Chin. Ann. Math.
37B(5), 2016, 767–776
DOI: 10.1007/s11401-016-1014-0

Chinese Annals of Mathematics, Series B

On 2-Adjacency Between Links*

Zhixiong TAO[1]

Abstract The author discusses 2-adjacency of two-component links and study the relations between the signs of the crossings to realize 2-adjacency and the coefficients of the Conway polynomial of two related links. By discussing the coefficient of the lowest m power in the Homfly polynomial, the author obtains some results and conditions on whether the trivial link is 2-adjacent to a nontrivial link, whether there are two links 2-adjacent to each other, etc. Finally, this paper shows that the Whitehead link is not 2-adjacent to the trivial link, and gives some examples to explain that for any given two-component link, there are infinitely many links 2-adjacent to it. In particular, there are infinitely many links 2-adjacent to it with the same Conway polynomial.

Keywords 2-Adjacency, Link, Conway polynomial, Jones polynomial, Homfly polynomial
2000 MR Subject Classification 57M25

1 Introduction

Since the concept of n-adjacency (see [1, 7]) was introduced as a specialization of Gusarov's notion of n-triviality (see [4]) and a generalization of being unknotting number one (see [12, 17], they coincide for $n = 1$), a lot of research about n-adjacency of knots (in particular for $n = 2$) has been done (see [1, 8, 18–21, 23–24]). Furthermore, this concept can be naturally extended to 2-adjacency between two links (see [23–25]). That is, a link L is called 2-adjacent to a link W, if L admits a projection D containing two crossings c_1, c_2 such that switching any $0 < s \leq 2$ of them yields a projection of W (see [1, 18, 23–24]).

In this paper, we are only concerned with 2-adjacency of two-component links. We study the relations between the signs of the crossings to realize 2-adjacency and the coefficients of the Conway polynomial of two related links. We give an expression of the Jones polynomial of the link obtained by opening two related crossings. We also study their Homfly polynomials and obtain some results and conditions on whether the trivial link is 2-adjacent to a nontrivial link, whether there are two links 2-adjacent to each other, etc. (see Sections 4–5). Finally, we show the Whitehead link not 2-adjacent to the trivial link, etc., and give some examples to explain that for any given link, there are infinitely many links 2-adjacent to it. In particular, there are infinitely many links 2-adjacent to it, which have the same Conway polynomial.

Unless otherwise stated, throughout this paper, our convention will be the following:

Manuscript received March 18, 2014. Revised September 15, 2015.
[1]Department of Mathematics, School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China. E-mail: tomzhx@163.com
*This work was supported by the Zhejiang Provincial Natural Science Foundation of China (No. LY12A01025).

Let a two-component link $L = L_1 \cup L_2$ be 2-adjacent to $W = W_1 \cup W_2$, c_1, c_2 the crossings to realize 2-adjacency, c_1 the crossing of L_1, and α (resp. β) the sign of c_1 (resp. c_2). Switching c_1 changes L_1 to W_1, so $L_2 = W_2$.

In this paper, we always assume a basic familiarity with Conway polynomial (defined by a skein relation, see [9, p. 19]), the Jones polynomial (see [11, p. 103]), Homfly polynomial (see [14]) and their properties. The reader may refer to [3, 9, 11, 13–14] for a more detailed exposition.

2 Conway Polynomial and 2-Adjacency of a Link

Proposition 2.1 (see [5–6, 9, 15]) *Let $L = L_1 \cup L_2 \cup \cdots \cup L_n$ be an oriented link with n components, and $l_{jk} = lk(L_j, L_k)$ for $j \neq k$.*

(1) $\nabla(L) = z^{n-1}(a_{n-1} + a_{n+1}z^2 + \cdots + a_{n-1+2m}z^{2m})$;

(2) *if $n = 1$, then $a_0 = 1$; if $n = 2$, then $a_1 = lk(L_1, L_2)$; if $n = 3$, then*

$$a_2 = l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23};$$

if $n = 4$, then

$$\begin{aligned} a_3(L) = &-l_{12}l_{13}l_{14} - l_{12}l_{23}l_{14} - l_{13}l_{23}l_{14} - l_{12}l_{13}l_{24} - l_{13}l_{14}l_{24} - l_{12}l_{23}l_{24} \\ &- l_{13}l_{23}l_{24} - l_{14}l_{23}l_{24} - l_{12}l_{13}l_{34} - l_{12}l_{14}l_{34} - l_{12}l_{23}l_{34} - l_{13}l_{23}l_{34} \\ &- l_{14}l_{23}l_{34} - l_{12}l_{24}l_{34} - l_{13}l_{24}l_{34} - l_{14}l_{24}l_{34}; \end{aligned}$$

(3) *if L_+ is an oriented knot and c is a positive crossing of L, then*

$$a_2(L_+) - a_2(L_-) = lk(L_0).$$

Here L_- and L_0 are obtained by switching and opening c respectively, and $a_j(X)$ indicates the coefficient of z^j in the Conway polynomial of a link X.

