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Mean Value of Kloosterman Sums over Short Intervals*

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Abstract This paper is concerned with a kind of mean value problem of Kloosterman sums, which will lead to a sum of Kloosterman sums over short intervals.

Keywords Mean value, Kloosterman sum, Short intervals
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1 Introduction

It is well known that various kinds of exponential sums play an important role in number theory, and one of these is the classical Kloosterman sum defined by

$$S(m,n;q)=\sum_{\substack{a \bmod q\\(a,q)=1}} e\Big(\frac{m\overline{a}+na}{q}\Big),$$

where $e(x)=\mathrm{e}^{2\pi \mathrm{i}x}$ and $\overline{a}$ satisfies $\overline{a}a\equiv 1 \pmod q$. A famous estimate of this sum is

$$S(m,n;q)\ll q^{\frac{1}{2}}(m,n,q)^{\frac{1}{2}}d(q), \tag{1.1}$$

with $d(q)$ being the divisor function, and (m,n,q) being the greatest common divisor of m, n and q.

A great deal of subjects involving this sum have been studied, and one of the interesting problems is to study the sum of Kloosterman sums. Many results have been given to date. For instance, when p is an odd prime, Kloosterman in [1] proved the identity

$$\sum_{a=1}^{p-1} S^4(a,1;p)=2p^3-3p^2-p-1.$$

Salié [2] and Davenport gave

$$\sum_{a=1}^{p-1} S^6(a,1;p)\ll p^4.$$

Although it seems that hardly any results have been given for the mean values over short intervals, this is indeed one of the problems that will be studied in this paper.

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This work is mainly concerned with the mean value of Kloosterman sums of the following type:

$$\sum_{\substack{c\le q\\(c,q)=1}} e\Big(\frac{lc}{q}\Big)S(a,c;q)S(b,c;q).$$

The main theorem can be described as follows.

Theorem 1.1 *Let $q\ge 3$ be an arbitrary integer, and a, b, u, v be four integers satisfying $u\mid q$, $v\mid q$, $(a,u)=(b,v)=1$. Then we have*

$$\sum_{l\le q-1}\frac{1}{\|\frac{l}{q}\|}\Big|\sum_{\substack{c\le q\\(c,q)=1}} e\Big(\frac{lc}{q}\Big)S(a,c;u)S(b,c;v)\Big| \ll q^{3-\frac{1}{2\omega(q)}}d^2(q)\log q,$$

where $\|x\|$ is the distance of x from the nearest integer, and $\omega(q)$ denotes the number of different prime divisors of q.

Based on this inequality, we will investigate the mean value properties of Kloosterman sums in short intervals, and obtain the following theorem.

Theorem 1.2 *Suppose that $q\ge 3$ is an integer, and $(a,q)=(b,q)=1$. For $N\le q$, we have*

$$\sum_{c\le N}{}' S(a,c;q)S(b,c;q) = N\sum_{d|q}\frac{\mu(d)}{d}\sum_{u\le q}{}'\sum_{\substack{v\le q\\u+v\equiv 0 \pmod{\frac{q}{d}}}}{}' e\Big(\frac{au+bv}{q}\Big)+O(q^{2-\frac{1}{2\omega(q)}}d^2(q)\log q),$$

where $\mu(n)$ is a Möbius function, and $\sum_n{}'$ abbreviates $\sum_{\substack{n\\(n,q)=1}}$.

A much simpler result can be obtained while $a=b$.

Corollary 1.1 *Suppose that $q\ge 3$ is an integer, which can be written in the form $q=Q_1Q_2$ with $(Q_1,Q_2)=1$, $\mu(Q_2)\ne 0$ and Q_1 being a squarefull number (i.e., $p\mid Q_1 \Longrightarrow p^2\mid Q_1$). Then for $(a,q)=1$ and $N\le q$,*

$$\sum_{c\le N}{}' S^2(a,c;q) = Nq\prod_{p\,|\,Q_1}\frac{p-1}{p}\prod_{\substack{p\,|\,Q_2\\p>2}}\frac{p^2-p-1}{p^2}+O(q^{2-\frac{1}{2\omega(q)}}d^2(q)\log q).$$

Especially when q is an odd prime, we have

$$\sum_{c\le N} S^2(a,c;p)=Np+O(p^{\frac{3}{2}}\log p).$$

From these we may reckon that the distribution of $S^2(a,c;q)$ ($(a,q)=1$) is quite "average" when c runs through the reduced residue class modulo q.

Remark 1.1 For a positive integer n, $\{a,n\}$ stands for the multiple inverse of a modulo n, that is

$$1\le\{a,n\}\le n \quad\text{and}\quad \{a,n\}a\equiv 1 \pmod n.$$

2 Auxiliary Lemmas

This section is devoted to the preparations for the proof of the theorems.

