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Estimates for Fourier Coefficients of Cusp Forms in Weight Aspect*

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Abstract Let f be a holomorphic Hecke eigenform of weight k for the modular group $\Gamma = SL_2(\mathbb{Z})$ and let $\lambda_f(n)$ be the n-th normalized Fourier coefficient. In this paper, by a new estimate of the second integral moment of the symmetric square L-function related to f, the estimate

$$\sum_{n\le x}\lambda_f(n^2) \ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log(x+k))^6$$

is established, which improves the previous result.

Keywords Fourier coefficients, Cusp forms, Symmetric square L-function
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1 Introduction

Let $H_k(\Gamma)$ be the space of Hecke-eigen cusp forms of even integral weight k for $\Gamma = SL(2,\mathbb{Z})$. Suppose that $f(z)$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e(nz),$$

where $e(x) := \mathrm{e}^{2\pi \mathrm{i}x}$ and the n-th normalized Fourier coefficient $\lambda_f(n)$ of f is the eigenvalue under the Hecke operator T_n. Then from the theory of Hecke operators, the following is nowadays widely known:

(i) $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)}\lambda_f\Big(\frac{mn}{d^2}\Big) \tag{1.1}$$

for all integers $m \ge 1$ and $n \ge 1$.

(ii) For all $n \ge 1$,

$$|\lambda_f(n)| \le d(n), \tag{1.2}$$

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where $d(n)$ is the divisor function. This is the well-known Petersson-Ramanujan conjecture which was proved by Deligne [2] in 1974. As a corollary, he proved that for any $\varepsilon > 0$,

$$\sum_{n\le x}\lambda_f(n) \ll_f x^{\frac{1}{3}+\varepsilon}.$$

Later, many authors considered the summation and the Sato-Tate conjecture implies that

$$\sum_{n\le x}\lambda_f(n) \ll_f \frac{x^{\frac{1}{3}}}{(\log x)^{\rho}}$$

with $\rho \approx 0.151$.

Let

$$S(x) = \sum_{n\le x}\lambda_f(n^2).$$

It was Ivić [7] who first considered oscillations of the Fourier coefficients over squares. Based on the prime number theorem, he successfully showed that

$$S(x) \ll_f x\exp(-A(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

In 2006, Fomenko [4] improved Ivić's result by proving

$$S(x) \ll_f x^{\frac{1}{2}}\log^3 x.$$

In 2006, Sankaranarayanan [13] proved that

$$S(x) \ll x^{\frac{3}{4}}(\log x)^{\frac{19}{2}}\log\log x \tag{1.3}$$

uniformly for $k \ll x^{\frac{1}{3}}(\log x)^{\frac{22}{3}}$, where the implied constant is absolute. Later, Lü [10] showed that, in fact, for $k \geqslant 2$,

$$S(x) \ll x^{\frac{1}{2}}k^{\frac{3}{4}}(\log x)^{\frac{19}{5}}\log\log x + x^{\frac{3}{5}}(\log x)^{\frac{42}{5}}(\log\log x)^{\frac{4}{5}}, \tag{1.4}$$

where the implied constant is absolute. Ichihara [6] obtained the best upper bound for x which states that

$$S(x) \ll x^{\frac{1}{2}}k^{\frac{3}{4}}(\log x)^{\frac{19}{2}}, \tag{1.5}$$

where the implied constant is effective.

The purpose of this paper is to improve the above results in the weight aspect. By a new bound for the second integral moment of the symmetric square L-function $L(s, \mathrm{sym}^2 f)$ at the critical line (see Proposition 2.1 in the next section), we get the following result.

Theorem 1.1 *Let f be a holomorphic Hecke eigenform of weight k for Γ and let $\lambda_f(n)$ be the n-th normalized Fourier coefficient. Then we have*

$$S(x) = \sum_{n\le x}\lambda_f(n^2) \ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log(x+k))^6,$$

where the implied constant is absolute and does not depend on f.

