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# Local Precise Large and Moderate Deviations for Sums of Independent Random Variables*

Fengyang CHENG[1] Minghua LI[1]

**Abstract** Let $\{X, X_k : k \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution $F$. In this paper, the authors establish some results on the local precise large and moderate deviation probabilities for partial sums $S_n = \sum\limits_{i=1}^{n} X_i$ in a unified form in which $X$ may be a random variable of an arbitrary type, which state that under some suitable conditions, for some constants $T > 0,\ a$ and $\tau > \frac{1}{2}$ and for every fixed $\gamma > 0$, the relation

$$P(S_n - na \in (x, x+T]) \sim nF((x+a, x+a+T])$$

holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$, that is,

$$\lim_{n\to+\infty} \sup_{x\geq\gamma n^\tau} \Big|\frac{P(S_n - na \in (x, x+T])}{nF((x+a, x+a+T])} - 1\Big| = 0.$$

The authors also discuss the case where $X$ has an infinite mean.

**Keywords** Local precise moderate deviation, Local precise large deviation, Intermediate regularly varying function, $O$-regularly varying function
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## 1 Introduction

Throughout this paper, let $\{X, X_k : k \geq 1\}$ be a sequence of independent and identically distributed (i.i.d) random variables (r.v.s) with a common distribution $F$. For some $T \in (0, \infty]$, let $\Delta = \Delta(T) = (0, T]$ if $T < \infty$ and $\Delta = \Delta(T) = (0, \infty)$ if $T = \infty$. In addition, for any real $x$, we write $x + \Delta = (x, x+T]$ if $T < \infty$ and $x + \Delta = (x, \infty)$ if $T = \infty$.

In this paper, we establish some results of the local precise moderate and large deviation probabilities for partial sums $S_n = \sum\limits_{i=1}^{n} X_i$, which state that under some suitable conditions, for some constants $T > 0,\ a$ and $\tau > \frac{1}{2}$ and for every fixed $\gamma > 0$, the relation

$$P(S_n - na \in x + \Delta) \sim nF(x + a + \Delta)$$

holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \Big|\frac{P(S_n - na \in x + \Delta)}{nF(x + a + \Delta)} - 1\Big| = 0.$$

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[1]Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, China.
E-mail: chengfy@suda.edu.cn Lmhy2006@163.com
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In the case $\tau = 1$, Doney [6], Baltrūnas [2] and Lin [8] established results of local large deviation probabilities for partial sums, in which $X$ was assumed to be an integer-valued r.v. with a finite mean. Yang et al. [10] gave a result in which $X$ was assumed to be an absolutely continuous r.v. with a finite mean. The results of local large deviation probabilities for which $X$ was assumed to be an integer-valued r.v. with an infinite mean were established by Doney [7]. Motivated by the above-mentioned literatures, we establish some results of local precise moderate and large deviation probabilities for partial sums in a unified form in which $X$ may be a type r.v. of an arbitrary. We also discuss the case where $X$ has an infinite mean.

This paper is organized as follows. Section 2 presents notations and definitions of some function classes. Main results are formulated in Section 3. In Section 4, we give some propositions as preparation. The proofs of the main results are presented in Section 5.

## 2 Definitions and Preliminaries

Throughout this section, $f$ will denote a nonnegative measurable function defined on $[0, \infty)$ or $(-\infty, \infty)$.

First, we introduce some classes of functions.

**Definition 2.1** *A function $f$ is said to be $O$-regularly varying (belonging to the class $\mathcal{OR}$) if $f$ is eventually positive (i.e., $f(x) > 0$ for sufficiently large $x$) and*

$$0 < \liminf_{x\to\infty} \frac{f(xy)}{f(x)} \le \limsup_{x\to\infty} \frac{f(xy)}{f(x)} < \infty$$

*for every fixed $y \ge 1$.*

If $f$ is an eventually positive function, then its upper and lower Matuszewska's indices are defined by

$$\alpha(f) = \lim_{y\to\infty} \frac{\log\left(\limsup\limits_{x\to\infty} \dfrac{f(xy)}{f(x)}\right)}{\log y}, \quad \beta(f) = \lim_{y\to\infty} \frac{\log\left(\liminf\limits_{x\to\infty} \dfrac{f(xy)}{f(x)}\right)}{\log y},$$

respectively. According to Theorem 2.1.7 in [4], if $f$ is an eventually positive function, then $f \in \mathcal{OR}$ if and only if its upper and lower Matuszewska's indices $\alpha(f)$ and $\beta(f)$ are both finite.

**Definition 2.2** *A function $f$ is said to be extended regularly varying (belonging to the class $\mathcal{ER}$) if $f$ is eventually positive and*

$$y^d \le \liminf_{x\to\infty} \frac{f(xy)}{f(x)} \le \limsup_{x\to\infty} \frac{f(xy)}{f(x)} \le y^c \tag{2.1}$$

*for all $y \ge 1$ and some constants $c$ and $d$, where $cd \ge 0$. In particular, if $c = d = \alpha$ in (2.1), then $f$ is said to be regularly varying (belonging to the class $\mathcal{R}$), and it is also said to be regularly varying with index $\alpha$ (belonging to the class $\mathcal{R}_\alpha$). If $f$ belongs to the class $\mathcal{R}_0$, then it is said to be slowly varying.*

Note that $f$ belongs to the class $\mathcal{R}_\alpha$ if and only if $f(x) = x^\alpha l(x)$, where $l(x)$ is a slowly varying function.