Moreover, $lk(L)$ denotes the total linking number of L, i.e., $lk(L) = \sum_{1 \le j < k \le n} l_{jk}$ (see p. 133 in [14]).

For the sake of convenience, let sx indicate switching crossing x, and ox opening x. According to the convention, L admits a diagram $D(c_1, c_2)$, such that switching the non-empty subset of $\{c_1, c_2\}$ yields a diagram of W respectively. By the definition (see [3, 9, 11]) of Conway polynomial, we have

$$\nabla(D(c_1, c_2)) - \nabla(D(sc_1, c_2)) = \alpha z \nabla(D(oc_1, c_2)), \tag{2.1}$$

$$\nabla(D(oc_1, c_2)) - \nabla(D(oc_1, sc_2)) = \beta z \nabla(D(oc_1, oc_2)), \tag{2.2}$$

$$\nabla(D(c_1, sc_2)) - \nabla(D(sc_1, sc_2)) = \alpha z \nabla(D(oc_1, sc_2)). \tag{2.3}$$

Here $D(u, v)$ has the same diagram as $D(c_1, c_2)$ except that u, v replace c_1, c_2 respectively and α (resp. β) is the sign of c_1 (resp. c_2) (see the above convention). Since $D(sc_1, c_2)$, $D(c_1, sc_2)$, $D(sc_1, sc_2)$ are W, from the above equalities, we have

$$\nabla(L) = \alpha\beta z^2 \nabla(D(oc_1, oc_2)) + \nabla(W). \tag{2.4}$$

From the above argument and Proposition 2.1, we obtain the following theorem.

Theorem 2.1 *If the notations and the conditions are as the convention, then* $lk(L) = lk(W)$ *and*

(1) *if* $a_3(L) \neq a_3(W)$, *then* $D(oc_1, oc_2)$ *is a two-component link;*

(2) *if* $a_3(L) = a_3(W)$, *then either* $lk(L) = 0$ *and* $D(oc_1, oc_2)$ *is a link with two components or* $D(oc_1, oc_2)$ *is a link with four components.*

From Theorem 2.1, it is easy to see that c_1 and c_2 are not the crossings between two components, since $lk(L) = lk(W)$.

In general, c_1, c_2 have the following two cases.

Case 1 If c_1, c_2 are the crossings of L_1, then $D(oc_1, oc_2)$ is a link with two or four components.

(a) If $D(oc_1, oc_2)$ has two components, then $lk(L) = lk(D(oc_1, oc_2))$, and $\alpha\beta$ can be got from $a_3(L) = \alpha\beta lk(L) + a_3(W)$.

However, if $lk(L) = 0$, the identity does not determine $\alpha\beta$. In this case, we need to consider the following: (i) Switching c_2 changes L_1 to W_2; (ii) switching c_2 (resp. c_1) changes L_1 to W_1 and switching c_1 (resp. c_2) changes W_1 to W_2. For these two cases, we obtain $W_2 = L_2 = W_1$. Hence, L_1 is 2-adjacent to W_1. It is similar to (2.1)–(2.4) that we have

$$\nabla(L_1) = \alpha\beta z^2 \nabla(\widehat{D}(oc_1, oc_2)) + \nabla(W_1), \tag{2.5}$$

where $\widehat{D}(oc_1, oc_2)$ is obtained from L_1 by opening c_1, c_2 and using Proposition 2.1. Thus, in both cases, $\alpha\beta$ can be got from the identity $a_2(L_1) = \alpha\beta + a_2(W_1)$.

(b) If $D(oc_1, oc_2)$ has four components, then $a_3(L) = a_3(W)$.

The discussion in (a) has shown that switching c_2 changes L_1 to W_1 and L_1 is 2-adjacent to W_1. Moreover, using (2.5) and Proposition 2.1, we know that opening c_1, c_2 changes L_1 to a three-component link, which is equivalent to $a_2(L_1) = a_2(W_1)$.

