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Exact Boundary Observability for a Kind of Second-Order Quasilinear Hyperbolic Systems*

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Abstract Based on the theory of semi-global classical solutions to quasilinear hyperbolic systems, the local exact boundary observability for a kind of second-order quasilinear hyperbolic systems is obtained by a constructive method.

Keywords First-order quasilinear hyperbolic systems, Second-order quasilinear hyperbolic systems, Exact boundary observability, Mixed initial-boundary value problem

2000 MR Subject Classification 35L51, 35L53, 93B07

1 Introduction

As a dual problem of controllability, the exact boundary observability for linear wave equations has been deeply studied (see [10–12, 18]). Based on the theory of semi-global classical solutions to quasilinear hyperbolic systems (see [6, 9]), by a constructive method, Li et al. [4, 7–8] obtained the exact boundary observability for quasilinear hyperbolic systems. Later, Li [3, 5] and Guo and Wang [1] discussed the exact boundary observability for autonomous and nonautonomous 1-D quasilinear wave equations, respectively, and showed the implicit dualities between the corresponding exact boundary controllability and the exact boundary observability. For the general 1-D quasilinear hyperbolic equation $u_{tt} + a(u, u_x, u_t)u_{tx} + b(u, u_x, u_t)u_{xx} = c(u, u_x, u_t)$, where u is the unknown function of (t, x) and $(a^2 - 4b)(0, 0, 0) > 0$, Shang and Zhuang [13] established the corresponding local exact boundary observability, including the 1-D quasilinear wave equation as its special case.

For second-order quasilinear hyperbolic systems, there are few results on the exact boundary observability. Yu [16] considered the second-order quasilinear hyperbolic system $u_{tt} + (A + B)(u, u_x, u_t)u_{tx} + AB(u, u_x, u_t)u_{xx} = F(u, u_x, u_t)$, where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), matrices A and B have only n positive eigenvalues and n negative eigenvalues, respectively. By a constructive method, she obtained the local exact boundary observability. Later, for a quasilinear coupled hyperbolic system

$$\begin{cases} u_{tt} + (\lambda + \mu)u_{tx} + \lambda\mu u_{xx} + c(\lambda - \nu)s_x = f_1, \\ s_t + \nu s_x = f_2, \end{cases}$$

where $\lambda(0) < 0$, $\mu(0) < 0$, $\nu(0) > 0$, she got the exact boundary observability by using similar constructive method and applied this result to a first-order quasilinear hyperbolic system of diagonal form and proved that the exact boundary observability is still valid even though the boundary conditions are not coupled (see [17]).

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Recently, for a kind of coupled system of 1-D quasilinear wave equations:

$$w_{itt} - a_i^2(w)w_{ixx} + \sum_{j=1}^{n} a_{ij}w_j = 0 \quad (i = 1, \cdots, n),$$

where $w = (w_1, \cdots, w_n)^{\mathrm{T}}$ and $a_i(0) > 0$ $(i = 1, \cdots, n)$, the authors of [2] discussed the local exact boundary observability with various types of boundary conditions and showed the implicit dualities between the exact boundary controllability and the exact boundary observability.

In this paper, we continue to consider the kind of second-order quasilinear hyperbolic systems proposed in [14]. Based on the known result on the existence and uniqueness of semi-global C^2 solution to this kind of systems (see [14]), by using a constructive method, we discuss the exact boundary observability and show the implicit dualities between it and the corresponding exact boundary controllability given in [14]. The conclusions in both [2] and [13] are of its special cases.

Consider the following kind of second-order quasilinear hyperbolic systems:

$$u_{tt} + A(u, u_x, u_t)u_{tx} + B(u, u_x, u_t)u_{xx} = C(u, u_x, u_t), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $A(u, v, w) = (a_{ij}(u, v, w))$ and $B(u, v, w) = (b_{ij}(u, v, w))$ $(i, j = 1, \cdots, n)$ are both $n\times n$ matrices with smooth entries, and have n real eigenvalues and a complete set of left eigenvectors on the domain under consideration, respectively. Suppose furthermore that

$$AB(u, v, w) = BA(u, v, w). \tag{1.2}$$

Thus, there exists an invertible $n \times n$ matrix $L(u, v, w)$ such that

$$LAL^{-1}(u, v, w) = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}, \tag{1.3}$$

$$LBL^{-1}(u, v, w) = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}, \tag{1.4}$$

where $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ are the real eigenvalues of matrices A and B, respectively, and $L = (l_{ij})$ is just the matrix composed by the common left eigenvectors of A and B. Moreover, we assume that on the domain under consideration

$$\mu_i(u, v, w) \neq 0 \quad (i = 1, \cdots, n) \tag{1.5}$$

and

$$\lambda_i^2 - 4\mu_i(u, v, w) > 0 \quad \text{when } \mu_i(u, v, w) > 0 \quad (i = 1, \cdots, n). \tag{1.6}$$

In addition, $C = C(u, v, w) = (c_1(u, v, w), \cdots, c_n(u, v, w))^{\mathrm{T}}$ is a smooth vector function with

$$C(0, 0, 0) = 0. \tag{1.7}$$

By [14], system (1.1) has $2n$ real eigenvalues

$$\widetilde{\lambda}_i^- = \frac{\lambda_i - \sqrt{\lambda_i^2 - 4\mu_i}}{2}, \quad \widetilde{\lambda}_i^+ = \frac{\lambda_i + \sqrt{\lambda_i^2 - 4\mu_i}}{2} \quad (i = 1, \cdots, n). \tag{1.8}$$

This paper is organized as follows. In Section 2, we recall the existence and uniqueness of semi-global C^2 solution to the second-order quasilinear hyperbolic system (1.1) under different cases. Then the two-sided and one-sided exact boundary observability are discussed in Section 3, respectively. Finally, in Section 4, we present an implicit duality between the exact boundary controllability and the exact boundary observability.

2 Existence and Uniqueness of Semi-global C^2 Solution

In this section, we recall briefly the result on the semi-global C^2 solution to the second-order quasilinear hyperbolic system (1.1) under different cases in [14].

For system (1.1), we give the following initial condition:

$$t = 0: \ u = \varphi(x), \quad u_t = \psi(x), \quad 0 \le x \le L, \tag{2.1}$$

where $\varphi = (\varphi_1, \cdots, \varphi_n)^{\mathrm{T}}$ is a given C^2 vector function, $\psi = (\psi_1, \cdots, \psi_n)^{\mathrm{T}}$ is a given C^1 vector function.

Let

$$D^{\pm} = \mathrm{diag}\Big\{\frac{\mu_1}{\widetilde{\lambda}_1^{\pm}}, \cdots, \frac{\mu_n}{\widetilde{\lambda}_n^{\pm}}\Big\} = \mathrm{diag}\{\widetilde{\lambda}_1^{\mp}, \cdots, \widetilde{\lambda}_n^{\mp}\}. \tag{2.2}$$

By [14], according to different signs of $\widetilde{\lambda}_i^{\pm}$ $(i = 1, \cdots, n)$ in a neighborhood of $(u, v, w) = (0, 0, 0)$, we need only to discuss the following three typical cases.

Case 1 System (1.1) has n positive eigenvalues $\widetilde{\lambda}_i^{+} > 0$ and n negative eigenvalues $\widetilde{\lambda}_i^{-} < 0$ $(i = 1, \cdots, n)$.