**Definition 2.3** *A function $f$ is said to be intermediate regularly varying (belonging to the class $\mathcal{IR}$) if $f$ is eventually positive and*

$$\lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{f(xy)}{f(x)}=\lim_{y\downarrow 1}\limsup_{x\to\infty}\frac{f(xy)}{f(x)}=1. \tag{2.2}$$

**Remark 2.1** By Corollary 2.2 I in [5], if $f$ is eventually positive, then the following are equivalent:

$$\begin{aligned}&\text{(i) } f\in\mathcal{IR};\\ &\text{(ii) } \lim_{\substack{y\to 1\\ x\to\infty}}\frac{f(xy)}{f(x)}=1;\\ &\text{(iii) } L_f^-=L_f^+=1,\end{aligned} \tag{2.3}$$

where

$$L_f^-=\lim_{\epsilon\downarrow 0}\liminf_{x\to\infty}\frac{\inf\limits_{(1-\epsilon)x\le z\le(1+\epsilon)x}f(z)}{f(x)},\quad L_f^+=\lim_{\epsilon\downarrow 0}\limsup_{x\to\infty}\frac{\sup\limits_{(1-\epsilon)x\le z\le(1+\epsilon)x}f(z)}{f(x)}. \tag{2.4}$$

If an eventually positive function $f$ satisfies (2.3), then it is also called regularly oscillating, which was introduced by [3].

By Corollary 1.2 in [5], it follows that

$$\mathcal{R}\subset\mathcal{ER}\subset\mathcal{IR}\subset\mathcal{OR},$$

and the inclusions are proper.

**Definition 2.4** *A function $f$ is said to be long tailed (belonging to the class $\mathcal{L}$) if $f$ is eventually positive and $\lim\limits_{x\to\infty}\frac{f(x+y)}{f(x)}=1$ for every fixed $y\in(-\infty,\infty)$.*

**Definition 2.5** *A sequence of nonnegative numbers $\{p_n: n=0,\pm1,\pm2,\cdots\}$ is said to belong to the class $\mathcal{R}$ (or $\mathcal{IR},\mathcal{L},\mathcal{ER},\mathcal{OR}$) if $p(x)$ belongs to the class $\mathcal{R}$ (or $\mathcal{IR},\mathcal{L},\mathcal{ER},\mathcal{OR}$), where $p(x)$ is defined by*

$$p(x)=p_n,\quad n\le x<n+1,\quad n=0,\pm1,\pm2,\cdots.$$

In the following, we give the definition of almost decreasing, which was introduced by Aljančić and Arandelović [1].

**Definition 2.6** *A function $f$ is said to be almost decreasing if*

$$\limsup_{x\to\infty}\frac{\sup\limits_{u\ge x}f(u)}{f(x)}<\infty. \tag{2.5}$$

Finally, we introduce some notations which will be used in the following sections. Let $a(n,x)$ and $b(n,x)$ be two positive functions ($n=1,2,\cdots,\ x\in(-\infty,\infty)$). We denote $a(n,x)\lesssim b(n,x)$ (or $b(n,x)\gtrsim a(n,x)$) which holds uniformly for all $x\in\Lambda$ as $n\to\infty$ if $\lim\limits_{n\to\infty}\sup\limits_{x\in\Lambda}\frac{a(n,x)}{b(n,x)}\le 1$, and we write $a(n,x) \sim b(n,x)$ which holds uniformly for all $x \in \Lambda$ as $n \to \infty$ if $\lim\limits_{n\to\infty} \sup\limits_{x\in\Lambda} |\frac{a(n,x)}{b(n,x)} - 1| = 0$.

Obviously, $a(n,x) \sim b(n,x)$ holds uniformly for all $x \in \Lambda$ as $n \to \infty$ if and only if both $a(n,x) \lesssim b(n,x)$ and $b(n,x) \lesssim a(n,x)$ hold uniformly for all $x \in \Lambda$ as $n \to \infty$.

## 3 Main Results

In this section, we will present the main results of this paper and give some corollaries. The proofs of the theorem and corollaries are arranged in Section 5.

**Theorem 3.1** *Let $\{X, X_k : k \geq 1\}$ be a sequence of* i.i.d.r.v.s. *with a common distribution $F$, and let $\tau > \frac{1}{2}$ be a constant. Suppose that $F_\Delta(x) = F(x+\Delta)$ is almost decreasing, and one of the following conditions holds:*

(i) *$\tau \geq 1$, $E|X|^{\frac{1}{\tau}} < \infty$ and $E(X^+)^p < \infty$ for some $p > \frac{1}{\tau}$, where $X^+ = \max(X, 0)$;*

*or*

(ii) *$\tau < 1$, and $E|X|^p < \infty$ for some $p > \frac{1}{\tau}$.*

*If $F(x+\Delta) \in \mathcal{OR}$, then for every fixed $\gamma > 0$, the relation*

$$nL^-_{F_\Delta} F(x+a+\Delta) \lesssim P(S_n - na \in x+\Delta) \lesssim nL^+_{F_\Delta} F(x+a+\Delta) \tag{3.1}$$

*holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$, where $a = EX$ if $\tau \leq 1$ and $a$ is any fixed constant if $\tau > 1$. Especially, if $F(x+\Delta) \in \mathcal{IR}$, then for every fixed $\gamma > 0$, the relation*

$$P(S_n - na \in x+\Delta) \sim nF(x+a+\Delta) \tag{3.2}$$

*holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$, where $a$ is defined as above.*

If $X$ is an integer-valued r.v., by taking $T = 1$, we have the following conclusion.