Lemma 2.1 *Let p be a prime number, and α and β be two integers with $\alpha \geq \beta \geq 1$. Then*

$$\sum_{c\leq p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big)S(a,c;p^\alpha)S(b,c;p^\alpha)\begin{cases} = 0, & \text{if } a \not\equiv b \pmod{p^{\alpha-\beta}},\\ \ll d(p^\alpha)p^{2\alpha-\frac{1}{2}\beta}(a,p^{2\alpha-\beta})^{\frac{1}{2}}, & \text{if } a\equiv b \pmod{p^{\alpha-\beta}},\end{cases}$$

where the dash indicates that the sum runs through integers coprime with p.

Proof From the definition of Kloosterman sums, we have

$$\begin{aligned}
&\sum_{c\leq p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big)S(a,c;p^\alpha)S(b,c;p^\alpha)\\
&= \sum_{c\leq p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big)\sum_{u\leq p^\alpha}{}' e\Big(\frac{a\overline{u}+cu}{p^\alpha}\Big)\sum_{v\leq p^\alpha}{}' e\Big(\frac{b\overline{v}+cv}{p^\alpha}\Big)\\
&= \sum_{u\leq p^\alpha}{}'\sum_{v\leq p^\alpha}{}' e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big)\sum_{c\leq p^\alpha}{}' e\Big(\frac{u+v+p^{\alpha-\beta}}{p^\alpha}c\Big)\\
&= p^\alpha \sum_{\substack{u\leq p^\alpha \ v\leq p^\alpha\\ u+v+p^{\alpha-\beta}\equiv 0 \ (\mathrm{mod}\ p^\alpha)}}{}'{}' e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big) - p^{\alpha-1}\sum_{\substack{u\leq p^\alpha \ v\leq p^\alpha\\ u+v+p^{\alpha-\beta}\equiv 0 \ (\mathrm{mod}\ p^{\alpha-1})}}{}'{}' e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big)\\
&= \frac{p^\alpha}{p^{2(\alpha-\beta)}}\sum_{\substack{u\leq p^{2\alpha-\beta} \ v\leq p^{2\alpha-\beta}\\ u+v+p^{\alpha-\beta}\equiv 0 \ (\mathrm{mod}\ p^\alpha)}}{}'{}' e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big) - \frac{p^{\alpha-1}}{p^{2(\alpha-\beta)}}\sum_{\substack{u\leq p^{2\alpha-\beta} \ v\leq p^{2\alpha-\beta}\\ u+v+p^{\alpha-\beta}\equiv 0 \ (\mathrm{mod}\ p^{\alpha-1})}}{}'{}' e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big)\\
&= \mathrm{A}_1 - \mathrm{A}_2, \qquad (2.1)
\end{aligned}$$

where, in the last equation but one, $\overline{u}$ and $\overline{v}$ mean the multiple inverse modulo $p^{2\alpha-\beta}$, i.e., $u\overline{u}\equiv v\overline{v}\equiv 1 \pmod{p^{2\alpha-\beta}}$.

A special case with $\alpha=\beta$ will be discussed first. In this circumstance,

$$\begin{aligned}
\mathrm{A}_1 &= p^\alpha \sum_{\substack{u\leq p^\alpha \ v\leq p^\alpha\\ u+1+\overline{v}\equiv 0 \ (\mathrm{mod}\ p^\alpha)}}{}'{}' e\Big(\frac{a\overline{u}+b}{p^\alpha}\overline{v}\Big) = p^\alpha\sum_{u\leq p^\alpha}{}' e\Big(\frac{a\overline{u}+b}{p^\alpha}(-u-1)\Big)\\
&= p^\alpha e\Big(-\frac{a+b}{p^\alpha}\Big)S(a,b;p^\alpha) \ll (\alpha+1)p^{\frac{3}{2}\alpha}(a,b,p^\alpha)^{\frac{1}{2}} \qquad (2.2)
\end{aligned}$$

in virtue of (1.1). The calculation of A_2 should be split into two cases. If $\alpha>1$, then

$$\begin{aligned}
\mathrm{A}_2 &= p^{\alpha-1}\sum_{\substack{u\leq p^\alpha \ v\leq p^\alpha\\ u+1+\overline{v}\equiv 0 \ (\mathrm{mod}\ p^{\alpha-1})}}{}'{}' e\Big(\frac{a\overline{u}+b}{p^\alpha}\overline{v}\Big)\\
&= p^{\alpha-1}\sum_{\substack{u\leq p^\alpha\\ p\nmid u+1}}{}'\sum_{l\leq p} e\Big(\frac{a\overline{u}+b}{p^\alpha}(-u-1+lp^{\alpha-1})\Big)\\
&= p^{\alpha-1}e\Big(-\frac{a+b}{p^\alpha}\Big)\sum_{l\leq p} e\Big(\frac{bl}{p}\Big)S((lp^{\alpha-1}-1)a,-b;p^\alpha)
\end{aligned}$$

