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Homogenization of Elliptic Problems with Quadratic Growth and Nonhomogenous Robin Conditions in Perforated Domains

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Abstract This paper deals with the homogenization of a class of nonlinear elliptic problems with quadratic growth in a periodically perforated domain. The authors prescribe a Dirichlet condition on the exterior boundary and a nonhomogeneous nonlinear Robin condition on the boundary of the holes. The main difficulty, when passing to the limit, is that the solution of the problems converges neither strongly in $L^2(\Omega)$ nor almost everywhere in Ω. A new convergence result involving nonlinear functions provides suitable weak convergence results which permit passing to the limit without using any extension operator. Consequently, using a corrector result proved in [Chourabi, I. and Donato, P., Homogenization and correctors of a class of elliptic problems in perforated domains, *Asymptotic Analysis*, **92**(1), 2015, 1–43, DOI: 10.3233/ASY-151288], the authors describe the limit problem, presenting a limit nonlinearity which is different for the two cases, that of a Neumann datum with a nonzero average and with a zero average.

Keywords Homogenization, Elliptic problems, Quadratic growth, Nonhomogeneous Robin boundary conditions, Perforated domains
2000 MR Subject Classification 17B40, 17B50

1 Introduction

In this paper, we study the homogenization of a class of a nonlinear elliptic problems containing a nonlinear term depending on the solution u_ε and on its gradient ∇u_ε with quadratic growth. The problem is posed in the perforated domain $\Omega_\varepsilon^* = \Omega \backslash T_\varepsilon$ obtained by removing from an open bounded set Ω of $\mathbb{R}^N (N \geq 2)$ a closed set T_ε representing a set of ε-periodic holes of size ε. We prescribe a Dirichlet condition on the exterior boundary Γ_0^ε and a nonhomogeneous nonlinear Robin condition on the boundary Γ_1^ε of the holes. More precisely, we study the asymptotic behavior, as ε tends to zero, of the bounded solution u_ε of the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) + \lambda u_\varepsilon = b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) + f & \text{in } \Omega_\varepsilon^*, \\ (A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) \cdot \nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \end{cases} \tag{1.1}$$

where $\lambda \geq 0$ and ν is the unit external normal vector with respect to Ω_ε^*.

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The function f belongs to $L^m(\Omega)$ with $m > \frac{N}{2}$ and $\rho(x) = \rho\left(\frac{x}{\varepsilon}\right)$, where ρ is a Y-periodic function that belongs to $L^\infty(\partial T)$, Y and T being the reference cell and the reference hole, respectively.

We assume that $A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right)$ is a bounded, uniformly elliptic, Y-periodic matrix field and that the function $b_\varepsilon(x,t,\xi)$ is a Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ with quadratic growth with respect to the third variable.

We also suppose that g_ε is defined by

$$g_\varepsilon(x) = \begin{cases} \varepsilon g\left(\dfrac{x}{\varepsilon}\right), & \text{if } \mathcal{M}_{\partial T}(g) \neq 0, \\ g\left(\dfrac{x}{\varepsilon}\right), & \text{if } \mathcal{M}_{\partial T}(g) = 0, \end{cases}$$

where g is a Y-periodic function in $L^r(\partial T)$ with $r > N-1$ and $\mathcal{M}_{\partial T}(g)$ denotes its mean over ∂T.

Let us mention that this type of equations appears in calculus of variations and stochastic control and the nonlinear term $b(y,t,\xi)$ appears in the Euler equation of certain functionals.

The homogenization of this kind of equation with a linear matrix field $A^\varepsilon(x)$ involving a term $b_\varepsilon(x,t,\xi)$ continuous in x variables and with quadratic growth with respect to ξ has been studied in [4–5] for a fixed domain and in [19] for perforated domains with Neumann boundary condition and $f = 0$. In [20] the case where $b_\varepsilon(x,t,\xi)$ is singular in t in a fixed domain has been studied.

The homogenization result of problem (1.1) is stated in Theorem 2.1. The main feature of this result is that the expression of the limit nonlinearity b_0 depends on the average of the nonhomogeneous boundary function g. This is due to the fact that, as proved in [10], the corrector results for the associated linear problem are different in the two cases $\mathcal{M}_{\partial T}(g) \neq 0$ and $\mathcal{M}_{\partial T}(g) = 0$.

More precisely, according to these two cases, we derive two different limit problems for the L^2-limit u_0 of the zero extension of u_ε:

(1) If $\mathcal{M}_{\partial T}(g) \neq 0$ or $g \equiv 0$, the function u_0 is a solution of the problem

$$\begin{cases} -\operatorname{div}(A^0(u_0)\nabla u_0) + \theta\lambda u_0 + c_\gamma h(u_0) = b_0(u_0, \nabla u_0) + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) + \theta f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\theta = \frac{|Y \setminus T|}{|Y|}$ and $A^0(t)$ is the homogenized matrix introduced in [1] (see also [2–3, 8]) for quasilinear problems with Neumann conditions in perforated domains and the constant c_γ is defined by

$$c_\gamma = \begin{cases} \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(\rho), & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases}$$

The function b_0 is given by

$$b_0(s,\xi) = \frac{1}{|Y|}\int_{Y\setminus T} b(y,s,C(y,s)\xi)\mathrm{d}y, \quad \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

where $\{C^\varepsilon(\cdot,s)\}$ is the usual corrector.

(2) If $\mathcal{M}_{\partial T}(g) = 0$ (with $g \not\equiv 0$) and A is independent of t, i.e., $A(y,t) = A(y)$ in Y, the function u_0 is a solution of the problem

$$\begin{cases} -\operatorname{div}(A^0\nabla u_0) + \theta\lambda u_0 + c_\gamma h(u_0) = b_0(u_0, \nabla u_0) + \theta f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where A^0 is the now classical constant homogenized matrix introduced in [18]. The function b_0 is defined by

$$b_0(t,\xi) = \frac{1}{|Y|}\int_{Y\setminus T} b(y, s, C(y)\xi + \nabla\widehat{\chi_g}(y))\mathrm{d}y \quad \text{for every } \ \xi \in \mathbb{R}^N,$$

where C^ε is the usual corrector (independent of t) and the function $\widehat{\chi_g}$ is the solution of the related problem posed in the reference cell with a nonhomogeneous Neumann data g.

As in [4–5, 19–20], the main tool in the homogenization process is a corrector result. To do that, we construct a suitable associated linear problem. The homogenization and the corrector results for this linear problem have been already proved in [10]. We use here the result about correctors proved therein and we show that the corrector for the linear problem is also a corrector for our nonlinear problem in both cases $\mathcal{M}_{\partial T}(g) \neq 0$ and $\mathcal{M}_{\partial T}(g) = 0$.

Here, the main difficulty in particular, when passing to the limit in the nonlinear problem, is due to the presence of the holes, and the solutions converge neither strongly in $L^2(\Omega)$ nor almost everywhere in Ω. To overcome this difficulty, we do not use the extension operators as done in [19] for the case of homogeneous Neumann condition. We apply here the periodic unfolding method introduced in [12] (see [13] for more details) and extended to perforated domains in [16–17] (see also [11] for more general situations). Using the fact that the unfolding operator T^*_ε for perforated domains transforms any function defined on Ω^*_ε into a function defined on a fixed domain, we prove a new technical convergence result involving nonlinear functions, stated in Theorem 4.1, which provides suitable weak convergence results. This technical tool allows to pass to the limit and to prove the corrector result simplifying the proofs and the presentation, even in the case studied in [19].

This paper is organized as follows: In Section 2 we present the problem and state the main results. In Section 3 we recall some preliminary results. Section 4 is devoted to the proof of the homogenization result.

2 Position of the Problem and Statement of the Main Results

In order to state the main results of this paper, we recall some general notations introduced in [11] (see also [12] for the unfolding periodic method in perforated domains).

We denote by

Ω an open bounded set of $\mathbb{R}^N$ $(N \geq 2)$ with $\partial\Omega$ being Lipschitz continuous;

$Y =]0, l_1[\times\cdots\times]0, l_N[$ the reference cell, where $l_i > 0$ for all $1 \leq i \leq N$;

T, the reference hole, a compact set contained in Y and $Y^* = Y\setminus T$ the perforated reference cell, with ∂T Lipschitz-continuous with a finite number of connected components;

$T_\varepsilon = \bigcup\limits_{k\in\Xi_\varepsilon} \varepsilon(k+T)$ the closed set of $\mathbb{R}^N$ representing the holes, where

$G = \big\{\xi \in \mathbb{R}^N : \xi = \sum\limits_{i=1}^N k_i b_i,\ (k_1, \cdots, k_N) \in \mathbb{Z}^N\big\}$, $\quad \Xi_\varepsilon = \{\xi \in G,\ \varepsilon(\xi+Y) \subset \Omega\}$, where $b = (b_1, \cdots, b_N)$ is a basis in $\mathbb{R}^N$;

$\Omega^*_\varepsilon = \Omega\setminus T_\varepsilon$ the perforated domain;

$\widehat{\Omega_\varepsilon} = \text{interior}\big(\bigcup\limits_{k\in\Xi_\varepsilon} \varepsilon(k+\overline{Y})\big)$ and $\widehat{\Omega^*_\varepsilon} = \widehat{\Omega_\varepsilon}\setminus T_\varepsilon$ the corresponding perforated set;

$\Lambda_\varepsilon = \Omega\setminus\widehat{\Omega}_\varepsilon$ and $\Lambda^*_\varepsilon = \Omega^*_\varepsilon\setminus\widehat{\Omega^*_\varepsilon}$ the corresponding perforated set.

