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On a Dual Risk Model Perturbed by Diffusion with Dividend Threshold*

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Abstract In the dual risk model, the surplus process of a company is a Lévy process with sample paths that are skip-free downwards. In this paper, the authors assume that the surplus process is the sum of a compound Poisson process and an independent Wiener process. The dual of the jump-diffusion risk model under a threshold dividend strategy is discussed. The authors derive a set of two integro-differential equations satisfied by the expected total discounted dividend until ruin. The cases where profits follow an exponential or mixtures of exponential distributions are solved. Applying the key method of the Laplace transform, the authors show how the integro-differential equations are solved. The authors also discuss the conditions for optimality and show how an optimal dividend threshold can be calculated as well.

Keywords Dual risk model, Threshold strategy, Stochastic optimal control, Smooth pasting condition
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1 Introduction

The optimal dividend problem proposed by de Finetti [10] in the XVth international congress of actuaries is to find the dividend-payment strategy that maximizes the expected discounted value of dividends which are paid to the shareholders until the company is ruined or bankrupt. He assumed that the annual gains of a stock company are independent and identically distributed random variables that only take the value -1 or $+1$. He also claimed that the optimal dividend strategy is a barrier strategy, that is, any surplus above a certain level should be paid to the shareholders immediately as dividends.

The optimal dividend problem in the classical compound Poisson model was first discussed by Bühlmann [8]. [2, 16] studied the problem with a bounded dividend rate in a Brownian motion model. They assumed that only dividend strategies with a ceiling for the dividend rate are admissible. They showed that the optimal dividend strategy is then a threshold strategy, that is, dividends should be paid to the shareholders at the maximal admissible rate once the surplus exceeds a certain threshold. A down-to-earth calculation can be found in Gerber and Shiu [14].

In insurance mathematics, the classical risk model has drawn the attention from researchers for decades. The surplus in the classical model at time t can be presented as

$$U(t) = u + ct - S(t), \tag{1.1}$$

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where u is the initial surplus, c is the premium rate, and $S(t)$ usually modeled by a compound Poisson process are the aggregate claims by time t. In this model, the optimal dividend strategy is not a barrier strategy in general (see [8]). In recent years, quite a few papers discussed the dual model to the classical insurance model. In the dual model, the surplus at time t is

$$U(t) = u - ct + S(t). \tag{1.2}$$

For example, Avanzi et al. [4] studied the expected total discounted dividends until ruin under the barrier strategy. Avanzi and Gerber [5] studied a dual model perturbed by diffusion and discussed how to determine the optimal value of the barrier. Ng [19] studied the optimal dividend problem under the bounded dividend rate constraint and calculated the optimal threshold by means of integro-differential equations.

In this paper, we discuss the dual risk model perturbed by diffusion with a bounded dividend rate constraint. Now the surplus at time t is

$$U(t) = u - ct + \sigma W(t) + S(t), \quad t \geqslant 0. \tag{1.3}$$

We assume $\mathbb{E}(S(1)) - c > 0$, which means the expected gain per unit time is positive. The diffusion term adds uncertainty to the expenses. It makes the model closer to the reality. Indeed, the expense rate can not be a constant in finance. Such a bounded dividend rate constraint makes our optimal strategy turn out to be a threshold strategy, that is, the company pays dividends at an admissible maximal rate when the surplus exceeds a threshold b. Comparing with the barrier strategy at the same level b, the time of ruin under the threshold strategy is longer, which makes the company prefer the threshold strategy. Indeed, the company with the barrier dividend policy will eventually go bankrupt, which is discrepant to the real cash flow.

This paper is inspired by [5, 19]. We formulate the problem and prove that the optimal strategy in our problem is a threshold strategy in Sections 2–3. In Section 4, we show that the expected total discounted dividends until ruin, denoted as $V(u;b)$, can be characterized as the solution of a set of two second order integro-differential equations in conjunction with three boundary conditions. It is shown that $V(u;b)$ can purely depend on the integro-differential equation satisfied by $0 \leqslant u \leqslant b$. We study the special cases where the profits distribution is an exponential or mixtures of exponential distributions. Section 5 introduces an alternative approach — the method of the Laplace transform — to obtain the expected total discounted dividends until ruin. Finally, we introduce the optimal threshold b^* in Section 6. With the help of $V(b^*;b^*) = \frac{c_2-c_1}{\delta} + \frac{1}{R_\delta^{(2)}}$ and $V_u(b^*;b^*) = 1$, b^* and $V(u;b^*)$ can be determined by the method of the Laplace transform.

2 Problem Formulation

We consider the dual risk model perturbed by diffusion. The surplus process $\{X(t)\}$ is given by

$$X(t) = u - ct + S(t) + \sigma W(t), \quad t \geqslant 0. \tag{2.1}$$

Here $u = X(0)$ is the initial surplus, and c is a positive constant which stands for the rate of expense. The aggregate profits process $\{S(t)\}$ is assumed to be a compound Poisson process, i.e.,

$$S(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $\{N(t)\}$ is a Poisson process with a Poisson parameter λ and individual profit amount Y_i's are independent and identically distributed with the probability density function $p(y)$, $y \geqslant 0$. $\{W(t)\}$ is a standard Wiener process which is independent of $\{S(t)\}$ and the volatility $\sigma > 0$ is a constant. The diffusion term adds uncertainty to the expenses which makes the model closer to the reality.

We now enrich the model. We assume that the dividends are paid to the shareholders according to some dividend strategies. Let $D(t)$ denote the aggregate dividends paid from time 0 to time t, and then the modified surplus process at time t is

$$U(t) = u - ct + S(t) + \sigma W(t) - D(t), \quad t \geqslant 0. \tag{2.2}$$

Let

$$T = \inf\{t \geqslant 0 \mid U(t) \leqslant 0\} \tag{2.3}$$

be the time of ruin and

$$D = \int_0^T \mathrm{e}^{-\delta t} \mathrm{d}D(t) \tag{2.4}$$

be the present value of all aggregate dividends until ruin, where $\delta > 0$ is the force of interest to discount the dividends. The company looks for a dividend strategy to maximize the expectation of the random variable D.

For a given $D(\cdot)$, the cost functional is defined by

$$V_D(u) = \mathbb{E}\Big[\int_0^T \mathrm{e}^{-\delta t} \mathrm{d}D(t) \mid U(0) = u\Big] = \mathbb{E}[D \mid U(0) = u]. \tag{2.5}$$

The value function is defined by

$$V(u) = \sup_{D(\cdot)\in\mathcal{A}} \mathbb{E}[D \mid U(0) = u], \quad u \geqslant 0, \tag{2.6}$$

where $\mathcal{A}$ is a class of processes called admissible controls which will be described below.