In this case, we prescribe the following nonlinear boundary conditions on the ends $x = 0$ and $x = L$, respectively:

$$x = 0: \ \begin{cases} G_p(u) = H_p(t) & (p = 1, \cdots, l), \\ G_q(u, u_x, u_t) = H_q(t) & (q = l + 1, \cdots, n), \end{cases} \tag{2.3}$$

$$x = L: \ \begin{cases} \overline{G}_r(u) = \overline{H}_r(t) & (r = 1, \cdots, m), \\ \overline{G}_s(u, u_x, u_t) = \overline{H}_s(t) & (s = m + 1, \cdots, n), \end{cases} \tag{2.4}$$

where G_p, H_p, $\overline{G}_r$ and $\overline{H}_r$ are all C^2 functions with respect to their arguments, G_q, H_q, $\overline{G}_s$ and $\overline{H}_s$ are all C^1 functions with respect to their arguments, and, without loss of generality, we may assume

$$\begin{aligned} &G_p(0) = 0, \quad \overline{G}_r(0) = 0, \quad G_q(0, 0, 0) = 0, \quad \overline{G}_s(0, 0, 0) = 0 \\ &(p = 1, \cdots, l; \ q = l + 1, \cdots, n; \ r = 1, \cdots, m; \ s = m + 1, \cdots, n). \end{aligned} \tag{2.5}$$

In what follows, the following assumptions will be imposed totally or partially in different situations:

$$\det\left|\begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial u_{1t}} & \cdots & \frac{\partial G_n}{\partial u_{nt}} \end{pmatrix}(L^{-1}D^{-}) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial u_{1x}} & \cdots & \frac{\partial G_n}{\partial u_{nx}} \end{pmatrix}(L^{-1})\right|_{(0,0,0)} \neq 0, \tag{2.6}$$

$$\det\left|\begin{pmatrix}\dfrac{\partial G_1}{\partial u_1} & \cdots & \dfrac{\partial G_1}{\partial u_n}\\ \vdots & & \vdots\\ \dfrac{\partial G_l}{\partial u_1} & \cdots & \dfrac{\partial G_l}{\partial u_n}\\ \dfrac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \dfrac{\partial G_{l+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \dfrac{\partial G_n}{\partial u_{1t}} & \cdots & \dfrac{\partial G_n}{\partial u_{nt}}\end{pmatrix}(L^{-1}D^+)-\begin{pmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \dfrac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \dfrac{\partial G_{l+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \dfrac{\partial G_n}{\partial u_{1x}} & \cdots & \dfrac{\partial G_n}{\partial u_{nx}}\end{pmatrix}(L^{-1})\right|_{(0,0,0)}\neq 0, \tag{2.7}$$

$$\det\left|\begin{pmatrix}\dfrac{\partial \overline{G}_1}{\partial u_1} & \cdots & \dfrac{\partial \overline{G}_1}{\partial u_n}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_m}{\partial u_1} & \cdots & \dfrac{\partial \overline{G}_m}{\partial u_n}\\ \dfrac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \dfrac{\partial \overline{G}_{m+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_n}{\partial u_{1t}} & \cdots & \dfrac{\partial \overline{G}_n}{\partial u_{nt}}\end{pmatrix}(L^{-1}D^-)-\begin{pmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \dfrac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \dfrac{\partial \overline{G}_{m+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_n}{\partial u_{1x}} & \cdots & \dfrac{\partial \overline{G}_n}{\partial u_{nx}}\end{pmatrix}(L^{-1})\right|_{(0,0,0)}\neq 0, \tag{2.8}$$

$$\det\left|\begin{pmatrix}\dfrac{\partial \overline{G}_1}{\partial u_1} & \cdots & \dfrac{\partial \overline{G}_1}{\partial u_n}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_m}{\partial u_1} & \cdots & \dfrac{\partial \overline{G}_m}{\partial u_n}\\ \dfrac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \dfrac{\partial \overline{G}_{m+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_n}{\partial u_{1t}} & \cdots & \dfrac{\partial \overline{G}_n}{\partial u_{nt}}\end{pmatrix}(L^{-1}D^+)-\begin{pmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \dfrac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \dfrac{\partial \overline{G}_{m+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \dfrac{\partial \overline{G}_n}{\partial u_{1x}} & \cdots & \dfrac{\partial \overline{G}_n}{\partial u_{nx}}\end{pmatrix}(L^{-1})\right|_{(0,0,0)}\neq 0. \tag{2.9}$$

For the convenience of statement, in Case 1 we denote that

$$\delta=\begin{cases}2 & \text{for } i=1,\cdots,l,\\ 1 & \text{for } i=l+1,\cdots,n,\end{cases}\qquad \overline{\delta}=\begin{cases}2 & \text{for } i=1,\cdots,m,\\ 1 & \text{for } i=m+1,\cdots,n.\end{cases}$$

Case 2 System (1.1) has d_1+d_2 positive eigenvalues $\widetilde{\lambda}_j^{\pm}>0$, $\widetilde{\lambda}_k^{+}>0$ and $2n-(d_1+d_2)$ negative eigenvalues $\widetilde{\lambda}_k^{-}<0$, $\widetilde{\lambda}_h^{\pm}<0$ $(j=1,\cdots,d_1;\ k=d_1+1,\cdots,d_2;\ h=d_2+1,\cdots,n)$, and, without loss of generality, we may assume

$$d_1 + d_2 \le n, \tag{2.10}$$

namely, the number of positive eigenvalues is less than or equal to that of negative ones.

In this case, we prescribe the following nonlinear boundary conditions on the ends $x = 0$ and $x = L$, respectively:

$$x = 0: \quad \begin{cases} G_p(u) = H_p(t) & (p = 1, \cdots, l), \\ G_q(u, u_x, u_t) = H_q(t) & (q = l+1, \cdots, d_1 + d_2), \end{cases} \tag{2.11}$$

$$x = L: \quad \begin{cases} \overline{G}_r(u) = \overline{H}_r(t) & (r = 1, \cdots, m), \\ \overline{G}_s(u, u_x, u_t) = \overline{H}_s(t) & (s = m+1, \cdots, 2n - (d_1 + d_2)), \end{cases} \tag{2.12}$$

where G_p, H_p, $\overline{G}_r$ and $\overline{H}_r$ are all C^2 functions with respect to their arguments, G_q, H_q, $\overline{G}_s$ and $\overline{H}_s$ are all C^1 functions with respect to their arguments, and, without loss of generality, we may assume

$$\begin{gathered} G_p(0) = 0, \quad \overline{G}_r(0) = 0, \quad G_q(0,0,0) = 0, \quad \overline{G}_s(0,0,0) = 0 \\ (p = 1, \cdots, l;\ q = l+1, \cdots, d_1 + d_2;\ r = 1, \cdots, m;\ s = m+1, \cdots, 2n - (d_1 + d_2)). \end{gathered} \tag{2.13}$$

In what follows, the following assumptions will be imposed totally or partially in different situations:

$$\det \left| \begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1t}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nt}} \end{pmatrix} ((L^{-1}D^-)_{\{1,d_2\}} \vdots (L^{-1}D^+)_{\{1,d_1\}}) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1x}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nx}} \end{pmatrix} ((L^{-1})_{\{1,d_2\}} \vdots (L^{-1})_{\{1,d_1\}}) \right|_{(0,0,0)} \neq 0, \tag{2.14}$$

$$\det\left|\begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1t}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nt}} \end{pmatrix}((L^{-1}D^-)_{\{d_2+1,d_2+d_1\}}\vdots(L^{-1}D^+)_{\{d_1+1,d_1+d_2\}})\right.$$
$$\left. - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1x}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nx}} \end{pmatrix}((L^{-1})_{\{d_2+1,d_2+d_1\}}\vdots(L^{-1})_{\{d_1+1,d_1+d_2\}})\right|_{(0,0,0)} \neq 0, \quad (2.15)$$

$$\det\left|\begin{pmatrix} \frac{\partial \overline{G}_1}{\partial u_1} & \cdots & \frac{\partial \overline{G}_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_m}{\partial u_1} & \cdots & \frac{\partial \overline{G}_m}{\partial u_n} \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{nt}} \end{pmatrix}((L^{-1}D^-)_{\{d_2+1,n\}}\vdots(L^{-1}D^+)_{\{d_1+1,n\}})\right.$$
$$\left. - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{nx}} \end{pmatrix}((L^{-1})_{\{d_2+1,n\}}\vdots(L^{-1})_{\{d_1+1,n\}})\right|_{(0,0,0)} \neq 0, \quad (2.16)$$

in which $(L^{-1}D^-)_{\{1,d_1\}}$ indicates the matrix composed of the first column to the d_1th column of matrix $(L^{-1}D^-)$, etc.