As in [8, 17], we decompose (see Figure 1) the boundary of the perforated domain Ω^*_ε as follows:

$$\partial\Omega^*_\varepsilon = \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_1, \quad \text{where } \Gamma^\varepsilon_1 = \partial\widehat{\Omega^*_\varepsilon} \cap \partial T_\varepsilon \text{ and } \Gamma^\varepsilon_0 = \partial\Omega^*_\varepsilon\setminus\Gamma^\varepsilon_1.$$

In the sequel, we also denote by

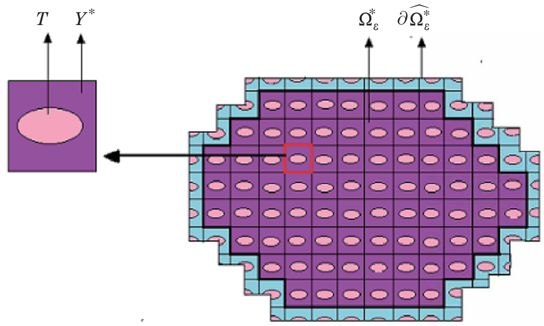

Figure 1 The perforated domain Ω_ε^* and the reference cell Y

χ_E the characteristic function of a subset E of $\mathbb{R}^N$;

$|E|$ the Lebesgue measure of a Lebesgue-measurable subset E of $\mathbb{R}^N$ and $|\partial E|$ the $(N-1)$-Hausdorff measure in $\mathbb{R}^N$ of its boundary ∂E;

$\mathcal{M}_{Y^*}(v) = \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} v(y)\,\mathrm{d}y$ the average of any function $v \in L^1(Y^*)$;

$\mathcal{M}_{\partial T}(v) = \dfrac{1}{|\partial T|}\displaystyle\int_{\partial T} v(y)\,\mathrm{d}\sigma_y$ the average of any function $v \in L^1(\partial T)$;

$\widetilde{v}$ or $(v)^\sim$ the extension by zero on Ω of any function defined on Ω_ε^*;

ν the unit external normal vector with respect to $Y\backslash T$ or Ω_ε^*;

$M(\alpha,\beta,\Omega)$ the set of matrix fields $A = (a_{ij})_{1\leq i,j\leq N} \in (L^\infty(\Omega))$ such that

$$\begin{cases}(A(x)\lambda,\lambda) \geq \alpha(|\lambda|)^2,\\ |A(x)\lambda| \leq \beta|\lambda|,\end{cases}$$

for x a.e in Ω, any $\lambda \in \mathbb{R}^N$ and $\alpha,\beta \in \mathbb{R}$, with $0<\alpha<\beta$;

$\theta = \dfrac{|Y^*|}{|Y|}$ the proportion of material;

c different strictly positive constants independent of ε.

Let us recall that

$$\chi_{\Omega_\varepsilon^*} \rightharpoonup \theta \quad \text{weakly* in } L^\infty(\Omega), \tag{2.1}$$

as ε tends to zero.

Our purpose is to study the asymptotic behavior, as ε tends to zero, of the following problem:

$$\begin{cases}-\mathrm{div}(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) + \lambda u_\varepsilon = b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + f & \text{in } \Omega_\varepsilon^*,\\ (A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon)\cdot\nu + \varepsilon^\gamma\rho_\varepsilon(x)h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon,\\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon,\end{cases} \tag{2.2}$$

where we suppose that

(H_1) $\lambda \geq 0$.

(H_2) $0 \leq f \in L^m(\Omega)$ with $m > \frac{N}{2}$, $\ f \not\equiv 0$.

(H_3) The real matrix field $A : (y,t) \in Y \times \mathbb{R} \longmapsto A(y,t) = \{a_{ij}(y,t)\}_{i,j=1,\cdots,N} \in \mathbb{R}^{N^2}$ satisfies the following conditions:

$$\begin{cases} (1)\ A(y,t) \text{ is } Y\text{-periodic for every } t \text{ and a Carathéodory function, i.e., } A(y,\cdot) \text{ is} \\ \qquad \text{continuous for almost every } y \in Y \text{ and } A(\cdot,t) \text{ is measurable for every } t \in \mathbb{R}; \\ (2)\ A(\cdot,t) \in M(\alpha,\beta,Y) \text{ for every } t \in \mathbb{R}; \\ (3)\ A \text{ is Lipschitz continuous with respect to the second variable, i.e.,} \\ \qquad \exists K \in \mathbb{R} : |A(y,t) - A(y,t_1)| \le K|t - t_1| \text{ a.e. } y \in Y, \text{ for } t \neq t_1 \in \mathbb{R}. \end{cases}$$

(H_4) The function h is an increasing and continuously differentiable function such that for some positive constant C and an exponent q, one has

$$\begin{cases} h(0) = 0, \\ \forall t \in \mathbb{R},\ |h'(t)| \le C(1 + |t|^{q-1}), \ \text{ with} \\ 1 \le q \le \infty \ \text{ if } \ N = 2 \ \text{ and } \ 1 \le q \le \frac{N}{N-2} \ \text{ if } \ N > 2. \end{cases}$$

(H_5) The function ρ is positive and Y-periodic, and it belongs to $L^\infty(\partial T)$.

(H_6) The function g is Y-periodic and belongs to $L^s(\partial T)$, where $s > N - 1$.

(H_7) The function $b : (y,t,\xi) \in Y \times \mathbb{R} \times \mathbb{R}^N \longmapsto b(y,t,\xi) \in \mathbb{R}$ satisfies the following conditions:

$$\begin{cases} (1)\ b(y,t,\xi) \text{ is } Y\text{-periodic for every } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \text{ and is a Carathéodory function, i.e.,} \\ \quad b(y,\cdot,\cdot) \text{ is continuous for a.e. } y \in Y \text{ and } b(\cdot,t,\xi) \text{ is measurable for every}(t,\xi) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}; \\ (2) \text{ for some positive constant } c_0, \text{ one has} \\ \quad |b(y,t,\xi)| \le c_0(1 + |\xi|^2) \text{ for a.e. } y \in Y,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^{\mathbb{N}}; \\ (3)\ d_1 \text{ and } d_2 \text{ are continuous increasing functions with } d_1(0) \ge 0 \text{ and } d_2(0) = 0, \text{ such that} \\ \quad |b(y,t,\xi) - b(y,t,\xi')| \le d_1(|t|)(1 + |\xi| + |\xi'|)|\xi - \xi'| \text{ for a.e. } y \in Y,\ \forall t \in \mathbb{R},\ \forall \xi,\xi' \in \mathbb{R}^{\mathbb{N}} \\ \quad \text{and} \\ \quad |b(y,t,\xi) - b(y,t',\xi)| \le d_2(|t - t'|)(1 + |\xi|^2) \text{ for a.e. } y \in Y,\ \forall t,t' \in \mathbb{R},\ \forall \xi \in \mathbb{R}^{\mathbb{N}}. \end{cases}$$

For almost every x in Ω, every t in $\mathbb{R}$ and every ξ in $\mathbb{R}^N$, we set

$$A^\varepsilon(x,t) = A\Big(\frac{x}{\varepsilon}, t\Big), \quad b_\varepsilon(x,t,\xi) = b\Big(\frac{x}{\varepsilon}, t, \xi\Big), \quad \rho_\varepsilon(x) = \rho\Big(\frac{x}{\varepsilon}\Big) \tag{2.3}$$

and

$$g_\varepsilon(x) = \begin{cases} \varepsilon g\Big(\dfrac{x}{\varepsilon}\Big), & \text{if } \mathcal{M}_{\partial T}(g) \neq 0, \\ g\Big(\dfrac{x}{\varepsilon}\Big), & \text{if } \mathcal{M}_{\partial T}(g) = 0. \end{cases} \tag{2.4}$$

We introduce now the space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon^*) : \ v = 0 \text{ on } \Gamma_0^\varepsilon\},$$

equipped with the norm

$$\|v\|_{V_\varepsilon} = \|\nabla v\|_{L^2(\Omega_\varepsilon^*)} \quad \text{for every } v \in V_\varepsilon,$$

and the variational formulation of problem (2.2)