2 Preliminaries

The behavior of $S(x)$ is intimately connected with the symmetric square L-function associated with f which is defined by

$$L(s, \mathrm{sym}^2 f) = \zeta(2s) \sum_{n\geq 1} \frac{\lambda_f(n^2)}{n^s}, \tag{2.1}$$

where $\Re s > 1$ and $\zeta(s)$ is the Riemann zeta function. For convenience, hereafter, we write $F = \mathrm{sym}^2 f$ which is a cuspidal automorphic form for $SL(3, \mathbb{Z})$ by the Gelbart-Jacquet lift (see [5]). The functional equation of $L(s, F)$ is given by

$$\Lambda(s, F) = L_\infty(s, F)L(s, F),$$

where

$$L_\infty(s, F) = \pi^{-\frac{3s}{2}} \Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(\frac{s+k-1}{2}\Big)\Gamma\Big(\frac{s+k}{2}\Big)$$

is the Archimedean local factor. It is known that $\Lambda(s, F)$ can be extended to an entire function and satisfies (see [8])

$$\Lambda(s, F) = \Lambda(1 - s, F). \tag{2.2}$$

Denote by $\lambda_F(n)$ the n-th coefficient of the Dirichlet series expansion of $L(s, F)$. This means that for $\Re s > 1$,

$$L(s, F) = \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s}.$$

From (2.1), we have

$$\lambda_F(n) = \sum_{\substack{m^2|n \\ m>0}} \lambda_f\Big(\Big(\frac{n}{m^2}\Big)^2\Big). \tag{2.3}$$

Comparing with another special $GL(3)$ L-function $\zeta^3(s)$,

$$\zeta^3(s) = \zeta(2s) \sum_{n\geq 1} \frac{d(n^2)}{n^s} = \sum_{n=1}^{\infty} \frac{d_3(n)}{n^s}, \quad \Re s > 1,$$

we have, by (1.2),

$$|\lambda_F(n)| \leq \sum_{\substack{m^2|n \\ m>0}} d\Big(\Big(\frac{n}{m^2}\Big)^2\Big) = d_3(n). \tag{2.4}$$

By Möbius inversion and (2.3), we have

$$\lambda_f(n^2) = \sum_{\substack{m^2|n \\ m>0}} \lambda_F\Big(\frac{n}{m^2}\Big)\mu(m),$$

where $\mu(m)$ is the Möbius function. As in Ichihara [6], we transform the question of estimating $S(x)$ into studying the sum $\sum_{n\leq x} \lambda_F(n)$ in the following way:

$$S(x) \leq \sum_{0<m\leq\sqrt{x}} \Big| \sum_{n\leq \frac{x}{m^2}} \lambda_F(n)\Big|.$$

Then Theorem 1.1 follows from the following result.

Proposition 2.1 *Let f be a holomorphic Hecke eigenform of weight k for Γ and $L(s,F)$ the symmetric square L-function associated with f. Denote by $\lambda_F(n)$ the n-th normalized Fourier coefficient of $L(s,F)$. Then we have*

$$\sum_{n\le x}\lambda_F(n)\ll x^{\frac12}k^{\frac12}(\log(x+k))^5,$$

where the implied constant is effective and does not depend on F.

To prove Proposition 2.1, we need the following four lemmas. The first one is related to the uniform convexity bound for $L(s,F)$. In order to give a new estimate for the mean square of $L(s,F)$ at the critical line, we introduce the approximate functional equation of $L(s,F)$ and a classical result due to Montgomery and Vaughan [11]. The difficulty is that the weight aspect should be considered.

Lemma 2.1 *Let $\tau=(|t|+1)(k+|t|)^2$. Then*

$$L(\sigma+\mathrm{i}t,F)\ll\tau^{\frac{1-\sigma}{2}}(\log\tau)^3 \tag{2.5}$$

holds for $-\frac{1}{\log\tau}\le\sigma\le1+\frac{1}{\log\tau}$.