In Case 2, we denote that

$$\widetilde{\delta} = \begin{cases} 2 & \text{for } i = 1, \cdots, l, \\ 1 & \text{for } i = l+1, \cdots, d_1 + d_2, \end{cases} \qquad \overline{\widetilde{\delta}} = \begin{cases} 2 & \text{for } i = 1, \cdots, m, \\ 1 & \text{for } i = m+1, \cdots, 2n - (d_1 + d_2). \end{cases}$$

Case 3 System (1.1) has $2n$ positive eigenvalues $\widetilde{\lambda}_i^{\pm} > 0$ $(i = 1, \cdots, n)$.

In this case, we need only $2n$ boundary conditions on the end $x = 0$:

$$x = 0: \ u = H(t), \quad u_x = \overline{H}(t), \tag{2.17}$$

where $H = (H_1, \cdots, H_n)^{\mathrm{T}}$ is a given C^2 vector function, $\overline{H} = (\overline{H}_1, \cdots, \overline{H}_n)^{\mathrm{T}}$ is a given C^1 vector function.

First of all, in Case 1 we give the following lemma on the existence and uniqueness of semi-global C^2 solution to system (1.1) (see [14]).

Lemma 2.1 *Suppose that* (2.6) *and* (2.9) *hold, and the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(0, L)$, *respectively. Then, for any given and possibly quite large* $T > 0$, *if the norms* $\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H_p, H_q)\|_{C^2[0,T]\times C^1[0,T]}$ *and* $\|(\overline{H}_r, \overline{H}_s)\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, n;\ r = 1, \cdots, m;\ s = m+1, \cdots, n)$ *are small enough, the forward mixed initial-boundary value problem* (1.1), (2.1) *and* (2.3)–(2.4) *admits a unique semi-global* C^2 *solution* $u = u(t, x)$ *on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T,\ 0 \le x \le L\}$ *with small* C^2 *norm, and*

$$\|u\|_{C^2[R(T)]} \le C\Big(\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]} + \sum_{i=1}^{n} \|(H_i, \overline{H}_i)\|_{C^{\delta}[0,T]\times C^{\overline{\delta}}[0,T]}\Big), \tag{2.18}$$

where C *is a positive constant.*

Corollary 2.1 *If* $\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]}$ *is suitably small, then the Cauchy problem* (1.1) *and* (2.1) *admits a unique global* C^2 *solution* $u = u(t, x)$ *on its whole maximum determinate domain with small* C^2 *norm, and*

$$\|u\|_{C^2} \le C\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]}, \tag{2.19}$$

where C *is a positive constant.*

Remark 2.1 If we give the following final condition

$$t = T: \ u = \Phi(x), \quad u_t = \Psi(x), \quad 0 \le x \le L, \tag{2.20}$$

where $\Phi = (\Phi_1, \cdots, \Phi_n)^{\mathrm{T}}$ is a given C^2 vector function, $\Psi = (\Psi_1, \cdots, \Psi_n)^{\mathrm{T}}$ is a given C^1 vector function. Suppose that (2.7)–(2.8) hold, and the conditions of C^2 compatibility are satisfied at the points $(t, x) = (T, 0)$ and (T, L), respectively. For any given and possibly quite large $T > 0$, if the norms $\|(\Phi, \Psi)\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H_p, H_q)\|_{C^2[0,T]\times C^1[0,T]}$ and $\|(\overline{H}_r, \overline{H}_s)\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, n;\ r = 1, \cdots, m;\ s = m+1, \cdots, n)$ are small enough, the backward mixed initial-boundary value problem (1.1), (2.20) and (2.3)–(2.4) admits a unique semi-global C^2 solution on the domain $R(T)$ with small C^2 norm, and

$$\|u\|_{C^2[R(T)]} \leq C\Big(\|(\Phi,\Psi)\|_{C^2[0,L]\times C^1[0,L]} + \sum_{i=1}^{n} \|(H_i,\overline{H}_i)\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}\Big), \tag{2.21}$$

where C is a positive constant.

In Case 2 and Case 3, we have the corresponding existence and uniqueness of semi-global C^2 solution, see Lemma 2.2 and Lemma 2.3, respectively (see [14]).

Lemma 2.2 *Suppose that* (2.14) *and* (2.16) *hold, and the conditions of* C^2 *compatibility are satisfied at the points* $(t,x)=(0,0)$ *and* $(0,L)$*, respectively. Then, for any given and possibly quite large* $T>0$*, if the norms* $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H_p,H_q)\|_{C^2[0,T]\times C^1[0,T]}$ *and* $\|(\overline{H}_r,\overline{H}_s)\|_{C^2[0,T]\times C^1[0,T]}$ $(p=1,\cdots,l;\ q=l+1,\cdots,d_1+d_2;\ r=1,\cdots,m;\ s=m+1,\cdots,2n-(d_1+d_2))$ *are small enough, the forward mixed initial-boundary value problem* (1.1), (2.1) *and* (2.11)–(2.12) *admits a unique semi-global* C^2 *solution* $u=u(t,x)$ *on the domain* $R(T)$ *with small* C^2 *norm, and*

$$\|u\|_{C^2[R(T)]} \leq C\Big(\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} + \sum_{i=1}^{d_1+d_2} \|H_i\|_{C^{\tilde{\delta}}[0,T]} + \sum_{i=1}^{2n-(d_1+d_2)} \|\overline{H}_i\|_{C^{\overline{\tilde{\delta}}}[0,T]}\Big), \tag{2.22}$$

where C *is a positive constant.*

Lemma 2.3 *For any given and possibly quite large* $T>0$*, if the norms*

$$\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} \quad \textit{and} \quad \|(H,\overline{H})\|_{C^2[0,T]\times C^1[0,T]}$$

are small enough, and the conditions of C^2 *compatibility are satisfied at the point* $(t,x)=(0,0)$*, the forward mixed initial-boundary value problem* (1.1), (2.1) *and* (2.17) *admits a unique semi-global* C^2 *solution* $u=u(t,x)$ *on the domain* $R(T)$ *with small* C^2 *norm, and*

$$\|u\|_{C^2[R(T)]} \leq C(\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} + \|(H,\overline{H})\|_{C^2[0,T]\times C^1[0,T]}), \tag{2.23}$$

where C *is a positive constant.*

3 Local Exact Boundary Observability in Case 1

Theorem 3.1 (Two-Sided Exact Boundary Observability) *Suppose that* a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ *are all* C^1 *functions with respect to their arguments, and* (2.6) *and* (2.9) *hold. Suppose furthermore that*

$$\frac{\partial(G_1,\cdots,G_l)}{\partial(u_1,\cdots,u_l)}\Big|_{u=0} \neq 0, \quad \frac{\partial(G_{l+1},\cdots,G_n)}{\partial(u_{(l+1)x},\cdots,u_{nx})}\Big|_{(0,0,0)} \neq 0 \tag{3.1}$$

and

$$\frac{\partial(\overline{G}_1,\cdots,\overline{G}_m)}{\partial(u_1,\cdots,u_m)}\Big|_{u=0} \neq 0, \quad \frac{\partial(\overline{G}_{m+1},\cdots,\overline{G}_n)}{\partial(u_{(m+1)x},\cdots,u_{nx})}\Big|_{(0,0,0)} \neq 0. \tag{3.2}$$