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\varphi \, \mathrm{d}x + \varepsilon^\gamma \int_{\Gamma_1^*} \rho_\varepsilon h(u_\varepsilon)\varphi \, \mathrm{d}\sigma + \lambda \int_{\Omega_\varepsilon^*} u_\varepsilon \varphi \, \mathrm{d}x \\ \displaystyle = \int_{\Omega_\varepsilon^*} b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi \, \mathrm{d}x + \int_{\Omega_\varepsilon^*} f\varphi \, \mathrm{d}x + \int_{\Gamma_1^*} g_\varepsilon \varphi \, \mathrm{d}\sigma, \quad \forall \varphi \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon^*). \end{cases} \tag{2.5}$$

The existence of a solution to problem (2.5), under the assumptions (H_1)–(H_7), has been proved in [9, Theorem 6.1] together with the boundedness of the solution and some uniform estimates for the sequence $\{u_\varepsilon\}_\varepsilon$. More precisely, it was proved that there exists a constant c such that

$$\|u_\varepsilon\|_{V_\varepsilon} \le c, \quad \|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon^*)} \le c, \tag{2.6}$$

where c is independent of ε and depends only on α, m, s and the Sobolev embedding constant, with the numbers α, m and s being defined by (H_2)–(H_3) and (H_6), respectively.

We recall that (see [1–2, 8, 10]) for every fixed $t \in \mathbb{R}$, the homogenized matrix $A^0(t)$ is defined by

$$A^0(t)\lambda = \frac{1}{|Y|} \int_{Y\setminus T} A(y,t)\nabla_y \widehat{w_\lambda}(y,t)\mathrm{d}y, \quad \forall \lambda \in \mathbb{R}^{\mathrm{N}}, \tag{2.7}$$

where

$$\widehat{w_\lambda}(y,t) = \lambda y - \widehat{\chi_\lambda}(y,t) \tag{2.8}$$

and for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{\mathrm{N}}$, the function $\widehat{\chi_\lambda}(\cdot,t)$ is the solution to the problem

$$\begin{cases} -\mathrm{div}(A(\cdot,t)\nabla_y \widehat{\chi_\lambda}(\cdot,t)) = -\mathrm{div}(A(\cdot,t)\lambda) & \text{in } Y\setminus T, \\ A(\cdot,t)(\lambda - \nabla_y \widehat{\chi_\lambda}(\cdot,t)) \cdot \nu = 0 & \text{on } \partial T, \\ \widehat{\chi_\lambda}(\cdot,t) \text{ is Y-periodic,} & \\ \frac{1}{|Y\setminus T|} \displaystyle\int_{Y\setminus T} \widehat{\chi_\lambda}(\cdot,t) \, \mathrm{d}y = 0. & \end{cases} \tag{2.9}$$

We also define for any ε and every fixed $t \in \mathbb{R}$, the corrector matrix $C^\varepsilon(\cdot,t) = (C^\varepsilon_{ij}(\cdot,t))_{1\le i,j\le n}$, given by (see [10])

$$\begin{cases} C^\varepsilon(x,t) = C\Big(\dfrac{x}{\varepsilon}, t\Big), & \text{a.e. in } \Omega_\varepsilon^*, \\ C_{ij}(y,t) = \dfrac{\partial \widehat{w_j}}{\partial y_i}(y,t), \quad i,j = 1,\cdots,N, & \text{a.e. on } Y, \end{cases} \tag{2.10}$$

where $\{e_j\}_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$.

We recall below its main properties.

Proposition 2.1 (see [10]) *Under the assumption* (H_3)*, let u_ε be the solution of problem* (2.2)*. Then, there exists a constant c_1, independent of ε, such that for some $r > 2$,*

$$\|C^\varepsilon(\cdot,t)\|_{L^r(\Omega_\varepsilon^*)} \le c_1 \quad \textit{for every } \varepsilon \textit{ and for any } t \in \mathbb{R}. \tag{2.11}$$

Moreover, the functions $C^\varepsilon(\cdot,u_\varepsilon)$ are equi-integrable and

$$\begin{cases}\text{(i) } C^\varepsilon(\cdot,u_\varepsilon)\rightharpoonup I \quad \textit{weakly in } (L^2(\Omega))^{N^2},\\ \text{(ii) } A^\varepsilon(\cdot,u_\varepsilon)C^\varepsilon(\cdot,u_\varepsilon)\rightharpoonup A^0(u_0) \quad \textit{weakly in } (L^2(\Omega))^{N^2}.\end{cases} \tag{2.12}$$

We can now state the main result of this paper.

Theorem 2.1 *Under the assumptions* (H_1)–(H_7), *let u_ε be the solution of problem* (2.2). *Then, there exists a subsequence $\{u_\varepsilon\}$ (still denoted by ε), a function $u_0\in H_0^1(\Omega)\cap L^\infty(\Omega)$ and a Carathéodory function $b_0:\ \mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ such that*

$$\begin{cases}\text{(i) } \widetilde{u_\varepsilon}\to\theta u_0 \quad \textit{weakly in } L^2(\Omega) \textit{ and weakly} * \textit{ in } L^\infty(\Omega),\\ \text{(ii) } (b_\varepsilon(\cdot,u_\varepsilon,\nabla u_\varepsilon))^\sim\to b_0(u_0,\nabla u_0) \quad \textit{on } D'(\Omega),\end{cases} \tag{2.13}$$

where θ is defined by (2.1).

The homogenized problems, depending on the mean of g, are the following ones:

(1) *If $\mathcal{M}_{\partial T}(g)\neq 0$ or $g\equiv 0$, the function u_0 is a solution of the problem*

$$\begin{cases}-\mathrm{div}(A^0(u_0)\nabla u_0)+\theta\lambda u_0+c_\gamma h(u_0)=b_0(u_0,\nabla u_0)+\dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)+\theta f & \textit{in } \Omega,\\ u_0=0 & \textit{on } \partial\Omega,\end{cases} \tag{2.14}$$

where the homogenized matrix $A^0(t)$ is given by (2.7) *for every fixed $t\in\mathbb{R}$, and the constant c_γ is defined by*

$$c_\gamma=\begin{cases}\dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(\rho), & \textit{if } \gamma=1,\\ 0, & \textit{if } \gamma>1.\end{cases} \tag{2.15}$$

The function b_0 is given by

$$b_0(s,\xi)=\frac{1}{|Y|}\int_{Y\setminus T} b(y,s,C(y,s)\xi)\mathrm{d}y, \quad \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N, \tag{2.16}$$

where the corrector matrix fields $\{C^\varepsilon(\cdot,s)\}$ are defined by (2.10). *Moreover,*

$$-\mathrm{div}(A^\varepsilon(\cdot,\widetilde{u_\varepsilon})\widetilde{\nabla u_\varepsilon})\to -\mathrm{div}(A^0(u_0)\nabla u_0) \quad \textit{strongly on } H^{-1}(\Omega). \tag{2.17}$$

(2) *If $\mathcal{M}_{\partial T}(g)=0$ (with $g\not\equiv 0$) and A is independent of t, i.e., $A(y,t)=A(y)$ in Y, the function u_0 is a solution of the problem*

$$\begin{cases}-\mathrm{div}(A^0\nabla u_0)+\theta\lambda u_0+c_\gamma h(u_0)=b_0(u_0,\nabla u_0)+\theta f & \textit{in } \Omega,\\ u_0=0 & \textit{on } \partial\Omega,\end{cases} \tag{2.18}$$

where the constant homogenized matrix A^0 is given by (2.7) (*independent of t*).

The function b_0 is defined by

$$b_0(t,\xi)=\frac{1}{|Y|}\int_{Y\setminus T} b(y,s,C(y)\xi+\nabla\widehat{\chi_g}(y))\mathrm{d}y \quad \textit{for every } \xi\in\mathbb{R}^N, \tag{2.19}$$

where $C^\varepsilon(\cdot)$ *is defined by* (2.10) (*independent of* t) *and the function* $\widehat{\chi_g}$ *is the solution of the following problem:*

$$\begin{cases} -\mathrm{div}(A\nabla\widehat{\chi_g}) = 0 & \text{in } Y\backslash T, \\ A\nabla\widehat{\chi_g}\cdot\nu = g & \text{on } \partial T, \\ \widehat{\chi_g} \text{ is } Y\text{-periodic}, \\ \mathcal{M}_{Y^*}(\widehat{\chi_g}) = 0. \end{cases} \tag{2.20}$$

Moreover,

$$-\mathrm{div}(A^\varepsilon\widetilde{\nabla u_\varepsilon}) \to -\mathrm{div}(A^0\nabla u_0) \quad \text{strongly on } H^{-1}(\Omega). \tag{2.21}$$

This result will be proved in Section 4.