Proof By (2.4), we have

$$\Big|L\Big(1+\frac{1}{\log\tau}+\mathrm{i}t,F\Big)\Big|\le\sum_{n\ge1}\frac{d_3(n)}{n^{1+\frac{1}{\log\tau}}}=\zeta^3\Big(1+\frac{1}{\log\tau}\Big)\ll(\log\tau)^3. \tag{2.6}$$

On the other hand, by the functional equation in (2.2), we have

$$L(s,F)=\chi(s,F)L(1-s,F),$$

where

$$\chi(s,F)=(2\pi^{\frac32})^{2s-1}\frac{\Gamma\big(1-\frac{s}{2}\big)\Gamma(k-s)}{\Gamma(\frac{s+1}{2})\Gamma(k+s-1)}.$$

In [13], Sankaranarayanan proved that for any $\epsilon>0$ and $-1+\epsilon\le\Re s=c<0$,

$$\Big|\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\Big|\ll(|t|+1)^{\frac12-c},\quad\Big|\frac{\Gamma(k-s)}{\Gamma(k+s-1)}\Big|\ll(k+|t|)^{1-2c}.$$

Then we have

$$\chi(c+\mathrm{i}t,F)\ll\tau^{\frac12-c}.$$

It follows that

$$\Big|L\Big(-\frac{1}{\log\tau}+\mathrm{i}t,F\Big)\Big|=\Big|\chi\Big(-\frac{1}{\log\tau}+\mathrm{i}t\Big)L\Big(1+\frac{1}{\log\tau}-\mathrm{i}t,F\Big)\Big|\ll\tau^{\frac12}(\log\tau)^3. \tag{2.7}$$

Replacing the formulas (3.4.1) and (3.4.2) in the paper of Sankaranarayanan [13] by (2.6)–(2.7), we complete the proof.

Lemma 2.2 *Let $s=\frac{1}{2}+\mathrm{i}t$, $T\le t\le 2T$ and $\varepsilon=\frac{1}{\log(T+k)}$. Then for any $Y\ge 2$, we have*

$$L(s,F)=\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^s}\mathrm{e}^{-\frac{n}{Y}}-\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z+O\Big(\frac{Y^{\frac{1}{2}+\varepsilon}}{\varepsilon^3T^{\frac{3}{2}}}+\frac{T+k}{\varepsilon^3TY^{\frac{1}{2}+\varepsilon}}\Big).$$

Proof By applying Mellin's inversion formula to $\Gamma(z)$, we have

$$\mathrm{e}^{-x}=\frac{1}{2\pi\mathrm{i}}\int_{(\frac{1}{2}+\varepsilon)}\Gamma(z)x^{-z}\mathrm{d}z,$$

where $x>0$ and (a) means the line $\Re z=a$. We put $x=\frac{n}{Y}$, multiply $\frac{\lambda_F(n)}{n^s}$ and sum over n on the both sides. Finally we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^s}\mathrm{e}^{-\frac{n}{Y}}&=\frac{1}{2\pi\mathrm{i}}\int_{(\frac{1}{2}+\varepsilon)}\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^{s+z}}Y^z\Gamma(z)\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int_{(\frac{1}{2}+\varepsilon)}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\&=\frac{1}{2\pi\mathrm{i}}\int_{\frac{1}{2}+\varepsilon-\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\&\quad+O\Big(\int_{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\infty}|L(s+z,F)Y^z\Gamma(z)|\mathrm{d}z\Big)\\&=\frac{1}{2\pi\mathrm{i}}\int_{\frac{1}{2}+\varepsilon-\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z+O\Big(\frac{Y^{\frac{1}{2}+\varepsilon}}{\varepsilon^3T^{\frac{3}{2}}}\Big),\end{aligned}\tag{2.8}$$

where we have used $\Gamma(a+\mathrm{i}b)\ll\mathrm{e}^{-\frac{\pi|b|}{2}}|b|^{a-\frac{1}{2}}$. Moving the line of integration to $\Re z=-\frac{1}{2}-\varepsilon$, we have