**Corollary 3.1** *Let $\{X, X_k : k \geq 1\}$ be a sequence of* i.i.d. *integer-valued* r.v.s. *with the mass $p_k = P(X = k)$, $k = 0, \pm 1, \pm 2, \cdots$ and let $\tau > \frac{1}{2}$. Suppose that $\{p_n : n \geq 1\}$ is almost decreasing and belongs to the class $\mathcal{IR}$. Moreover, Assume that one of the following conditions holds:*

(i) *$\tau \geq 1$, $E|X|^{\frac{1}{\tau}} < \infty$ and $E(X^+)^p < \infty$ for some $p > \frac{1}{\tau}$;*

*or*

(ii) *$\tau < 1$ and $E|X|^p < \infty$ for some $p > \frac{1}{\tau}$.*

*Then, for every $\gamma > 0$, the relation*

$$P(m - 1 < S_n - na \leq m) \sim np_m \tag{3.3}$$

*holds uniformly for all $m \geq \gamma n^\tau$ as $n \to \infty$, where $a = EX$ if $\tau \leq 1$ and $a$ is any fixed constant if $\tau > 1$. Especially, if $a = 0$, then for every $\gamma > 0$, the relation*

$$P(S_n = m) \sim np_m$$

*holds uniformly for all $m \geq \gamma n^\tau$ as $n \to \infty$.*

**Remark 3.1** Note that $\liminf_{n\to\infty}\left(-\frac{\log R_n}{\log n}\right) > 1$, where $R_n = \sum_{k>n} p_k,\ n = 1, 2, \cdots$, implies that $E(X^+)^p < \infty$ for some $p > 1$. Hence, Corollary 3.1 covers Theorem 3.1 in [8].

If $X$ is an absolutely continuous r.v., then the following conclusion is immediately obtained.

**Corollary 3.2** *Let* $\{X, X_k : k \geq 1\}$ *be a sequence of* i.i.d.r.v.s. *with a common almost decreasing density function* $f$. *Assume that* $\mu = EX$ *is finite and* $E(X^+)^r < \infty$ *for some* $r > 1$. *Let* $\gamma$ *and* $T$ *be any fixed positive numbers. If* $f \in \mathcal{OR}$, *then the relation*

$$L^-_{F_\Delta} nF(x+\mu+\Delta) \lesssim P(S_n - n\mu \in x+\Delta) \lesssim L^+_{F_\Delta} nF(x+\mu+\Delta) \tag{3.4}$$

*holds uniformly for all* $x \geq \gamma n$ *as* $n \to \infty$, *where* $F_\Delta(x) = \int_x^{x+T} f(y)\mathrm{d}y$.

In what follows, we present an example to show that there is a distribution $F$ with a density $f$, such that $F(x+\Delta) \in \mathcal{OR}$ for some $T > 0$ but $f \notin \mathcal{OR}$.

**Example 3.1** Let $\alpha > 0$ and define

$$f(x) = \begin{cases} c\alpha x^{-1-\alpha}, & 2n-1 \leq x \leq 2n, \\ 0, & \text{otherwise}, \end{cases} \qquad n = 1, 2, \cdots,$$

where $c$ is a constant such that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$. Then, it is clear that $f \notin \mathcal{OR}$. Let $T = 2$, and then simple calculations yield that

$$F(x+\Delta) = \begin{cases} c(x^{-\alpha} - (x+2)^{-\alpha} + (2n+1)^{-\alpha} - (2n)^{-\alpha}), & 2n-1 \leq x \leq 2n, \\ c((2n+1)^{-\alpha} - (2n+2)^{-\alpha}), & 2n < x < 2n+1, \end{cases}$$

$n = 1, 2, \cdots$. It follows that $F(x+\Delta) \sim c\alpha x^{-\alpha-1}$ as $x \to \infty$. Thus $F(x+\Delta) \in \mathcal{R} \subset \mathcal{IR} \subset \mathcal{OR}$.

**Remark 3.2** Example 3.1 and Propositions 4.3–4.5 below show that Theorem 3.1 has a wider range of applications than Theorem 3.1 in [10] even if $X$ has a density function $f$.

Similarly, there is a lattice distribution $F$, which has the mass $p_n = F\{n\},\ n = 1, 2, \cdots$, such that $F(x+\Delta) \in \mathcal{OR}$ for some $T > 0$ but $\{p_n, n = 1, 2, \cdots\}$ does not belong to $\mathcal{OR}$.

**Example 3.2** Let $\alpha > 0$ and $0 < r < 1$. Let $F$ be a distribution with the mass $p_n,\ n = 1, 2, \cdots$, where $p_n$ is defined by

$$p_n = \begin{cases} c\alpha n^{-1-\alpha}, & n = 2m, \\ cr^n, & n = 2m-1, \end{cases} \qquad m = 1, 2, \cdots$$

for some constant $c$ satisfying $\sum_{n=1}^{\infty} p_n = 1$. Then, it is clear that $\{p_n, n = 1, 2, \cdots\}$ does not belong to $\mathcal{OR}$, and hence, it does not belong to $\mathcal{R}$. Let $T = 2$, and then simple calculations yield that

$$F(x+\Delta) = \begin{cases} cr^{2n+1} + c\alpha(2n)^{-1-\alpha}, & 2n-1 \leq x < 2n, \\ cr^{2n+1} + c\alpha(2n+2)^{-1-\alpha}, & 2n \leq x < 2n+1, \end{cases} \qquad n = 1, 2, \cdots.$$

It follows that $F(x+\Delta) \sim c\alpha x^{-\alpha-1}$ as $x \to \infty$, and hence, $F(x+\Delta) \in \mathcal{R} \subset \mathcal{IR} \subset \mathcal{OR}$.

## 4 Some Propositions

In this section, we give some propositions to investigate the relationships between $f \in \mathcal{OR}$ and $F(x+\Delta) \in \mathcal{OR}$ and the relationships among $L_f^-$, $L_{F_\Delta}^-$, $L_f^+$, $L_{F_\Delta}^+$ if a distribution $F$ has a density function $f$.

First we present some properties of the class $\mathcal{OR}$, which play important roles in the following discussions and can be found in [4].

**Proposition 4.1** *Assume that a function $f \in \mathcal{OR}$. Then, for every $\alpha > \alpha(f)$, there exist positive constants $C_\alpha$ and $x_\alpha$, such that*

$$\frac{f(y)}{f(x)} \le C_\alpha \Big(\frac{y}{x}\Big)^\alpha, \quad y \ge x \ge x_\alpha. \tag{4.1}$$

*Similarly, for every $\beta < \beta(f)$, there exist positive constants $C_\beta$ and $x_\beta$, such that*

$$\frac{f(y)}{f(x)} \ge C_\beta \Big(\frac{y}{x}\Big)^\beta, \quad y \ge x \ge x_\beta. \tag{4.2}$$

The following proposition was established by Aljančić and Arandelović [1].