Remark 2.1 Let us point out the main novelty in this result. It concerns the fact that the presence of g_ε in the nonhomogeneus boundary condition of problem (2.2) gives rise to two different limit nonlinearities b_0 in the problem, according to the case $\mathcal{M}_{\partial T}(g) \neq 0$ or $\mathcal{M}_{\partial T}(g) = 0$.

This is due to the fact that the corrector results for the associated linear problem are different in the two cases, as recalled in Theorem 3.2.

We end this section by stating the following result, which shows that C^ε is a corrector for the nonlinear problem (2.5).

Corollary 2.1 *Under the assumptions of Theorem* 2.1 *and the notation therein, we have the following assertions:*

(1) *If* $\mathcal{M}_{\partial T}(g) \neq 0$ *or* $g = 0$*, then*

$$\lim_{\varepsilon\to 0}\|\nabla u_\varepsilon - C^\varepsilon(\cdot, u_\varepsilon)\nabla u_0\|_{L^1(\Omega_\varepsilon^*)^N} = 0.$$

(2) *If* $\mathcal{M}_{\partial T}(g) = 0$ *and* A *is independent of* t*, i.e.,* $A(y,t) = A(t)$ *in* Y*, then*

$$\lim_{\varepsilon\to 0}\Big\|\nabla u_\varepsilon - C^\varepsilon(\cdot)\nabla u_0 - \nabla_y\widehat{\chi_g}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^1(\Omega_\varepsilon^*)^N} = 0.$$

Proof This corollary is a straightforward consequence of Theorem 3.4 and Theorem 4.1 proved in Sections 3 and 4, respectively.

3 Some Preliminary Results

In this section, we recall some homogenization and corrector results proved in [10].

To do that, we introduce as in [10] a linear operator $\mathcal{L}_\varepsilon$ from $H^{-1}(\Omega)$ to $(V_\varepsilon)'$ verifying the following assumption:

(H_8) If $\{\psi_\varepsilon\}$ is a sequence such that

$$\|\psi_\varepsilon\|_{V_\varepsilon} \leq c \quad \text{and} \quad \widetilde{\psi_\varepsilon} \rightharpoonup \theta\psi_0 \text{ weakly in } L^2(\Omega), \tag{3.1}$$

then

$$\lim_{\varepsilon\to 0}\langle\mathcal{L}_\varepsilon(Z), \psi_\varepsilon\rangle_{V_\varepsilon', V_\varepsilon} = \langle Z, \psi_0\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{3.2}$$

Remark 3.1 Let us point out that there exist many operators $\mathcal{L}_\varepsilon$ verifying the assumption (H_8), which can be constructed in different ways.

For instance, the assumption (H_8) is satisfied for $\mathcal{L}_\varepsilon = P^*_\varepsilon$ (see Remark 4.3 of [10]), where P^*_ε is the adjoint of the linear extension operators P_ε introduced by Cioranescu D. and Saint Jean Paulin J. in [18]. Let us recall that for any sequence $\{v_\varepsilon\}_\varepsilon$ in V_ε, we have that

$$\{\|v_\varepsilon\|_{V_\varepsilon} \le c \text{ and } \widetilde{v_\varepsilon} \rightharpoonup \theta v_0 \text{ weakly in } L^2(\Omega)\} \;\Leftrightarrow\; P_\varepsilon v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H^1_0(\Omega). \tag{3.3}$$

Another (different) operator can be defined by using the periodic unfolding method as done in [21] when studying the correctors for the wave equation in perforated domains via the above method. We refer to [10, Remark 2.3] for more details and comments.

We recall first the following homogenization result.

Theorem 3.1 (see [10]) *Under the assumptions* (H_1)–(H_6) *and* (H_8), *let* Z *be given in* $H^{-1}(\Omega)$ *and let* v_ε *be the unique solution of problem*

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x,z_\varepsilon)\nabla v_\varepsilon) + \lambda v_\varepsilon = \mathcal{L}_\varepsilon(Z) & \text{in } \ \Omega^*_\varepsilon, \\ (A^\varepsilon(x,z_\varepsilon)\nabla v_\varepsilon)\cdot\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(z_\varepsilon) = g_\varepsilon & \text{on } \ \Gamma^\varepsilon_1, \\ v_\varepsilon = 0 & \text{on } \ \Gamma^\varepsilon_0, \end{cases} \tag{3.4}$$

where the sequence $\{z_\varepsilon\}_\varepsilon$ *belongs to* V_ε *and* A^ε, ρ_ε *and* g_ε *are given by* (2.3) *and* (2.4).

Suppose that the sequence $\{z_\varepsilon\}_\varepsilon$ *satisfies* (3.1), *that is*

$$\|z_\varepsilon\|_{V_\varepsilon} \le c \quad \text{and} \quad \widetilde{z_\varepsilon} \rightharpoonup \theta z_0 \ \text{weakly in } L^2(\Omega),$$

with $z_0 \in H^1_0(\Omega)$ *and* θ *given by* (2.1). *Then, as* ε *tends to* 0, *we have the following convergences:*

$$\begin{cases} \text{(i) } \widetilde{v_\varepsilon} \rightharpoonup \theta v_0 & \text{weakly in } L^2(\Omega), \\ \text{(ii) } A^\varepsilon(\cdot,z_\varepsilon)\widetilde{\nabla v_\varepsilon} \rightharpoonup A^0(z_0)\nabla v_0 & \text{weakly in } (L^2(\Omega))^N. \end{cases} \tag{3.5}$$

The function v_0 *is the unique solution of the problem*

$$\begin{cases} -\mathrm{div}(A^0(z_0)\nabla v_0) + \theta\lambda v_0 = -\,c_\gamma h(z_0) + Z + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where the homogenized matrix $A^0(t)$ *and the constant* c_γ *are defined by* (2.7) *and* (2.15) *respectively.*

Let us recall now the results concerning the convergence of the energies and the corrector, proved in [10] where we distinguish the two cases given in (2.4).

Theorem 3.2 (see [10]) *Under the assumptions of Theorem* 3.1, *let* A^0 *be defined by* (2.7) *and* $\widehat{\chi_g}$ *defined by* (2.20). *Let* v_ε *and* v_0 *be the solutions of problems* (3.6) *and* (3.4), *respectively.*

(1) *If* $\mathcal{M}_{\partial T}(g) \neq 0$ *or* $g \equiv 0$, *then*

$$A^\varepsilon(\cdot,z_\varepsilon)\widetilde{\nabla v_\varepsilon}\widetilde{\nabla v_\varepsilon} \rightharpoonup A^0(z_0)\nabla v_0 \nabla v_0 \quad \text{weakly in } L^1(\Omega).$$

Moreover, if $\{C^\varepsilon(\cdot,z_\varepsilon)\}$ *is defined by* (2.10), *we have*

$$\lim_{\varepsilon\to 0}\|\nabla v_\varepsilon - C^\varepsilon(\cdot,z_\varepsilon)\nabla v_0\|_{L^1(\Omega^*_\varepsilon)^N} = 0.$$

(2) *If* $\mathcal{M}_{\partial T}(g) = 0$ (*with* $g \not\equiv 0$) *and* A *is independent of* t, *i.e.*, $A(y,t) = A(y)$ *in* Y, *then*

$$A^\varepsilon(\cdot)\Big(\widetilde{\nabla v_\varepsilon} - \nabla_y \widehat{\chi_g}\Big(\frac{\cdot}{\varepsilon}\Big)\Big)\widetilde{\nabla v_\varepsilon} \rightharpoonup (A^0 \nabla v_0 + \mathcal{M}_{Y^*}(A\nabla\widehat{\chi_g}))\nabla v_0 \quad \textit{weakly in } L^1(\Omega).$$

Moreover, if $C^\varepsilon(\cdot)$ *is defined by* (2.10) (*independent of* t), *we have*

$$\lim_{\varepsilon\to 0}\Big\|\nabla v_\varepsilon - C^\varepsilon(\cdot)\nabla v_0 - \nabla_y\widehat{\chi_g}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^1(\Omega_\varepsilon^*)^N} = 0.$$

The proof of the corrector result given in [10] is based on the proposition below, which will be needed in the sequel.

Proposition 3.1 (see [10]) *Under the assumptions of Theorem* 3.1 *and with the notations therein, we have the following assertions:*

(1) *If* $\mathcal{M}_{\partial T}(g) \neq 0$ *or* $g \equiv 0$, *then*

$$\limsup_{\varepsilon\to 0}\|\nabla v_\varepsilon - C^\varepsilon(\cdot, z_\varepsilon)\Phi\|_{L^2(\Omega_\varepsilon^*)^N} \le c\|\nabla v_0 - \Phi\|_{L^2(\Omega)^N}, \quad \forall \Phi \in (C_0^\infty(\Omega))^N.$$

(2) *If* $\mathcal{M}_{\partial T}(g) = 0$ (*with* $g \not\equiv 0$) *and* A *is independent of* t, *i.e.*, $A(y,t) = A(t)$ *in* Y, *then*

$$\limsup_{\varepsilon\to 0}\Big\|\nabla v_\varepsilon - C^\varepsilon(\cdot)\Phi - \nabla_y\widehat{\chi_g}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^2(\Omega_\varepsilon^*)^N} \le c\|\nabla v_0 - \Phi\|_{L^2(\Omega)^N}, \quad \forall \Phi \in (C_0^\infty(\Omega))^N.$$

In both cases, $c = C(\alpha,\beta)$ *is a constant independent of* Φ.