$$
\begin{aligned}
&- p^{\alpha-1} e\Big(-\frac{a+b}{p^{\alpha}}\Big)\sum_{l\le p} e\Big(\frac{bl}{p}\Big)\cdot\frac{1}{p}\sum_{j\le p} e\Big(\frac{j}{p}\Big) S((lp^{\alpha-1}-1)a, jp^{\alpha-1}-b; p^{\alpha})\\
&\ll (\alpha+1)p^{\frac{3}{2}\alpha-2}\sum_{l\le p}\sum_{j\le p}\{((lp^{\alpha-1}-1)a,-b,p^{\alpha})^{\frac12}+((lp^{\alpha-1}-1)a, jp^{\alpha-1}-b,p^{\alpha})^{\frac12}\}\\
&\ll (\alpha+1)p^{\frac{3}{2}\alpha}(a,p^{\alpha})^{\frac12}. \qquad (2.3)
\end{aligned}
$$

For $\alpha=\beta=1$, we have

$$
\mathrm{A}_2 = \sum_{u\le p}{}' e\Big(\frac{au}{p}\Big)\sum_{v\le p}{}' e\Big(\frac{bv}{p}\Big) \ll \begin{cases} p, & \text{if } p\nmid a,\\ p^2, & \text{if } p\mid a.\end{cases} \qquad (2.4)
$$

Combining (2.1)–(2.4), we complete the proof for $\alpha=\beta$.

Now we suppose that $1\le\beta<\alpha$. We still begin from the calculation of A_1.

$$
\begin{aligned}
\mathrm{A}_1 &= \frac{p^{\alpha}}{p^{2(\alpha-\beta)}}\sum_{\substack{u\le p^{2\alpha-\beta}\\ u+1+p^{\alpha-\beta}\overline{v}\equiv 0 \ (\mathrm{mod}\ p^{\alpha})}}{}'\ \sum_{v\le p^{2\alpha-\beta}}{}' e\Big(\frac{a\overline{u}+b}{p^{\alpha}}\overline{v}\Big)\\
&= p^{2\beta-\alpha}\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta}\|u+1}}{}'\ \sum_{l\le p^{2(\alpha-\beta)}} e\Big(\frac{a\overline{u}+b}{p^{\alpha}}\Big(-\frac{u+1}{p^{\alpha-\beta}}+lp^{\beta}\Big)\Big)\\
&= p^{2\beta-\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta}\|u+1}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\sum_{l\le p^{2(\alpha-\beta)}} e\Big(\frac{a\overline{u}+b}{p^{\alpha-\beta}}l\Big)\\
&= p^{2\beta-\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta}\|u+1}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\sum_{l\le p^{2(\alpha-\beta)}} e\Big(\frac{b-a}{p^{\alpha-\beta}}l\Big),
\end{aligned}
$$

so $\mathrm{A}_1=0$ when $a\not\equiv b \ (\mathrm{mod}\, p^{\alpha-\beta})$. Otherwise,

$$
\begin{aligned}
\mathrm{A}_1 &= p^{\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta}\|u+1}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\\
&= p^{\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta}|u+1}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big) - p^{\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{\substack{u\le p^{2\alpha-\beta}\\ p^{\alpha-\beta+1}|u+1}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\\
&= p^{\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{u\le p^{2\alpha-\beta}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\cdot\frac{1}{p^{\alpha-\beta}}\sum_{j\le p^{\alpha-\beta}} e\Big(\frac{u+1}{p^{\alpha-\beta}}j\Big)\\
&\quad - p^{\alpha} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{u\le p^{2\alpha-\beta}}{}' e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big)\cdot\frac{1}{p^{\alpha-\beta+1}}\sum_{j\le p^{\alpha-\beta+1}} e\Big(\frac{u+1}{p^{\alpha-\beta+1}}j\Big)\\
&= p^{\beta} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{j\le p^{\alpha-\beta}} e\Big(\frac{j}{p^{\alpha-\beta}}\Big) S\Big(a, b-p^{\alpha}j; p^{2\alpha-\beta}\Big)\\
&\quad - p^{\beta-1} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big)\sum_{j\le p^{\alpha-\beta+1}} e\Big(\frac{j}{p^{\alpha-\beta+1}}\Big) S(a, b-p^{\alpha-1}j; p^{2\alpha-\beta})\\
&\ll (2\alpha-\beta+1)p^{\alpha+\frac12\beta-1}\Big(p\sum_{j\le p^{\alpha-\beta}}(a, b-p^{\alpha}j, p^{2\alpha-\beta})^{\frac12}+\sum_{j\le p^{\alpha-\beta+1}}(a, b-p^{\alpha-1}j, p^{2\alpha-\beta})^{\frac12}\Big)
\end{aligned}
$$

$$\ll d(p^\alpha)p^{2\alpha-\frac{1}{2}\beta}(a,p^{2\alpha-\beta})^{\frac{1}{2}}. \tag{2.5}$$