Let

$$T > L \max_{i=1,\cdots,n} \Big\{ \frac{1}{\widetilde{\lambda}_i^+(0,0,0)},\ \frac{1}{|\widetilde{\lambda}_i^-(0,0,0)|} \Big\}. \tag{3.3}$$

For any given initial data $(\varphi(x), \psi(x))$ *and boundary functions* $(H(t), \overline{H}(t))$ *with small norms* $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}$, *such that the conditions of* C^2 *compatibility are satisfied at the points* $(t,x)=(0,0)$ *and* $(0,L)$, *respectively, if we have the observed values* $u_q = \overline{k}_q(t)$, $u_{px} = \overline{k}_p(t)$ $(p=1,\cdots,l;\ q=l+1,\cdots,n)$ *at* $x=0$ *and* $u_s = \overline{\overline{k}}_s(t)$, $u_{rx} = \overline{\overline{k}}_r(t)$ $(r=1,\cdots,m;\ s=m+1,\cdots,n)$ *at* $x=L$ *on the interval* $[0,T]$, *then the initial data* $(\varphi(x),\psi(x))$ *can be uniquely determined by these observed values and* $(H(t),\overline{H}(t))$. *Moreover, we have the following observability inequality:*

$$\begin{aligned}\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} \le C\Big(&\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]} + \sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]} + \sum_{r=1}^{m}\|\overline{\overline{k}}_r\|_{C^1[0,T]}\\ &+ \sum_{s=m+1}^{n}\|\overline{\overline{k}}_s\|_{C^2[0,T]} + \|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}\Big),\end{aligned} \tag{3.4}$$

where C *is a positive constant.*

Proof Since $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}$ are small, by Lemma 2.1, the mixed initial-boundary value problem (1.1), (2.1) and (2.3)–(2.4) admits a unique C^2 solution on the domain $R(T)$ with small C^2 norm. Thus, the corresponding C^2 norms or C^1 norms of the observed values $u_q = \overline{k}_q(t)$, $u_{px} = \overline{k}_p(t)$ $(p=1,\cdots,l;\ q=l+1,\cdots,n)$ at $x=0$, and $u_s = \overline{\overline{k}}_s(t)$, $u_{rx} = \overline{\overline{k}}_r(t)$ $(r=1,\cdots,m;\ s=m+1,\cdots,n)$ at $x=L$ are all small.

By (3.1), in a neighborhood of $(u,v,w)=(0,0,0)$, the boundary condition (2.3) at $x=0$ can be equivalently rewritten as

$$x=0:\ \begin{cases} u_p = g_p(H_1,\cdots,H_l,u_{l+1},\cdots,u_n) & (p=1,\cdots,l),\\ u_{qx} = g_q(H_{l+1},\cdots,H_n,u,u_t,u_{1x},\cdots,u_{lx}) & (q=l+1,\cdots,n),\end{cases} \tag{3.5}$$

where g_p $(p=1,\cdots,l)$ are C^2 functions, g_q $(q=l+1,\cdots,n)$ are C^1 functions, and by (2.5), we have

$$g_p(0,\cdots,0) = g_q(0,\cdots,0) = 0 \quad (p=1,\cdots,l;\ q=l+1,\cdots,n). \tag{3.6}$$

Then, the values $\overline{u}_i(t)$ of u_i $(i=1,\cdots,n)$ at $x=0$ can be uniquely determined by the observed values $u_q = \overline{k}_q(t)$ $(q=l+1,\cdots,n)$ at $x=0$ as follows:

$$x=0:\ \begin{cases} \overline{u}_p(t) = g_p(H_1(t),\cdots,H_l(t),\overline{k}_{l+1}(t),\cdots,\overline{k}_n(t)) & (p=1,\cdots,l),\\ \overline{u}_q(t) = \overline{k}_q(t) & (q=l+1,\cdots,n)\end{cases} \tag{3.7}$$

and

$$\|\overline{u}\|_{C^2[0,T]} \le C\Big(\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]} + \sum_{p=1}^{l}\|H_p\|_{C^2[0,T]}\Big). \tag{3.8}$$

On the other hand, the values $\overline{u}_{ix}(t)$ of u_{ix} $(i=1,\cdots,n)$ at $x=0$ can be uniquely determined by the observed values $u_{px}=\overline{k}_p(t)$ $(p=1,\cdots,l)$ at $x=0$ as follows:

$$x=0:\quad \begin{cases}\overline{u}_{px}(t)=\overline{k}_p(t) & (p=1,\cdots,l),\\ \overline{u}_{qx}(t)=g_q(H_{l+1}(t),\cdots,H_n(t),\overline{u}(t),\overline{u}'(t),\overline{k}_1(t),\cdots,\overline{k}_l(t)) & (q=l+1,\cdots,n)\end{cases} \tag{3.9}$$

and

$$\|\overline{u}_x\|_{C^1[0,T]}\le C\Big(\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]}+\sum_{i=1}^{n}\|H_i\|_{C^\delta[0,T]}\Big). \tag{3.10}$$

Similarly, by (3.2), in a neighborhood of $(u,v,w)=(0,0,0)$, the boundary condition (2.4) at $x=L$ can be equivalently rewritten as

$$x=L:\ \begin{cases}u_r=\overline{g}_r(\overline{H}_1,\cdots,\overline{H}_m,u_{m+1},\cdots,u_n) & (r=1,\cdots,m),\\ u_{sx}=\overline{g}_s(\overline{H}_{m+1},\cdots,\overline{H}_n,u,u_t,u_{1x},\cdots,u_{mx}) & (s=m+1,\cdots,n),\end{cases} \tag{3.11}$$

where $\overline{g}_r$ $(r=1,\cdots,m)$ are C^2 functions, $\overline{g}_s$ $(s=m+1,\cdots,n)$ are C^1 functions, and by (2.5), we have

$$\overline{g}_r(0,\cdots,0)=\overline{g}_s(0,\cdots,0)=0\quad (r=1,\cdots,m;\ s=m+1,\cdots,n). \tag{3.12}$$

Then, the values $\overline{\overline{u}}_i(t)$ of u_i $(i=1,\cdots,n)$ at $x=L$ can be uniquely determined by the observed values $u_s=\overline{\overline{k}}_s(t)$ $(s=m+1,\cdots,n)$ at $x=L$ as follows:

$$x=L:\ \begin{cases}\overline{\overline{u}}_r(t)=\overline{g}_r(\overline{H}_1(t),\cdots,\overline{H}_m(t),\overline{\overline{k}}_{m+1}(t),\cdots,\overline{\overline{k}}_n(t)) & (r=1,\cdots,m),\\ \overline{\overline{u}}_s(t)=\overline{\overline{k}}_s(t) & (s=m+1,\cdots,n)\end{cases} \tag{3.13}$$

and

$$\|\overline{\overline{u}}\|_{C^2[0,T]}\le C\Big(\sum_{s=m+1}^{n}\|\overline{\overline{k}}_s\|_{C^2[0,T]}+\sum_{r=1}^{m}\|\overline{H}_r\|_{C^2[0,T]}\Big). \tag{3.14}$$