We also recall the following property.

Lemma 3.1 (see [5]) *Let* $\{g_\varepsilon\}_\varepsilon$ *be a sequence of functions which converges weakly in* $L^1(\Omega)$ *to a function* g_0 *and let* $\{t_\varepsilon\}_\varepsilon$ *be a sequence of equibounded and measurable functions, which converges almost pointwise in* Ω *to a function* t_0. *Then*

$$\lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon t_\varepsilon \mathrm{d}x = \int_\Omega g_0 t_0 \mathrm{d}x.$$

We end this section by stating the following result.

Proposition 3.2 *Under assumption* (H_7), *let* $\{b_\varepsilon\}_\varepsilon$ *be the sequence of the Carathéodory functions given by* (2.3). *Then, the function* b_0 *given by* (2.16) *or* (2.19) *satisfies* (H_7) *and for any* ϕ *in* $(C_0^\infty(\Omega))^N$ *and* φ_0 *in* $L^\infty(\Omega)$, *one has the following assertions:*

(1) *If* $\mathcal{M}_{\partial T}(g) = 0$ *or* $g = 0$, *then*

$$[b_\varepsilon(x, \varphi_0, C^\varepsilon\phi)]^\sim \rightharpoonup b_0(\varphi_0, \phi) \quad \textit{weakly in } L^1(\Omega), \tag{3.7}$$

where $\{C^\varepsilon(\cdot, z_\varepsilon)\}$ *is defined by* (2.10).

(2) *If* $\mathcal{M}_{\partial T}(g) = 0$ *and* A *is independent of* t, *i.e.*, $A(y,t) = A(y)$ *in* Y,

$$[b_\varepsilon(x, \varphi_0, C^\varepsilon\phi + \nabla\widehat{\chi_g})]^\sim \rightharpoonup b_0(\varphi_0, \phi) \quad \textit{weakly in } L^1(\Omega), \tag{3.8}$$

where $C^\varepsilon(\cdot)$ *is defined by* (2.10) (*independent of* t) *and the function* $\widehat{\chi_g}$ *is the solution of problem* (2.20).

Proof The convergence (3.7) is a simple consequence of Theorem 2.6 of [19] (see also [5] for the case of a fixed domain) and the convergence (3.8) can be deduced by the same arguments as those used to prove (3.7).

4 Proof of the Main Result

4.1 A technical result

In this section, we give a preliminary tool which will play an essential role in proving the corrector result stated in Theorem 4.2, and seems interesting by itself.

To prove it, we use the periodic unfolding method, introduced in [12] (see [13] for a general presentation and detailed proofs) and extended to perforated domains in [16–17] (see [11] for more general situations and a comprehensive presentation).

Theorem 4.1 *Let $\{\psi_\varepsilon\}$ be a sequence satisfying* (3.1).

(1) *The following convergences hold:*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*}|\psi_\varepsilon-\psi_0|^2\mathrm{d}x=0,\quad \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*}|\psi_\varepsilon|^2\mathrm{d}x=\theta\int_\Omega|\psi_0|^2\mathrm{d}x. \tag{4.1}$$

(2) *Let $p\in[1,+\infty)$ and $\{h_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that*

$$h_\varepsilon \rightharpoonup h_0 \quad \textit{weakly in } L^p(\Omega), \tag{4.2}$$

for some h_0 in $L^p(\Omega)$. Suppose further that $F:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $F(\psi_\varepsilon)\in L^q(\Omega)$, with

$$\begin{cases} q\in(p',+\infty) & \textit{with } \dfrac{1}{p}+\dfrac{1}{p'}=1, \textit{ if } \ p>1,\\ q=+\infty, & \textit{if } \ p=1.\end{cases} \tag{4.3}$$

If

$$\|F(\psi_\varepsilon)\|_{L^q(\Omega)}\le c \tag{4.4}$$

for some positive constant c independent of ε, then

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*}h_\varepsilon F(\psi_\varepsilon)\,\mathrm{d}x=\int_\Omega h_0F(\psi_0)\,\mathrm{d}x. \tag{4.5}$$

Moreover, if

$$h_\varepsilon\to h_0 \quad \textit{strongly in } L^p(\Omega), \tag{4.6}$$

then

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*}h_\varepsilon F(\psi_\varepsilon)\,\mathrm{d}x=\theta\int_\Omega h_0F(\psi_0)\,\mathrm{d}x. \tag{4.7}$$

In particular,

$$F(\psi_\varepsilon)\rightharpoonup\theta F(\psi_0), \tag{4.8}$$

weakly in $L^{p'}(\Omega)$ if $p>1$ and weakly $*$ in $L^\infty(\Omega)$ if $p=1$.

Proof Convergences (4.1) follow from Corollary 1.13, Corollary 1.19 and Theorem 2.13 of [11].

In order to show (4.5), let us first recall (see [11]) that for any Lebesgue-measurable function ϕ on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as

$$\mathcal{T}_\varepsilon^*(\phi)(x,y)=\begin{cases}\phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y+\varepsilon y\Big) & \text{a.e. for } (x,y)\in\widehat{\Omega}_\varepsilon\times Y^*,\\ 0 & \text{a.e. for } (x,y)\in\Lambda_\varepsilon\times Y^*.\end{cases}$$

In view of Proposition 1.14 of [11], the assumption (4.2) implies that there exists some function $\widehat{h}_0$ in $L^p(\Omega \times Y^*)$ such that

$$\mathcal{T}_\varepsilon^*(h_\varepsilon) \rightharpoonup \widehat{h}_0 \quad \text{weakly in } L^p(\Omega \times Y^*), \tag{4.9}$$

with

$$\frac{1}{|Y|}\int_{Y^*} \widehat{h}_0(\cdot, y)\, \mathrm{d}y = h_0. \tag{4.10}$$

Using again Theorem 2.13 of [11], we derive that

$$\mathcal{T}_\varepsilon^*(\psi_\varepsilon) \to \psi_0 \quad \text{strongly in } L^2(\Omega, H^1(Y^*)).$$

Then, there exists a subsequence (still denoted by $\{\varepsilon\}$) such that

$$\mathcal{T}_\varepsilon^*(\psi_\varepsilon) \to \psi_0 \quad \text{a.e. in } \ \Omega \times Y^*,$$

so that in view of the continuity of F, we get (for a subsequence)

$$\mathcal{T}_\varepsilon^*(F(\psi_\varepsilon)) = F(\mathcal{T}_\varepsilon^*(\psi_\varepsilon)) \to F(\psi_0) \quad \text{a.e. in } \ \Omega \times Y^*. \tag{4.11}$$

Moreover, using the properties of $\mathcal{T}_\varepsilon$ (see [11, Proposition 1.12 and Corollary 1.13]), we deduce that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} h_\varepsilon F(\psi_\varepsilon)\, \mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(h_\varepsilon)(x,y)\mathcal{T}_\varepsilon^*(F(\psi_\varepsilon))(x,y)\, \mathrm{d}x\, \mathrm{d}y. \tag{4.12}$$

On the other hand, if $p > 1$, thanks to (4.11), the Hölder inequality provides the equintegrability of $|\mathcal{T}_\varepsilon^*(F(\psi_\varepsilon)) - F(\psi_0)|^{p'}$. Then, using (4.4) we can apply the Vitali's theorem to obtain

$$\mathcal{T}_\varepsilon^*(F(\psi_\varepsilon)) \to F(\psi_0) \quad \text{strongly in } L^{p'}(\Omega \times Y^*),$$

and this convergence holds for the whole sequence, since the limit is uniquely determined.

Consequently, from (4.9), we have

$$\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(h_\varepsilon)(x,y)\mathcal{T}_\varepsilon^*(F(\psi_\varepsilon))(x,y)\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{|Y|}\int_{\Omega\times Y^*} \widehat{h}_0(x,y)F(\psi_0)\, \mathrm{d}x\, \mathrm{d}y. \tag{4.13}$$

Since $F(\psi_0)$ is independent of y, in view of (4.10) this gives the result for $p > 1$.

