Chin. Ann. Math.
37B(6), 2016, 853–864
DOI: 10.1007/s11401-016-0148-4

Chinese Annals of Mathematics, Series B

Asymptotic Behavior of the Incompressible Navier-Stokes Fluid with Degree of Freedom in Porous Medium*

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Abstract The authors study the asymptotic behavior of the incompressible Navier-Stokes fluid with degree of freedom in the porous medium in $\mathbb{R}^n$ with $n = 2$ or 3. They derive the Darcy law as ε, the character size of the hole, tends to zero. Moreover, the authors obtain the expression of the degree of freedom from the homogenized model.

Keywords Homogenization, Navier-Stokes fluid, Darcy law, Degree of freedom
2000 MR Subject Classification 76D05, 76M50, 74Q10

1 Introduction

Homogenization is a mathematical tool that allows changing the scale in problems containing several characteristic scales. Typical examples of its utilization are finding effective models for composite materials, in optimal shape design, etc. Another important example, which we are interested in, is the fluid mechanics of the flow through porous medium.

In porous medium, there are at least two length scales: A microscopic scale and a macroscopic scale. Quite often, the partial differential equations describing a physical phenomenon are posed at the microscopic level whereas only macroscopic quantities are of interest for the engineers or the physicists. Therefore, effective or homogenized equations should be derived from the microscopic ones by an asymptotic analysis. To this end, it is convenient to assume that the porous medium has a periodic structure.

A number of known laws from the dynamics of fluids in porous media were derived using homogenization. The most well-known example is Darcy law, being the homogenized equation for one-phase flow through a rigid porous medium. Its formal derivation by two-scale expansion goes back to the classical paper by Sanchez-Palencia [1], Keller [2] and the classical book Bensoussan [3]. It was rigorously derived by using oscillating functions by Tartar [4]. In other cases of periodic porous media, we refer the readers to the papers by Allaire [5–7] and Mikelic [8–9]. Other works can be seen in [10–11] and the references therein.

Besides the Darcy law, Brinkman [12] introduced a new set of equations, which is called the Brinkman law, an intermediate between the Darcy and Stokes equations. The so-called Brinkman law is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity (see [6]).

Manuscript received May 7, 2015. Revised September 26, 2015.
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*This work was supported by the Research of High Schools of Guangzhou (No. 1201420925) and Guangdong Natural Science Foundation (No. 2014A030313526).

In this paper, we are interested in obtaining the homogenized result for the Navier-Stokes fluid with degree of freedom in porous medium. This model problem was proposed by Lions [13], where he proved the existence and the regularity of the solutions.

Before stating the system, let us recall the domain we consider. A porous medium is defined as the periodic repetition of an elementary cell of size ε $\big($we assume $\frac{1}{\varepsilon}$ to be an integral$\big)$ in a bounded domain Ω of $\mathbb{R}^n$ with $n = 2, 3$. The solid part of the porous medium is also taken of size ε. The domain Ω_ε is then defined as the intersection of Ω with the fluid part. We consider an incompressible fluid governed by the Navier-Stokes equations with degree of freedom. So, we have the following equations:

$$\begin{aligned}
&\varepsilon^2 \frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla) u^\varepsilon - \mu \triangle u^\varepsilon + \nabla p^\varepsilon = f + w^\varepsilon \times u^\varepsilon \qquad && \text{in } \ \Omega_\varepsilon \times (0, T), \\
&\frac{\partial w^\varepsilon}{\partial t} + \mathrm{div}(u^\varepsilon \otimes w^\varepsilon) + \kappa w^\varepsilon = m && \text{in } \ \Omega_\varepsilon \times (0, T), \\
&\mathrm{div}\, u^\varepsilon = 0 && \text{in } \ \Omega_\varepsilon \times (0, T),
\end{aligned} \tag{1.1}$$

where $u^\varepsilon,\ p^\varepsilon,\ w^\varepsilon$ are the unknown quantities velocity, pressure and degree of freedom of the fluid, respectively, $f \in L^2(\Omega \times (0, T))$ is the external force.

The system is supplemented with the boundary condition and initial conditions as follows:

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0, T) \tag{1.2}$$

and

$$u^\varepsilon|_{t=0} = u_0^\varepsilon, \quad w^\varepsilon|_{t=0} = w_0^\varepsilon, \tag{1.3}$$

where $u_0^\varepsilon, w_0^\varepsilon$ are bounded in $L^2(\Omega_\varepsilon)$.