On the other hand, the values $\overline{\overline{u}}_{ix}(t)$ of u_{ix} $(i=1,\cdots,n)$ at $x=L$ can be uniquely determined by the observed values $u_{rx}=\overline{\overline{k}}_r(t)$ $(r=1,\cdots,m)$ at $x=L$ as follows:

$$x=L:\quad \begin{cases}\overline{\overline{u}}_{rx}(t)=\overline{\overline{k}}_r(t) & (r=1,\cdots,m),\\ \overline{\overline{u}}_{sx}(t)=\overline{g}_s(\overline{H}_{m+1}(t),\cdots,\overline{H}_n(t),\overline{\overline{u}}(t),\overline{\overline{u}}'(t),\overline{\overline{k}}_1(t),\cdots,\overline{\overline{k}}_m(t)) & (s=m+1,\cdots,n)\end{cases} \tag{3.15}$$

and

$$\|\overline{\overline{u}}_x\|_{C^1[0,T]}\le C\Big(\sum_{r=1}^{m}\|\overline{\overline{k}}_r\|_{C^1[0,T]}+\sum_{s=m+1}^{n}\|\overline{\overline{k}}_s\|_{C^2[0,T]}+\sum_{i=1}^{n}\|\overline{H}_i\|_{C^{\overline{\delta}}[0,T]}\Big). \tag{3.16}$$

Changing the role of t and x, we consider the rightward Cauchy problem for system (1.1) with the initial condition

$$x=0:\quad u_i=\overline{u}_i(t),\quad u_{ix}=\overline{u}_{ix}(t),\quad 0\le t\le T\quad (i=1,\cdots,n). \tag{3.17}$$

By Corollary 2.1 and noting (3.8) and (3.10), this Cauchy problem admits a unique C^2 solution $u=\widetilde{u}(t,x)$ on its whole maximum determinate domain, and

$$\|\widetilde{u}\|_{C^2}\le C\Big(\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]}+\sum_{i=1}^{n}\|H_i\|_{C^\delta[0,T]}\Big). \tag{3.18}$$

Similarly, the leftward Cauchy problem for system (1.1) with the final condition

$$x=L:\quad u_i=\overline{\overline{u}}_i(t),\quad u_{ix}=\overline{\overline{u}}_{ix}(t),\quad 0\le t\le T\quad (i=1,\cdots,n) \tag{3.19}$$

admits a unique C^2 solution $u=\widetilde{\widetilde{u}}(t,x)$ on its whole maximum determinate domain, and

$$\|\widetilde{\widetilde{u}}\|_{C^2}\le C\Big(\sum_{r=1}^{m}\|\overline{\overline{k}}_r\|_{C^1[0,T]}+\sum_{s=m+1}^{n}\|\overline{\overline{k}}_s\|_{C^2[0,T]}+\sum_{i=1}^{n}\|\overline{H}_i\|_{C^{\overline{\delta}}[0,T]}\Big). \tag{3.20}$$

Obviously, both $u=\widetilde{u}(t,x)$ and $u=\widetilde{\widetilde{u}}(t,x)$ are the restrictions of the solution $u=u(t,x)$ to the original mixed problem on the corresponding domains, respectively.

Noting (3.3), these two maximum determinate domains must intersect each other. Then, there exists T_0 $(0<T_0<T)$ such that the value $(\overline{\Phi}(x),\overline{\Psi}(x))$ of (u,u_t) at $t=T_0$ can be uniquely determined by $u=\widetilde{u}(t,x)$ and $u=\widetilde{\widetilde{u}}(t,x)$. Noting (3.18) and (3.20), we have

$$\begin{aligned}\|(\overline{\Phi},\overline{\Psi})\|_{C^2[0,L]\times C^1[0,L]}\le C\Big(&\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]}+\sum_{r=1}^{m}\|\overline{\overline{k}}_r\|_{C^1[0,T]}\\&+\sum_{s=m+1}^{n}\|\overline{\overline{k}}_s\|_{C^2[0,T]}+\|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}\Big).\end{aligned} \tag{3.21}$$

We now consider the backward mixed initial-boundary value problem for system (1.1) with

$$t=T_0:\quad (u,u_t)=(\overline{\Phi}(x),\overline{\Psi}(x)),\quad 0\le x\le L, \tag{3.22}$$

$$x=0:\quad u=\overline{u}_i(t)\quad (i=1,\cdots,n), \tag{3.23}$$

$$x=L:\quad u=\overline{\overline{u}}_i(t)\quad (i=1,\cdots,n) \tag{3.24}$$

on the domain $R(T_0)=\{(t,x)\mid 0\le t\le T_0,\ 0\le x\le L\}$. By Remark 2.1, this backward mixed problem admits a unique C^2 solution $u=u_b(t,x)$, which is the restriction of the original C^2 solution $u=u(t,x)$ on the domain $R(T_0)$, thus we have

$$\|u\|_{C^2[R(T_0)]}\le C(\|(\overline{\Phi},\overline{\Psi})\|_{C^2[0,L]\times C^1[0,L]}+\|(\overline{u},\overline{\overline{u}})\|_{C^2[0,T]\times C^2[0,T]}). \tag{3.25}$$

By (2.1) and noting (3.8), (3.14) and (3.21), we get the desired observability inequality (3.4).

Theorem 3.2 (One-Sided Exact Boundary Observability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ are all C^1 functions with respect to their arguments, and* (2.6), (2.8)–(2.9) *and* (3.1) *hold. Let*

$$T>L\Big(\max_{i=1,\cdots,n}\frac{1}{\widetilde{\lambda}_i^+(0,0,0)}+\max_{i=1,\cdots,n}\frac{1}{|\widetilde{\lambda}_i^-(0,0,0)|}\Big). \tag{3.26}$$

For any given initial data $(\varphi(x),\psi(x))$ and boundary functions $(H(t),\overline{H}(t))$ with small norms $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}$, such that the conditions of C^2 compatibility are satisfied at the points $(t,x)=(0,0)$ and $(0,L)$, respectively, if we have the observed values $u_q=\overline{k}_q(t)$, $u_{px}=\overline{k}_p(t)$ $(p=1,\cdots,l;\ q=l+1,\cdots,n)$ at $x=0$ on the interval $[0,T]$, then the initial data $(\varphi(x),\psi(x))$ can be uniquely determined by these observed values and $(H(t),\overline{H}(t))$. Moreover, we have the following observability inequality:

$$\begin{aligned}&\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}\\ \le\ & C\Big(\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]}+\|(H,\overline{H})\|_{C^\delta[0,T]\times C^{\overline{\delta}}[0,T]}\Big),\end{aligned}\tag{3.27}$$

where C is a positive constant.

Proof Changing the role of t and x, we consider the rightward Cauchy problem for system (1.1) with the initial condition (3.17), which admits a unique C^2 solution $u=\widetilde{u}(t,x)$ on its whole maximum determinate domain and (3.18) holds. Obviously, $u=\widetilde{u}(t,x)$ is the restriction of the C^2 solution $u=u(t,x)$ to the original mixed problem on the corresponding domain.