Let $p = 1$ now. Then, thanks to (4.3)–(4.4), (4.9) and (4.11), we can still pass to the limit in the right-hand side of (4.12) by applying Lemma 3.1 in $\Omega \times Y^*$, with $g_\varepsilon = \mathcal{T}_\varepsilon^*(h_\varepsilon)$ and $t_\varepsilon = \mathcal{T}_\varepsilon^*(\psi_\varepsilon)$. We obtain again (4.13) and conclude as in the previous case.

Finally, observe that if (4.6) holds, from Corollary 1.19 of [11], one has $\widehat{h}_0 = h_0$ which implies that

$$\frac{1}{|Y|}\int_{\Omega\times Y^*} \widehat{h}_0(x,y)F(\psi_0)\, \mathrm{d}x\, \mathrm{d}y = \theta\int_\Omega h_0 F(\psi_0)\, \mathrm{d}x,$$

since $\theta = \frac{|Y^*|}{|Y|}$. This gives (4.7) and in particular (4.8) (taking $h_\varepsilon = h_0$), which concludes the proof.

4.2 A corrector result for the nonlinear problem

In this section we prove Theorem 2.1. To do that, we adapt some arguments introduced in [4–5] for the case of oscillating coefficients in a fixed domain, and extended for the linear case in the periodically perforated domains in [19]. We make here an essential use of Theorem 4.1, proved in the previous section.

The main idea is to define a suitable linear problem associated to a weak cluster point of the sequence of the solutions of problem (2.2) and then prove that the corrector for this linear problem is also a corrector for the original nonlinear problem.

Observe first that from (2.6) there exists a subsequence $\{u_\varepsilon\}$ (still denoted by ε), which will be fixed from now on, and a function $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\widetilde{u_\varepsilon} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega) \text{ and weakly} * \text{in } L^\infty(\Omega), \tag{4.14}$$

as ε tends to zero, where θ is defined by (2.1).

Then, in the present situation, the suitable linear problem associated to problem (2.2) is the following one:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon) + \lambda v_\varepsilon & \\ = \mathcal{L}_\varepsilon\Big(-\mathrm{div}(A^0(u_0)\nabla u_0) + \theta\lambda u_0 + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big) & \text{in } \Omega_\varepsilon^*, \\ (A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon)\cdot\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \\ v_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \end{cases} \tag{4.15}$$

where $\mathcal{L}_\varepsilon$ is a linear operator satisfying (H_8).

Its variational formulation is

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon \nabla\phi \,\mathrm{d}x + \lambda\int_{\Omega_\varepsilon^*} v_\varepsilon\phi\,\mathrm{d}x + \varepsilon^\gamma\int_{\Gamma_1^\varepsilon}\rho_\varepsilon h(u_\varepsilon)\phi\,\mathrm{d}\sigma \\ \displaystyle = \int_{\Gamma_1^\varepsilon} g_\varepsilon\phi\,\mathrm{d}\sigma + \Big\langle \mathcal{L}_\varepsilon\Big(-\mathrm{div}(A^0(u_0)\nabla u_0) + \theta\lambda u_0 \\ \displaystyle +c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \phi\Big\rangle_{V_\varepsilon', V_\varepsilon}, \quad \forall \phi \in V_\varepsilon. \end{cases} \tag{4.16}$$

In view of Theorem 3.1, written for $z_\varepsilon = u_\varepsilon$ and

$$Z = -\mathrm{div}(A^0(u_0)\nabla u_0) + \theta\lambda u_0 + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g),$$

using (4.14) we deduce that

$$\widetilde{v_\varepsilon} \rightharpoonup \theta v_0 \quad \text{weakly in } L^2(\Omega),$$

as ε tends to zero, where θ is defined by (2.1) and $v_0 \in H_0^1(\Omega)$ is the unique solution of the equation

$$-\mathrm{div}(A^0(x,u_0)\nabla v_0) + \theta\lambda v_0 = -\mathrm{div}(A^0(x,u_0)\nabla u_0) + \theta\lambda u_0.$$

Hence, $v_0 = u_0$, so that

$$\widetilde{v_\varepsilon} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega). \tag{4.17}$$

We approximate now the function $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ by a sequence $\{u_n\} \subset D(\Omega)$ such that

$$\begin{cases} \text{(i)} \quad u_n \to u_0 \quad \text{strongly in } H_0^1(\Omega), \text{ as } n \to +\infty, \\ \text{(ii)} \quad \|u_n\|_{L^\infty(\Omega)} \le c, \end{cases} \tag{4.18}$$

for any n, where c is independent of n.

Let us introduce, for any $n \in \mathbb{N}$, the sequence $\{v_{n,\varepsilon}\}_\varepsilon$ where the function $v_{n,\varepsilon}$ is the solution of the problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x,u_\varepsilon)\nabla v_{n,\varepsilon}) + \lambda v_{n,\varepsilon} = \mathcal{L}_\varepsilon\Big(-\mathrm{div}(A^0(u_0)\nabla u_n) \\ \qquad\qquad + \theta\lambda u_n + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big) & \text{in } \Omega_\varepsilon^*, \\ (A^\varepsilon(x,u_\varepsilon)\nabla v_{n,\varepsilon})\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \\ v_{n,\varepsilon} = 0 & \text{on } \Gamma_0^\varepsilon, \end{cases}$$

whose variational formulation is

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_{n,\varepsilon}\nabla\phi \,\mathrm{d}x + \lambda\int_{\Omega_\varepsilon^*} v_{n,\varepsilon}\phi\,\mathrm{d}x + \varepsilon^\gamma\int_{\Gamma_1^\varepsilon}\rho_\varepsilon h(u_\varepsilon)\phi\,\mathrm{d}\sigma \\ \displaystyle = \int_{\Gamma_1^\varepsilon} g_\varepsilon\phi\,\mathrm{d}\sigma + \Big\langle \mathcal{L}_\varepsilon\Big(-\mathrm{div}(A^0(u_0)\nabla u_n) + \theta\lambda u_n \\ \displaystyle + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \phi\Big\rangle_{V_\varepsilon', V_\varepsilon}, \quad \forall \phi \in V_\varepsilon. \end{cases} \tag{4.19}$$

Then, for any n, we apply again Theorem 3.1, written here for $z_\varepsilon = u_\varepsilon$ and for

$$Z = -\mathrm{div}(A^0(u_0)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g).$$

The same argument used to prove (4.17) gives

$$\widetilde{v_{n,\varepsilon}} \rightharpoonup \theta u_n \quad \text{weakly in } L^2(\Omega), \tag{4.20}$$

as ε tends to zero, where θ is defined by (2.1). Moreover,

$$\|v_{n,\varepsilon}\|_{V_\varepsilon} \le c, \tag{4.21}$$

where c is independent of n and ε, and from the classical results of Stampacchia [24], for any fixed n, we have

$$\|v_{n,\varepsilon}\|_{L^\infty(\Omega_\varepsilon^*)} \le c_n, \tag{4.22}$$

where c_n is a constant independent of ε.

The following theorem is the essential tool to prove the corrector result for the nonlinear problem.

Theorem 4.2 *Under the assumptions* (H_1)–(H_7)*, let* u_ε *be a sequence of solution of problem* (2.5) *and* v_ε *be a solution of problem* (4.19)*. Then, up to a subsequence, we have*

$$\lim_{\varepsilon\to 0}\|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{(L^2(\Omega_\varepsilon))^N} = 0.$$

Proof Let Φ_ε be the function defined by

$$\Phi_\varepsilon = \xi_\mu(u_\varepsilon - v_{n,\varepsilon}) \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon^*), \tag{4.23}$$

where μ is a suitable positive constant to be chosen later on, and

$$\xi_\mu(s) = s\mathrm{e}^{\mu s^2}, \quad \forall s \in \mathbb{R}^N.$$

Taking Φ_ε as a test function in (2.5) and (4.19), after subtraction of two identities, we obtain

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla(u_\varepsilon - v_{n,\varepsilon})\nabla(u_\varepsilon - v_{n,\varepsilon})\xi_\mu'(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x + \lambda\int_{\Omega_\varepsilon^*}(u_\varepsilon - v_{n,\varepsilon})^2 \mathrm{e}^{\mu(u_\varepsilon - v_{n,\varepsilon})^2}\,\mathrm{d}x$$
$$= \int_{\Omega_\varepsilon^*} b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x + \int_{\Omega_\varepsilon^*} f\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x$$
$$- \left\langle \mathcal{L}_\varepsilon\Big(- \mathrm{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \xi_\mu(u_\varepsilon - v_{n,\varepsilon})\right\rangle_{V_\varepsilon',V_\varepsilon}.$$