The estimation of A_2 is just a bit more complicated.

$$A_2 = \frac{p^{\alpha-1}}{p^{2(\alpha-\beta)}} \mathop{\sum\nolimits' \sum\nolimits'}_{\substack{u\le p^{2\alpha-\beta}\ v\le p^{2\alpha-\beta} \\ u+v+p^{\alpha-\beta}\equiv 0 \pmod{p^{\alpha-1}}}} e\Big(\frac{a\overline{u}+b\overline{v}}{p^\alpha}\Big),$$

and we consider two different situations:

(i) $\beta = 1$. Then

$$\begin{aligned} A_2 &= \frac{1}{p^{\alpha-1}} \sideset{}{'}\sum_{\substack{u\le p^{2\alpha-1} \\ u+1\equiv 0 \pmod{p^{\alpha-1}}}} \sideset{}{'}\sum_{v\le p^{2\alpha-1}} e\Big(\frac{a\overline{u}+b}{p^\alpha}\overline{v}\Big) \\ &= p^{2\alpha-1} \sideset{}{'}\sum_{\substack{u\le p^{\alpha} \\ u+1\equiv 0 \pmod{p^{\alpha-1}} \\ au+b\equiv 0 \pmod{p^\alpha}}} 1 \ - \ p^{2(\alpha-1)} \sideset{}{'}\sum_{\substack{u\le p^{\alpha} \\ u+1\equiv 0 \pmod{p^{\alpha-1}} \\ au+b\equiv 0 \pmod{p^{\alpha-1}}}} 1, \end{aligned}$$

so that A_2 vanishes when $a \not\equiv b \pmod{p^{\alpha-1}}$. Otherwise, write $a = b + mp^{\alpha-1}$ and we obtain

$$A_2 = p^{2\alpha-1} \sideset{}{'}\sum_{\substack{u\le p^{\alpha} \\ u+1\equiv 0 \pmod{p^{\alpha-1}} \\ mp^{\alpha-1}u+b(u+1)\equiv 0 \pmod{p^\alpha}}} 1 \ - \ p^{2\alpha-1}.$$

In order to make the summation on the right side of the last equation not vanishing, one of the following two conditions has to hold:

$$(a,p) = (b,p) = 1 \quad \text{and} \quad p \mid (a,b,m),$$

whence

$$A_2 \ll p^{2\alpha-\frac{1}{2}}(a,p)^{\frac{1}{2}}. \tag{2.6}$$

(ii) $\beta > 1$. Analogous to the estimate for A_1, we have

$$\begin{aligned} A_2 &= \frac{p^{\alpha-1}}{p^{2(\alpha-\beta)}} \mathop{\sum\nolimits' \sum\nolimits'}_{\substack{u\le p^{2\alpha-\beta}\ v\le p^{2\alpha-\beta} \\ u+1+p^{\alpha-\beta}\overline{v}\equiv 0 \pmod{p^{\alpha-1}}}} e\Big(\frac{a\overline{u}+b}{p^\alpha}\overline{v}\Big) \\ &= p^{2\beta-\alpha-1} \sideset{}{'}\sum_{\substack{u\le p^{2\alpha-\beta} \\ p^{\alpha-\beta}||u+1}} \sum_{l\le p^{2(\alpha-\beta)+1}} e\Big(\frac{a\overline{u}+b}{p^\alpha}\Big(-\frac{u+1}{p^{\alpha-\beta}}+lp^{\beta-1}\Big)\Big) \\ &= p^{2\beta-\alpha-1} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big) \sideset{}{'}\sum_{\substack{u\le p^{2\alpha-\beta} \\ p^{\alpha-\beta}||u+1}} e\Big(-\frac{a\overline{u}+bu}{p^{2\alpha-\beta}}\Big) \sum_{l\le p^{2(\alpha-\beta)+1}} e\Big(\frac{a\overline{u}+b}{p^{\alpha-\beta+1}}l\Big). \end{aligned}$$