**Proposition 4.2** *Assume that a function $f \in \mathcal{OR}$. Then, for any fixed $0 < a < b < \infty$,*

$$0 < \liminf_{x\to\infty} \inf_{a\le y\le b} \frac{f(xy)}{f(x)} \le \limsup_{x\to\infty} \sup_{a\le y\le b} \frac{f(xy)}{f(x)} < \infty. \tag{4.3}$$

**Remark 4.1** It is obvious that (4.3) is equivalent to

$$0 < \liminf_{x\to\infty} \frac{\inf\limits_{ax\le z\le bx} f(z)}{f(x)} \le \limsup_{x\to\infty} \frac{\sup\limits_{ax\le z\le bx} f(z)}{f(x)} < \infty.$$

The next proposition shows that condition $f \in \mathcal{OR}$ is stronger than condition $F(x+\Delta) \in \mathcal{OR}$.

**Proposition 4.3** *Assume that a distribution $F$ has a density function $f$ and $f \in \mathcal{OR}$. Then, for every fixed $T > 0$, $F(x+\Delta) \in \mathcal{OR}$.*

**Proof** If $T = \infty$, then $\overline{F}(x) = F(x,\infty) \in \mathcal{OR}$ follows from Lemma 4.3 of Yang et al. [10]. Hence, we only need to discuss the case $T < \infty$. For any fixed $y > 1$, by $f \in \mathcal{OR}$, combining with Proposition 4.1, simple calculations yield that

$$\limsup_{x\to\infty} \frac{F(xy+\Delta)}{F(x+\Delta)} < \infty. \tag{4.4}$$

Hence, we only need to estimate the lower bound: Suppose that $n-1 < y \le n$ holds for some integer $n > 1$. By $f \in \mathcal{OR}$ and Proposition 4.2, there exist constants $C_3$ and $x_1 > T$, such that

$$\begin{aligned}\int_x^{x+T} f(u)\mathrm{d}u &\le \sum_{k=1}^n \int_{x+(k-1)\frac{T}{y}}^{x+k\frac{T}{y}} f(u)\mathrm{d}u = \sum_{k=1}^n \int_x^{x+\frac{T}{y}} f\Big(u + \frac{(k-1)T}{y}\Big)\mathrm{d}u \\ &\le \sum_{k=1}^n \int_x^{x+\frac{T}{y}} \sup_{u\le t\le 2u} f(t)\mathrm{d}u \le nC_3 \int_x^{x+\frac{T}{y}} f(u)\mathrm{d}u\end{aligned}$$

holds for any $x > x_1$, where the last but one step is obtained by $u \le u + \frac{(k-1)T}{y} \le 2u$ for all $u \ge x \ge T$ and $1 \le k \le n$. Hence, we have that

$$\begin{aligned} F(xy+\Delta) &= \int_{xy}^{xy+T} f(u)\mathrm{d}u = y\int_{x}^{x+\frac{T}{y}} f(uy)\mathrm{d}u \\ &\ge C_1 y\int_{x}^{x+\frac{T}{y}} f(u)\mathrm{d}u \ge \frac{C_1 y}{nC_3}\int_{x}^{x+T} f(u)\mathrm{d}u = \frac{C_1 y}{nC_3}F(x+\Delta) \end{aligned}$$

holds for all $x > x_1$, which yields that

$$\liminf_{x\to\infty} \frac{F(xy+\Delta)}{F(x+\Delta)} > 0. \tag{4.5}$$

It is obvious that $F(x+\Delta) \in \mathcal{OR}$ follows from (4.4)–(4.5).

**Proposition 4.4** *Assume that a distribution $F$ has a density function $f$. If $f \in \mathcal{OR}$, then, for every fixed $T > 0$,*

$$0 < L_f^- \le L_{F_\Delta}^- \le L_{F_\Delta}^+ \le L_f^+ < \infty. \tag{4.6}$$

*Moreover, if $f \in \mathcal{IR}$, then $F(x+\Delta) \in \mathcal{IR}$ for every fixed $T > 0$.*

**Proof** By Proposition 4.2, it follows immediately that $L_f^- > 0$ and $L_f^+ < \infty$. In what follows, we prove that $L_f^- \le L_{F_\Delta}^-$: By the definition of $L_f^-$, for every fixed $\delta > 0$, there exist constants $\epsilon_0 > 0$ and $x_0 > 0$ such that

$$\inf_{(1-\epsilon)x \le z \le (1+\epsilon)x} f(z) > (L_f^- - \delta) f(x)$$

holds for any $\epsilon < \epsilon_0$ and $x > x_0$. Hence, for any $x > x_0$, $\epsilon < \epsilon$ and $(1-\epsilon)x \le z \le (1+\epsilon)x$, we have

$$\begin{aligned} F(z+\Delta) &= \int_{z}^{z+T} f(u)\mathrm{d}u = \int_{x}^{x+T} f(t+z-x)\mathrm{d}t \\ &\ge \int_{x}^{x+T} \inf_{(1-\epsilon)t \le u \le (1+\epsilon)t} f(u)\mathrm{d}t \ge (L_f^- - \delta)\int_{x}^{x+T} f(t)\mathrm{d}t \\ &= (L_f^- - \delta)F(x+\Delta). \end{aligned}$$

Therefore,

$$\lim_{\epsilon\downarrow 0}\liminf_{x\to\infty} \frac{\inf\limits_{(1-\epsilon)x \le z \le (1+\epsilon)x} F(z+\Delta)}{F(x+\Delta)} \ge L_f^- - \delta.$$

By the arbitrariness of $\delta$, we immediately obtain that $L_{F_\Delta}^- \ge L_f^-$. The proof of $L_{F_\Delta}^+ \le L_f^+$ is similar to that of $L_{F_\Delta}^- \ge L_f^-$, so it is omitted.