Our aim here is to investigate the asymptotic behavior of $u^\varepsilon,\ p^\varepsilon,\ w^\varepsilon$ as $\varepsilon \to 0^+$ under the assumptions mentioned above. The main difficulty here is how to pass the limit in the momentum equations. To overcome this obstacle, we need to revise the estimates on the inertia and extend the pressure to the whole domain. It is resolved by using the general Poincaré's equality in porous medium (see [11]).

The paper is organized as follows. In Section 2, we list some useful results and state the main results in this paper. In Section 3, we give priori estimates of the unknowns and extend them to the whole domain. In Section 4, we prove the main result in this paper.

2 Notations, Preliminaries and Main Results

The structure of a porous medium is standard (see [4–5, 8]). To give a good understanding for the readers, we write it detail again. Let Ω be an open bounded subset of $\mathbb{R}^n$ with $n = 2$ or 3 and define $\mathcal{Y} = [0, 1]^n$ to be the unit open cube of $\mathbb{R}^n$. Let $\mathcal{Y}_s$ be a closed smooth subset of $\mathcal{Y}$ with a strictly positive measure. The fluid part is then defined by $\mathcal{Y}_f = \mathcal{Y} - \mathcal{Y}_s$. Let $\theta = |\mathcal{Y}_f|$. The constant θ is called the porosity of the porous medium. We assume that $0 < \theta < 1$.

Repeating the domain $\mathcal{Y}_f$ by $\mathcal{Y}$-periodicity, we get the whole fluid domain D_f, and we can write it as

$$D_f = \{x \in \mathbb{R}^n \mid \exists k \in Z^n \text{ such that } x - k \in \mathcal{Y}_f\}.$$

Then the solid part is defined by $D_s = \mathbb{R}^n - D_f$. It is easy to see that D_f is a connected domain, while D_s is formed by separated smooth subsets. In the sequel, we denote for all $k \in Z^n$,

$\mathcal{Y}^k = \mathcal{Y} + k$ and then $\mathcal{Y}_f^k = \mathcal{Y}_f + k$. For all ε, we define the domain Ω_ε as the intersection of Ω with the fluid domain scaled by ε, namely, $\Omega_\varepsilon = \Omega \cap \varepsilon D_f$. To get a smooth connected domain, we will not remove the solid part of the cells which intersect with the boundary of Ω. Now, the fluid domain can be also defined by

$$\Omega_\varepsilon = \Omega - \cup\{\varepsilon\mathcal{Y}_s^k,\ k \in Z^n,\ \varepsilon\mathcal{Y}^k \subset \Omega\}.$$

Throughout this paper, we denote by $L^p(0,T;L^q(X))$ the time-space Lebesgue spaces, where X would be Ω or Ω_ε. $W^{s,p}(X)$ will be the classical Sobolev space with all functions, whose all derivatives up to order s belong to L^p and $H^s(X) = W^{s,2}(X)$. $W_0^{1,p}(X)$ is the subset of $W^{1,p}(X)$ with trace 0 on X. We also denote by $W^{-s,p'}(X)$ the dual space of $W_0^{s,p}(X)$, where p' is the conjugate exponent of p. C will be constants that may differ from one place to another. Throughout this paper, we will use $\|\cdot\|_X$ to denote the modules for all vectors or matrices if there is no confusion.

Due to the presence of the holes, the domain Ω_ε depends on ε and hence to study the convergence of $\{u^\varepsilon, \rho_\varepsilon, p_\varepsilon\}$, we have to extend the functions defined in Ω_ε to the whole domain. This can be done in two different possible ways.

Definition 2.1 (see [10]) *For any fixed $\varphi \in L^1(\Omega_\varepsilon)$, we define*

$$\widetilde{\varphi} = \begin{cases} \varphi & in\ \Omega, \\ 0 & in\ \Omega - \Omega_\varepsilon \end{cases}$$

as the null extension and

$$\widehat{\varphi} = \begin{cases} \varphi & in\ \Omega_\varepsilon, \\ \dfrac{1}{|\varepsilon\mathcal{Y}_f^k|}\displaystyle\int_{\varepsilon\mathcal{Y}_f^k} \varphi(x)\mathrm{d}x & in\ \Omega \cap \varepsilon\mathcal{Y}_s^k \end{cases}$$

as the mean value extension.

The relation between the weak limits of both types of extensions is given by the following lemma.

Lemma 2.1 (see [10]) *For all $\omega^\varepsilon \in L^p(\Omega_\varepsilon)$, $p \geq 1$, the following two assertions are equivalent:*

$$(1)\ \widehat{\omega}^\varepsilon \rightharpoonup \omega \quad in\ L^p(\Omega); \quad (2)\ \widetilde{\omega}^\varepsilon \rightharpoonup \theta\omega \quad in\ L^p(\Omega).$$

A very important property of the porous medium is a variant of the Poincaré's inequality. Due to the presence of the holes in Ω_ε, the Poincaré's inequality is given by the following lemma.