So $A_2 = 0$ while $a \not\equiv b \pmod{p^{\alpha-\beta}}$. Otherwise, we have

$$A_2 = p^{2\beta-\alpha-1} e\Big(-\frac{a+b}{p^{2\alpha-\beta}}\Big) \sum_{l\le p^{2(\alpha-\beta)+1}} e\Big(\frac{bl}{p^{\alpha-\beta+1}}\Big) \sideset{}{'}\sum_{u\le p^{2\alpha-\beta}} e\Big(-\frac{a(1-p^{\alpha-1}l)\overline{u}+bu}{p^{2\alpha-\beta}}\Big)$$

$$\times \Big\{ \frac{1}{p^{\alpha-\beta}} \sum_{j\le p^{\alpha-\beta}} e\Big(\frac{u+1}{p^{\alpha-\beta}} j\Big) - \frac{1}{p^{\alpha-\beta+1}} \sum_{j\le p^{\alpha-\beta+1}} e\Big(\frac{u+1}{p^{\alpha-\beta+1}} j\Big)\Big\}$$
$$\ll d(p^\alpha) p^{2\alpha-\frac{1}{2}\beta}(a, p^{2\alpha-\beta})^{\frac{1}{2}}$$

by following the similar way as in (2.5). This, with (2.5)–(2.6), gives the lemma.

Lemma 2.2 *Suppose that p is a prime number, and α, β, γ, δ are four integers satisfying $1 \le \beta, \gamma, \delta \le \alpha$ and $\delta \le \gamma$. Then*

$$\sum_{c\le p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big) S(a,c;p^\gamma) S(b,c;p^\delta) \begin{cases} = 0, & \text{if } \beta > \gamma, \\ \ll d(p^\gamma) p^{\alpha+\gamma-\frac{1}{2}\beta}(a, p^{2\gamma-\beta})^{\frac{1}{2}}, & \text{if } \beta \le \gamma. \end{cases}$$

Proof We write

$$\mathrm{A} = \sum_{c\le p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big) S(a,c;p^\gamma) S(b,c;p^\delta).$$

If we write $\overline{u}$ and $\overline{v}$ as the integers satisfying $\overline{u}u \equiv 1 \pmod{p^\gamma}$ and $\overline{v}v \equiv 1 \pmod{p^\delta}$ respectively, we will get

$$\begin{aligned}
\mathrm{A} &= \sum_{c\le p^\alpha}{}' e\Big(\frac{c}{p^\beta}\Big) \sum_{u\le p^\gamma}{}' e\Big(\frac{a\overline{u}+cu}{p^\gamma}\Big) \sum_{v\le p^\delta}{}' e\Big(\frac{b\overline{v}+cv}{p^\delta}\Big) \\
&= \sum_{u\le p^\gamma}{}' \sum_{v\le p^\delta}{}' e\Big(\frac{a\overline{u}+p^{\gamma-\delta}b\overline{v}}{p^\gamma}\Big) \sum_{c\le p^\alpha}{}' e\Big(\frac{up^{\alpha-\gamma}+vp^{\alpha-\delta}+p^{\alpha-\beta}}{p^\alpha} c\Big) \\
&= p^\alpha \sum_{\substack{u\le p^\gamma \quad v\le p^\delta \\ up^{\alpha-\gamma}+vp^{\alpha-\delta}+p^{\alpha-\beta}\equiv 0 \pmod{p^\alpha}}}{}' \sum{}' e\Big(\frac{a\overline{u}+p^{\gamma-\delta}b\overline{v}}{p^\gamma}\Big) \\
&\quad - p^{\alpha-1} \sum_{\substack{u\le p^\gamma \quad v\le p^\delta \\ up^{\alpha-\gamma}+vp^{\alpha-\delta}+p^{\alpha-\beta}\equiv 0 \pmod{p^{\alpha-1}}}}{}' \sum{}' e\Big(\frac{a\overline{u}+p^{\gamma-\delta}b\overline{v}}{p^\gamma}\Big) \\
&= \mathrm{A}_3 - \mathrm{A}_4. \qquad (2.7)
\end{aligned}$$

Obviously, $\mathrm{A}_3 = \mathrm{A}_4 = 0$ for $\beta > \gamma$, so we may assume $\beta \le \gamma$ in the following discussion. Two cases should be considered:

(i) $\gamma = \delta$. Then Lemma 2.1 implies that

$$\begin{aligned}
\mathrm{A} &= p^{\alpha-\gamma} \sum_{c\le p^\gamma}{}' e\Big(\frac{c}{p^\beta}\Big) S(a,c;p^\gamma) S(b,c;p^\gamma) \\
&\ll p^{\alpha-\gamma} d(p^\gamma) p^{2\gamma-\frac{1}{2}\beta}(a,p^{2\gamma-\beta})^{\frac{1}{2}} = d(p^\gamma) p^{\alpha+\gamma-\frac{1}{2}\beta}(a,p^{2\gamma-\beta})^{\frac{1}{2}}.
\end{aligned}$$