Noting (3.26), this maximum determinate domain must intersect $x=L$. Then, there exists T_0 $(0<T_0<T)$ such that the value $(\overline{\Phi}(x),\overline{\Psi}(x))$ of (u,u_t) at $t=T_0$ can be uniquely determined by $u=\widetilde{u}(t,x)$. Noting (3.18), we have

$$\|(\overline{\Phi},\overline{\Psi})\|_{C^2[0,L]\times C^1[0,L]}\le C\Big(\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{q=l+1}^{n}\|\overline{k}_q\|_{C^2[0,T]}+\|H\|_{C^\delta[0,T]}\Big).\tag{3.28}$$

We consider the backward mixed initial-boundary value problem for system (1.1) with the final condition (3.22) and boundary conditions (3.23) and (2.4) on the domain $R(T_0)$. By Remark 2.1, this backward mixed problem admits a unique C^2 solution $u=u_b(t,x)$ on the domain $R(T_0)$, which is just the restriction of the original C^2 solution $u=u(t,x)$ on the domain $R(T_0)$, thus we have

$$\|u\|_{C^2[R(T_0)]}\le C(\|(\overline{\Phi},\overline{\Psi})\|_{C^2[0,L]\times C^1[0,L]}+\|\overline{u}\|_{C^2[0,T]}+\|\overline{H}\|_{C^{\overline{\delta}}[0,T]}).\tag{3.29}$$

By (2.1) and noting (3.8) and (3.28), we get the desired observability inequality (3.27).

Remark 3.1 In Case 1, if the boundary conditions are particularly given as

$$x=0:\ u_i=H_i(t)\quad(i=1,\cdots,n),\tag{3.30}$$

$$x=L:\ u_i=\overline{H}_i(t)\quad(i=1,\cdots,n),\tag{3.31}$$

it is easy to see that assumptions (3.1)–(3.2) are automatically satisfied.

Remark 3.2 In Case 1, since the number of positive eigenvalues for system (1.1) is equal to that of negative eigenvalues, similar result holds if we take observed values at $x=L$ instead of at $x=0$, and hypotheses (2.6), (2.8)–(2.9) and (3.1) are replaced by (2.6)–(2.7), (2.9) and (3.2).

4 Local Exact Boundary Observability in Case 2 and Case 3

Let

$$\alpha = m + (d_1 + d_2) - n. \tag{4.1}$$

Assume that $\alpha \geq 0$.

Theorem 4.1 (Two-Sided Exact Boundary Observability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j = 1,\cdots,n)$ are all C^1 functions with respect to their arguments, and* (2.10), (2.14) *and* (2.16) *hold. Suppose furthermore that*

$$\frac{\partial(G_1,\cdots,G_l)}{\partial(u_1,\cdots,u_l)}\Big|_{u=0} \neq 0, \qquad \frac{\partial(G_{l+1},\cdots,G_{d_1+d_2})}{\partial(u_{(l+1)x},\cdots,u_{(d_1+d_2)x})}\Big|_{(0,0,0)} \neq 0 \tag{4.2}$$

and

$$\frac{\partial(\overline{G}_1,\cdots,\overline{G}_m)}{\partial(u_1,\cdots,u_m)}\Big|_{u=0} \neq 0, \quad \frac{\partial(\overline{G}_{m+1},\cdots,\overline{G}_{2n-(d_1+d_2)})}{\partial(u_{(\alpha+1)x},\cdots,u_{nx})}\Big|_{(0,0,0)} \neq 0. \tag{4.3}$$

Let

$$T > L \max_{i=1,\cdots,n} \frac{1}{|\widetilde{\lambda}_i^{\pm}(0,0,0)|}. \tag{4.4}$$

For any given initial data $(\varphi(x),\psi(x))$ and boundary functions $(H(t),\overline{H}(t))$ with small norms $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(H,\overline{H})\|_{C^{\widetilde{\delta}}[0,T]\times C^{\overline{\widetilde{\delta}}}[0,T]}$, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, respectively, if we have the observed values $u_{\widetilde{q}} = \overline{k}_{\widetilde{q}}(t)$, $u_{px} = \overline{k}_p(t)$, $u_{\widetilde{p}x} = \overline{\overline{\widetilde{k}}}_{\widetilde{p}}(t)$ $(\widetilde{q} = l+1,\cdots,n;\ p = 1,\cdots,l;\ \widetilde{p} = (d_1+d_2)+1,\cdots,n)$ at $x = 0$ and $u_{\widetilde{s}} = \overline{\overline{\overline{k}}}_{\widetilde{s}}(t)$, $u_{\gamma x} = \overline{\overline{k}}_\gamma(t)$ $(\widetilde{s} = m+1,\cdots,n;\ \gamma = 1,\cdots,\alpha)$ at $x = L$ on the interval $[0,T]$, then the initial data $(\varphi(x),\psi(x))$ can be uniquely determined by these observed values and $(H(t),\overline{H}(t))$. Moreover, we have the following observability inequality:

$$\begin{aligned}\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} \leq C\Big(&\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]} + \sum_{\widetilde{p}=(d_1+d_2)+1}^{n}\|\overline{\overline{\widetilde{k}}}_{\widetilde{p}}\|_{C^1[0,T]} \\ &+ \sum_{\widetilde{q}=l+1}^{n}\|\overline{k}_{\widetilde{q}}\|_{C^2[0,T]} + \sum_{\gamma=1}^{\alpha}\|\overline{\overline{k}}_\gamma\|_{C^1[0,T]} + \sum_{\widetilde{s}=m+1}^{n}\|\overline{\overline{\overline{k}}}_{\widetilde{s}}\|_{C^2[0,T]} \\ &+ \|(H,\overline{H})\|_{C^{\widetilde{\delta}}[0,T]\times C^{\overline{\widetilde{\delta}}}[0,T]}\Big),\end{aligned} \tag{4.5}$$

where C is a positive constant.

Proof The proof of Theorem 4.1 is similar to that of Theorem 3.1. By (4.2), in a neighborhood of $(u,v,w) = (0,0,0)$, the boundary condition (2.11) at $x = 0$ can be equivalently rewritten as

$$x = 0: \begin{cases} u_p = g_p(H_1,\cdots,H_l,u_{l+1},\cdots,u_n) \quad (p = 1,\cdots,l), \\ u_{qx} = g_q(H_{l+1},\cdots,H_{d_1+d_2},u,u_t,u_{1x},\cdots,u_{lx},u_{((d_1+d_2)+1)x},\cdots,u_{nx}) \\ \qquad\qquad\qquad\qquad\qquad\qquad (q = l+1,\cdots,d_1+d_2), \end{cases}$$

where g_p $(p=1,\cdots,l)$ are C^2 functions, g_q $(q=l+1,\cdots,d_1+d_2)$ are C^1 functions, and by (2.13), we have

$$g_p(0,\cdots,0)=g_q(0,\cdots,0)=0 \quad (p=1,\cdots,l;\ q=l+1,\cdots,d_1+d_2).$$

Then, the values $\overline{u}_i(t)$ of u_i $(i=1,\cdots,n)$ at $x=0$ can be uniquely determined by the observed values $u_{\widetilde{q}}=\overline{k}_{\widetilde{q}}(t)$ $(\widetilde{q}=l+1,\cdots,n)$ at $x=0$ as follows:

$$x=0:\ \begin{cases}\overline{u}_p(t)=g_p(H_1(t),\cdots,H_l(t),\overline{k}_{l+1}(t),\cdots,\overline{k}_n(t)) & (p=1,\cdots,l),\\ \overline{u}_{\widetilde{q}}(t)=\overline{k}_{\widetilde{q}}(t) & (\widetilde{q}=l+1,\cdots,n)\end{cases}$$

and

$$\|\overline{u}\|_{C^2[0,T]}\le C\Big(\sum_{\widetilde{q}=l+1}^{n}\|\overline{k}_{\widetilde{q}}\|_{C^2[0,T]}+\sum_{p=1}^{l}\|H_p\|_{C^2[0,T]}\Big).$$