We study this stochastic optimal control problem under the constraint that only dividend strategies with a dividend rate bounded by a ceiling are admissible. We call a dividend strategy admissible if it is non-negative, non-decreasing, absolutely continuous and rate bounded. Thus, we assume

$$\mathrm{d}D(t) \leqslant \alpha \mathrm{d}t, \tag{2.7}$$

where $\alpha < \infty$ is the dividend rate ceiling.

3 Dynamic Programming

3.1 HJB equation

Proposition 3.1 *The value function V defined by* (2.6) *satisfies the Hamilton-Jacobi-Bellman* (HJB *for short*) *functional equation*

$$\begin{aligned} &\max_{0\leqslant r\leqslant\alpha}\{r - (c + r)V'(u)\} + \frac{\sigma^2}{2}V''(u) - (\lambda + \delta)V(u) \\ &+ \lambda \int_0^\infty V(u + y)p(y)\mathrm{d}y = 0, \quad u \geqslant 0. \end{aligned} \tag{3.1}$$

Proof We use Bellman's dynamic programming principle to prove (3.1). We consider a small time interval $[0,\epsilon]$, $\epsilon > 0$. Suppose that dividends are paid at rate r between time 0 and time ϵ, and then continue optimally. By conditioning on whether a jump occurs at the time interval $[0,\epsilon]$ and on the amount of the jump, we can obtain that the expectation of the present value of all dividends until ruin is

$$r\epsilon + \mathrm{e}^{-\delta\epsilon}\mathbb{E}\Big[(1-\lambda\epsilon)V(u-(c+r)\epsilon+\sigma W(\epsilon)) + \lambda\epsilon\int_0^\infty V(u+y)p(y)\mathrm{d}y\Big] + o(\epsilon). \tag{3.2}$$

Since

$$\mathrm{e}^{-\delta\epsilon} = 1 - \delta\epsilon + o(\epsilon),$$
$$\mathbb{E}[V(u-(c+r)\epsilon+\sigma W(\epsilon))] = V(u) - V'(u)(c+r)\epsilon + \frac{\sigma^2}{2}\epsilon V''(u) + o(\epsilon),$$

the expression (3.2) is equal to

$$\Big[r + \frac{\sigma^2}{2}V''(u) - (c+r)V'(u) - (\lambda+\delta)V(u) + \lambda\int_0^\infty V(u+y)p(y)\mathrm{d}y\Big]\epsilon + V(u) + o(\epsilon). \tag{3.3}$$

Because $V(u)$ is the optimal value, it must be equal to the maximum value of the expression (3.3), where $r \in [0,\alpha]$. Thus, we obtain the functional equation (3.1).

On the left-hand side of the equation (3.1), the expression to be maximized is

$$r[1 - V'(u)]$$

for $r \in [0,\alpha]$. Thus, the optimal dividend rate at time 0 is

$$\begin{aligned} r &= 0 \quad \text{if } V'(u) > 1,\\ r &= \alpha \quad \text{if } V'(u) < 1. \end{aligned}$$

Then, at time $t \in [0,T]$, the optimal dividend rate is

$$\begin{aligned} r &= 0 \quad \text{if } V'(U(t)) > 1,\\ r &= \alpha \quad \text{if } V'(U(t)) < 1. \end{aligned} \tag{3.4}$$

Such a dividend strategy has the character of a bang-bang strategy.

Remark 3.1 (3.4) can be interpreted as follows: When $V'(U(t)) > 1$, the company can be considered efficient, so it is best to leave all the funds with the company and pay no dividend. On the other hand, when $V'(U(t)) < 1$, the company is inefficient, it is advantageous to pay out as many dividends as allowable. The problem of decision between dividend payout and plowback is a classical problem in corporate finance.

3.2 Verification of optimality

We have shown that the value function V satisfies the HJB equation (3.1). However, this does not guarantee that any solution of the HJB equation (3.1) is the value function. The following theorem (the verification theorem) shows that a strategy is indeed an optimal strategy if its corresponding cost functional satisfies the HJB equation (3.1).

Theorem 3.1 *Suppose that $v(u)$ satisfies the* HJB *equation* (3.1). *Then for all $u \geqslant 0$, suppose that v is a C^2-function satisfying the* HJB *equation* (3.1)*, and then, for all $u \geqslant 0$ and*

$D(\cdot) \in \mathcal{A}$, we have that $v(u) \geqslant V_D(u)$. Consequently, if there exists a $D^*(\cdot) \in \mathcal{A}$ such that $v(u) = V_{D^*}(u)$, then

$$v(u) = \sup_{D(\cdot)\in\mathcal{A}} \mathbb{E}[D \mid U(0) = u] = V(u).$$

Proof Consider any admissible dividend strategies with dividend rate $r(t)$ and surplus $U(t)$ at time t. We claim that

$$\mathbb{E}\Big[\int_0^T \mathrm{e}^{-\delta t} r(t)\mathrm{d}t \mid U(0) = u\Big] \leqslant v(u). \tag{3.5}$$

To prove (3.5), we consider the compensated process

$$\Big\{\mathrm{e}^{-\delta t} v(U(t)) - \int_0^t \mathrm{e}^{-\delta\tau}\kappa(\tau)\mathrm{d}\tau\Big\}, \quad 0 \leqslant t \leqslant T, \tag{3.6}$$

where

$$\kappa(\tau) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[\mathrm{e}^{-\delta\epsilon} v(U(\tau+\epsilon)) - v(U(\tau)) \mid U(\tau)]. \tag{3.7}$$

Note that $\kappa(\tau)$ is the generator of the Itô diffusion, and then (3.6) is a martingale (see [20, Theorem 7.3.3, p. 123]). We have

$$\mathbb{E}\Big[\mathrm{e}^{-\delta(t\wedge T)} v(U(t\wedge T)) - \int_0^{t\wedge T} \mathrm{e}^{-\delta\tau}\kappa(\tau)\mathrm{d}\tau \mid U(0) = u\Big] = v(u), \tag{3.8}$$

which implies

$$\mathbb{E}\Big[-\int_0^{t\wedge T} \mathrm{e}^{-\delta\tau}\kappa(\tau)\mathrm{d}\tau \mid U(0) = u\Big] \leqslant v(u). \tag{3.9}$$