$$\begin{aligned}\mathrm{I}=:&\ \frac{1}{2\pi\mathrm{i}}\int_{\frac{1}{2}+\varepsilon-\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\=&\ L(s,F)+\frac{1}{2\pi\mathrm{i}}\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\&+O\Big(\int_{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\Big)\\=&\ L(s,F)+\frac{1}{2\pi\mathrm{i}}\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\&+O\Big(\frac{1}{T^{\frac{3}{2}}}\int_{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}^{\frac{1}{2}+\varepsilon+\mathrm{i}\log T}|L(s+z,F)Y^z|\mathrm{d}z\Big)\\=&\ L(s,F)+\frac{1}{2\pi\mathrm{i}}\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\\&+O\Big(\frac{Y^{\frac{1}{2}+\varepsilon}}{\varepsilon^3T^{\frac{3}{2}}}+\frac{T+k}{\varepsilon^3TY^{\frac{1}{2}+\varepsilon}}\Big),\end{aligned}\tag{2.9}$$

where we have used Lemma 2.1 in the last estimate. Following from (2.8)–(2.9), we complete the proof of this lemma.

Lemma 2.3 *Let $\{a_i\}_{i=1}^{\infty}$ be a set of arbitrarily complex numbers. Then*

$$\int_T^{2T}\Big|\sum_{n\le N}a_nn^{\mathrm{i}t}\Big|^2\mathrm{d}t=T\sum_{n\le N}|a_n|^2+O\Big(\sum_{n\le N}n|a_n|^2\Big).$$

The above formula also remains valid if $N=\infty$, provided that the series on the right-hand side converge. Furthermore,

$$\int_T^{2T}\Big|\sum_{n>N}a_n n^{\mathrm{i}t}\Big|^2\mathrm{d}t=T\sum_{n>N}|a_n|^2+O\Big(\sum_{n>N}n|a_n|^2\Big),$$

provided that the summations in the formula converge.

Proof See Theorem 5.2 of Ivić [7].

Lemma 2.4 *Let $s=\frac{1}{2}+\mathrm{i}t$, $T\le t\le 2T$. Then for sufficiently large $T>2$, we have*

$$\int_T^{2T}|L(s,F)|^2\mathrm{d}t\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7.$$

Proof The approximate functional equation of $L(s,F)$ states that

$$L(s,F)=\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^s}\mathrm{e}^{-\frac{n}{Y}}-\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z+O\Big(\frac{Y^{\frac{1}{2}+\varepsilon}}{\varepsilon^3T^{\frac{3}{2}}}+\frac{T+k}{\varepsilon^3TY^{\frac{1}{2}+\varepsilon}}\Big),$$

where $s=\frac{1}{2}+\mathrm{i}t,\ T\le t\le 2T$ and $Y\ge 2$. Hence it is sufficient to prove

$$\mathrm{I}_1:=\int_T^{2T}\Big|\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^s}\mathrm{e}^{-\frac{n}{Y}}\Big|^2\mathrm{d}t\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7,\tag{2.10}$$

$$\mathrm{I}_2:=\int_T^{2T}\Big|\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}L(s+z,F)Y^z\Gamma(z)\mathrm{d}z\Big|^2\mathrm{d}t\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7\tag{2.11}$$

and

$$\mathrm{I}_3:=\int_T^{2T}\Big|O\Big(\frac{Y^{\frac{1}{2}+\varepsilon}}{\varepsilon^3T^{\frac{3}{2}}}+\frac{T+k}{\varepsilon^3TY^{\frac{1}{2}+\varepsilon}}\Big)\Big|^2\mathrm{d}t\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7.$$

Taking $Y=T^{\frac{1}{2}}(T+k)$ and using Lemma 2.3, we have

$$\begin{aligned}\mathrm{I}_1&=T\sum_{n=1}^{\infty}\frac{\lambda_F^2(n)}{n}\mathrm{e}^{-\frac{2n}{Y}}+O\Big(\sum_{n=1}^{\infty}\lambda_F^2(n)\mathrm{e}^{-\frac{2n}{Y}}\Big)\\&\ll T\Big(\sum_{n\le Y}\frac{\lambda_F^2(n)}{n}+Y\sum_{n>Y}\frac{\lambda_F^2(n)}{n^2}\Big)+\sum_{n\le Y}\lambda_F^2(n)+Y^2\sum_{n>Y}\frac{\lambda_F^2(n)}{n^2}\\&\ll T\log^6k\log Y+Y\log^6k\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7,\end{aligned}$$

where we have used the partial summation formula and the estimate (see [15])

$$\sum_{n\le x}\lambda_F^2(n)\ll|L(1,F)L(1,\mathrm{sym}^4f)|x\ll x\log^6k.\tag{2.12}$$

Hence the estimate (2.10) follows. Trivially, we also have $\mathrm{I}_3\ll(T+k)(\log(T+k))^6$ because of the choice of Y. Thus it only remains to prove (2.11).