Lemma 2.2 (see [11]) *Let $1 \leq p,q < \infty$ and $u \in W_0^{1,p}(\Omega_\varepsilon)$, then*

$$\|u\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{1+n(\frac{1}{q}-\frac{1}{p})}\|\nabla u\|_{L^p(\Omega_\varepsilon)},$$

where C depends only on $\mathcal{Y}_f$ and p,q satisfies

(1) $1 \leq p < n,\ p \leq q \leq p^* = \frac{np}{n-p}$;

(2) $p \geq n,\ p \leq q < \infty$.

Especially, if $p = q$, we have the standard inequality

$$\|u\|_{L^p(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla u\|_{L^p(\Omega_\varepsilon)}.$$

We introduce the restriction operator by the following lemma.

Lemma 2.3 (see [4]) *There exists an operator $\mathcal{R}_\varepsilon$ with the following properties:*
(1) $\mathcal{R}_\varepsilon$ *is a bounded linear operator on* $W_0^{1,p}(\Omega)$ *ranging in* $W_0^{1,p}(\Omega_\varepsilon)$, $p \geq 2$;
(2) $\mathcal{R}_\varepsilon[\varphi] = \varphi|_{\Omega_\varepsilon}$ *provides* $\varphi = 0$ *in* $\Omega - \Omega_\varepsilon$;
(3) $\operatorname{div}_x \varphi = 0$ *in* Ω *implies* $\operatorname{div}_x \mathcal{R}_\varepsilon[\varphi] = 0$ *in* Ω_ε;
(4) $\|\mathcal{R}_\varepsilon[\varphi]\|_{L^p(\Omega_\varepsilon)} + \varepsilon\|\nabla\mathcal{R}_\varepsilon[\varphi]\|_{L^p(\Omega_\varepsilon)} \leq C(\|\varphi\|_{L^p(\Omega)} + \varepsilon\|\nabla\varphi\|_{L^p(\Omega)})$.

In addition, we can find the restriction operator $\mathcal{R}_\varepsilon$ satisfies a compatibility relation with the extension operator introduced in Definition 2.1, namely,

$$\langle \nabla\widehat{\omega}, \varphi\rangle = -\int_\Omega \widehat{\omega} \operatorname{div}\varphi \mathrm{d}x = -\int_{\Omega_\varepsilon} \omega \operatorname{div} R_\varepsilon[\varphi]\mathrm{d}x, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Finally, we define the permeability matrix $\mathcal{A}$. For $1 \leq i \leq n$, let $(\omega_i, \pi_i) \in H^1(\mathcal{Y}_f) \times L^2(\mathcal{Y}_f)/\mathbb{R}$ be the unique solution of the following system:

$$\begin{cases} -\triangle\omega_i + \nabla\pi_i = e_i & \text{in } \mathcal{Y}_f, \\ \operatorname{div}\omega_i = 0 & \text{in } \mathcal{Y}_f, \\ \omega_i = 0 & \text{on } \partial\mathcal{Y}_s, \end{cases}$$

where ω_i, π_i are $\mathcal{Y}$-periodic, e_i is the standard basis of $\mathbb{R}^n$. Set $\omega_i^\varepsilon = \omega_i\left(\frac{x}{\varepsilon}\right)$, $\pi_i^\varepsilon = \pi_i\left(\frac{x}{\varepsilon}\right)$. Then we get the cell problem

$$\begin{cases} -\varepsilon^2\triangle\omega_i^\varepsilon + \varepsilon\nabla\pi_i^\varepsilon = e_i & \text{in } \varepsilon\mathcal{Y}_f, \\ \operatorname{div}\omega_i^\varepsilon = 0 & \text{in } \varepsilon\mathcal{Y}_f, \\ \omega_i^\varepsilon = 0 & \text{on } \partial(\varepsilon\mathcal{Y}_s), \end{cases}$$

where $\omega_i^\varepsilon, \pi_i^\varepsilon$ are $\varepsilon\mathcal{Y}$-periodic.

Lemma 2.4 (see [4, 10]) *Let $\omega_i^\varepsilon, \pi_i^\varepsilon$ be the solution to the cell problem and be extended to zero outside Ω_ε. Then the following estimates hold:*

$$\|\omega_i^\varepsilon\|_{[L^q(\Omega_\varepsilon)]^n} \leq C, \quad \|\pi_i^\varepsilon\|_{L^q(\Omega_\varepsilon)/R} \leq C, \quad \varepsilon\|\nabla\omega_i^\varepsilon\|_{[L^q(\Omega_\varepsilon)]^n} \leq C$$

for any $1 \leq q \leq +\infty$, C only depends on q and $\mathcal{Y}_f$.