Suppose that opening c_1 (resp. c_2) changes L_1 to $G_1 \cup H$ (resp. $G \cup G_3$), and opening c_2 (resp. c_1) changes H (resp. G) to $G_2 \cup G_3$ (resp. $G_1 \cup G_2$). Denote L_2 and $lk(G_j, G_k)$ by G_4 and l_{jk} $(1 \leq j < k \leq 4)$ respectively. Since $a_2(L_1) = a_2(W_1)$, then $lk(G_1 \cup H) = 0$ and $lk(G \cup G_3) = 0$, i.e., $l_{12} = -l_{13} = l_{23}$. In general, α, β can be determined by the identity $a_4(L_1) - a_4(W_1) = -\alpha\beta(lk(\widetilde{L}))^2$ (see [21]), $\widetilde{L} = \widehat{D}(oc_1, oc_2) = G_1 \cup G_2 \cup G_3$ and $lk(\widetilde{L}) = l_{12}$.

From the identity $0 = a_3(L) - a_3(W) = \alpha a_2(G_1 \cup H \cup L_2)$ and Proposition 2.1, we have

$$0 = (l_{12} + l_{13}) lk(L) + l_{14}(l_{24} + l_{34}).$$

Similarly, $0 = (l_{13} + l_{23}) lk(L) + l_{34}(l_{14} + l_{24})$.

So $l_{14}(l_{24} + l_{34}) = 0,\ l_{34}(l_{14} + l_{24}) = 0$.

Notice that $lk(L) = l_{14} + l_{24} + l_{34}$. If $l_{24} = 0$, then $lk(L) = l_{14}$ or $lk(L) = l_{34}$. If $l_{24} \neq 0$, then $lk(L) = l_{24}$ (and $l_{14} = l_{34} = 0$) or $lk(L) = l_{34} (= l_{14} = -l_{24})$. Therefore, $lk(L)$ is always equal to the linking number between L_2 and one of the other components of $D(oc_1, oc_2)$ and $lk(D(oc_1, oc_2)) = l_{12} + lk(L)$.

We see that if either L or W has a trivial linking number (e.g. the Whitehead link, etc.), then $lk(L) = 0$, $|lk(D(oc_1, oc_2))| = \sqrt{|a_4(L_1) - a_4(W_1)|}$, $l_{14} = l_{24} = l_{34} = 0$, i.e., $a_3(D(oc_1, oc_2)) = 0$. In general, we have the following theorem.

Theorem 2.2 *If the notations and the conditions are as the convention, c_1, c_2 are the crossings of L_1, and $D(oc_1, oc_2)$ has four components, then*

$$\sqrt{|a_4(L_1) - a_4(W_1)|} = |lk(D(oc_1, oc_2)) - lk(L)|,$$

and $a_3(D(oc_1, oc_2))$ is $(lk(D(oc_1, oc_2)) - 2lk(L))^2 lk(L)$ if all linking numbers of L_2 with any other component of $D(oc_1, oc_2)$ are not zero, and is $|a_4(L_1) - a_4(W_1)|lk(L)$ otherwise.

Proof Since $l_{12} = -l_{13} = l_{23}$, by Proposition 2.1, $a_3(D(oc_1, oc_2)) = l_{12}^2 lk(L) - 2l_{12}l_{14}l_{34} - l_{14}l_{24}l_{34}$. If $l_{24} \neq 0$ and $lk(L) = l_{34} = l_{14} = -l_{24}$, since $l_{12} = lk(D(oc_1, oc_2)) - lk(L)$, then $a_3(D(oc_1, oc_2)) = [l_{12}^2 - 2l_{12}lk(L) + (lk(L))^2]lk(L) = (lk(D(oc_1, oc_2)) - 2lk(L))^2 lk(L)$. The rest of the proof is obvious.

Case 2 If c_1, c_2 are the crossings of the different components, i.e., c_1 in L_1 and c_2 in L_2, then $D(oc_1, oc_2)$ is a four-component link.

It is similar to Case 1 to prove that after switching c_2, although L_2 may become W_1 or W_2, L_j is equal to W_j, $j = 1, 2$.

Now, let $G_1 \cup G_2$ (resp. $G_3 \cup G_4$) be a link obtained from L_1 (resp. L_2) by opening c_1 (resp. c_2), and $l_{jk} = lk(G_j, G_k)$, $1 \leq j < k \leq 4$, so $lk(L) = l_{13} + l_{14} + l_{23} + l_{24}$. From Proposition 2.1 and the identity $0 = a_3(L) - a_3(W) = \alpha a_2(D(oc_1, c_2)) = \alpha a_2(G_1 \cup G_2 \cup L_2) = \beta a_2(L_1 \cup G_3 \cup G_4)$, we have

$$l_{12}lk(L) + (l_{13} + l_{14})(l_{23} + l_{24}) = 0, \quad l_{34}lk(L) + (l_{13} + l_{23})(l_{14} + l_{24}) = 0.$$

For the case of $lk(L) = 0$ (e.g. either L or W is the trivial link, the Whitehead link, etc.), by the above two equalities and $lk(L) = l_{13} + l_{14} + l_{23} + l_{24}$, we have $l_{13} = l_{14} = l_{23} = l_{24} = 0$, so $a_3(D(oc_1, oc_2)) = 0$.