Finally, if $f \in \mathcal{IR}$, then $F(x+\Delta) \in \mathcal{IR}$ follows immediately from (4.6) and Remark 2.1.

The following proposition shows, the fact that $f$ is almost decreasing implies that $F(x+\Delta)$ is almost decreasing.

**Proposition 4.5** *Assume that a distribution $F$ has a density function $f$, and $f$ is almost decreasing. Then for every fixed $T > 0$, $F(x + \Delta)$ is almost decreasing.*

**Proof** Since $f$ is almost decreasing, there exist constants $C$ and $x_0$, such that $\sup_{z\geq x} f(z) \leq Cf(x)$ for all $x > x_0$. We have that

$$\begin{aligned}\sup_{z\geq x} F(z+\Delta) &= \sup_{z\geq x}\int_z^{z+T} f(u)\mathrm{d}u = \sup_{z\geq x}\int_x^{x+T} f(t+z-x)\mathrm{d}t\\ &\leq \int_x^{x+T}\sup_{u\geq t} f(u)\mathrm{d}t \leq C\int_x^{x+T} f(t)\mathrm{d}t = CF(x+\Delta)\end{aligned}$$

holds for all $x > x_0$. Hence, $F(x + \Delta)$ is almost decreasing.

## 5 Proofs of Main Results

In this section, we give the proofs of Theorem 3.1 and corollaries. First, we estimate the lower bound of $P(S_n - na \in x + \Delta)$ under slightly weaker conditions than those in Theorem 3.1, which will be generalized as Lemma 5.1.

**Lemma 5.1** *Let $\{X, X_k : k \geq 1\}$ be a sequence of* i. i. d. r. v. s. *with a common distribution $F$. Suppose that $E|X|^{\frac{1}{\tau}} < \infty$ for some $\tau > \frac{1}{2}$ and $F_\Delta(x) = F(x + \Delta) \in \mathcal{OR}$. Then for arbitrarily fixed $\gamma > 0$, the relation*

$$P(S_n - na \in x + \Delta) \gtrsim nL^-_{F_\Delta} F(x + a + \Delta) \tag{5.1}$$

*holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$, where $a = EX$ if $\tau \leq 1$ and $a$ is any fixed constant if $\tau > 1$.*

**Proof** Without loss of generality, hereafter we assume that $a = 0$. Since $X_1, X_2, \cdots, X_n, \cdots$ are i. i. d. r. v. s., for any fixed $\epsilon \in (0, \frac{1}{2})$, we have

$$\begin{aligned}P(S_n \in x+\Delta) &\geq P\Big(\bigcup_{i=1}^{n}(S_n \in x+\Delta, X_i > \epsilon x, X_j \leq \epsilon x, j \neq i, j \leq n)\Big)\\ &= nP(S_{n-1} + X_n \in x+\Delta, X_n > \epsilon x, M_{n-1} \leq \epsilon x)\\ &\geq n\int_{-\epsilon x}^{\epsilon x} P(X_n \in x-y+\Delta)P(S_{n-1} \in \mathrm{d}y, M_{n-1} \leq \epsilon x)\\ &\geq n\inf_{|y|\leq\epsilon x} F(x-y+\Delta)P(M_{n-1} \leq \epsilon x, |S_{n-1}| \leq \epsilon x),\end{aligned} \tag{5.2}$$

where $M_n = \max\{X_1, X_2, \cdots, X_n\}$ for all $n \geq 1$. Since $EX = 0$ if $\tau \leq 1$, by the strong law of large numbers for i. i. d. r. v. s., it follows from $E|X|^{\frac{1}{\tau}} < \infty$ that $\frac{S_{n-1}}{n^\tau} \overset{\text{a.s}}{\to} 0$ and $\frac{M_{n-1}}{n^\tau} \overset{\text{a.s}}{\to} 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty}\inf_{x\geq\gamma n^\tau} P(|S_{n-1}| \leq \epsilon x) = 1$ and $\lim_{n\to\infty}\inf_{x\geq\gamma n^\tau} P(|M_{n-1}| \leq \epsilon x) = 1$. Combining with (5.2), it follows that

$$\lim_{n\to\infty}\inf_{x\geq\gamma n^\tau}\frac{P(S_n \in x+\Delta)}{nF(x+\Delta)} \geq \lim_{n\to\infty}\inf_{x\geq\gamma n^\tau}\frac{\inf_{(1-\epsilon)x\leq z\leq(1+\epsilon)x} F(z+\Delta)}{F(x+\Delta)}$$

$$\geq \liminf_{x\to\infty} \frac{\inf\limits_{(1-\epsilon)x\leq z\leq(1+\epsilon)x} F(z+\Delta)}{F(x+\Delta)}.$$

By the arbitrariness of $\epsilon$, we immediately obtain that (5.1) holds uniformly for all $x \geq \gamma n^\tau$ as $n \to \infty$.

**Remark 5.1** The conditions in Lemma 5.1 are slightly weaker than those in Theorem 3.1. In Lemma 5.1, we do not need the condition that $F(x + \Delta)$ is almost decreasing.

Now we stand in the position to prove Theorem 3.1.