By the functional equation of $L(s,F)$, we obtain

$$\begin{aligned}\mathrm{I}_2=\int_T^{2T}(\log(T+k))\Big|&\int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T}\chi(s+z,F)\\&\cdot L(1-s-z,F)Y^z\Gamma(z)\mathrm{d}z(\log(T+k))\Big|^2\mathrm{d}t.\end{aligned}\tag{2.13}$$

Next, we split $L(1-s-z,F)$ into two parts. Then

$$\begin{aligned}
\mathrm{I}_2 &= \int_T^{2T} \Big| \int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T} \chi(s+z,F)\Big(\sum_{n\le Y}\frac{\lambda_F(n)}{n^{1-s-z}}+\sum_{n>Y}\frac{\lambda_F(n)}{n^{1-s-z}}\Big)Y^z\Gamma(z)\mathrm{d}z\Big|^2\mathrm{d}t \\
&\ll \int_T^{2T} \Big| \int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T} \chi(s+z,F)\Big(\sum_{n\le Y}\frac{\lambda_F(n)}{n^{1-s-z}}\Big)Y^z\Gamma(z)\mathrm{d}z\Big|^2\mathrm{d}t \\
&\quad + \int_T^{2T} \Big| \int_{-\frac{1}{2}-\varepsilon-\mathrm{i}\log T}^{-\frac{1}{2}-\varepsilon+\mathrm{i}\log T} \chi(s+z,F)\Big(\sum_{n>Y}\frac{\lambda_F(n)}{n^{1-s-z}}\Big)Y^z\Gamma(z)\mathrm{d}z\Big|^2\mathrm{d}t \\
&=: \mathrm{I}_{21}+\mathrm{I}_{22}.
\end{aligned}$$

By the Cauchy's inequality and Lemma 2.3, we obtain

$$\begin{aligned}
\mathrm{I}_{22} &\ll \frac{T(T+k)^2\log T}{Y}\int_{T-\log T}^{2T+\log T}\Big|\sum_{n>Y}\frac{\lambda_F(n)}{n^{1+\varepsilon-\mathrm{i}t}}\Big|^2\mathrm{d}t \\
&\ll \frac{T(T+k)^2\log T}{Y}\Big(T\sum_{n>Y}\frac{\lambda_F^2(n)}{n^{2+2\varepsilon}}+\sum_{n>Y}\frac{\lambda_F^2(n)}{n^{1+2\varepsilon}}\Big) \\
&\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7. \qquad (2.14)
\end{aligned}$$

Here we have used the partial summation and $Y=T^{\frac{1}{2}}(T+k)$.

For I_{21}, it is slightly different from the estimation of I_{22}. Moving the inner integration to the parallel segment with $\Re z=-\frac{1}{6}$, we have

$$\mathrm{I}_{21}=\int_T^{2T}\Big|\int_{-\frac{1}{6}-\mathrm{i}4\log T}^{-\frac{1}{6}+\mathrm{i}4\log T}\chi(s+z,F)\Big(\sum_{n\le Y}\frac{\lambda_F(n)}{n^{1-s-z}}\Big)Y^z\Gamma(z)\mathrm{d}z\Big|^2\mathrm{d}t+O(T+k).$$

Next, following the step of the evaluation of I_{22}, we get

$$\begin{aligned}
\mathrm{I}_{21} &\ll \frac{T\log T}{Y^{\frac{1}{3}}}\int_{T-4\log T}^{2T+4\log T}\Big|\sum_{n\le Y}\frac{\lambda_F(n)}{n^{\frac{2}{3-\varepsilon-\mathrm{i}t}}}\Big|^2\mathrm{d}t+O(T+k) \\
&\ll \frac{T^{\frac{1}{3}}(k+T)^{\frac{2}{3}}\log T}{Y^{\frac{1}{3}}}\Big(T\sum_{n\le Y}\frac{\lambda_F^2(n)}{n^{\frac{4}{3}-2\varepsilon}}+\sum_{n\le Y}\frac{\lambda_F^2(n)}{n^{\frac{1}{3}-2\varepsilon}}\Big)+O(T+k) \\
&\ll T^{\frac{1}{2}}(T+k)(\log(T+k))^7. \qquad (2.15)
\end{aligned}$$

This completes the proof.