On the other hand, the values $\overline{u}_{ix}(t)$ of u_{ix} $(i=1,\cdots,n)$ at $x=0$ can be uniquely determined by the observed values $u_{px}=\overline{k}_p(t)$, $u_{\widetilde{p}x}=\overline{\overline{k}}_{\widetilde{p}}(t)$ $(p=1,\cdots,l;\ \widetilde{p}=(d_1+d_2)+1,\cdots,n)$ at $x=0$ as follows:

$$x=0:\ \begin{cases}\overline{u}_{px}(t)=\overline{k}_p(t),\ \overline{u}_{\widetilde{p}x}(t)=\overline{\overline{k}}_{\widetilde{p}}(t) & (p=1,\cdots,l;\ \widetilde{p}=(d_1+d_2)+1,\cdots,n),\\ \overline{u}_{qx}(t)=g_q(H_{l+1}(t),\cdots,H_{d_1+d_2}(t),\overline{u}(t),\overline{u}'(t),\overline{k}_p(t),\overline{\overline{k}}_{\widetilde{p}}(t)) & (q=l+1,\cdots,d_1+d_2)\end{cases}$$

and

$$\|\overline{u}_x\|_{C^1[0,T]}\le C\Big(\sum_{p=1}^{l}\|\overline{k}_p\|_{C^1[0,T]}+\sum_{\widetilde{p}=(d_1+d_2)+1}^{n}\|\overline{\overline{k}}_{\widetilde{p}}\|_{C^1[0,T]}+\sum_{\widetilde{q}=l+1}^{n}\|\overline{k}_{\widetilde{q}}\|_{C^2[0,T]}+\|H\|_{C^{\widetilde{\delta}}[0,T]}\Big).$$

The observed values at $x=L$ depend on the value of α, which is divided into two subcases.

(a) $\alpha=0$, namely, $m=n-(d_1+d_2)$.

By (4.3), in a neighborhood of $(u,v,w)=(0,0,0)$, the boundary condition (2.12) at $x=L$ can be equivalently rewritten as

$$x=L:\ \begin{cases}u_r=\overline{g}_r(\overline{H}_1,\cdots,\overline{H}_m,u_{m+1},\cdots,u_n) & (r=1,\cdots,m),\\ u_{ix}=\overline{g}_i(\overline{H}_{m+1},\cdots,\overline{H}_{m+n},u,u_t) & (i=1,\cdots,n),\end{cases}$$

where $\overline{g}_r$ $(r=1,\cdots,m)$ are C^2 functions, $\overline{g}_i$ $(i=1,\cdots,n)$ are C^1 functions, and by (2.13), we have

$$\overline{g}_r(0,\cdots,0)=\overline{g}_i(0,\cdots,0)=0 \quad (r=1,\cdots,m;\ i=1,\cdots,n).$$

Then, the values $\overline{\overline{u}}_i(t)$ of u_i and the values $\overline{\overline{u}}_{ix}(t)$ of u_{ix} $(i=1,\cdots,n)$ at $x=L$ can be uniquely determined by the observed values $u_{\widetilde{s}}=\overline{\overline{k}}_{\widetilde{s}}(t)$ $(\widetilde{s}=m+1,\cdots,n)$ at $x=L$ as follows:

$$x=L:\ \begin{cases}\overline{\overline{u}}_r(t)=\overline{g}_r(\overline{H}_1(t),\cdots,\overline{H}_m(t),\overline{\overline{k}}_{m+1}(t),\cdots,\overline{\overline{k}}_n(t)) & (r=1,\cdots,m),\\ \overline{\overline{u}}_{\widetilde{s}}(t)=\overline{\overline{k}}_{\widetilde{s}}(t) & (\widetilde{s}=m+1,\cdots,n),\\ \overline{\overline{u}}_{ix}(t)=\overline{g}_i(\overline{H}_{m+1}(t),\cdots,\overline{H}_{m+n}(t),\overline{\overline{u}}(t),\overline{\overline{u}}'(t)) & (i=1,\cdots,n)\end{cases}$$

and

$$\|(\overline{\overline{u}}, \overline{\overline{u}}_x)\|_{C^2[0,T]\times C^1[0,T]} \le C\Big(\sum_{\widetilde{s}=m+1}^{n} \|\overline{\overline{k}}_{\widetilde{s}}\|_{C^2[0,T]} + \|\overline{H}\|_{C^{\overline{\delta}}[0,T]}\Big).$$

(b) $\alpha > 0$, namely, $m > n - (d_1 + d_2)$.

By (4.3), in a neighborhood of $(u, v, w) = (0, 0, 0)$, the boundary condition (2.12) at $x = L$ can be equivalently rewritten as

$$x = L: \begin{cases} u_r = \overline{g}_r(\overline{H}_1, \cdots, \overline{H}_m, u_{m+1}, \cdots, u_n) & (r = 1, \cdots, m), \\ u_{\beta x} = \overline{g}_\beta(\overline{H}_{m+1}, \cdots, \overline{H}_{2n-(d_1+d_2)}, u, u_t, u_{1x}, \cdots, u_{\alpha x}) & (\beta = \alpha + 1, \cdots, n), \end{cases}$$

where $\overline{g}_r$ $(r = 1, \cdots, m)$ are C^2 functions, $\overline{g}_\beta$ $(\beta = \alpha + 1, \cdots, n)$ are C^1 functions, and by (2.13), we have

$$\overline{g}_r(0, \cdots, 0) = \overline{g}_\beta(0, \cdots, 0) = 0 \quad (r = 1, \cdots, m;\ \beta = \alpha + 1, \cdots, n).$$

Then, the values $\overline{\overline{u}}_i(t)$ of u_i $(i = 1, \cdots, n)$ at $x = L$ can be uniquely determined by the observed values $u_{\widetilde{s}} = \overline{\overline{k}}_{\widetilde{s}}(t)$ $(\widetilde{s} = m + 1, \cdots, n)$ at $x = L$ as follows:

$$x = L: \begin{cases} \overline{\overline{u}}_r(t) = \overline{g}_r(\overline{H}_1(t), \cdots, \overline{H}_m(t), \overline{\overline{k}}_{m+1}(t), \cdots, \overline{\overline{k}}_n(t)) & (r = 1, \cdots, m), \\ \overline{\overline{u}}_{\widetilde{s}}(t) = \overline{\overline{k}}_{\widetilde{s}}(t) & (\widetilde{s} = m + 1, \cdots, n) \end{cases}$$

and

$$\|\overline{\overline{u}}\|_{C^2[0,T]} \le C\Big(\sum_{\widetilde{s}=m+1}^{n} \|\overline{\overline{k}}_{\widetilde{s}}\|_{C^2[0,T]} + \sum_{r=1}^{m} \|\overline{H}_r\|_{C^2[0,T]}\Big).$$

On the other hand, the values $\overline{\overline{u}}_{ix}(t)$ of u_{ix} $(i = 1, \cdots, n)$ at $x = L$ can be uniquely determined by the observed values $u_{\gamma x} = \overline{\overline{k}}_\gamma(t)$ $(\gamma = 1, \cdots, \alpha)$ at $x = L$ as follows:

$$x = L: \begin{cases} \overline{\overline{u}}_{\gamma x}(t) = \overline{\overline{k}}_\gamma(t) & (\gamma = 1, \cdots, \alpha), \\ \overline{\overline{u}}_{\beta x}(t) = \overline{g}_\beta(\overline{H}_{m+1}(t), \cdots, \overline{H}_{2n-(d_1+d_2)}(t), \overline{\overline{u}}(t), \overline{\overline{u}}'(t), \overline{\overline{k}}_\gamma(t)) & (\beta = \alpha + 1, \cdots, n) \end{cases}$$

and

$$\|\overline{\overline{u}}_x\|_{C^2[0,T]} \le C\Big(\sum_{\widetilde{s}=m+1}^{n} \|\overline{\overline{k}}_{\widetilde{s}}\|_{C^2[0,T]} + \sum_{\gamma=1}^{\alpha} \|\overline{\overline{k}}_\gamma\|_{C^1[0,T]} + \|\overline{H}\|_{C^{\overline{\delta}}[0,T]}\Big).$$

The rest of the proof is similar to the proof of Theorem 3.1 and can be omitted.