Therefore, from the above discussion and Theorem 2.1, we obtain the following theorem.

Theorem 2.3 *If the notations and the conditions are as the convention, $lk(L) = 0$ and $D(oc_1, oc_2)$ has four components, then $a_5(L) = a_5(W)$.*

3 Jones Polynomial and 2-Adjacency

Let $V(X; t)$ indicate Jones polynomial (see [11]) of link X.

Proposition 3.1 (see [9, 11, 13]) *Suppose that $V(G; t)$ is the Jones polynomial of a link G with $c = c(G)$ components.*

(1) *If $c(G) = 1$, then $V'(G; 1) = 0$, $V''(G; 1) = -6a_2(G)$.*

(2) *If $c(G) > 1$, then $V'(G; 1) = -3(-2)^{c(G)-2}lk(G)$.*

(3) $V(G; 1) = (-2)^{c(G)-1}$.

It is similar to the discussions of (2.1)–(2.4) that we have the following theorem.

Theorem 3.1 *If the notations and the conditions are as the convention, then*

$$V(D(oc_1, oc_2); t) = \alpha\beta t^{1-\alpha-\beta}(V(L; t) - (t^{2\alpha} + t^{2\beta} - t^{2(\alpha+\beta)})V(W; t))(1 - t)^{-2}.$$

Corollary 3.1 *The notations and the conditions are as the convention. $D(oc_1, oc_2)$ is a link with four components if and only if $a_2(L_1) = a_2(W_1)$; $D(oc_1, oc_2)$ is a link with two components if and only if $a_2(L_1) = \alpha\beta + a_2(W_1)$.*

Proof According to Theorem 3.1, we have

$$V(D(oc_1, oc_2); 1) = \frac{\alpha\beta(8\alpha\beta V(W;1) + V''(L;1) - V''(W;1))}{2}.$$

Since $V(W;1) = -2$, $lk(L) = lk(W)$, $a_2(L_2) = a_2(W_2)$, using corrected Murakami's formula (see [16, 22]), we have

$$V(D(oc_1, oc_2); 1) = -8 + 6\alpha\beta(a_2(L_1) - a_2(W_1)).$$

Thus, the results follows from Proposition 3.1.

4 Homfly Polynomial and 2-Adjacency

We use $P(G)(l,m) = \sum p_j(l)m^j$ to represent the Homfly polynomial (see [14]) of a link G (sometimes we replace $p_j(l)$ by $p_j(G)$ or $p_j(G)(l)$). i always denotes $\sqrt{-1}$. Lickorish W. B. R. and Millett K. C. gave the following proposition.

Proposition 4.1 (see [14]). *Let link G have $c(G)$ components and the other notations be as above. We have the following properties:*

(1) *If G is a link with $c(G) \geq 2$ components, then*

$$\lim_{l \to \mathrm{i}} [(-(l + l^{-1})^{2-c(G)} p_{3-c(G)}(l)] = lk(G)\mathrm{i}.$$

The exponent of the lowest power of m which appears in the Homfly polynomial of L is precisely $1 - c(G)$. It has a coefficient

$$p_{1-c(G)}(l) = (-l^2)^{-lk(G)}[-(l + l^{-1})]^{c(G)-1} \prod_{j=1}^{c(G)} p_0^j(l). \tag{4.1}$$

(2) *If G is a knot, then $p_2(\mathrm{i}) = -a_2(G)$, $p_0(\mathrm{i}) = 1$, $p_0'(\mathrm{i}) = 0$, $p_0''(\mathrm{i}) = 8a_2(G)$.*

The notations and the conditions are as the convention. Then it is similar to the above argument in (2.1)–(2.4) that

$$m^2 P(D(oc_1, oc_2)) = P(W)(l^{-\alpha-\beta} + l^{\alpha-\beta} + l^{\beta-\alpha}) + l^{\alpha+\beta} P(L). \tag{4.2}$$

So we have the following theorem.

Theorem 4.1 *If the notations and the conditions are as above, then*

$$m^2 P(D(oc_1, oc_2)) = \begin{cases} P(W)(2 + l^{-2\alpha}) + l^{2\alpha} P(L), & \text{if } \alpha = \beta = \pm 1; \\ P(W)(l^{-2} + 1 + l^2) + P(L), & \text{if } \alpha\beta = -1. \end{cases}$$

Furthermore, by Theorem 2.1 and (4.1), $D(oc_1, oc_2)$ has two or four components depending on $p_{-3}(D(oc_1, oc_2))$ to be zero or not. So the following corollary is obtained easily.