By a calculation similar to that by which we obtained (3.3), we see

$$\begin{aligned}\kappa(\tau) = {} & [-c - r(\tau)]v'(U(\tau)) + \frac{\sigma^2}{2}v''(U(\tau)) - (\lambda+\delta)v(U(\tau)) \\ & + \lambda\int_0^\infty v(U(\tau)+y)p(y)\mathrm{d}y.\end{aligned} \tag{3.10}$$

Because the function $v(u)$ satisfies the HJB equation (3.1), the sum of $r(\tau)$ and the expression (3.10) can not be positive. That is,

$$r(\tau) + \kappa(\tau) \leqslant 0. \tag{3.11}$$

Together with (3.9), we have

$$\mathbb{E}\Big[\int_0^{t\wedge T} \mathrm{e}^{-\delta\tau} r(\tau)\mathrm{d}\tau \mid U(0) = u\Big] \leqslant v(u). \tag{3.12}$$

Finally, (3.5) can be obtained by taking the limit $t \to \infty$. Then we have $v(u) \geqslant V_D(u)$. The remainder of the theorem is obvious, so the proof is completed.

4 Integro-Differential Equations

If the solution of the HJB equation (3.1) has the property that $V'(x) > 1$ for $x < b$ and $V'(x) < 1$ for $x > b$, for some number b, then the optimal dividend strategy is particularly appealing: When $0 < U(t) < b$, no dividends are paid, otherwise, when $U(t) > b$, dividends are paid at the maximal rate α. Such a dividend strategy is called a threshold strategy. After a threshold dividend strategy with a threshold level b applied, the dynamics of the surplus process $U(t)$ become

$$\mathrm{d}U(t) = -\widetilde{c}(U(t)) + \sigma \mathrm{d}W(t) + \mathrm{d}S(t), \quad U(0) = u, \tag{4.1}$$

where

$$\widetilde{c}(x) = \begin{cases} c_1 = c, & \text{when } 0 \leqslant x < b, \\ c_2 = c + \alpha, & \text{when } x > b. \end{cases}$$

The present value of all aggregate dividends until ruin is

$$D(b) = (c_2 - c_1) \int_0^T \mathrm{e}^{-\delta t} I(U(t) > b) \mathrm{d}t,$$

where I is the identity operator. The expected present value of all aggregate dividends until ruin is

$$V(u; b) = \mathbb{E}[D(b) \mid U(0) = u].$$

Since the surplus $U(t)$ has different paths for $0 \leqslant U(t) < b$ and $U(t) > b$, then we define

$$V(u; b) = \begin{cases} V_1(u), & \text{when } 0 \leqslant u < b, \\ V_2(u), & \text{when } u > b. \end{cases}$$

We derive a set of integro-differential equations satisfied by $V(u; b)$ in the following proposition.

Proposition 4.1 *The expectation of the discounted dividend $V(u; b)$ satisfies the following integro-differential equations: When $0 \leqslant u < b$,*

$$\begin{aligned} &(\lambda + \delta) V_1(u) + c_1 V_1'(u) - \frac{\sigma^2}{2} V_1''(u) \\ &= \lambda \int_0^{b-u} V_1(u + y) p(y) \mathrm{d}y + \lambda \int_{b-u}^{\infty} V_2(u + y) p(y) \mathrm{d}y, \end{aligned} \tag{4.2}$$

and when $u > b$,

$$(\lambda + \delta) V_2(u) + c_2 V_2'(u) - \frac{\sigma^2}{2} V_2''(u) = \lambda \int_0^{\infty} V_2(u + y) p(y) \mathrm{d}y + c_2 - c_1, \tag{4.3}$$

with the initial condition $V_1(0) = 0$, and continuity conditions $V_1(b) = V_2(b)$ and $V_1'(b) = V_2'(b)$.

Proof Firstly, we consider the case where $u > b$ and fix a small enough time τ such that $u - c_2\tau + \sigma W(\tau) > b$. By conditioning on whether a jump occurs and on the amount of the jump at the time interval $[0, \tau]$, it follows that

$$V_2(u) = (c_2 - c_1) \frac{1 - \mathrm{e}^{-\delta\tau}}{\delta} + \mathbb{E}[\mathrm{e}^{-(\delta+\lambda)\tau} V_2(u - c_2\tau + \sigma W(\tau))]$$

$$+\mathbb{E}\Big[\int_0^\tau \lambda e^{-(\lambda+\delta)t}\int_0^\infty V_2(u-c_2t+\sigma W(t)+y)p(y)\mathrm{d}y\mathrm{d}t\Big]. \tag{4.4}$$

Combining with the identities

$$e^{-(\delta+\lambda)\tau}=1-(\delta+\lambda)\tau+o(\tau),\quad e^{-\delta\tau}=1-\delta\tau+o(\tau)$$

and

$$\mathbb{E}[V_2(u-c_2\tau+\sigma W(\tau))]=V_2(u)-c_2\tau V_2'(u)+\frac{\sigma^2}{2}\tau V_2''(u)+o(\tau),$$

we subtract $V_2(u)$ on both sides of (4.4), divide by τ and let $\tau\to 0$. This induces (4.3).

Using a similar argument, we can also derive the corresponding integro-differential equation satisfied by $V_1(u)$.

Since ruin occurs immediately if the initial surplus is zero and no dividend is paid, the initial condition holds.

Remark 4.1 Though $V(u;b)$ and $V_u(u;b)$ are continuous at b, it may not be the case for $V_{uu}(u;b)$. Indeed, from (4.2),

$$(\lambda+\delta)V_1(b-)+c_1V_1'(b-)-\frac{\sigma^2}{2}V_1''(b-)=\lambda\int_0^\infty V_2(b+y)p(y)\mathrm{d}y,$$

while from (4.3),

$$(\lambda+\delta)V_2(b+)+c_2V_2'(b+)-\frac{\sigma^2}{2}V_2''(b+)=\lambda\int_0^\infty V_2(b+y)p(y)\mathrm{d}y+c_2-c_1.$$

As a result,

$$c_2V_2'(b+)-c_1V_1'(b-)-\frac{\sigma^2}{2}[V_2''(b+)-V_1''(b-)]=c_2-c_1, \tag{4.5}$$

which implies that

$$V_1''(b-)\neq V_2''(b+),$$

unless $V_1'(b-)=V_2'(b+)=1$. This fact will be used later in the determination of the optimal level of threshold.