Let us define

$$\mathcal{A} = (\mathcal{A}_{i,j})_{i,j=1}^n, \quad \mathcal{A}_{i,j} = \frac{1}{|\varepsilon\mathcal{Y}_f|}\int_{\varepsilon\mathcal{Y}_f}(\omega_i^\varepsilon)_j\mathrm{d}x = \frac{1}{|\mathcal{Y}_f|}\int_{\mathcal{Y}_f}(\omega_i)_j\mathrm{d}x. \tag{2.1}$$

The periodic lemma (see [1]) shows that $(\omega_i^\varepsilon)_j$ converges weakly (or weakly $\star$ for $p = +\infty$) to its average on $\varepsilon\mathcal{Y}_f$ in $L^p(\Omega_\varepsilon)$ for $1 \leq p \leq +\infty$. It is easy to see that $\mathcal{A}$ is a symmetric positive defined matrix. The form of the permeability matrix has different form if $\mathcal{Y}_s$ has different form. For more information about $\mathcal{A}$, we refer the interested readers to [6] for detail.

Now we introduce the definition of weak solution to the system (1.1)–(1.3).

Definition 2.2 *We shall say that a trio $\{u^\varepsilon, p^\varepsilon, w^\varepsilon\}$ is a weak solution of* (1.1)–(1.3)*, supplemented with the boundary and initial conditions* (1.4) *and* (1.5) *if and only if*
(1) $u^\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon)) \cap L^2(0,T;\mathcal{H}(\Omega_\varepsilon))$, *and*

$$\int_0^T\int_{\Omega_\varepsilon}(\varepsilon^2 u^\varepsilon\cdot\varphi_t + u^\varepsilon\otimes u^\varepsilon : \nabla\varphi - \mu\nabla u^\varepsilon : \nabla\varphi)\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^T\int_{\Omega_\varepsilon}(w^\varepsilon\times u^\varepsilon)\cdot\varphi\mathrm{d}x\mathrm{d}t - \int_{\Omega_\varepsilon}\varepsilon^2 u_0^\varepsilon\cdot\varphi(x,0)\mathrm{d}x - \int_0^T\int_{\Omega_\varepsilon} f\cdot\varphi\mathrm{d}x\mathrm{d}t$$

holds for any $\varphi \in C_0^\infty([0,T) \times \overline{\Omega}_\varepsilon)$ *with* $\operatorname{div} \varphi = 0$. $\mathcal{H}(X) = \{u \mid u \in H_0^1(X), \operatorname{div} u = 0\}$.

(2) $w^\varepsilon \in L^\infty(0,T; L^2(\Omega_\varepsilon)) \cap L^2(\Omega_\varepsilon \times (0,T))$, *and the integral identity*

$$\int_0^T \int_{\Omega_\varepsilon} (w^\varepsilon \psi_t + w^\varepsilon u^\varepsilon \cdot \nabla \psi - \kappa w^\varepsilon \psi) \mathrm{d}x\mathrm{d}t = -\int_0^T \int_{\Omega_\varepsilon} m\psi \mathrm{d}x\mathrm{d}t - \int_{\Omega_\varepsilon} w_0^\varepsilon \psi(x,0)\mathrm{d}x$$

holds for any $\psi \in C_0^\infty([0,T) \times \overline{\Omega}_\varepsilon)$.

The existence of weak solutions with finite energy is the following.

Theorem 2.1 (see [13]) *Under the above conditions and assumptions, for any fixed* $\varepsilon > 0$, *there exists a global solution* $(u^\varepsilon, p^\varepsilon, w^\varepsilon)$ *of the system* (1.1)–(1.3) *in the sense of Definition* 2.2.

In this paper, we always assume that

$$\mathcal{H}: \ u_0^\varepsilon \to u_0 \text{ strongly in } L^2(\Omega_\varepsilon), \qquad w_0^\varepsilon \to w_0 \text{ strongly in } L^2(\Omega_\varepsilon). \tag{2.2}$$

With all the preparation above, we are now in the position to state our main result in this paper.