Corollary 4.1 *If the notations and the conditions are as above, and $D(oc_1, oc_2)$ has two components, then*

(1) $\alpha = \beta = \pm 1 \Leftrightarrow p_{-1}(L) = -p_{-1}(W)(l^{-4\alpha} + 2l^{-2\alpha})$;

(2) $\alpha\beta = -1 \Leftrightarrow p_{-1}(L) = -p_{-1}(W)(l^{-2} + 1 + l^2)$.

Corollary 4.2 *The notations and conditions are as the convention.*

(1) *If $D(oc_1, oc_2)$ has two components, then $p_{-1}(L)(1)$ is an odd multiple of 6, or $p_0(1)$ of one component of L can be divisible by 3 and the other is trivial.*

(2) *If both components of L have trivial $p_0(l)$ (e.g. the trivial link, the Whitehead link, the Hopf link, etc.), then one component of W is the same as one of the components of L and the other has trivial a_2.*

Proof (1) It follows from (4.1) and the property that for any knot Q, $p_0(Q)(1)$ is an odd number.

(2) Assume $a_2(W_1) \neq 0$. From Corollary 3.1, we know that $D(oc_1, oc_2)$ is a two-component link, i.e., $p_{-3}(D(oc_1, oc_2)) = 0$. Calculating $p_{-1}(L)$, by (4.1)–(4.2), we have

$$(-l^2)^{-lk(L)}(l + l^{-1}) = p_{-1}(W)(l^{-2(\alpha+\beta)} + l^{-2\beta} + l^{-2\alpha}).$$

It is impossible.

Corollary 4.3 *The notations and conditions are as the convention. If $a_3(L) \neq a_3(W)$, then 2-adjacency is one-way at most.*

Proof Since $a_3(L) \neq a_3(W)$, by Theorem 2.1, $D(oc_1, oc_2)$ is a two-component link. Furthermore, by Corollary 3.1, $a_2(L_1) = \alpha\beta + a_2(W_1)$.

If W is also 2-adjacent to L, we can prove similarly that $a_2(W_1) - a_2(L_1) = \text{sign}(\overline{c}_1)\text{sign}(\overline{c}_2)$, where $\overline{c}_1, \overline{c}_2$ are the crossings to realize 2-adjacency, W_1 is 2-adjacent to L_1 and $D(o\overline{c}_1, o\overline{c}_2)$ has two components. Hence, $\text{sign}(\overline{c}_1)\text{sign}(\overline{c}_2) = -\alpha\beta$.

(1) If $\alpha\beta = 1$, then by Corollary 4.1, $p_{-1}(L) = -p_{-1}(W)(l^{-4\alpha} + 2l^{-2\alpha})$ and $p_{-1}(W) = -p_{-1}(L)(l^{-2} + 1 + l^2)$, i.e.,

$$p_{-1}(L) = p_{-1}(L)(l^{-4\alpha} + 2l^{-2\alpha})(l^{-2} + 1 + l^2).$$

However, it is impossible.

(2) It is similar to prove the case of $\alpha\beta = -1$.

5 2-Adjacency of the Trivial Link

Corollary 5.1 *The notations and the conditions are as the convention. If L is the trivial link and is 2-adjacent to W, then $D(oc_1, oc_2)$ has four components. Furthermore, if $a_4(W_1) = 0$, then $lk(D(oc_1, oc_2)) = 0$ and there exists an integer n such that $\prod_{j=1}^{2} p_0(W_j)(1) = 1 + 8n$, $\prod_{j=1}^{4} p_0^j(l) = 1 + 6n$. In particular, if c_1, c_2 are in the different components, then each component of $D(oc_1, oc_2)$ has trivial $p_0(l)$.*

Proof If the trivial link is 2-adjacent to W, then according to Corollary 4.2, the trivial link can not be 2-adjacent to a link with a component whose a_2 is not zero. By Corollary 3.1, $D(oc_1, oc_2)$ has four components. If c_1, c_2 are in L_1, then we know that L_1 is adjacent to W_1. Since $a_4(W_1) = 0$, by Theorem 2.2 and its proof, we obtain $lk(D(oc_1, oc_2)) = 0$ and all linking numbers between any two components of $D(oc_1, oc_2)$ are zero.