Remark 4.2 For further reference, it is useful to rewrite the integro-differential equations as follows: When $0\leqslant u\leqslant b$,

$$(\lambda+\delta)V_1(u)+c_1V_1'(u)-\frac{\sigma^2}{2}V_1''(u)=\lambda\int_u^b V_1(x)p(x-u)\mathrm{d}x+\lambda\int_b^\infty V_2(x)p(x-u)\mathrm{d}x, \tag{4.6}$$

and when $u>b$,

$$(\lambda+\delta)V_2(u)+c_2V_2'(u)-\frac{\sigma^2}{2}V_2''(u)=\lambda\int_u^\infty V_2(x)p(x-u)\mathrm{d}x+c_2-c_1. \tag{4.7}$$

Remark 4.3 When $\sigma=0$, for $0\leqslant u\leqslant b$,

$$(\lambda+\delta)V_1(u)+c_1V_1'(u)=\lambda\int_0^{b-u}V_1(u+y)p(y)\mathrm{d}y+\lambda\int_{b-u}^\infty V_2(u+y)p(y)\mathrm{d}y,$$

and for $u>b$,

$$(\lambda+\delta)V_2(u)+c_1V_2'(u)=\lambda\int_0^\infty V_2(u+y)p(y)\mathrm{d}y+c_2-c_1.$$

These results were obtained in [19].

4.1 Explicit results for exponentially distributed profits

In this section, we obtain the explicit solution of (4.6)–(4.7) when jump amounts are exponentially distributed.

Let profits Y_i's follow an exponential distribution with $p(y) = \beta e^{-\beta y}$ for $y > 0$. Substituting the distribution density function into (4.7), we have, for $u > b$,

$$(\lambda+\delta)V_2(u) + c_2V_2'(u) - \frac{\sigma^2}{2}V_2''(u) = \lambda\beta\int_u^\infty V_2(y)e^{-\beta y}dy + (c_2 - c_1).$$

Applying the operator $\left(\frac{d}{du} - \beta\right)$ to both sides, we get

$$\frac{\sigma^2}{2}V_2'''(u) - \Big(\frac{\beta\sigma^2}{2} + c_2\Big)V_2''(u) + (c_2\beta - \lambda - \delta)V_2'(u) + \beta\delta V_2(u) - \beta(c_2 - c_1) = 0.$$

The third-order linear differential equation above has a particular solution $\frac{c_2-c_1}{\delta}$. Since the characteristic equation of the differential equation

$$\frac{1}{2}\sigma^2 r^3 - \Big(\frac{\beta\sigma^2}{2} + c_2\Big)r^2 + (c_2\beta - \lambda - \delta)r + \beta\delta = 0$$

has a negative root r_1 and two positive roots r_2, r_3 ($r_1 < 0 < r_2 < \beta < r_3$), we obtain

$$V_2(u) = D_1e^{r_1u} + D_2e^{r_2u} + D_3e^{r_3u} + \frac{c_2 - c_1}{\delta},$$

where D_1, D_2 and D_3 are undetermined coefficients. From

$$(c_2 - c_1)\int_0^T e^{-\delta t}I(U(t) > b)dt < (c_2 - c_1)\int_0^\infty e^{-\delta t}dt = \frac{c_2 - c_1}{\delta},$$

it is clear that

$$0 \leqslant V(u;b) \leqslant \frac{c_2 - c_1}{\delta}.$$

We have $D_3 \leqslant 0$. To prove that $D_3 = 0$, we consider the derivative of $V_2(u)$:

$$V_2'(u) = D_1r_1e^{r_1u} + D_2r_2e^{r_2u} + D_3r_3e^{r_3u}.$$

If $D_3 < 0$, then $V_2'(u) < 0$ for sufficiently large values of u, which contradicts the fact that V_2 is increasing in u. Thus, $D_3 = 0$ and $D_1 < 0$. Using exactly the same argument, we can also derive that $D_2 = 0$. Therefore,

$$V_2(u) = D_1e^{r_1u} + \frac{c_2 - c_1}{\delta}. \tag{4.8}$$

To solve V_1, we substitute the expression for $V_2(u)$ above into (4.6) and obtain

$$\begin{aligned}(\lambda+\delta)V_1(u) + c_1V_1'(u) - \frac{\sigma^2}{2}V_1''(u) = \lambda\beta e^{\beta u}\int_u^b V_1(y)e^{-\beta y}dy + \frac{\lambda\beta D_1}{\beta - r_1}e^{(r_1-\beta)b+\beta u}\\ + \frac{\lambda(c_2 - c_1)}{\delta}e^{-\beta(b-u)}\end{aligned}$$

for $0 \leqslant u \leqslant b$. Applying the operator $\left(\frac{d}{du} - \beta\right)$ to both sides, we have

$$\frac{\sigma^2}{2}V_1'''(u) - \Big(\frac{\beta\sigma^2}{2} + c_1\Big)V_1''(u) + (c_1\beta - \lambda - \delta)V_1'(u) + \beta\delta V_1(u) = 0.$$

Hence

$$V_1(u) = E_1 e^{s_1 u} + E_2 e^{s_2 u} + E_3 e^{s_3 u}, \tag{4.9}$$

where E_1, E_2 and E_3 are undetermined coefficients. s_1, s_2 and s_3 ($s_1 < 0 < s_2 < \beta < s_3$) are the solutions of the characteristic equation

$$\frac{1}{2}\sigma^2 s^3 - \Big(\frac{\beta\sigma^2}{2} + c_1\Big)s^2 + (c_1\beta - \lambda - \delta)s + \beta\delta = 0.$$

Since $V_1(0) = 0$, we have

$$E_1 + E_2 + E_3 = 0. \tag{4.10}$$

On the other hand, with the continuity condition: $V_1(b-) = V_2(b+)$, we have

$$E_1 e^{s_1 b} + E_2 e^{s_2 b} + E_3 e^{s_3 b} = D_1 e^{r_1 b} + \frac{c_2 - c_1}{\delta}. \tag{4.11}$$

With the first-order continuity condition: $V_1'(b-) = V_2'(b+)$, we have

$$E_1 s_1 e^{s_1 b} + E_2 s_2 e^{s_2 b} + E_3 s_3 e^{s_3 b} = D_1 r_1 e^{r_1 b}. \tag{4.12}$$