Theorem 2.2 *Let* $\{u^\varepsilon, p^\varepsilon, w^\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the system* (1.1)–(1.3). *We also assume that* $\mathcal{H}$ *is satisfied. Then, there exist three functions* u, p, w *such that*

$$\begin{aligned} &\widehat{p^\varepsilon} \to p && \textit{weakly in } L^2(0,T; H^1(\Omega)), \\ &w^\varepsilon \to w && \textit{weakly in } L^2(\Omega \times (0,T)), \\ &\frac{u^\varepsilon}{\varepsilon^2} \to u && \textit{weakly in } L^2(\Omega \times (0,T)), \end{aligned} \tag{2.3}$$

where $\{u, p, w\}$ *satisfies the following homogenized system:*

$$\begin{aligned} &\operatorname{div} u = 0 && \textit{in } \ \Omega \times (0,T), \\ &\mu u = \mathcal{A}(-\nabla p + f) && \textit{in } \ \Omega \times (0,T), \\ &w = w_0 \mathrm{e}^{-\kappa t} + \int_0^t m(x,\tau) \mathrm{e}^{-\kappa(t-\tau)} \mathrm{d}\tau && \textit{in } \ \Omega \times (0,T), \end{aligned} \tag{2.4}$$

where $\mathcal{A}$ *is the so-called permeability matrix, which is defined by* (2.1). *Moreover,* u, w *satisfy the following initial conditions:*

$$u|_{t=0} = u_0, \qquad w|_{t=0} = w_0, \quad \forall x \in \Omega, \tag{2.5}$$

and $u|_{\partial\Omega} = 0$ *for any* $t \in (0,T)$.

Remark 2.1 The relationship of u and ∇p is often called the linear Darcy law. If we assume that $\lim\limits_{t \to +\infty} m(x,t) = M(x)$, we find that the degree of freedom w is determined only by M and κ for t is large enough.

3 Uniform Bounds

In this section, we collect all available bounds on the family $\{u^\varepsilon, p^\varepsilon, w^\varepsilon\}$. Let us begin with the basic estimates.

3.1 Priori estimates for $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{w}^\varepsilon$

In this subsection, we will obtain some estimates for the solutions to the system (1.1)–(1.3) which are independent of ε.

Lemma 3.1 *Let $\{u^\varepsilon, w^\varepsilon\}$ be the solution pair to (1.1)–(1.3). Under the conditions in Theorem 2.2, for $\varepsilon \in (0,1)$ small enough, the following estimates hold:*

$$\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \varepsilon^{-1}\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leq C,$$
$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \|w^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leq C,$$

where C does not depend on ε.

Firstly, multiplying (1.2) by w^ε and integrating over $\Omega_\varepsilon \times (0,t)$ for any $t \in [0,T]$, we have

$$\int_{\Omega_\varepsilon} |w^\varepsilon|^2 \mathrm{d}x + 2\kappa \int_0^t \int_{\Omega_\varepsilon} |w^\varepsilon|^2 \mathrm{d}x\mathrm{d}\tau \leq \int_{\Omega_\varepsilon} |w_0^\varepsilon|^2 \mathrm{d}x + 2\int_0^t \int_{\Omega_\varepsilon} m w^\varepsilon \mathrm{d}x\mathrm{d}\tau.$$

We have

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \|w^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leq C, \tag{3.1}$$

where C does not depend on ε.

Secondly, multiplying the momentum equations by u^ε and integrating over $\Omega_\varepsilon \times (0,t)$ for any $t \in [0,T]$, we have

$$\begin{aligned}&\int_{\Omega_\varepsilon} |u^\varepsilon|^2 \mathrm{d}x + 2\mu\varepsilon^{-2} \int_0^t \int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \mathrm{d}x\mathrm{d}\tau \\ &\leq \int_{\Omega_\varepsilon} |u_0^\varepsilon|^2 \mathrm{d}x + 2\int_0^t \int_{\Omega_\varepsilon} f\cdot u^\varepsilon \mathrm{d}x\mathrm{d}\tau + 2\int_0^t \int_{\Omega_\varepsilon} (w^\varepsilon \times u^\varepsilon)\cdot u^\varepsilon \mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

By Lemma 2.2, the force term can be estimated by

$$\int_0^t \int_{\Omega_\varepsilon} f\cdot u^\varepsilon \mathrm{d}x\mathrm{d}\tau \leq C\int_0^T \int_{\Omega_\varepsilon} |f|^2 \mathrm{d}x\mathrm{d}t + \mu\varepsilon^{-2}\int_0^T \int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \mathrm{d}x\mathrm{d}t.$$

By Lemma 2.2 and (3.1), the last term can be estimated by

$$\begin{aligned}\Big|2\int_0^t \int_{\Omega_\varepsilon} (w^\varepsilon \times u^\varepsilon)\cdot u^\varepsilon \mathrm{d}x\mathrm{d}\tau\Big| &\leq 2\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}\|u^\varepsilon\|^2_{L^2(0,T;L^4(\Omega_\varepsilon))} \\ &\leq 2C\varepsilon^{\frac{1}{2}}\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}\|\nabla u^\varepsilon\|^2_{L^2(\Omega_\varepsilon\times(0,T))} \\ &\leq \mu\varepsilon^{-2}\|\nabla u^\varepsilon\|^2_{L^2(\Omega_\varepsilon\times(0,T))}\end{aligned}$$

for $\varepsilon \in (0,1)$ small enough.