Using (H_1), (H_3) and the property (2) of (H_7), since $\xi_\mu' \geq 0$, we get

$$\alpha\int_{\Omega_\varepsilon^*}|\nabla(u_\varepsilon - v_{n,\varepsilon})|^2\xi_\mu'(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x \leq c_0\int_{\Omega_\varepsilon^*}(1+|\nabla u_\varepsilon|^2)|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|\,\mathrm{d}x$$
$$+ \int_{\Omega_\varepsilon^*} f\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x - \langle\mathcal{L}_\varepsilon(-\mathrm{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n), \xi_\mu(u_\varepsilon - v_{n,\varepsilon})\rangle_{V_\varepsilon',V_\varepsilon}$$
$$\leq c_0\int_{\Omega_\varepsilon^*}(1 + 2|\nabla(u_\varepsilon - v_{n,\varepsilon})|^2 + 2|\nabla v_{n,\varepsilon}|^2)|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|\,\mathrm{d}x + \int_{\Omega_\varepsilon^*} f\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x$$
$$- \left\langle \mathcal{L}_\varepsilon\Big(- \mathrm{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \xi_\mu(u_\varepsilon - v_{n,\varepsilon})\right\rangle_{V_\varepsilon',V_\varepsilon}.$$

Let us take $\mu = \frac{c_0^2}{\alpha^2}$. For this choice, one gets $\alpha\xi_\mu'(s) - 2c_0|\xi_\mu(s)| \geq \frac{\alpha}{2}$ for any s, so that using again (H_3), we get

$$\frac{\alpha}{2}\int_{\Omega_\varepsilon^*}|\nabla(u_\varepsilon - v_{n,\varepsilon})|^2\,\mathrm{d}x \leq \int_{\Omega_\varepsilon^*}|\nabla(u_\varepsilon - v_{n,\varepsilon})|^2(\alpha\xi_\mu'(u_\varepsilon - v_{n,\varepsilon}) - 2c_0|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|)\,\mathrm{d}x$$
$$\leq c_0\int_{\Omega_\varepsilon^*}\Big(1 + \frac{2}{\alpha}A^\varepsilon(x,u_\varepsilon)\nabla v_{n,\varepsilon}\nabla v_{n,\varepsilon}\Big)|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|\,\mathrm{d}x + \int_{\Omega_\varepsilon^*} f\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\,\mathrm{d}x$$
$$- \left\langle \mathcal{L}_\varepsilon\Big(- \mathrm{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \xi_\mu(u_\varepsilon - v_{n,\varepsilon})\right\rangle_{V_\varepsilon',V_\varepsilon}. \quad (4.24)$$

From (2.6), (4.21)–(4.22), the function $u_\varepsilon - v_{n,\varepsilon}$ satisfies (3.1) and the sequence $\{\xi_\mu(u_\varepsilon - v_{n,\varepsilon})\}$ is bounded in $L^\infty(\Omega)$ (for fixed n).

Hence, applying Theorem 4.1 to $\psi_\varepsilon = u_\varepsilon - v_{n,\varepsilon}$ and $p = m$, first with $h = f$ and $F = \xi_\mu$, and then with $h = c_0$ and $F = |\xi_\mu|$, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} f\xi_\mu(u_\varepsilon - v_{n,\varepsilon}) + c_0|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|\,\mathrm{d}x$$
$$= \theta\int_\Omega f\xi_\mu(u_0 - v_n)\,\mathrm{d}x + \theta c_0\int_\Omega|\xi_\mu(u_0 - v_n)|\,\mathrm{d}x. \quad (4.25)$$

Observe now that in view of Theorem 3.2 we can apply Theorem 4.1 for $p = 1$ to the functions

$$h_\varepsilon = A^\varepsilon(x,u_\varepsilon)\widetilde{\nabla v_{n,\varepsilon}}\widetilde{\nabla v_{n,\varepsilon}}, \quad \psi_\varepsilon = u_\varepsilon - v_{n,\varepsilon}, \quad F = |\xi_\mu|.$$

We get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_{n,\varepsilon}\nabla v_{n,\varepsilon}|\xi_\mu(u_\varepsilon - v_{n,\varepsilon})|\,\mathrm{d}x = \int_\Omega \theta A^0(u_0)\nabla v_n\nabla v_n|\xi_\mu(u_0 - v_n)|\,\mathrm{d}x. \quad (4.26)$$

On the other hand, from (2.6) and (4.22)–(4.23), using again Theorem 4.1, we deduce that $z_\varepsilon = \xi_\mu(u_\varepsilon - v_{n,\varepsilon})$ satisfies (3.1) with $z_0 = \xi_\mu(u_0 - v_n)$.

Then, the assumption (H_8) given in Section 3 implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big\langle \mathcal{L}_\varepsilon\Big(-\operatorname{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\Big), \xi_\mu(u_\varepsilon - v_{n,\varepsilon})\Big\rangle_{V'_\varepsilon, V_\varepsilon} \\
&= \Big\langle -\operatorname{div}(A^0(u_n)\nabla u_n) + \theta\lambda u_n + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g), \xi_\mu(u_0 - v_n)\Big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&\doteq I_n. \qquad (4.27)
\end{aligned}$$

Collecting (4.24)–(4.27), we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{\alpha}{2}\int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_{n,\varepsilon})|^2 \, \mathrm{d}x \\
&\le 2\theta\,\frac{c_0}{\alpha}\int_\Omega A^0(u_0)\nabla v_n \nabla v_n |\xi_\mu(u_0 - v_n)| \, \mathrm{d}x \\
&\quad + \theta\int_\Omega (|f| + c_0)|\xi_\mu(u_0 - v_n)| \, \mathrm{d}x + c\|u_n - u_0\|_{H_0^1(\Omega)} + I_n. \qquad (4.28)
\end{aligned}$$

Hence, using (H_2)–(H_3), (4.18) and (4.23) we deduce that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0} \frac{\alpha}{2}\int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_{n,\varepsilon})|^2 \, \mathrm{d}x = 0. \qquad (4.29)$$

On the other hand, taking $v_{n,\varepsilon} - v_\varepsilon$ as test function in (4.16) and (4.19), subtracting the two identities and passing to the limit, one can easily deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} |\nabla(v_{n,\varepsilon} - v_\varepsilon)|^2 \, \mathrm{d}x &\le \lim_{\varepsilon\to 0} \alpha \int_{\Omega_\varepsilon^*} (A^\varepsilon(x, u_\varepsilon)\nabla(v_{n,\varepsilon} - v_\varepsilon)\nabla(v_{n,\varepsilon} - v_\varepsilon)) \, \mathrm{d}x \\
&= \int_\Omega (A^0(u_n)\nabla u_n \nabla u_n - A^0(u_0)\nabla u_n \nabla u_0) \, \mathrm{d}x \\
&\le c\|u_n - u_0\|_{H_0^1(\Omega)},
\end{aligned}$$

where the right-hand side goes to zero as $n \to \infty$, by virtue of (4.18).

This, together with (4.29), gives the result, since

$$\int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_\varepsilon)|^2 \, \mathrm{d}x \le 2\int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_{n,\varepsilon})|^2 \, \mathrm{d}x + 2\int_{\Omega_\varepsilon^*} |\nabla(v_{n,\varepsilon} - v_\varepsilon)|^2 \, \mathrm{d}x.$$

4.3 Proof of Theorem 2.1

We distinguish here the two cases given by (2.4).

Let us treat first the case where $\mathcal{M}_{\partial T}(g) \neq 0$ or $g \equiv 0$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $(C_0^\infty(\Omega))^N$ such that

$$\phi_n \to \nabla u_0 \quad \text{strongly in } (L^2(\Omega))^N. \qquad (4.30)$$

Then, for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, using (3) of ($\mathrm{H}_7$), (2.6) and (2.16), we get

$$
\begin{aligned}
&\Big| \int_\Omega \Big[(b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))^\sim \varphi - \int_\Omega b_0(u_0, \nabla u_0)\varphi \Big] \,\mathrm{d}x \Big| \\
&\le \int_{\Omega_\varepsilon^*} |b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n)||\varphi \mathrm{d}x| \\
&\quad + \Big| \int_\Omega [(b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n))^\sim - b_0(u_0, \phi_n)]\varphi \,\mathrm{d}x \Big| \\
&\quad + \int_\Omega |b_0(u_0, \phi_n) - b_0(u_0, \nabla u_0)||\varphi|\mathrm{d}x \\
&\le d_1(c) \int_{\Omega_\varepsilon^*} (1 + |\nabla u_\varepsilon| + |C^\varepsilon \phi_n|)|\nabla u_\varepsilon - C^\varepsilon \phi_n||\varphi|\mathrm{d}x \\
&\quad + \Big| \int_\Omega [(b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n))^\sim - b_0(u_0, \phi_n)]\varphi \,\mathrm{d}x \Big| \\
&\quad + c d_1(c) \int_\Omega (1 + |\phi_n| + |\nabla u_0|)|\phi_n - \nabla u_0||\varphi|\mathrm{d}x \\
&\doteq I_n^\varepsilon + J_n^\varepsilon + J_n.
\end{aligned} \tag{4.31}
$$

Let us pass to the limit as ε tends to zero in the right-hand side of this inequality.