Substituting back the solution for $V_1(u)$ and $V_2(u)$ into (4.6), we have

$$\begin{aligned}
&(\lambda+\delta)(E_1 e^{s_1 u} + E_2 e^{s_2 u} + E_3 e^{s_3 u}) + c_1(s_1 E_1 e^{s_1 u} + s_2 E_2 e^{s_2 u} + s_3 E_3 e^{s_3 u}) \\
&- \frac{\sigma^2}{2}(s_1^2 E_1 e^{s_1 u} + s_2^2 E_2 e^{s_2 u} + s_3^2 E_3 e^{s_3 u}) \\
= {}& \lambda\beta e^{\beta u}\int_u^b [E_1 e^{s_1 y} + E_2 e^{s_2 y} + E_3 e^{s_3 y}] e^{-\beta y} dy \\
&+ \lambda\beta e^{\beta u}\int_b^\infty \Big[D_1 e^{r_1 y} + \frac{c_2 - c_1}{\delta}\Big] e^{-\beta y} dy.
\end{aligned}$$

Since the expression above must be satisfied for all $0 < u < b$, the sum of the coefficients of $e^{\beta u}$ is zero, i.e.,

$$\frac{E_1 e^{s_1 b}}{s_1 - \beta} + \frac{E_2 e^{s_2 b}}{s_2 - \beta} + \frac{E_3 e^{s_3 b}}{s_3 - \beta} + \frac{D_1 e^{r_1 b}}{\beta - r_1} + \frac{c_2 - c_1}{\beta\delta} = 0. \tag{4.13}$$

We have (4.10)–(4.13) to solve for E_1, E_2, E_3 and D_1. Then the solution for $V(u;b)$ can be expressed as

$$V(u;b) = \begin{cases} E_1 e^{s_1 u} + E_2 e^{s_2 u} + E_3 e^{s_3 u} & \text{for } 0 \leqslant u \leqslant b; \\ D_1 e^{r_1 u} + \dfrac{c_2 - c_1}{\delta} & \text{for } u > b. \end{cases}$$

4.2 $V(u;b)$ as a function of $V_1(u)$

When profits follow an exponential distribution, we have

$$V_2(u) = D_1 e^{r_1 u} + \frac{c_2 - c_1}{\delta}$$

for $D_1 < 0$. In this section, we show that the presentation above holds for any other profit distribution, which implies that $V(u;b)$ can be expressed as a function of $V_1(u)$. So it is not necessary to solve $V_2(u)$ explicitly.

Firstly, we need to deduce the generalized Lundberg fundamental equation, which plays an important role in the risk theory.

Lemma 4.1 *Consider a compound Poisson jump-diffusion dual model, where no dividend-distribution policy is imposed and expenses are paid continuously at a constant rate c,*

$$\widetilde{U}(t) = u - ct + \sigma W(t) + S(t).$$

Then the Laplace transform of the time of ruin, $\phi(u)$, is given by

$$\phi(u) = \mathbb{E}[\mathrm{e}^{-\delta T} \mid \widetilde{U}(0) = u] = \mathrm{e}^{R_\delta u},$$

where R_δ is the unique non-positive root of the generalized Lundberg fundamental equation

$$-\delta - \theta c + \lambda(M_Y(\theta) - 1) + \frac{1}{2}\sigma^2\theta^2 = 0, \tag{4.14}$$

where M_Y is the moment-generating function of Y_i.

Proof Consider a process $\{Z_\theta(t) : t \geqslant 0\}$ defined by $Z_\theta(t) = \mathrm{e}^{-\delta t + \theta \widetilde{U}(t)}$. Since $\{\widetilde{U}(t)\}$ has independent and stationary increments, $\{Z_\theta(t)\}$ is a martingale if and only if $\mathbb{E}(Z_\theta(t)) = Z_\theta(0)$. This condition is equivalent to

$$\exp\Big(- \delta t + \theta u - \theta ct + \lambda t(M_Y(\theta) - 1) + \frac{1}{2}\sigma^2\theta^2 t\Big) = \exp(\theta u).$$

Then, we obtain the generalized Lundberg fundamental equation

$$-\delta - \theta c + \lambda(M_Y(\theta) - 1) + \frac{1}{2}\sigma^2\theta^2 = 0.$$

Let R_δ be the unique non-positive root of (4.14), and note that $0 < Z_{R_\delta}(t) \leqslant 1$ for $0 \leqslant t \leqslant T$ gives that $\{Z_{R_\delta}(t \wedge T)\}$ is a bounded martingale. An application of the optional sampling theorem shows $\mathbb{E}(Z_{R_\delta}(t \wedge T)) = Z_{R_\delta}(0)$ for every t. By applying the dominated convergence theorem, we can obtain $\mathbb{E}(Z_{R_\delta}(T)) = Z_{R_\delta}(0)$, which is the result asserted.

Remark 4.4 By considering the slope of $\lambda(M_Y(\theta) - 1) - \theta c + \frac{1}{2}\sigma^2\theta^2$, it can be observed that $R_\delta = 0$ if and only if $\delta = 0$ and $c \geqslant \lambda\mu$, where μ is the mean of Y_i. Since we assume $\delta > 0$, R_δ will be strictly less than 0 no matter the drift of $\{\widetilde{U}(t)\}$ is positive or not.

Theorem 4.1 *For $u > b$,*

$$V(u; b) = (c_2 - c_1)\frac{1 - \mathrm{e}^{R_\delta^{(2)}(u-b)}}{\delta} + \mathrm{e}^{R_\delta^{(2)}(u-b)}V(b; b), \tag{4.15}$$

where $R_\delta^{(2)}$ is the unique negative root of the generalized Lundberg fundamental equation

$$\lambda(M_Y(\theta) - 1) - c_2\theta + \frac{1}{2}\sigma^2\theta^2 = \delta.$$

Proof Let $\chi = u - b$ and denote the first passage time until the surplus process descends χ units by $T_{-\chi}^{(2)}$. We consider a life status with the failure time $T_{-\chi}^{(2)}$. Dividends are paid at the rate $(c_2 - c_1)$ until $T_{-\chi}^{(2)}$. A life insurance of 1 payable at time $T_{-\chi}^{(2)}$ discounted at a continuously compounded rate of δ has the expected present value $\overline{A} = \mathrm{e}^{R_\delta^{(2)}\chi}$ according to Lemma 4.1. With the relation $\overline{A} + \delta\overline{a} = 1$, we have $\overline{a} = \frac{1-\mathrm{e}^{R_\delta^{(2)}\chi}}{\delta}$. Since the total discounted dividends until ruin are the sum of the continuous annuity payable until $T_{-\chi}^{(2)}$ with a payment rate $(c_2 - c_1)$ and the discounted dividends until ruin after the first downcrossing level b,

$$V(b + \chi; b) = (c_2 - c_1)\overline{a} + \overline{A}V(b; b)$$

$$= (c_2 - c_1)\frac{1 - \mathrm{e}^{R_\delta^{(2)}\chi}}{\delta} + \mathrm{e}^{R_\delta^{(2)}\chi} V(b; b), \tag{4.16}$$

which is the result asserted. This method is discussed in [19].