By the initial conditions, we immediately deduce

$$\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \varepsilon^{-1}\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leq C, \tag{3.2}$$

where C does not depend on ε. By Lemma 2.2, we also have

$$\varepsilon^{-2}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leq C. \tag{3.3}$$

3.2 Extensions of $u^\varepsilon, w^\varepsilon$ and p^ε

Note that Ω_ε will vary as ε tends to 0^+. We need to extend the unknowns $u^\varepsilon, w^\varepsilon$ and p^ε to the whole domain Ω. It is reasonable to take null extensions for u^ε and w^ε since the velocity on the solid part is 0. That is, we can define

$$\widetilde{u}^\varepsilon = \begin{cases} u^\varepsilon & \text{for } x \in \Omega_\varepsilon, \\ 0 & \text{for } x \in \Omega - \Omega_\varepsilon, \end{cases} \qquad \widetilde{w}^\varepsilon = \begin{cases} w^\varepsilon & \text{for } x \in \Omega_\varepsilon, \\ 0 & \text{for } x \in \Omega - \Omega_\varepsilon. \end{cases} \tag{3.4}$$

We will still denote the extensions by u^ε and w^ε if there are no confusions.

The extension of the pressure p^ε is different from u^ε and w^ε. The reason is that the pressure on the solid part will not disappear even if the velocity is 0 on it. To give the extension on p^ε, we define a function in the following way:

$$\begin{aligned} &\langle \mathcal{F}, \varphi \rangle_{\Omega\times(0,T),\Omega\times(0,T)} = \langle \nabla p^\varepsilon, \mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,T)} \\ &= \Big\langle f + w^\varepsilon \times u^\varepsilon - \varepsilon^2 \frac{\partial u^\varepsilon}{\partial t} - (u^\varepsilon\cdot\nabla)u^\varepsilon + \mu\triangle u^\varepsilon, \mathcal{R}_\varepsilon\varphi \Big\rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,T)} \end{aligned} \tag{3.5}$$

for $\varphi \in C_0^\infty(\Omega\times(0,T))$. $\mathcal{R}_\varepsilon$ is defined by Lemma 2.3.

Now, we give estimates on the right-hand side. We only consider the case $n = 3$ because $n = 2$ is much easier.

$$\begin{aligned} |\langle f, \mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)}| &\le \|f\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\mathcal{R}_\varepsilon\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))} \\ &\le C(\|\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))} + \varepsilon\|\nabla\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))}). \end{aligned}$$

By Lemma 2.2, Lemma 2.3 and estimations in (3.2), we have

$$\begin{aligned} &\|\langle w^\varepsilon \times u^\varepsilon, \mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)}| \\ &\le \|w^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\|u^\varepsilon\|_{L^2(0,T;L^6(\Omega_\varepsilon))}\|\mathcal{R}_\varepsilon\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))} \\ &\le C\varepsilon^{1+3(\frac{1}{6}-\frac{1}{2})}\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\mathcal{R}_\varepsilon\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))} \\ &\le C\varepsilon(\|\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))} + \varepsilon\|\nabla\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))}). \end{aligned}$$

To the third term, we have

$$\begin{aligned} \Big|\Big\langle \varepsilon^2 \frac{\partial u^\varepsilon}{\partial t}, \mathcal{R}_\varepsilon\varphi \Big\rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)}\Big| &\le \varepsilon^2\|u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\mathcal{R}_\varepsilon\varphi_t\|_{L^2(\Omega_\varepsilon\times(0,T))} \\ &\le C\varepsilon^4(\|\varphi_t\|_{L^2(\Omega_\varepsilon\times(0,T))} + \varepsilon\|\nabla\varphi_t\|_{L^2(\Omega_\varepsilon\times(0,T))}). \end{aligned}$$

By using Lemmas 2.2–2.3 and (3.2) again, we have

$$\begin{aligned} &|\langle (u^\varepsilon\cdot\nabla)u^\varepsilon, \mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)}| \\ &\le \|u^\varepsilon\|_{L^2(0,T;L^6(\Omega_\varepsilon))}\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\mathcal{R}_\varepsilon\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))} \\ &\le C\varepsilon^2(\|\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))} + \varepsilon\|\nabla\varphi\|_{L^\infty(0,T;L^3(\Omega_\varepsilon))}). \end{aligned}$$