**Proof of Theorem 3.1** Without loss of generality, we assume that $a = 0$. By Lemma 5.1, it suffices to prove that

$$\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{P(S_n \in x+\Delta)}{nF(x+\Delta)} \leq L^+_{F_\Delta}. \tag{5.3}$$

Let $v = v(x) = -\log F(x + \Delta)$. It is obvious that $v(x)$ is a slowly varying function since $F(x+\Delta) \in \mathcal{OR}$ and $\lim\limits_{x\to\infty} v(x) = \infty$. In fact, for every fixed positive $y$,

$$\frac{v(xy)}{v(x)} = 1 - \frac{\log\Big(\dfrac{F(xy+\Delta)}{F(x+\Delta)}\Big)}{v(x)} \to 1, \quad x \to \infty.$$

Using the notations similar to Yang et al. [10], we denote

$$\widetilde{X} = XI\Big(X \leq \frac{x}{v^2}\Big), \quad \widetilde{X_i} = X_i I\Big(X_i \leq \frac{x}{v^2}\Big), \quad i = 1, 2, \cdots, \quad \widetilde{S_n} = \sum_{i=1}^n \widetilde{X_i}, \quad n = 1, 2, \cdots,$$

where $I(A)$ is the indicator function of the set $A$, and let $\eta = \eta(n, x) = \sum\limits_{i=1}^n I\big(X_i > \frac{x}{v^2}\big)$, i.e., $\eta$ is the (random) number of summands $X_i$ $(1 \leq i \leq n)$ in the sum $S_n = \sum\limits_{i=1}^n X_i$, such that $X_i \geq \frac{x}{v^2}$. Our starting point is the decomposition

$$P(S_n \in x + \Delta) = \mathrm{J}_0 + \mathrm{J}_1 + \mathrm{J}_2, \tag{5.4}$$

where

$$\mathrm{J}_i = \mathrm{J}_i(n, x) = P(S_n \in x + \Delta, \eta = i), \quad i = 0, 1$$

and

$$\mathrm{J}_2 = \mathrm{J}_2(n, x) = P(S_n \in x + \Delta, \eta \geq 2).$$

We will estimate (5.4) by three steps.

**Step 1** Estimation of $\mathrm{J}_0$

We use an idea of Tang [9] to deal with $\mathrm{J}_0$. For a positive number $h = h(x)$ which will be specified later, by Chebyshev's inequality, we have

$$\mathrm{J}_0 = P(\widetilde{S_n} \in x + \Delta, \eta = 0)$$

$$\leq P(\widetilde{S_n} > x) \leq \mathrm{e}^{-hx} E\mathrm{e}^{h\widetilde{S_n}} = \mathrm{e}^{-hx}(E\mathrm{e}^{h\widetilde{X}})^n. \tag{5.5}$$

To estimate the upper bound of $E\mathrm{e}^{h\widetilde{X}}$, we will discuss two cases according to $\tau > 1$ and $\tau \leq 1$, respectively. When $\tau > 1$, let $q = \min\{p, 1\} > \frac{1}{\tau}$. By virtue of the monotonicity in $x \in (0, \infty)$ of $\frac{\mathrm{e}^x - 1}{x^q}$, we obtain that

$$\begin{aligned}
E\mathrm{e}^{h\widetilde{X}} &\leq \overline{F}\Big(\frac{x}{v^2}\Big) + \int_{-\infty}^{0} F(\mathrm{d}u) + \int_0^{\frac{x}{v^2}} \mathrm{e}^{hu} F(\mathrm{d}u) \\
&= 1 + \int_0^{\frac{x}{v^2}} (\mathrm{e}^{hu} - 1) F(\mathrm{d}u) \\
&\leq 1 + \frac{\mathrm{e}^{\frac{hx}{v^2}} - 1}{\Big(\frac{x}{v^2}\Big)^q} \int_0^{\frac{x}{v^2}} u^q F(\mathrm{d}u) \\
&\leq \exp\Big\{ B_q \frac{\mathrm{e}^{\frac{hx}{v^2}} - 1}{\Big(\frac{x}{v^2}\Big)^q} \Big\},
\end{aligned}$$

where $B_q = \int_0^\infty u^q F(\mathrm{d}u) < \infty$. Hence, we have

$$\frac{\mathrm{J}_0}{F(x + \Delta)} \leq \exp\Big\{ -hx + v + \frac{B_q n(\mathrm{e}^{\frac{hx}{v^2}} - 1)}{\Big(\frac{x}{v^2}\Big)^q} \Big\}.$$

Taking $h = \frac{2v}{x}$ in the above equation, it yields that

$$\frac{\mathrm{J}_0}{F(x + \Delta)} \leq \exp\Big\{ -v + \frac{B_q n(\mathrm{e}^{\frac{2}{v}} - 1)}{\Big(\frac{x}{v^2}\Big)^q} \Big\}.$$

Since $v(x)$ is slowly varying and $v(x) \to \infty$ as $x \to \infty$, combining with $q\tau > 1$, it follows that

$$\lim_{n\to\infty} \sup_{x \geq \gamma n^\tau} \frac{\mathrm{J}_0}{F(x + \Delta)} = 0. \tag{5.6}$$

When $\tau \leq 1$, we split $\mathrm{e}^{h\widetilde{X}}$ into the following several parts:

$$\begin{aligned}
E\mathrm{e}^{h\widetilde{X}} &= \int_{-\infty}^{0} (\mathrm{e}^{hu} - hu - 1) F(\mathrm{d}u) + \int_0^{\frac{x}{v^2}} (\mathrm{e}^{hu} - hu - 1) F(\mathrm{d}u) + h \int_{-\infty}^{\frac{x}{v^2}} u F(\mathrm{d}u) + 1 \\
&\leq \int_{-\infty}^{0} (\mathrm{e}^{hu} - hu - 1) F(\mathrm{d}u) + \int_0^{\frac{x}{v^2}} (\mathrm{e}^{hu} - hu - 1) F(\mathrm{d}u) + 1 \\
&\cong \mathrm{I}_1 + \mathrm{I}_2 + 1,
\end{aligned} \tag{5.7}$$

where the last step but one is obtained by $EX = 0$. Let

$$q = \begin{cases} 1, & \text{if } \tau = 1, \\ \min\{p, 2\}, & \text{if } \tau < 1. \end{cases}$$