(ii) $\gamma > \delta$. Then we could also find that $\mathrm{A} = 0$ when $\beta < \gamma$. While $\beta = \gamma$, we have

$$\begin{aligned}
\mathrm{A}_3 &= p^\alpha \sum_{\substack{u\le p^\gamma \quad v\le p^\delta \\ u+p^{\gamma-\delta}v+1\equiv 0 \pmod{p^\gamma}}}{}' \sum{}' e\Big(\frac{a\overline{u}+p^{\gamma-\delta}b\overline{v}}{p^\gamma}\Big) = p^\alpha \sum_{\substack{u\le p^\gamma \quad v\le p^\delta \\ 1+p^{\gamma-\delta}v+\overline{u}\equiv 0 \pmod{p^\gamma}}}{}' \sum{}' e\Big(\frac{a+p^{\gamma-\delta}b\overline{v}}{p^\gamma}\overline{u}\Big) \\
&= p^\alpha \sum_{v\le p^\delta}{}' e\Big(\frac{a+p^{\gamma-\delta}b\overline{v}}{p^\gamma}(-1-p^{\gamma-\delta}v)\Big)
\end{aligned}$$

$$= p^{\alpha} e\Big(-\frac{a + p^{2(\gamma-\delta)}b}{p^{\gamma}}\Big) S(a, b; p^{\delta}). \tag{2.8}$$

Besides,

$$\mathrm{A}_4 = p^{\alpha-1} \underset{1+p^{\gamma-\delta}v+\overline{u}\equiv 0 \pmod{p^{\gamma-1}}}{\sideset{}{'}\sum_{u\le p^{\gamma}} \sideset{}{'}\sum_{v\le p^{\delta}}} e\Big(\frac{a + p^{\gamma-\delta}b\overline{v}}{p^{\gamma}}\overline{u}\Big).$$

Let $\overline{u} = -1 - p^{\gamma-\delta}v + lp^{\gamma-1}$, and we have

$$\mathrm{A}_4 = p^{\alpha-1} e\Big(-\frac{a + p^{2(\gamma-\delta)}b}{p^{\gamma}}\Big) S(a, b; p^{\delta}) \sum_{l\le p} e\Big(\frac{al}{p}\Big).$$

With (2.8), this gives

$$\mathrm{A} \begin{cases} \ll d(p^{\delta}) p^{\alpha+\frac{\delta}{2}}, & \text{if } (a,p) = 1, \\ = 0, & \text{if } p \mid a. \end{cases}$$

The conclusion of the lemma holds at once.

3 Proof of the Theorems

Proof of Theorem 1.1 Suppose that the standard prime factorization of q is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, and put

$$p^{\alpha} = \max\{p_j^{\alpha_j} : 1 \le j \le r\} \quad \text{and} \quad q = p^{\alpha} q_1.$$

Assume $p^{\gamma}||u$ and $p^{\delta}||v$, and write $u = p^{\gamma}u',\ v = p^{\delta}v'$. Without loss of generality, we may also assume $\gamma \ge \delta$.

Notice that $S(a, c; q)$ is multiplicative for q, and therefore

$$\sum_{l\le q-1} \frac{1}{\|\frac{l}{q}\|} \Big| \sum_{\substack{c\le q \\ (c,q)=1}} e\Big(\frac{lc}{q}\Big) S(a, c; u) S(b, c; v)\Big|$$

$$= \sum_{l\le q-1} \frac{1}{\|\frac{l}{q}\|}$$

$$\times \Big| \sum_{\substack{c\le q \\ (c,q)=1}} e\Big(\frac{lc}{q}\Big) S(a\{u', p^{\gamma}\}^2, c; p^{\gamma}) S(a\{p^{\gamma}, u'\}^2, c; u') S(b\{v', p^{\delta}\}, c; p^{\delta}) S(b\{p^{\delta}, v'\}^2, c; v')\Big|$$

$$= \sum_{l\le q-1} \frac{1}{\|\frac{l}{q}\|} \Big| \sum_{\substack{c_1\le p^{\alpha} \\ (c_1,p)=1}} e\Big(\frac{lc_1}{p^{\alpha}}\Big) S(a\{u', p^{\gamma}\}^2, c_1q_1; p^{\gamma}) S(b\{v', p^{\delta}\}, c_1q_1; p^{\delta})\Big|$$

$$\times \Big| \sum_{\substack{c_2\le q_1 \\ (c_2,q_1)=1}} e\Big(\frac{lc_2}{q_1}\Big) S(a\{p^{\gamma}, u'\}^2, c_2p^{\alpha}; u') S(b\{p^{\delta}, v'\}^2, c_2p^{\alpha}; v')\Big|$$