If c_1, c_2 are in different components, then W_1, W_2 are trivial. By (4.1)–(4.2),

$$l^{-\alpha-\beta} + l^{\alpha-\beta} + l^{\beta-\alpha} + l^{\alpha+\beta} = (-l^2)^{-lk(D(oc_1,oc_2))}(l+l^{-1})^2 \prod_{j=1}^{4} p_0^j(l), \tag{5.1}$$

i.e., $l^{-2} + 2 + l^2 = (-l^2)^{-lk(D(oc_1,oc_2)}(l+l^{-1})^2 \prod_{j=1}^{4} p_0^j(l)$, and also,

$$1 = (-l^2)^{-lk(D(oc_1,oc_2)} \prod_{j=1}^{4} p_0^j(l).$$

Hence, $p_0^j(l) = 1,\ j = 1, 2, 3, 4$, and $lk(D(oc_1, oc_2)) = 0$.

Next, taking $l = 1$, by (4.1)–(4.2), we have $3 \prod_{j=1}^{2} p_0(W_j)(1) + 1 = 4 \prod_{j=1}^{4} p_0^j(1)$. Assume $k = \prod_{j=1}^{2} p_0(W_j)(1) - \prod_{j=1}^{4} p_0^j(1)$. Obviously, $\prod_{j=1}^{2} p_0(W_j)(1) = 1 + 4k$, and $\prod_{j=1}^{4} p_0^j(1) = 1 + 3k$. Furthermore, since $p_0^j(1)$ is always an odd number, the conclusion is true.

Corollary 5.2 *Let the notations and the conditions be as the convention, L be the trivial link, c_1, c_2 be the crossings of L_1, $a_4(W_1) = 0$ and $D(oc_1, oc_2)$ have four components K_1, K_2, K_3, K_4 $(= L_2 = W_2)$. Then*

(1) $p_0'''(W_1)(\mathrm{i}) = 0$;

(2) $p_0^{(4)}(W_1)(\mathrm{i}) = 384 \sum_{j=1}^{3} a_2(K_j)$;

(3) $V''(D(oc_1, oc_2); 1) = 48 \sum_{j=1}^{3} a_2(K_j) - 6$;

(4) $V^{(3)}(W; 1) = \frac{3}{2}$, $V^{(4)}(W; 1) = -\frac{45}{8} - 576\alpha\beta \sum_{j=1}^{3} a_2(K_j)$.

Proof From Corollary 5.1 and its proof, we know that $lk(D(oc_1, oc_2)) = 0$ and $lk(K_j, K_k) = 0$, for any $j \neq k$. Since L is the trivial link and $D(oc_1, oc_2)$ has four components, by Corollary 3.1, $a_2(W_1) = 0$, i.e., $p_0''(W_1)(\mathrm{i}) = 0$, and by (4.1)–(4.2),

$$p_0(W_1)(l)(l^{-\alpha-\beta} + l^{\alpha-\beta} + l^{\beta-\alpha}) + l^{\alpha+\beta} = (l+l^{-1})^2 \prod_{j=1}^{4} p_0^j(l). \tag{5.2}$$

Thus, the result of (1) (resp. (2)) is easily obtained by calculating the values of the third (resp. the fourth) derivatives of both sides of (5.2) at $l = \sqrt{-1}$ and by using Proposition 4.1. Using the corrected Murakami's formula (see [16, 22]) and L'Hospital's rule, we conclude with the result (3)–(4) can be obtained by calculating the values of $V'(D(oc_1, oc_2); 1)$, $V''(D(oc_1, oc_2); 1)$ and by using (3), Proposition 3.1, L'Hospital's rule and corrected Murakami's formula again.

It is not difficult for the reader to find infinitely many two-component links, with one trivial component and the other nontrivial while $a_2 = 0$. For instance, these links can be constructed by using 8_{14}, whose a_2 is zero. In fact, there are many knots like 8_{14}, such as 10_{33}, 10_{67}, 10_{82}, 10_{108}, 10_{116}, 10_{118}, 10_{146}, etc. In [21], it has been proven that the trivial knot is not 2-adjacent to them. So by the proof of Corollary 5.1, the trivial link can not be 2-adjacent to the link constructed by the trivial knot and one of these knots. However, we do not know whether it is true in general.

6 Applications and Examples

Using the above results, we give two examples.

Example 6.1 The Whitehead link and the trivial link are not 2-adjacent to each other.

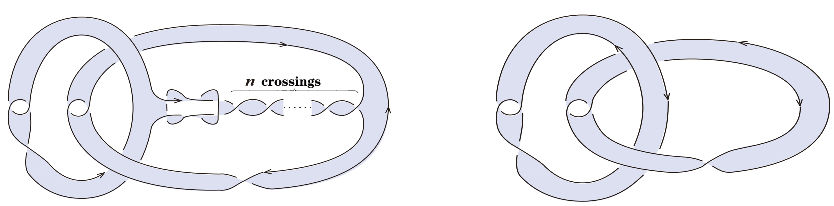
Proof We consider whether the trivial link is 2-adjacent to the Whitehead link. Since their a_3 are not equal, by Theorem 2.1, $D(oc_1, oc_2)$ is a two-component link. However, from Corollary 3.1, we know that $D(oc_1, oc_2)$ has four components. Hence, the trivial link is not 2-adjacent to the Whitehead link.

