lie algebra $(\mathfrak{g}_0, [, \cdots,]_1)$ is as follows:

$$\begin{cases} [x^0, x_2, \cdots, x_{n+1}]_1 = x_1, \\ [x^0, x_{n+2}, x_2, \cdots, \widehat{x}_i, \cdots, x_{n+1}]_1 = x_i, \quad 2 \le i \le n+1, \quad Z(\mathfrak{g}_0) = \mathbb{F}x_1 \oplus \mathbb{F}x^{-1}, \\ [x_{n+2}, x_2, \cdots, x_{n+1}]_1 = x_1 + \lambda x^{-1}. \end{cases}$$

For every symmetric bilinear form $B : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathbb{F}$ satisfying $B|_{\mathfrak{g} \otimes \mathfrak{g}} = B_g$ and Equation (2.2), we have $B(x_1, \mathfrak{g}_0) = 0$, that is, B is degenerate. Therefore, there does not exist a metric on the $(n+1)$-Lie algebra $(\mathfrak{g}_0, [, \cdots,]_1)$.

(b) Let $(\mathfrak{g}, [, \cdots,], B_g)$ be isomorphic to the case (r^2). Then by Equation (5.2), $Z(\mathfrak{g}) = \mathbb{F}x_{n+2}$ is non-isotropic.

($\mathrm{i_b}$) Let c not be contained in $\mathfrak{g}$. Set

$$\mathfrak{L} = \mathfrak{g} \oplus \mathbb{F}c$$

and

$$f \in \mathfrak{g}^*, \quad f(x_{n+2}) = 1$$

and

$$f(x_k) = 0, \quad 1 \le k \le n+1.$$

Then by Theorem 3.2 and Equations (3.1) and (5.2), the multiplication and the metric of the $(n+3)$-dimensional metric $(n+1)$-Lie algebra $(\mathfrak{L}, [, \cdots,]_{cf}, B)$ are as follows:

$$\begin{cases} [x_1, \ \cdots, \ x_{n+1}]_{cf} = c, \\ [x_{n+2}, x_1, \ \cdots, \ \widehat{x}_i, \cdots, , x_{i_{n+1}}]_{cf} = x_i, \quad 1 \le i \le n+1, \\ [c, x_{i_1}, \ \cdots, \ x_{i_n}]_{cf} = 0, \quad 0 \le i_1, \cdots, i_{n+1} \le n+2, \end{cases}$$

$$\begin{cases} B(x_i, x_j) = (-1)^{n-i+1} \lambda \delta_{ij}, \quad 1 \le i, j \le n+1, \ \lambda \in \mathbb{F}, \ \lambda \ne 0, \\ B(x_{n+2}, x_{n+2}) = \mu, \quad \mu \in \mathbb{F}, \ \mu \ne 0, \\ B(c, x_{n+2}) = (-1)^{n+1} \lambda, \\ B(c, x_k) = B(c, c) = 0, \quad 1 \le k \le n+1. \end{cases}$$

The center $Z(\mathfrak{L}) = \mathbb{F}c$ is isotropic.

(ii$_\mathrm{b}$) Let x^0, x^{-1} not be contained in $\mathfrak{g}$. Set

$$\mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{F}x^0 \oplus \mathbb{F}x^{-1}$$

and

$$f \in \mathfrak{g}^*, \quad f(x_{n+2}) = 1$$

and

$$f(x_k) = 0, \quad 1 \le k \le n+1.$$

Then by Theorem 4.3 and Equations (4.3)–(4.4) and (5.2), the multiplication and the metric of $(\mathfrak{g}_0, [, \cdots,]_1, B)$ are as follows:

$$\begin{cases} [x^0, x_1, \ \cdots, \ \widehat{x}_i, \cdots, x_{n+1}]_1 = x_i, \quad 1 \le i \le n+1, \\ [x_{n+2}, x_1, \ \cdots, \ \widehat{x}_i, \cdots, x_{n+1}]_1 = x_i, \quad 1 \le i \le n+1, \\ [x_1, \ \cdots, \ x_{n+1}]_1 = \lambda x^{-1}, \\ [x^{-1}, x_{i_1}, \cdots, x_{i_n}]_1 = 0, \quad 0 \le i_1, \ \cdots, \ i_{n+1} \le n+2, \end{cases}$$

$$\begin{cases} B(x_i, x_j) = (-1)^{n-i+1} \lambda \delta_{ij}, \quad 1 \le i, j \le n+1, \ \lambda \in \mathbb{F}, \ \lambda \ne 0, \\ B(x_{n+2}, x_{n+2}) = \mu, \quad \mu \in \mathbb{F}, \ \mu \ne 0, \\ B(x^{-1}, x_{n+2}) = (-1)^{n+1} \lambda, \\ B(x^{-1}, x^0) = (-1)^{n+1} \lambda, \\ B(x^{-1}, x_k) = 0, \quad 1 \le k \le n+1, \\ B(x^{-1}, x^{-1}) = B(x^0, x^0) = B(x^0, x_k) = 0, \quad 1 \le k \le n+2, \end{cases}$$

and $Z(\mathfrak{g}_0) = \mathbb{F}x^{-1} + \mathbb{F}(x_{n+2} - x^0)$ is non-isotropic.

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