It is obvious that for $1 \leq q \leq 2$, the inequality $\mathrm{e}^x - x - 1 \leq |x|^q$ holds for all $x < 0$. Therefore, by $\int_{-\infty}^0 |u|^q F(\mathrm{d}u) < \infty$ and the dominated convergence theorem, it follows that

$$\mathrm{I}_1 = o(h^q) \quad \text{as} \;\; h \downarrow 0. \tag{5.8}$$

In what follows, we estimate $\mathrm{I}_2$. By virtue of the monotonicity in $x \in (0,\infty)$ of $\frac{\mathrm{e}^x - x - 1}{x^p}$, we have

$$\mathrm{I}_2 \le \frac{\mathrm{e}^{\frac{hx}{v^2}} - \frac{hx}{v^2} - 1}{\left(\frac{x}{v^2}\right)^p} \int_0^{\frac{x}{v^2}} u^p F(\mathrm{d}u) \le \frac{\mathrm{e}^{\frac{hx}{v^2}} - \frac{hx}{v^2} - 1}{\left(\frac{x}{v^2}\right)^p} B_p, \tag{5.9}$$

where $B_p = E(X^+)^p < \infty$. Substituting (5.8) and (5.9) into (5.7), it follows that

$$\begin{aligned} E\mathrm{e}^{h\widetilde{X}} &\le 1 + o_1(h^q) + \frac{\mathrm{e}^{\frac{hx}{v^2}} - \frac{hx}{v^2} - 1}{\left(\frac{x}{v^2}\right)^p} B_p \\ &\le \exp\Big\{ B_p \frac{\mathrm{e}^{\frac{hx}{v^2}} - 1}{\left(\frac{x}{v^2}\right)^p} + o_1(h^q) \Big\}, \end{aligned} \tag{5.10}$$

where $o_1(h)$ is a real function of $h > 0$ satisfying $\frac{o_1(h)}{h} \to 0$ as $h \to 0+$. Taking $h = \frac{2v}{x}$, it follows from (5.5) and (5.10) that

$$\begin{aligned} \frac{\mathrm{J}_0}{F(x+\Delta)} &\le \exp\Big\{ -hx + v + nB_p \frac{\mathrm{e}^{\frac{hx}{v^2}} - 1}{\left(\frac{x}{v^2}\right)^p} + n\cdot o_1(h^q) \Big\} \\ &\le \exp\Big\{ -v + \frac{B_p n(\mathrm{e}^{\frac{2}{v}} - 1)}{\left(\frac{x}{v^2}\right)^p} + n\cdot o_1\Big(\Big(\frac{2v}{x}\Big)^q\Big) \Big\}. \end{aligned}$$

Hence, (5.6) follows since $v(x)$ is slowly varying and $v(x) \to \infty$.

**Step 2** Estimation of $\mathrm{J}_2$

Since $X_1, X_2, \cdots, X_n, \cdots$ are i. i. d. r. v. s., for $n \ge 2$ and for sufficiently large $x$, we have that

$$\begin{aligned} \mathrm{J}_2 &\le 2 \sum_{1\le i<j\le n} P\Big(S_n \in x+\Delta, X_i > \frac{x}{v^2}, X_j > \frac{x}{v^2}\Big) \\ &= n(n-1) P\Big(S_n \in x+\Delta, X_{n-1} > \frac{x}{v^2}, X_n > \frac{x}{v^2}\Big) \\ &\le n^2 P\Big(S_{n-2} + X_{n-1} + X_n \in x+\Delta, X_{n-1} > \frac{x}{v^2}, X_n > \frac{x}{v^2}\Big) \\ &= n^2 \int_{-\infty}^{\infty} \int_{\frac{x}{v^2}}^{\infty} P\Big(X_n \in x-y-z+\Delta, X_n > \frac{x}{v^2}\Big) P(S_{n-2} \in \mathrm{d}y, X_{n-1} \in \mathrm{d}z) \\ &\le n^2 \sup_{u > \frac{x}{2v^2}} F(u+\Delta) \overline{F}\Big(\frac{x}{v^2}\Big), \end{aligned} \tag{5.11}$$

where the last step is obtained by the fact that $u+\Delta \cap \big[\frac{x}{v^2}, \infty\big) = \emptyset$ holds for all $u \le \dfrac{x}{2v^2}$ and sufficiently large $x$ if $T < \infty$, while the inequality is obvious if $T = \infty$. Since $F(x+\Delta)$ is almost decreasing, there exist constants $A > 0$ and $x_1 > 0$, for all $x > x_1$ such that

$$\sup_{u > \frac{x}{2v^2}} F(u+\Delta) \le AF\Big(\frac{x}{2v^2} + \Delta\Big).$$

According to Proposition 4.1, for some $\beta < \min\{\beta(F_\Delta), \beta(\overline{F}), 0\}$, there exist constants $C_\beta$ and $x_\beta > 0$, such that

$$\frac{F\Big(\dfrac{x}{2v^2}+\Delta\Big)}{F(x+\Delta)} \le C_\beta v^{-2\beta}$$

and

$$\frac{\overline{F}(\frac{x}{v^2})}{\overline{F}(x)} \le C_\beta v^{-2\beta}$$

hold for all $x$ satisfying $\frac{x}{v^2} > 2x_\beta$. Combining with (5.11), for sufficiently large $x$, we have

$$\frac{\mathrm{J}_2}{nF(x+\Delta)} \le A(C_\beta)^2 v^{-4\beta} n\overline{F}(x).$$

Note that $E|X|^p < \infty$ implies that $x^p\overline{F}(x) \to 0$, and that the function $v$ is slowly varying and $\tau > \frac{1}{p}$ yield that $x^{\frac{1}{\tau}-p}v^{-4\beta} \to 0$ as $x \to \infty$. Hence, it follows that

$$\lim_{n\to\infty} \sup_{x\ge\gamma n^\tau} \frac{\mathrm{J}_2}{nF(x+\Delta)} = 0, \tag{5.12}$$

since $nv^{-4\beta}\overline{F}(x) \le \gamma^{-\frac{1}{\tau}} x^{\frac{1}{\tau}}\overline{F}(x)v^{-4\beta} = \gamma^{-\frac{1}{\tau}} x^p\overline{F}(x)(x^{\frac{1}{\tau}-p}v^{-4\beta})$ holds for $x \ge \gamma n^\tau$.