Remark 2.1 Sankaranarayanan [13] pointed out that mean value theorems play an important role in L-function theory and he established the following result:

$$\int_T^{2T}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,F\Big)\Big|^2\mathrm{d}t\ll (T+k)^{\frac{3}{2}}(\log(T+k))^{17}$$

holds for sufficiently large T. By the observation of Γ-functions, we obtained $T^{\frac{1}{2}}k$ instead of $k^{\frac{3}{2}}$, which implies the convexity bound in the k-aspect, i.e., $L\big(\frac{1}{2}+\mathrm{i}t,F\big)\ll_t k^{\frac{1}{2}}(\log k)^3$. If one can reduce the power of T, the subconvexity bound of $L(s,F)$ in the t-aspect will be given obviously. Another way is to evaluate the integral in short intervals. Recently, Li [9] proved that

$$\int_T^{T+H} \Big|L\Big(\frac{1}{2}+\mathrm{i}t,F\Big)\Big|^2\mathrm{d}t \ll_k T^{1+\varepsilon}H$$

for $H=T^{\frac{3}{8}}$, which implies that $L\big(\frac{1}{2}+\mathrm{i}t,F\big)\ll(|t|+1)^{\frac{11}{16}+\varepsilon}$. Unfortunately, the subconvexity bound in k-aspect is rarely given. There has been no other result up to now, except one for the weak subconvexity of Soundararajan [14].

3 Proof of Proposition 2.1

Without loss of generality, we assume that x is not an integer. By Perron's formula (see Davenport [1, p. 105]), we have

$$\sum_{n\le x}\lambda_F(n)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}L(s,F)\frac{x^s}{s}\mathrm{d}s+O\Big(\sum_{n\ge1}|\lambda_F(n)|\Big(\frac{x}{n}\Big)^c\min\Big\{1,\Big|T\log\Big(\frac{x}{n}\Big)\Big|^{-1}\Big\}\Big),$$

where $c=1+\frac{1}{\log(x+k)}$ and $T\le x$ is a parameter to be chosen later. Following from the argument of Ramachandra and Sankaranarayanan [12], we have

$$\sum_{n\le x}\lambda_F(n)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+iT}L(s,F)\frac{x^s}{s}\mathrm{d}s+O\Big(x^{\epsilon}+\frac{x}{T}\Big(\log(x+k)\Big)^3\Big),$$

where $\epsilon>0$ can be arbitrarily small. Taking $T=x^{\frac{1}{2}}$ and moving the line of integration in (3.1) to $\Re s=\frac{1}{2}$, by the residue theorem, we get

$$\sum_{n\le x}\lambda_F(n)=\frac{1}{2\pi\mathrm{i}}\int_{\frac{1}{2}-\mathrm{i}T}^{\frac{1}{2}+\mathrm{i}T}L(s,F)\frac{x^s}{s}\mathrm{d}s+O\Big(\Big|\int_{\frac{1}{2}+\mathrm{i}T}^{c+\mathrm{i}T}L(s,F)\frac{x^s}{s}\mathrm{d}s\Big|+x^{\frac{1}{2}}(\log(x+k))^3\Big). \quad (3.1)$$

For the integral in the O-term, we have

$$\begin{aligned}\Big|\int_{\frac{1}{2}+\mathrm{i}T}^{c+\mathrm{i}T}L(s,F)\frac{x^s}{s}\mathrm{d}s\Big| &\ll \frac{1}{T}\max_{\frac{1}{2}\le\sigma\le c}x^{\sigma}|L(\sigma+\mathrm{i}T,F)|\\ &\ll \frac{x+x^{\frac{1}{2}}(T+1)^{\frac{1}{4}}(k+T)^{\frac{1}{2}}}{T}(\log(T+k))^3\\ &\ll \Big(x^{\frac{1}{2}}k^{\frac{1}{2}}+\frac{x}{T}\Big)(\log(T+k))^3\\ &\ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log(T+k))^3, \qquad (3.2)\end{aligned}$$

where we have used Lemma 2.1.