$$= \sum_{\beta=0}^{\alpha} \sum_{\substack{l\le q-1 \\ p^{\alpha-\beta}\|l}} \frac{1}{\|\frac{l}{q}\|} \Big| \sum_{\substack{c_1\le p^{\alpha} \\ (c_1,p)=1}} e\Big(\frac{lc_1}{p^{\alpha}}\Big) S(a\{u', p^{\gamma}\}^2 q_1, c_1; p^{\gamma}) S(b\{v', p^{\delta}\} q_1, c_1; p^{\delta})\Big|$$

$$\times \Big| \sum_{\substack{c_2\le q_1\\(c_2,q_1)=1}} e\Big(\frac{lc_2}{q_1}\Big) S(a\{p^\gamma,u'\}^2p^\alpha,c_2;u')S(b\{p^\delta,v'\}^2p^\alpha,c_2;v')\Big|$$

$$\ll q_1^2d^2(q_1)\sum_{\beta=0}^{\alpha}\sum_{\substack{l\le(\frac{q}{p^{\alpha-\beta}})-1\\(l,p)=1}} \frac{1}{\Big\|\frac{l}{\frac{q}{p^{\alpha-\beta}}}\Big\|}$$

$$\times \Big| \sum_{\substack{c_1\le p^\alpha\\(c_1,p)=1}} e\Big(\frac{lc_1}{p^\beta}\Big) S(a\{u',p^\gamma\}^2q_1,c_1;p^\gamma)S(b\{v',p^\delta\}q_1,c_1;p^\delta)\Big|.$$

Corresponding to the case $\beta\gamma\delta=0$ and $\beta\gamma\delta\neq 0$, we use (1.1) and Lemma 2.2 respectively, so

$$\sum_{l\le q-1}\frac{1}{\|\frac{l}{q}\|}\Big|\sum_{\substack{c\le q\\(c,q)=1}} e\Big(\frac{lc}{q}\Big)S(a,c;u)S(b,c;v)\Big|$$

$$\ll q_1^2d^2(q_1)\sum_{\beta=0}^{\gamma}\frac{q\log q}{p^{\alpha-\beta}}d(p^\gamma)p^{\alpha+\gamma-\frac{1}{2}\beta}$$

$$\ll qq_1^2d^2(q_1)d(p^\gamma)p^\gamma\log q\sum_{\beta=0}^{\gamma}p^{\frac{\beta}{2}}$$

$$\ll (\gamma+1)qq_1^2d^2(q_1)d(p^\gamma)p^{\frac{3}{2}\gamma}\log q \ll q^{3-\frac{1}{2\omega(q)}}d^2(q)\log q$$

in virtue of $\gamma\le\alpha$ and $p^\alpha\ge q^{\frac{1}{\omega(q)}}$. This completes the proof.

Proof of Theorem 1.2 Our starting point is calculating the following sum in two different ways:

$$S=\sum_{l\le q-1}\Big(\sum_{m\le N} e\Big(-\frac{l}{q}m\Big)\Big)\sum_{\substack{c\le q\\(c,q)=1}} e\Big(\frac{l}{q}c\Big)S(a,c;q)S(b,c;q).$$

Theorem 1.1 implies that

$$S\ll q^{3-\frac{1}{2\omega(q)}}d^2(q)\log q. \tag{3.1}$$

On the other hand, changing the order of summations, we have

$$\begin{aligned} S &= \sum_{m\le N}\sum_{c\le q}{}' S(a,c;q)S(b,c;q)\sum_{l\le q-1} e\Big(\frac{c-m}{q}l\Big)\\ &= q\sum_{c\le N}{}' S(a,c;q)S(b,c;q) - N\sum_{c\le q}{}' S(a,c;q)S(b,c;q). \end{aligned}\tag{3.2}$$

Note that

$$\begin{aligned} \sum_{c\le q}{}' S(a,c;q)S(b,c;q) &= \sum_{c\le q}{}'\sum_{u\le q}{}' e\Big(\frac{a\overline{u}+cu}{q}\Big)\sum_{v\le q}{}' e\Big(\frac{b\overline{v}+cv}{q}\Big)\\ &= \sum_{u\le q}{}'\sum_{v\le q}{}' e\Big(\frac{a\overline{u}+b\overline{v}}{q}\Big)\sum_{c\le q}{}' e\Big(\frac{u+v}{q}c\Big)\\ &= \sum_{d\mid q}\mu(d)\sum_{u\le q}{}'\sum_{v\le q}{}' e\Big(\frac{a\overline{u}+b\overline{v}}{q}\Big)\sum_{c\le \frac{q}{d}} e\Big(\frac{u+v}{\frac{q}{d}}c\Big) \end{aligned}$$

$$= q\sum_{d\mid q}\frac{\mu(d)}{d}\sideset{}{'}\sum_{\substack{u\le q\\ u+v\equiv 0 \pmod{\frac{q}{d}}}}\sideset{}{'}\sum_{v\le q} e\Big(\frac{au+bv}{q}\Big). \tag{3.3}$$

Theorem 1.2 follows immediately from (3.1)–(3.3).