Similarly, we can prove that the Whitehead link is not 2-adjacent to the trivial link.

Example 6.2 $L7a2$ and $L7n1$ (see [2]) are not 2-adjacent to each other.

Proof Choosing the directions of their components such that their linking numbers are 2. $\nabla(L7a2) = 3z^3+2z$, $P(L7a2) = (l^{-9}+3l^{-7}+2l^{-5})m^{-1}-(3l^{-7}+4l^{-5}-l^{-3})m+(2l^{-5}-l^{-3})m^3$, $\nabla(L7n1) = z^5+4z^3+2z$, $P(L7n1) = (l^{-9}+3l^{-7}+2l^{-5})m^{-1}-(4l^{-7}+6l^{-5})m+(l^{-7}+5l^{-5})m^3-l^{-5}m^5$, $\nabla(L7n1!) = z^3+2z$, $P(L7n1!) = (2l^{-3}+3l^{-1}+l)m^{-1}-(l^{-3}+4l^{-1}+l)m+l^{-1}m^3$ (here $L7n1!$ is the mirror image of $L7n1$ and its linking number is -2). Using (4.1)–(4.2), we know that $D(oc_1, oc_2)$ is always a four-component link. However, their a_3 are different. By Theorem 2.1, $L7a2$ is not 2-adjacent to $L7n1$.

Similarly, we can prove that $L7n1$ is not 2-adjacent to $L7a2$.



(a) (b)

Figure 1 The links L_n and H

It is easy to check that the link in Figure 1(a) is 2-adjacent to the trivial link and for any $n \in \mathbb{N}$, its Conway polynomial is always zero. If the links in Figure 1 are denoted by L_n and H respectively, using the relation between bracket polynomial and Jones polynomial (see [10–11]), we have

$$\begin{aligned}
\langle L\rangle &= A^{-1}\langle L_{n-1}\rangle + A(-A^3)^{n-1}\langle H \sqcup \bigcirc\rangle \\
&= A^{-n}\langle L_0\rangle + \langle H \sqcup \bigcirc\rangle \sum_{j=0}^{n-1} A^{1-j}(-A^3)^{n-1-j} \\
&= A^{-n}\langle L_0\rangle + A(-A^3)^{n-1}\frac{1-(-A^{-4})^n}{1+A^{-4}}\langle H \sqcup \bigcirc\rangle, \\
(-A^3)^{-n+6}\langle L_n\rangle &= (-A^3)^{-n}A^{-n}[(-A^3)^6\langle L_0\rangle] \\
&\quad + A(-A^3)^{n-1}(-A^3)^{-n}\frac{1-(-A^{-4})^n}{1+A^{-4}}(-A^3)^6\langle H \sqcup \bigcirc\rangle, \\
V(L_n;t) &= (-1)^n t^n V(L_0;t) + (1-(-t)^n)V(H;t),
\end{aligned}$$

$$V(L_0;t) = -t^{-\frac{17}{2}}(1+t)(-1+5t-14t^2+28t^3-44t^4+57t^5-64t^6+64t^7 \\ -55t^8+42t^9-26t^{10}+13t^{11}-5t^{12}+t^{13}),$$

$$V(H;t) = t^{-1\frac{3}{2}}(1+t)(1-3t+5t^2-6t^3+6t^4-6t^5+4t^6-3t^7+t^8).$$

So, the highest-power term of $V(L_n;t)$ is $(-1)^{n+1}t^{n+\frac{11}{2}}$, i.e., L_n $(n = 1, 2, \cdots)$ are different from each other. In other words, there exist infinitely many links 2-adjacent to the trivial link and their Conway polynomials are 0.

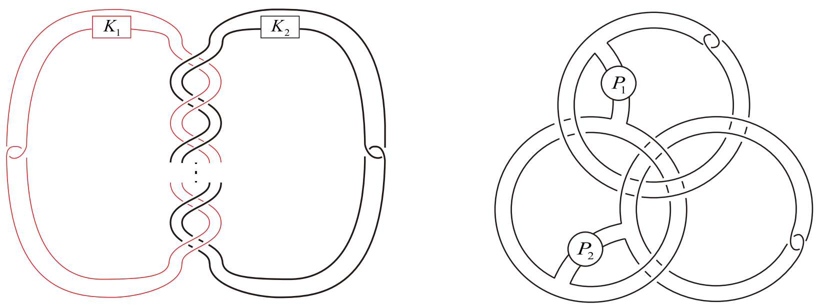


Figure 2 The two examples

The two examples in Figure 2 tell us that for any split link (such as $K_1 \cup K_2$ and $P_1 \cup P_2$ in Figure 2), we can find infinitely many links 2-adjacent to it.