In view of Theorem 4.1, we do not need to solve $V_2(u)$ to obtain $V(u; b)$ when $0 \leqslant u \leqslant b$. Instead, we can directly substitute (4.16) into (4.2) to obtain an integro-differential equation to solve $V(u; b)$ when $0 \leqslant u \leqslant b$:

$$\begin{aligned}
&(\delta + \lambda)V(u; b) + c_1 V_u(u; b) - \frac{\sigma^2}{2} V_{uu}(u; b)\\
=& \lambda \int_0^{b-u} V(u + y; b)p(y)\mathrm{d}y + \frac{\lambda(c_2 - c_1)}{\delta}\int_{b-u}^{\infty} p(y)\mathrm{d}y\\
&+ \lambda\Big[V(b; b) - \frac{c_2 - c_1}{\delta}\Big]\int_{b-u}^{\infty} \mathrm{e}^{R_\delta^{(2)}(u-b+y)} p(y)\mathrm{d}y.
\end{aligned} \tag{4.17}$$

The equation above is analogous to Equation (2.4) in [5] with the exception of the term involving $R_\delta^{(2)}$.

Remark 4.5 When profits follow an exponential distribution, we have already proved

$$V_2(u) = D_1 \mathrm{e}^{r_1 u} + \frac{c_2 - c_1}{\delta}.$$

Together with (4.15), we can verify that

$$\begin{cases} r_1 = R_\delta^{(2)}, \\ D_1 = \Big(V(b; b) - \dfrac{c_2 - c_1}{\delta}\Big)\mathrm{e}^{-R_\delta^{(2)} b}. \end{cases} \tag{4.18}$$

4.3 Mixtures of exponential distributions

In this section we show how $V(u; b)$ can be calculated when

$$p(y) = \sum_{i=1}^{n} A_i \beta_i \mathrm{e}^{-\beta_i y}, \quad y > 0, \tag{4.19}$$

where $\beta_1 < \beta_2 < \beta_3 < \cdots < \beta_n$, $A_i > 0$, and $A_1 + A_2 + \cdots + A_n = 1$.

For notational convenience, we write

$$V_2(u) = G\mathrm{e}^{ru} + \frac{c_2 - c_1}{\delta}, \tag{4.20}$$

where $G = [V(b; b) - \frac{c_2 - c_1}{\delta}]\mathrm{e}^{-R_\delta^{(2)} b}$ and $r = R_\delta^{(2)}$.

The substitution of (4.19) in (4.6) induces

$$\begin{aligned}
&(\delta + \lambda)V_1(u) + c_1 V_1'(u) - \frac{\sigma^2}{2} V_1''(u)\\
=& \lambda \sum_{i=1}^{n} A_i \beta_i \mathrm{e}^{\beta_i u} \int_u^b V_1(y)\mathrm{e}^{-\beta_i y}\mathrm{d}y\\
&+ \frac{\lambda(c_2 - c_1)}{\delta} \sum_{i=1}^{n} A_i \mathrm{e}^{\beta_i (b-u)} + \lambda G \sum_{i=1}^{n} \frac{A_i \beta_i}{\beta_i - r}\mathrm{e}^{-(\beta_i - r)b + \beta_i u}.
\end{aligned} \tag{4.21}$$

Applying the operator $\left(\frac{\mathrm{d}}{\mathrm{d}u}-\beta_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}-\beta_2\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}u}-\beta_n\right)$ to both sides, we obtain an $(n+2)$-th order homogeneous linear differential equation (with undetermined coefficients) for $V(u;b)$. We assume that the roots of the corresponding characteristic equation are distinct. Hence, we get

$$V_1(u)=\sum_{k=0}^{n+1}C_k\mathrm{e}^{s_k u},\quad 0\leqslant u\leqslant b, \tag{4.22}$$

where $s_0<s_1<\cdots<s_{n+1}$ and C_1, $C_2,\cdots$, C_{n+1} are undetermined coefficients.

Substituting $V_1(u)$ into the original integro-differential equation (4.2), we obtain

$$\begin{aligned}&\sum_{k=0}^{n+1}\Big(\lambda+\delta+c_1s_k-\frac{1}{2}\sigma^2s_k^2\Big)C_k\mathrm{e}^{s_k u}\\&=\lambda\sum_{i=1}^{n}\sum_{k=0}^{n+1}\frac{A_i\beta_iC_k}{\beta_i-s_k}(\mathrm{e}^{s_k u}-\mathrm{e}^{-(\beta_i-s_k)b+\beta_i u})\\&\quad+\lambda G\sum_{i=1}^{n}\frac{A_i\beta_i}{\beta_i-r}\mathrm{e}^{-(\beta_i-r)b+\beta_i u}+\frac{\lambda(c_2-c_1)}{\delta}\sum_{i=1}^{n}A_i\mathrm{e}^{-\beta_i(b-u)}.\end{aligned}$$

Since the expression above holds for all $0\leqslant u\leqslant b$, by comparing the coefficients of $\mathrm{e}^{s_k u}$ and $\mathrm{e}^{\beta_i u}$, we obtain

$$\delta+\lambda+c_1s_k-\frac{1}{2}\sigma^2s_k^2=\lambda\sum_{i=1}^{n}\frac{A_i\beta_i}{\beta_i-s_k},\quad k=0,1,2,\cdots,n+1 \tag{4.23}$$

and

$$\sum_{k=0}^{n+1}\frac{C_k\mathrm{e}^{s_k b}}{\beta_i-s_k}=\frac{G\mathrm{e}^{rb}}{\beta_i-r}+\frac{c_2-c_1}{\beta_i\delta},\quad i=1,2,3,\cdots,n. \tag{4.24}$$