Next, we put $T_0=\max\{\mathrm{e}^{30},8k\}$ and split the first integral in (3.1) into three pieces, i.e.,

$$\int_{\frac{1}{2}-\mathrm{i}T}^{\frac{1}{2}+\mathrm{i}T}L(s,F)\frac{x^s}{s}\mathrm{d}s=\Big\{\int_{\frac{1}{2}-\mathrm{i}T_0}^{\frac{1}{2}+\mathrm{i}T_0}+\int_{\frac{1}{2}-\mathrm{i}T}^{\frac{1}{2}-\mathrm{i}T_0}+\int_{\frac{1}{2}+\mathrm{i}T_0}^{\frac{1}{2}+\mathrm{i}T}\Big\}L(s,F)\frac{x^s}{s}\mathrm{d}s.$$

By Cauchy's inequality and Lemma 2.4, we have

$$\begin{aligned}\int_{\frac{1}{2}-\mathrm{i}T_0}^{\frac{1}{2}+\mathrm{i}T_0}L(s,F)\frac{x^s}{s}\mathrm{d}s &\ll x^{\frac{1}{2}}k^{\frac{1}{2}}\log^3k+x^{\frac{1}{2}}\log T_0\max_{T_1\le T_0}\frac{1}{T_1}\int_{\frac{T_1}{2}}^{T_1}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,F\Big)\Big|\mathrm{d}t\\ &\ll x^{\frac{1}{2}}k^{\frac{1}{2}}\log^3k+x^{\frac{1}{2}}\log T_0\max_{T_1\le T_0}\frac{1}{T_1}T_1^{\frac{1}{2}}\Big\{\int_{\frac{T_1}{2}}^{T_1}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,F\Big)\Big|^2\mathrm{d}t\Big\}^{\frac{1}{2}}\\ &\ll x^{\frac{1}{2}}k^{\frac{1}{2}}\log^3k+x^{\frac{1}{2}}k^{\frac{1}{2}}(\log k)^{\frac{9}{2}}\ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log k)^{\frac{9}{2}}. \qquad (3.3)\end{aligned}$$

For the other two integrals, we follow the argument of Ichihara [6]. Consider

$$\int_L L(s,F)\frac{x^s}{s}\mathrm{d}s,$$

where the integral interval L means two segments which satisfy $\sigma = \frac{1}{2}$ and $T_0 \le |t| \le T$. Divide the interval L into L_j $(0 \le j \le J)$ with J satisfying $\frac{T}{2^{J+1}} \le T_0 \le \frac{T}{2^J}$. L_j $(0 \le j \le J-1)$ denotes the interval $\frac{T}{2^{j+1}} < |t| \le \frac{T}{2^j}$ and L_J is $T_0 < |t| \le \frac{T}{2^J}$. The argument of the first case implies that

$$\int_{L_J} L(s,F)\frac{x^s}{s}\mathrm{d}s \ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log(x+k))^3.$$

Furthermore, Section 4 of Ichihara [6] gives the bound of the integral over L_j $(0 \le j \le J-1)$, i.e.,

$$\int_{L_j} L(s,F)\frac{x^s}{s}\mathrm{d}s \ll x^{\frac{1}{2}}(\log(x+k))^4. \tag{3.4}$$

The only difference is that we have used (2.12) instead of the estimate

$$\sum_{n\le x}\lambda_F^2(n) \ll x\log^{15} x.$$

Obviously, $J \ll \log x$. Combining the above estimates, we finally get

$$\sum_{n\le x}\lambda_F(n) \ll x^{\frac{1}{2}}k^{\frac{1}{2}}(\log(k+x))^5.$$

This proves Proposition 2.1.

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