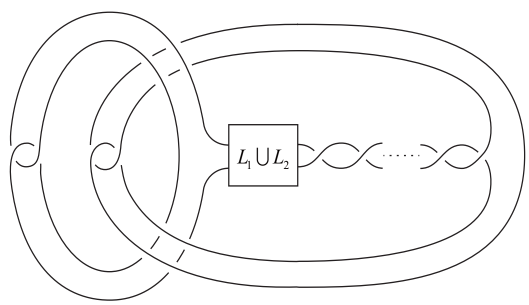
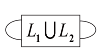


Figure 3 The links 2-adjacent to $L_1 \cup L_2$

Here  is a non-split link. Clearly, the example in Figure 3 shows the following fact that for any non-split link, we can find infinitely many links 2-adjacent to it, which have the same Conway polynomial and α, β can be chosen as you want.

References

[1] Askitas, N. and Kalfagianni, E., On knot adjacency, *Topology and Its Applications*, **126**, 2002, 63–81.

[2] Bar-Natan, D., The Thistlethwaite link table, http://katlas.math.toronto.edu/wiki/The_Thistlethwaite_Link_Table

[3] Burde, G. and Zieschang, H., Knots, de Gruyter, Berlin, New York, 1985.

[4] Gusarov, M. N., On n-equivalence of knots and invariants of finite degree, *Advances in Soviet Math.*, **18**, 1994, 173–192.

[5] Hoste, J., The Arf invariant of a totally proper link, *Topology and Its Applications*, **18**, 1984, 163–177.

[6] Hoste, J., The first coefficient of the Conway polynomial, *Proceedings of A. M. S.*, **95**(2), 1985, 299–302.

[7] Howards, H. and Luecke, J., Strongly n-trivial knots, *Bull. London Math. Soc.*, **34**, 2002, 431–437.

[8] Kalfagianni, E. and Lin, X. S., Knot adjacency, genus and essential tori, *Pacific J. Math.*, **228**(2), 2006, 251–275.

[9] Kauffman, L. H., On Knots, Annals of Mathematics Studies, Vol. 115, Princeton, University Press, Princeton, New Jersey, 1987.

[10] Kauffman, L. H., State models and the Jones polynomial, *Topology*, **26**, 1987, 395–497.

[11] Kawauchi, A., A survey of knot theory, Birkhäuser, Basel, Boston, Berlin, 1996.

[12] Lickorish, W. B. R., The unknotting number of a classical knot, *Contemporary Mathematics*, **44**, 1985, 117–119.

[13] Lickorish, W. B. R. and Millett, K. C., Some evaluations of link polynomials, *Commemt. Math. Helv.*, **61**, 1986, 349–359.

[14] Lickorish, W. B. R. and Millett, K. C., A polynomial invariant of oriented links, *Topology*, **26**(1), 1987, 107–141.

[15] Masbaum, G. and Vaintrobw, A., A new matrix tree theorem, http://arxiv.org/pdf/math.CO /0109104.pdf

[16] Murakami, H., On derivatives of the Jones polynomial, *Kobe J. Math.*, **3**, 1986, 61–64,

[17] Scharlemann, M., Unknotting number one knots are prime, *Invent. Math.*, **82**, 1985, 37–55.

[18] Stoimenow, A., On unknotting numbers and knot trivadjacency, *Mathematica Scandinavica*, **94**(2), 2004, 227–248.

[19] Tao, Z. X., On 2-adjacency of classical pretzel knots, *Journal of Knot Theory and Its Ramifications*, **22**(11), 2013, 1350066 (13 pages).

[20] Tao, Z. X., Conway polynomial of 2-adjacent knot, *Journal of Zhejiang University*, **32**(1), 2005, 17–20.

[21] Tao, Z. X., 2-Adjacency between knots, *Journal of Knot Theory and Its Ramifications*, **24**(11), 2015, 1550054.

[22] Tao, Z. X., An Evaluation Property of Jones Polynomial of a Link, *Journal of Zhejiang University*, **41**(5), 2014, 509–511.

[23] Torisu, I., On 2-adjacency relation of two-bridge knots and links, *Journal of the Australian Mathematical Society*, **84**(1), 2008, 139–144.

[24] Torisu, I., On 2-adjacency Relation of Links, Proceedings of the International Workshop on Knot Theory for Scientific Objects, 2007, 277–284.

[25] Tsutsumi, Y., Strongly n-trivial links are boundary links, *Tokyo J. Math.*, **30**(2), 2007, 343–350.