Hence, s_0, s_1, s_2, $\cdots$, s_{n+1} are the roots of

$$\delta+\lambda+c_1s-\frac{1}{2}\sigma^2s^2=\lambda\sum_{i=1}^{n}\frac{A_i\beta_i}{\beta_i-s}$$

and

$$s_0<0<s_1<\beta_1<s_2<\beta_2<\cdots<\beta_{n-1}<s_n<\beta_n<s_{n+1}.$$

Finally, combining (4.24) with the initial condition and the continuity conditions

$$\sum_{k=0}^{n}C_k=0,\quad \sum_{k=0}^{n}C_k\mathrm{e}^{s_k b}=G\mathrm{e}^{rb}+\frac{c_2-c_1}{\delta},\quad \sum_{k=0}^{n}C_ks_k\mathrm{e}^{s_k b}=Gr\mathrm{e}^{rb},$$

we have a system of $n+3$ equations to solve C_0, C_1, $\cdots$, C_{n+1} and G. Then the solution for $V(u;b)$ can be expressed as

$$V(u;b)=\begin{cases}\displaystyle\sum_{k=0}^{n}C_k\mathrm{e}^{s_k u} & \text{for } 0\leqslant u\leqslant b;\\ G\mathrm{e}^{ru}+\dfrac{c_2-c_1}{\delta} & \text{for } u>b.\end{cases} \tag{4.25}$$

5 Laplace Transforms

In the last section, we have already calculated $V(u;b)$ purely depending on the integro-differential equation satisfied by $0 \leqslant u \leqslant b$ and have illustrated the result explicitly for the exponential distribution and mixtures of exponential distributions. Now, we introduce an alternative method for the general cases.

Considering the convolution form in the integral part of (4.2)–(4.3), we apply the Laplace transform to solve $V(u;b)$. In order to discuss the domain of the Laplace transform, we replace the variable u by $z = b - u$ for $0 < u \leqslant b$. z denotes the distance between the threshold and the initial surplus, and we define W by

$$W(z;b) = V(b-z;b), \quad 0 \leqslant z \leqslant b.$$

In particular,

$$W(0;b) = V(b;b)$$

and

$$W(b;b) = 0.$$

In terms of the function $W(z;b)$, the integro-differential equation (4.17) becomes

$$\begin{aligned}&(\delta+\lambda)W(z;b) + c_1 W_z(z;b) - \frac{\sigma^2}{2}W_{zz}(z;b)\\ =& \lambda\int_0^z W(y;b)p(z-y)\mathrm{d}y + \frac{\lambda(c_2-c_1)}{\delta}\int_z^\infty p(y)\mathrm{d}y\\ &+\lambda\Big[W(0;b) - \frac{c_2-c_1}{\delta}\Big]\int_z^\infty \mathrm{e}^{R_\delta^{(2)}(y-z)}p(y)\mathrm{d}y \end{aligned}\tag{5.1}$$

with the initial condition $W(0;b) = V(b;b)$ and the boundary condition $W(b;b) = 0$.

We extend the definition of W by (5.1) to $z \geqslant 0$ and denote the resulting function by w. Then, the equation (5.1) becomes

$$\begin{aligned}&(\delta+\lambda)w(z) + c_1 w'(z) - \frac{\sigma^2}{2}w''(z)\\ =& \lambda\int_0^z w(y)p(z-y)\mathrm{d}y + \frac{\lambda(c_2-c_1)}{\delta}\int_z^\infty p(y)\mathrm{d}y\\ &+\lambda\Big[w(0) - \frac{c_2-c_1}{\delta}\Big]\int_z^\infty \mathrm{e}^{R_\delta^{(2)}(y-z)}p(y)\mathrm{d}y \end{aligned}\tag{5.2}$$

with two constraints $w(0) = V(b,b)$ and $w(b) = 0$.

Let $\widehat{w}$ and $\widehat{p}$ be the Laplace transform of w and the density p, respectively. Namely,

$$\widehat{w}(\xi) = \int_0^\infty \mathrm{e}^{-\xi z}w(z)\mathrm{d}z, \quad \widehat{p}(\xi) = \int_0^\infty \mathrm{e}^{-\xi y}p(y)\mathrm{d}y.$$

Taking the Laplace transforms in the equation (5.2) for $w(z)$ and $p(y)$, we obtain

$$\begin{aligned}&(\lambda+\delta) + c_1 w(0) - c_1\xi\widehat{w}(\xi) - \frac{1}{2}\sigma^2 w'(0) + \frac{1}{2}\sigma^2\xi w(0) - \frac{1}{2}\sigma^2\xi^2\widehat{w}(\xi)\\ =& \lambda\widehat{w}(\xi)\widehat{p}(\xi) + \frac{\lambda(c_2-c_1)}{\delta}\frac{1-\widehat{p}(\xi)}{\xi} + \lambda\Big[w(0) - \frac{c_2-c_1}{\delta}\Big]\frac{\widehat{p}(-R_\delta^{(2)}) - \widehat{p}(\xi)}{\xi + R_\delta^{(2)}}\end{aligned}\tag{5.3}$$

and hence

$$\widehat{w}(\xi) = \frac{A(\xi)}{c_1\xi + \frac{1}{2}\sigma^2\xi^2 + \lambda\widehat{p}(\xi) - (\lambda+\delta)}, \tag{5.4}$$

where

$$A(\xi) = c_1 w(0) - \frac{1}{2}\sigma^2 w'(0) + \frac{1}{2}\sigma^2\xi w(0) + \lambda w(0)\frac{\widehat{p}(\xi) - \widehat{p}(-R_\delta^{(2)})}{\xi + R_\delta^{(2)}} + \frac{\lambda(c_2 - c_1)}{\delta}\frac{R_\delta^{(2)}[\widehat{p}(\xi) - 1] + \xi[\widehat{p}(-R_\delta^{(2)}) - 1]}{\xi(\xi + R_\delta^{(2)})}.$$

We thus have the following procedure for determining $V(u;b)$ and b for a given value of $w(0) = V(b;b)$ and $w'(0) = V_u(b;b)$. Firstly, determine $w(z)$ by the inversion of (5.4). Then the underlying value of b follows from the condition that $w(b) = 0$. Finally, $V(u;b) = w(b-u)$, $0 \leqslant u \leqslant b$.