Proof of Corollary 1.1 We only need to calculate

$$\sum_{d\mid q}\frac{\mu(d)}{d}\sideset{}{'}\sum_{\substack{u\le q\\ u+v\equiv 0 \pmod{\frac{q}{d}}}}\sideset{}{'}\sum_{v\le q} e\Big(\frac{u+v}{q}\Big).$$

It is easy to verify that

$$\begin{aligned}\sideset{}{'}\sum_{\substack{u\le q\\ u+v\equiv 0 \pmod{\frac{q}{d}}}}\sideset{}{'}\sum_{v\le q} e\Big(\frac{u+v}{q}\Big) &= \sideset{}{'}\sum_{u\le q}\sideset{}{'}\sum_{v\le q} e\Big(\frac{u+v}{q}\Big)\cdot\frac{d}{q}\sum_{j\le\frac{q}{d}} e\Big(\frac{u+v}{\frac{q}{d}}j\Big)\\ &= \frac{d}{q}\sum_{j\le\frac{q}{d}}\Big(\sideset{}{'}\sum_{u\le q} e\Big(\frac{dj+1}{q}u\Big)\Big)^2\\ &= \frac{d}{q}\sum_{j\le\frac{q}{d}}\Big|\mu\Big(\frac{q}{(dj+1,q)}\Big)\Big|\phi^2(q)\Big(\phi\Big(\frac{q}{(dj+1,q)}\Big)\Big)^{-2}\\ &= \frac{d\phi^2(q)}{q}\sum_{\substack{s\mid q\\ (s,d)=1}}\frac{|\mu(\frac{q}{s})|}{\phi^2(\frac{q}{s})}\sum_{\substack{j\le\frac{q}{d}\\ (dj+1,q)=s}} 1.\end{aligned}$$

Writing $dj+1=ks$ implies

$$\begin{aligned}\sum_{\substack{j\le\frac{q}{d}\\ (dj+1,q)=s}} 1 &= \sum_{\substack{k\le\frac{q}{s}\\ (k,\frac{q}{s})=1\\ ks\equiv 1 \pmod d}} 1 = \sum_{\substack{l\mid(\frac{q}{s})\\ (l,d)=1}}\mu(l)\sum_{\substack{k\le\frac{q}{sl}\\ ksl\equiv 1 \pmod d}} 1\\ &= \frac{q}{ds}\sum_{\substack{l\mid(\frac{q}{s})\\ (l,d)=1}}\frac{\mu(l)}{l} = \frac{\phi(\frac{q}{s})}{\phi(d)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{d\mid q}\frac{\mu(d)}{d}\sideset{}{'}\sum_{\substack{u\le q\\ u+v\equiv 0 \pmod{\frac{q}{d}}}}\sideset{}{'}\sum_{v\le q} e\Big(\frac{u+v}{q}\Big)\\ &= \frac{\phi^2(q)}{q}\sum_{s\mid q}\frac{|\mu(\frac{q}{s})|}{\phi(\frac{q}{s})}\sum_{\substack{d\mid q\\ (d,s)=1}}\frac{\mu(d)}{\phi(d)} = \frac{\phi^2(q)}{q}\sum_{s\mid q}\frac{|\mu(s)|}{\phi(s)}\prod_{\substack{p\mid q\\ p\nmid(\frac{q}{s})}}\frac{p-2}{p-1}\\ &= \frac{\phi^2(q)}{q}\Big(\sum_{s_1\mid Q_1}\frac{|\mu(s_1)|}{\phi(s_1)}\prod_{\substack{p\mid Q_1\\ p\nmid(\frac{Q_1}{s_1})}}\frac{p-2}{p-1}\Big)\Big(\sum_{s_2\mid Q_2}\frac{|\mu(s_2)|}{\phi(s_2)}\prod_{\substack{p\mid Q_2\\ p\nmid(\frac{Q_2}{s_2})}}\frac{p-2}{p-1}\Big)\\ &= \frac{\phi^2(q)}{q}\Big(\sum_{s_1\mid Q_1}\frac{|\mu(s_1)|}{\phi(s_1)}\Big)\Big(\sum_{\substack{s_2\mid Q_2\\ 2\nmid s_2}}\frac{|\mu(s_2)|}{\phi(s_2)}\prod_{p\mid s_2}\frac{p-2}{p-1}\Big)\end{aligned}$$

$$= q \prod_{p \mid Q_1} \frac{p-1}{p} \prod_{\substack{p \mid Q_2 \\ p>2}} \frac{p^2-p-1}{p^2},$$

which suffices.

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