The last term is estimated by

$$\begin{aligned} |\langle \mu\triangle u^\varepsilon, \mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)}| &= -\langle \mu\nabla u^\varepsilon, \nabla\mathcal{R}_\varepsilon\varphi \rangle_{\Omega_\varepsilon\times(0,T),\Omega_\varepsilon\times(0,t)} \\ &\le C\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\nabla\mathcal{R}_\varepsilon\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))} \\ &\le C(\|\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))} + \varepsilon\|\nabla\varphi\|_{L^2(\Omega_\varepsilon\times(0,T))}). \end{aligned}$$

If we choose the test function φ such that $\mathrm{div}\varphi = 0$, we have

$$\langle \mathcal{F}, \varphi\rangle_{\Omega\times(0,T),\Omega\times(0,T)} = 0.$$

Thus, $\mathcal{F}$, being orthogonal to divergence free functions, is the gradient of some functions ϑ. A result from [14] shows that up to a constant, we have

$$\vartheta = p^\varepsilon, \quad \text{then } \mathcal{F} = \nabla\vartheta = \nabla p^\varepsilon \quad \text{in } \Omega_\varepsilon.$$

At this moment, we can say that we have extended p^ε to the whole domain. We denote the extension function by P^ε. It remains to determine the expression of P^ε on the solid part. Suppressing the t dependence, following the steps in [15], we choose a smooth test function φ in (3.5), with compact support in one of the solid parts $\mathcal{Y}_s$, and we have

$$P^\varepsilon = \text{constant} \quad \text{in } \mathcal{Y}_s.$$

Next, we choose a smooth test function in (3.5), with compact support in the entire cell $\mathcal{Y}_f$. Integrating by parts, we have

$$P^\varepsilon = \widehat{p}^\varepsilon = \frac{1}{|\mathcal{Y}_f|}\int_{\mathcal{Y}_f} p^\varepsilon \mathrm{d}x \quad \text{in } \mathcal{Y}_s.$$

In fact, we have proved the following lemma.

Lemma 3.2 *The extension of p^ε, denoted by $\widehat{p}^\varepsilon$, has the form*

$$\widehat{p}^\varepsilon = \begin{cases} p^\varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{1}{|\mathcal{Y}_f|}\displaystyle\int_{\mathcal{Y}_f} p^\varepsilon \mathrm{d}x & \text{in } \mathcal{Y}_s. \end{cases} \tag{3.6}$$

Moreover,

$$\begin{aligned}\nabla\widehat{p}^\varepsilon \in\ & L^2(\Omega\times(0,T)) + \varepsilon L^1(0,T;L^{\frac{3}{2}}(\Omega)) + \varepsilon^2 L^1(0,T;W^{-1,\frac{3}{2}}(\Omega)) \\ & + \varepsilon L^2(0,T;H^{-1}(\Omega)) + \varepsilon^4 H^{-1}(0,T;L^2(\Omega)) + \varepsilon^5 H^{-1}(\Omega\times(0,T)).\end{aligned} \tag{3.7}$$

4 Proof of the Main Result

In this section, we focus on the proof of Theorem 2.2. It contains three parts:
(1) The convergence results in (2.3);
(2) Recover the system (2.4);
(3) Determine the initial and boundary conditions (2.5).

Proof of Theorem 2.2 Note that the velocity $\varepsilon^{-2}u^\varepsilon$ and the degree of freedom w^ε are both bounded in $L^2(\Omega\times(0,T))$. By using the standard compactness theorem, we can extract subsequences, still denoted by itself, such that

$$\begin{aligned} &\varepsilon^{-2}u^\varepsilon \rightharpoonup u \quad && \text{weakly in } L^2(\Omega\times(0,T)), \\ &w^\varepsilon \rightharpoonup w && \text{weakly in } L^2(\Omega\times(0,T)). \end{aligned}$$

Due to (3.7), we can decompose $\nabla\widehat{p}^\varepsilon$ as following:

$$\nabla\widehat{p}^\varepsilon = \nabla\widehat{p}_1^\varepsilon + \varepsilon\nabla\widehat{p}_2^\varepsilon + \varepsilon^2\nabla\widehat{p}_3^\varepsilon + \varepsilon\nabla\widehat{p}_4^\varepsilon + \varepsilon^4\nabla\widehat{p}_5^\varepsilon + \varepsilon^4\nabla\widehat{p}_6^\varepsilon,$$

where $\nabla\widehat{p}_i^\varepsilon,\ i=1,2,\cdots,6$ is bounded uniformly in the corresponding space respectively. As above, we assume that $\nabla\widehat{p}_1^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega\times(0,T))$ and $\nabla\widehat{p}_i^\varepsilon \rightharpoonup \nabla p_i$ weakly in the corresponding space respectively for $i=2,3,\cdots,6$. Let $\varphi\in C_0^\infty(\Omega\times(0,T))$. We have