Remark 5.1 It is easy to see from the graph of

$$-\frac{1}{2}\sigma^2\theta^2 + \theta c_2 + \lambda + \delta = \lambda M_Y(\theta)$$

that $R_\delta^{(2)} \to 0$ when $c_2 \to \infty$. As a result, the limit of the final term in $A(\xi)$ is

$$\begin{aligned}
&\lim_{c_2\to\infty} \frac{\lambda(c_2 - c_1)}{\delta}\frac{R_\delta^{(2)}[\widehat{p}(\xi) - 1] + \xi[\widehat{p}(-R_\delta^{(2)}) - 1]}{\xi(\xi + R_\delta^{(2)})}\\
&= \lambda \lim_{c_2\to\infty} \frac{c_2}{\delta}\frac{R_\delta^{(2)}[\widehat{p}(\xi) - 1] + \xi[\widehat{p}(-R_\delta^{(2)}) - 1]}{\xi(\xi + R_\delta^{(2)})}\\
&= \lambda \lim_{R_\delta^{(2)}\to 0} \frac{\lambda[M_Y(R_\delta^{(2)}) - 1] + \frac{1}{2}\sigma^2(R_\delta^{(2)})^2 - \delta}{\delta\xi(\xi + R_\delta^{(2)})}\frac{R_\delta^{(2)}[\widehat{p}(\xi) - 1] + \xi[\widehat{p}(-R_\delta^{(2)}) - 1]}{R_\delta^{(2)}}\\
&= \lambda \lim_{R_\delta^{(2)}\to 0} -\frac{1}{\xi^2}\Big\{[\widehat{p}(\xi) - 1] + \frac{\xi[\widehat{p}(-R_\delta^{(2)})] - \widehat{p}(0)}{R_\delta^{(2)}}\Big\}\\
&= -\frac{\lambda}{\xi^2}[\widehat{p}(\xi) - 1 - \xi\widehat{p}'(0)].
\end{aligned}$$

It means that when $c_2 \to \infty$,

$$\widehat{w}(\xi) \to \frac{c_1 w(0) - \frac{1}{2}\sigma^2 w'(0) + \frac{1}{2}\sigma^2\xi w(0) + \frac{\lambda}{\xi}w(0)[\widehat{p}(\xi) - 1] - \frac{\lambda}{\xi^2}[\widehat{p}(\xi) - 1 - \xi\widehat{p}'(0)]}{c_1\xi + \frac{1}{2}\sigma^2\xi^2 + \lambda\widehat{p}(\xi) - (\lambda+\delta)},$$

which is the Laplace transform of the corresponding w for the barrier strategy (see (4.5) in [5]). Thus, the barrier strategy can be viewed as a limiting case of the threshold strategy.

Remark 5.2 For $\sigma = 0$, an equivalent result to (7.3) in [4] can be obtained directly.

6 Calculating the Optimal Threshold

In this section, we study the problem of the determination of the optimal threshold. For a particular value of the initial surplus u, we want to find an optimal threshold level b^* such that the expected total discounted dividends until ruin $V(u;b)$ are maximized. Thus

$$\frac{\partial V(u;b)}{\partial b}\Big|_{b=b*} = 0. \tag{6.1}$$

We show how b^* and $V(u;b^*)$ can be calculated below.

If we differentiate the identity $V_u(b-;b) = V_u(b+;b)$, we obtain another identity:

$$V_{uu}(b-;b) + \frac{\partial^2 V(u;b)}{\partial u \partial b}\Big|_{u=b-} = V_{uu}(b+;b) + \frac{\partial^2 V(u;b)}{\partial u \partial b}\Big|_{u=b+}. \tag{6.2}$$

For $b = b^*$, the second term of both sides vanishes because of (6.1). It follows that

$$V_{uu}(b^*-;b^*) = V_{uu}(b^*+;b^*), \tag{6.3}$$

and thus $V_{uu}(u;b^*)$ is continuous at b^*. This phenomenon is named the high contact condition in finance literature and the smooth pasting condition in the literature on optimal stopping problems. By Remark 4.1 in Section 4, we have

$$V_u(b^*;b^*) = V_u(b^*-;b^*) = V_u(b^*+;b^*) = 1. \tag{6.4}$$

It follows that

$$V_{uu}(b^*;b^*) = V_{uu}(b^*-;b^*) = V_{uu}(b^*+;b^*) = 0. \tag{6.5}$$

By using (4.15) and the above equations, we have

$$V_u(u;b) = -\frac{(c_2-c_1)R_\delta^{(2)}}{\delta}\mathrm{e}^{R_\delta^{(2)}(u-b)} + R_\delta^{(2)}\mathrm{e}^{R_\delta^{(2)}(u-b)}V(b;b), \tag{6.6}$$

and hence

$$V(b^*;b^*) = \frac{c_2-c_1}{\delta} + \frac{1}{R_\delta^{(2)}}. \tag{6.7}$$

In fact, the form of $V(b^*;b^*)$ in the exponential case holds for all profit distributions.

Remark 6.1 For $\sigma = 0$, $R_\delta^{(2)}$ shares exactly the same definition as Theorem 2 in [19]. Thus, (6.7) coincides with (25) in [19].

Remark 6.2 By using the definition of $R_\delta^{(2)}$, we can rewrite $V(b^*;b^*)$ as

$$\begin{aligned} V(b^*;b^*) &= \frac{(c_2-c_1)R_\delta^{(2)} + \delta}{\delta R_\delta^{(2)}} \\ &= \frac{\lambda(M_Y(R_\delta^{(2)}) - 1) - \frac{1}{2}\sigma^2(R_\delta^{(2)})^2}{\delta R_\delta^{(2)}} - \frac{c_1}{\delta}. \end{aligned}$$

Since $\lim\limits_{c_2\to\infty} R_\delta^{(2)} = 0$, we obtain

$$\lim_{c_2\to\infty} V(b^*;b^*) = \frac{\lambda\mu - c_1}{\delta}, \tag{6.8}$$

which coincides with (5.9) in [4] and (5.4) in [5]. A similar result has been obtained by Gerber [11] for the pure diffusion model.

Now we turn to the problem of determining $V(u;b^*)$ and b^*. Formula (6.7) is crucial for implementing the method of the Laplace transform described in Section 5. Formula (6.7) is equivalent to $W(0;b^*) = \frac{c_2-c_1}{\delta} + \frac{1}{R_\delta^{(2)}}$. Thus we proceed as follows. In (5.4) we set $w(0) =$

$\frac{c_2-c_1}{\delta}+\frac{1}{R_\delta^{(2)}}$ and $w'(0)=1$, so we obtain the function $w(z)$ by the inversion of its Laplace transform. Then b^* is the zero of $w(z)$, and

$$V(u;b^*)=w(b^*-u), \quad 0\leqslant u\leqslant b^*. \tag{6.9}$$

Together with (4.15), we can deduce the optimal level of threshold and $V(u;b^*)$ for $u\geqslant 0$.

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