Similarly to Theorem 3.2, we have the following theorem.

Theorem 4.2 (One-Sided Exact Boundary Observability) *Suppose that* a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i, j = 1, \cdots, n)$ *are all* C^1 *functions with respect to their arguments, and* (2.10), (2.14)–(2.16) *and* (4.2) *hold. Let*

$$T > L\Big(\max_{\substack{j=1,\cdots,d_1 \\ k=d_1+1,\cdots,d_2}} \Big\{\frac{1}{\widetilde{\lambda}_j^{\pm}(0,0,0)}, \frac{1}{\widetilde{\lambda}_k^{+}(0,0,0)}\Big\}$$

$$+ \max_{\substack{k=d_1+1,\cdots,d_2\\ h=d_2+1,\cdots,n}} \left\{ \frac{1}{|\widetilde{\lambda}_k^-(0,0,0)|}, \frac{1}{|\widetilde{\lambda}_h^{\pm}(0,0,0)|} \right\} \Big). \tag{4.6}$$

For any given initial data $(\varphi(x), \psi(x))$ *and boundary functions* $(H(t), \overline{H}(t))$ *with small norms* $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(H,\overline{H})\|_{C^{\widetilde{\delta}}[0,T]\times C^{\overline{\widetilde{\delta}}}[0,T]}$, *such that the conditions of* C^2 *compatibility are satisfied at the points* $(t,x)=(0,0)$ *and* $(0,L)$, *respectively, if we have the observed values* $u_{\widetilde{q}} = \overline{k}_{\widetilde{q}}(t)$, $u_{px} = \overline{k}_p(t)$, $u_{\widetilde{p}x} = \overline{\overline{k}}_{\widetilde{p}}(t)$ $(\widetilde{q} = l+1,\cdots,n;\ p=1,\cdots,l;\ \widetilde{p} = (d_1+d_2)+1,\cdots,n)$ *at* $x=0$, *then the initial data* $(\varphi(x),\psi(x))$ *can be uniquely determined by these observed values and* $(H(t),\overline{H}(t))$. *Moreover, we have the following observability inequality:*

$$\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} \le C\Big(\sum_{p=1}^{l} \|\overline{k}_p\|_{C^1[0,T]} + \sum_{\widetilde{p}=(d_1+d_2)+1}^{n} \|\overline{\overline{k}}_{\widetilde{p}}\|_{C^1[0,T]} + \sum_{\widetilde{q}=l+1}^{n} \|\overline{k}_{\widetilde{q}}\|_{C^2[0,T]} + \|(H,\overline{H})\|_{C^{\widetilde{\delta}}[0,T]\times C^{\overline{\widetilde{\delta}}}[0,T]} \Big), \tag{4.7}$$

where C *is a positive constant.*

Remark 4.1 In Case 2, suppose that the boundary conditions are particularly given as

$$x=0: \begin{cases} (\mathrm{I}_{d_2},0)L(0)u = H(t), \\ (\mathrm{I}_{d_1},0)L(0)u_x = \widetilde{H}(t), \end{cases} \tag{4.8}$$

$$x=L: \begin{cases} (0,\mathrm{I}_{n-d_2})L(0)u = \overline{H}(t), \\ (0,\mathrm{I}_{n-d_1})L(0)u_x = \overline{\widetilde{H}}(t). \end{cases} \tag{4.9}$$

By Laplace theorem of determinant (see [15]), for the invertible matrix $L(0)$, there exists a nonsingular subdeterminant composed of the elements of the intersections of, for instance, the first row to the d_2th row with the first column to the d_2th column, and we denote this d_2-subdeterminant of $L(0)$ as $\left|L(0)\binom{1,\cdots,d_2}{1,\cdots,d_2}\right| \neq 0$. Meanwhile, the $(n-d_2)$-algebraic cofactor of this d_2-subdeterminant satisfies $\left|L(0)\binom{d_2+1,\cdots,n}{d_2+1,\cdots,n}\right| \neq 0$. Thus it is easy to see that the assumption (4.2) is automatically satisfied. Similarly, we can also get (4.3).

In Case 3, we need only to consider the local one-sided exact boundary observability at $x=L$.

Theorem 4.3 (One-Sided Exact Boundary Observability at $x=L$) *Suppose that* a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ *are all* C^1 *functions with respect to their arguments. Let*

$$T > L \max_{i=1,\cdots,n} \frac{1}{\widetilde{\lambda}_i^-(0,0,0)}. \tag{4.10}$$

For any given initial data $(\varphi(x),\psi(x))$ *and boundary functions* $(H(t),\overline{H}(t))$ *with small norms* $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(H,\overline{H})\|_{C^2[0,T]\times C^1[0,T]}$, *such that the conditions of* C^2 *compatibility are satisfied at the point* $(t,x)=(0,0)$, *if we have the observed values* $(u,u_x) = (\overline{u}(t),\overline{u}_x(t))$ *at* $x=L$, *then the initial data* $(\varphi(x),\psi(x))$ *can be uniquely determined by these observed values and* $(H(t),\overline{H}(t))$. *Moreover, we have the following observability inequality:*

$$\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]} \le C\left(\|(\overline{u},\overline{u}_x)\|_{C^2[0,T]\times C^1[0,T]} + \|(H,\overline{H})\|_{C^2[0,T]\times C^1[0,T]} \right), \tag{4.11}$$

where C is a positive constant.

5 Implicit Duality Between Controllability and Observability

Comparing the observability discussed above with the controllability obtained in [14], we may find an implicit duality between the exact boundary controllability and the exact boundary observability for this kind of second-order quasilinear hyperbolic systems.

For the two-sided control, we have

(i) The controllability time is equal to the observability time, and both of them are sharp. The restriction on the controllability time essentially means that the two maximum determinate domains for the forward and backward Cauchy problems do not intersect each other, while, the restriction on the observability time essentially means that the two maximum determinate domains for the leftward and rightward Cauchy problems must intersect each other.

(ii) Both the number of boundary controls and the number of boundary observed values are equal to $2n$, which is the number of all positive eigenvalues and negative eigenvalues.

For the one-sided control, we have

(i) The controllability time is still equal to the observability time, and both of them are sharp. The restriction on the controllability time essentially means that the two maximum determinate domains for the forward and backward one-sided mixed problems do not intersect each other, while, the restriction on the observability time essentially means that the maximum determinate domain for the rightward Cauchy problems must intersect $x = L$.

(ii) Both the number of boundary controls and the number of boundary observed values are equal to the maximum value between the number of positive eigenvalues and that of negative eigenvalues.

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