**Step 3** Estimation of $\mathrm{J}_1$

For every fixed $\epsilon \in (0,1)$, we split $\mathrm{J}_1$ into three parts as

$$\begin{aligned}
\mathrm{J}_1 &= \sum_{i=1}^n P\Big(S_n \in x+\Delta, X_i > \frac{x}{v^2}, X_j \le \frac{x}{v^2}, j \ne i, j \le n\Big)\\
&= nP\Big(S_{n-1}+X_n \in x+\Delta, X_n > \frac{x}{v^2}, M_{n-1} \le \frac{x}{v^2}\Big)\\
&= \Big(\int_{-\infty}^{-\epsilon x} + \int_{-\epsilon x}^{\epsilon x} + \int_{\epsilon x}^{\infty}\Big) nP\Big(X_n \in x-y+\Delta, X_n > \frac{x}{v^2}\Big)P\Big(S_{n-1}\in \mathrm{d}y, M_{n-1} \le \frac{x}{v^2}\Big)\\
&= \mathrm{J}_{11} + \mathrm{J}_{12} + \mathrm{J}_{13}.
\end{aligned} \tag{5.13}$$

First, we estimate $\mathrm{J}_{11}$. Since $F(x+\Delta)$ is almost decreasing and $x - y > (1+\epsilon)x > x$ for all $y < -\epsilon x$, there exist constants $A$ and $x_0$, such that

$$\begin{aligned}
\mathrm{J}_{11} &= n\int_{-\infty}^{-\epsilon x} P(X_n \in x-y+\Delta)P\Big(S_{n-1}\in \mathrm{d}y, M_{n-1} \le \frac{x}{v^2}\Big)\\
&\le AnF(x+\Delta)P(S_{n-1} \le -\epsilon x)
\end{aligned}$$

holds for all $x > x_0$. By the strong law of large numbers of i. i. d. r. v. s., it follows that

$$\lim_{n\to\infty} \sup_{x\ge\gamma n^\tau} \frac{\mathrm{J}_{11}}{nF(x+\Delta)} \le \lim_{n\to\infty} AP(S_{n-1} \le -\epsilon\gamma n^\tau) = 0. \tag{5.14}$$

Next, we deal with $\mathrm{J}_{13}$. Clearly,

$$\mathrm{J}_{13} \le nP\Big(S_{n-1} > \epsilon x, M_{n-1} \le \frac{x}{v^2}\Big) = nP(\widetilde{S}_{n-1} > \epsilon x).$$

Hence, it follows from (5.5)–(5.6) that

$$\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{\mathrm{J}_{13}}{nF(x+\Delta)} = 0. \tag{5.15}$$

Finally, we discuss $\mathrm{J}_{12}$. Since $(1-\epsilon)x \leq x-y \leq (1+\epsilon)x$ for all $|y| < \epsilon x$, it follows that

$$\begin{aligned}
\mathrm{J}_{12} &= n\int_{-\epsilon x}^{\epsilon x} P\Big(X_n \in x-y+\Delta, X_n > \frac{x}{v^2}\Big)P\Big(S_{n-1}\in \mathrm{d}y, M_{n-1} \leq \frac{x}{v^2}\Big)\\
&\leq n \sup_{|y|\leq\epsilon x} F(x-y+\Delta)P\Big(|S_{n-1}| \leq \epsilon x, M_{n-1} \leq \frac{x}{v^2}\Big)\\
&\leq n \sup_{(1-\epsilon)x\leq z\leq(1+\epsilon)x} F(z+\Delta).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{\mathrm{J}_{12}}{nF(x+\Delta)} &\leq \lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{\sup\limits_{(1-\epsilon)x\leq z\leq(1+\epsilon)x} F(z+\Delta)}{F(x+\Delta)}\\
&\leq \limsup_{x\to\infty} \frac{\sup\limits_{(1-\epsilon)x\leq z\leq(1+\epsilon)x} F(z+\Delta)}{F(x+\Delta)}.
\end{aligned}$$

Letting $\epsilon \downarrow 0$, it follows that

$$\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{\mathrm{J}_{12}}{nF(x+\Delta)} \leq L^+_{F_\Delta}. \tag{5.16}$$

From (5.13)–(5.16), we have

$$\lim_{n\to\infty} \sup_{x\geq\gamma n^\tau} \frac{\mathrm{J}_{1}}{nF(x+\Delta)} \leq L^+_{F_\Delta}. \tag{5.17}$$

Consequently, (5.3) follows from (5.4), (5.6), (5.12) and (5.17). This completes the proof of the first part of Theorem 3.1. The second part of Theorem 3.1 follows immediately from the first part since $F(x+\Delta) \in \mathcal{IR}$ implies that $L^-_{F_\Delta} = L^+_{F_\Delta} = 1$.

**Proof of Corollary 3.1** Taking $T = 1$ and $x = m-1$ in Theorem 3.1, we have that

$$P(m-1 < S_n - na \leq m) \sim np_{[m+a]}$$

holds uniformly for $m \geq \gamma n^\tau$ as $n \to \infty$, where $[y]$ denotes the largest integer no more than $y$. Corollary 3.1 follows since the fact that the sequence $\{p_k : k = 0, \pm1, \pm2, \cdots\}$ belongs to the class $\mathcal{IR}$ implies that the sequence belongs to the class $\mathcal{L}$.

**Proof of Corollary 3.2** Corollary 3.2 follows immediately from Theorem 3.1 and Propositions 4.3 and 4.5.

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