Concerning the first term, by a similar argument as in [5, 19], using (2.6), (2.12), (4.30), and Proposition 3.1, we have

$$
\begin{aligned}
\limsup_{\varepsilon\to0} I_n^\varepsilon &\le d_1(c) \limsup_{\varepsilon\to0} \int_{\Omega_\varepsilon^*} (1 + 2|\nabla u_\varepsilon| + |C^\varepsilon \phi_n - \nabla u_\varepsilon|)|\nabla u_\varepsilon - C^\varepsilon \phi_n||\varphi|\mathrm{d}x \\
&\le \limsup_{\varepsilon\to0} (\|C^\varepsilon \phi_n - \nabla u_\varepsilon\|_{L^2(\Omega)} + \|C^\varepsilon \phi_n - \nabla u_\varepsilon\|_{L^2(\Omega)}^2) \\
&\le c(\|\phi_n - \nabla u_0\|_{L^2(\Omega)} + \|\phi_n - \nabla u_0\|_{L^2(\Omega)}^2).
\end{aligned}
$$

This, together with (4.30), implies

$$
\lim_{n\to\infty} \limsup_{\varepsilon\to0} I_n^\varepsilon = 0. \tag{4.32}
$$

For the third term J_n (independent of ε), we have

$$
\lim_{n\to\infty} \limsup_{\varepsilon\to0} J_n \le c \lim_{n\to\infty} \|\phi_n - \nabla u_0\|_{L^2(\Omega)} = 0. \tag{4.33}
$$

It remains to pass to the limit in the second term. To this aim, we write

$$
\begin{aligned}
J_n^\varepsilon &\le \Big| \int_\Omega [(b_\varepsilon(x, u_0, C^\varepsilon \phi_n))^\sim - b_0(u_0, \phi_n)]\varphi \,\mathrm{d}x \Big| \\
&\quad + \Big| \int_\Omega [(b_\varepsilon(x, u_0, C^\varepsilon \phi_n))^\sim - (b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n))^\sim] \,\mathrm{d}x \Big| \\
&\le \Big| \int_\Omega [(b_\varepsilon(x, u_0, C^\varepsilon \phi_n))^\sim - b_0(u_0, \phi_n)]\varphi \,\mathrm{d}x \Big| + c \int_{\Omega_\varepsilon^*} |u_\varepsilon - u_0| \,\mathrm{d}x,
\end{aligned}
$$

where we used the assumption (H_7) and again (2.12) and (4.30).

Consequently, by Proposition 3.2 and Theorem 4.1,

$$
\lim_{n\to\infty} \limsup_{\varepsilon\to0} J_n^\varepsilon = 0. \tag{4.34}
$$

Hence, by (4.31)–(4.34) for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\lim_{\varepsilon \to 0} \left| \int_\Omega \left[(b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))^{\sim} \varphi - \int_\Omega b_0(u_0, \nabla u_0)\varphi \right] \mathrm{d}x \right| = 0. \tag{4.35}$$

Moreover, from (2.1) and (4.14), we derive

$$\lim_{\varepsilon \to 0} \lambda \int_{\Omega_\varepsilon^*} u_\varepsilon \varphi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \lambda \int_\Omega \widetilde{u_\varepsilon} \varphi \, \mathrm{d}x = \lambda\theta \int_\Omega u_0 \varphi \, \mathrm{d}x \tag{4.36}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} f\varphi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega f \chi_{\Omega_\varepsilon^*} \varphi \, \mathrm{d}x = \theta \int_\Omega f\varphi \, \mathrm{d}x. \tag{4.37}$$

On the other hand, using the unfolding periodic method and arguing as in [10] (see also [8]), we get

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_1^\varepsilon} \rho_\varepsilon(x) h(u_\varepsilon) \varphi \, \mathrm{d}\sigma_x = c_\gamma \int_\Omega h(u) \varphi \, \mathrm{d}x, \tag{4.38}$$

where c_γ is defined by (2.15) and

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\varepsilon} g_\varepsilon(x) \varphi \, \mathrm{d}\sigma_x = \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_\Omega \varphi \, \mathrm{d}x. \tag{4.39}$$

Let us prove now (2.17). For any $\psi \in H_0^1(\Omega)$, from Theorem 4.2, the assumption (H_3) and the Hölder inequality, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \operatorname{div} \left\langle A^\varepsilon(x, \widetilde{u_\varepsilon}) \widetilde{\nabla u_\varepsilon}, \psi \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi \, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \psi \, \mathrm{d}x + \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla (u_\varepsilon - v_\varepsilon) \nabla \psi \, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \psi \, \mathrm{d}x,
\end{aligned} \tag{4.40}$$

where v_ε is the solution of problem (4.15).

On the other hand, thanks to (3.5)(ii) and (4.17), we can apply Theorem 3.1 (written for $z_\varepsilon = u_\varepsilon$ and $z_0 = v_0 = u_0$) to problem (4.15) and we have

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla v_\varepsilon \nabla \psi \, \mathrm{d}x = \int_\Omega A^0(u_0) \nabla u_0 \nabla \psi \, \mathrm{d}x.$$

This together with (4.40) gives (2.17).

Hence, using (2.17) and (4.35)–(4.39), we can pass to the limit in (2.5) for any φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and obtain

$$\begin{cases}
\displaystyle\int_{\Omega_\varepsilon^*} A^0(u_0) \nabla u_0 \nabla \varphi \, \mathrm{d}x + \theta\lambda \int_\Omega u_0 \varphi \, \mathrm{d}x + c_\gamma \int_\Omega u_0 \varphi \, \mathrm{d}x \\
\displaystyle = \int_{\Omega_\varepsilon^*} b_0(u_0, \nabla u_0) \varphi \, \mathrm{d}x + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_\Omega \varphi \, \mathrm{d}x + \theta \int_\Omega f\varphi \, \mathrm{d}x, \\
\forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega),
\end{cases} \tag{4.41}$$

where b_0 is given by (2.16), i.e., the variational formulation of the homogenized problem (2.18).

Let us consider now the case $\mathcal{M}_{\partial T}(g) = 0$ and that A is independent of t. Using (3) of (H_3), for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we write here

$$\begin{aligned}
&\Big|\int_\Omega (b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))^\sim \varphi \, \mathrm{d}x - \int_\Omega b_0(u_0, \nabla u_0)\varphi \, \mathrm{d}x\Big| \\
&\leq \int_{\Omega_\varepsilon^*} |b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n + \nabla \widehat{\chi}_g)||\varphi| \mathrm{d}x \\
&\quad + \Big|\int_\Omega [(b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n + \nabla \chi_\varepsilon))^\sim - b_0(u_0, \phi_n)]\varphi \, \mathrm{d}x\Big| + \int_\Omega |b_0(u_0, \phi_n) - b_0(u_0, \nabla u_0)||\varphi| \mathrm{d}x \\
&\leq b_1(c) \int_{\Omega_\varepsilon^*} (1 + |\nabla u_\varepsilon| + |C^\varepsilon \phi_n + \nabla \chi_\varepsilon||\nabla u_\varepsilon - C^\varepsilon \phi_n - \nabla \widehat{\chi}_g||\varphi| \mathrm{d}x \\
&\quad + \Big|\int_\Omega [(b_\varepsilon(x, u_\varepsilon, C^\varepsilon \phi_n + \nabla \widehat{\chi}_g))^\sim - b_0(u_0, \phi_n)]\varphi \, \mathrm{d}x\Big| \\
&\quad + cb_1(c) \int_\Omega (1 + |\phi_n| + |\nabla u_0||\phi_n - \nabla u_0||\varphi|) \mathrm{d}x.
\end{aligned}$$

Then, arguing as before and using here the results corresponding to this case, we have

$$(b_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon))^\sim \rightharpoonup b_0(u_0, \nabla u_0) \quad \text{weakly in } L^1(\Omega), \tag{4.42}$$

where b_0 is now given by (2.19).

On the other hand, (4.36)–(4.38) still hold true and from Proposition 3.8 of [10],

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\varepsilon} g_\varepsilon(x)\varphi \, \mathrm{d}\sigma_x = 0.$$

Finally, the convergence (2.21) follows as before by using (4.40). Hence, again we can pass to the limit in (2.5) for this case, to obtain

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^0(u_0)\nabla u_0 \nabla \varphi \, \mathrm{d}x + \theta\lambda \int_\Omega u_0 \varphi \, \mathrm{d}x + c_\gamma \int_\Omega u_0 \varphi \, \mathrm{d}x \\ \displaystyle = \int_{\Omega_\varepsilon^*} b_0(u_0, \nabla u_0)\varphi \, \mathrm{d}x + \theta \int_\Omega f\varphi \, \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{4.43}$$

with b_0 given by (2.19), which is the variational formulation of the homogenized problem (2.18). This ends the proof of Theorem 2.1.

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