$$\begin{aligned}&\int_0^T\int_\Omega \nabla\widehat{p}^\varepsilon\cdot\varphi \mathrm{d}x\mathrm{d}t\\ =&\int_0^T\int_\Omega (\nabla\widehat{p}_1^\varepsilon+\varepsilon\nabla\widehat{p}_2^\varepsilon+\varepsilon^2\nabla\widehat{p}_3^\varepsilon+\varepsilon\nabla\widehat{p}_4^\varepsilon+\varepsilon^4\nabla\widehat{p}_5^\varepsilon+\varepsilon^4\nabla\widehat{p}_6^\varepsilon)\cdot\varphi \mathrm{d}x\mathrm{d}t\\ \to&\int_0^T\int_\Omega \nabla p\cdot\varphi \mathrm{d}x\mathrm{d}t \quad \text{as } \varepsilon\to0^+,\end{aligned}$$

which implies

$$\nabla\widehat{p}^\varepsilon \rightharpoonup \nabla p \quad \text{weakly in} \ \ L^2(\Omega\times(0,T)).$$

Finally, by using the Nečas's inequality (see [16–17]), we obtain $p\in L^2(0,T;H^1(\Omega))$ and

$$\widehat{p}^\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

In the sequel, we derive the homogenized model of (1.1)–(1.3). Note that

$$\varepsilon^{-2}\mathrm{div}\, u^\varepsilon=0.$$

Let

$$\varphi\in C_0^\infty(\Omega\times(0,T)).$$

We have

$$\begin{aligned}0=&\int_0^T\int_\Omega \varepsilon^{-2}\mathrm{div}\, u^\varepsilon\ \varphi \mathrm{d}x\mathrm{d}t=-\int_0^T\int_\Omega \varepsilon^{-2}u^\varepsilon\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t\\ &\to-\int_0^T\int_\Omega u\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega \mathrm{div}\, u\ \varphi \mathrm{d}x\mathrm{d}t,\end{aligned}$$

which implies $\mathrm{div}\, u=0$ in $\Omega\times(0,T)$.

Let $\varphi\in C_0^\infty(\Omega\times(0,T))$. Taking $\omega_i^\varepsilon\varphi$ as a test function in the momentum equations, where ω_i^ε is defined in Lemma 2.4 and has been extended 0 on $\partial\Omega$, we have

$$\begin{aligned}&\int_0^T\int_\Omega \varepsilon^2\frac{\partial u^\varepsilon}{\partial t}\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega (u^\varepsilon\cdot\nabla)u^\varepsilon\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t-\mu\int_0^T\int_\Omega \triangle u^\varepsilon\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \nabla p^\varepsilon\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega f\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega w^\varepsilon\times u^\varepsilon\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now we compute the limit of each term in above equality,

$$\begin{aligned}\Big|\int_0^T\int_\Omega \varepsilon^2\frac{\partial u^\varepsilon}{\partial t}\cdot\omega_i^\varepsilon\varphi \mathrm{d}x\mathrm{d}t\Big|&=\Big|\varepsilon^2\int_0^T\int_\Omega u^\varepsilon\cdot\omega_i^\varepsilon\varphi_t \mathrm{d}x\mathrm{d}t\Big|\\ &\le C\varepsilon^2\|u^\varepsilon\|_{L^2(\Omega\times(0,T))}\to0 \quad \text{as } \varepsilon\to0^+.\end{aligned}$$

By Lemma 2.2 and Lemma 2.4, we have

$$
\begin{aligned}
&\Big|\int_0^T\int_\Omega (u^\varepsilon\cdot\nabla)u^\varepsilon\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t\Big|\\
=&\Big|-\int_0^T\int_\Omega u^\varepsilon\times u^\varepsilon:\nabla\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega u^\varepsilon\times u^\varepsilon:\omega_i^\varepsilon\otimes\nabla\varphi\mathrm{d}x\mathrm{d}t\Big|\\
\le& C\|\nabla\omega_i^\varepsilon\|_{L^2(\Omega\times(0,T))}\|u^\varepsilon\|^2_{L^2(0,T;L^4(\Omega))}+C\|\omega_i^\varepsilon\|_{L^2(\Omega\times(0,T))}\|u^\varepsilon\|^2_{L^2(0,T;L^4(\Omega))}\\
\le& C\varepsilon^{-1}\varepsilon^{2+6(\frac14-\frac12)}\|\nabla u^\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}+C\varepsilon^{2+6(\frac14-\frac12)}\|\nabla u^\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}\\
\le& C\varepsilon^{\frac32}\to 0\quad\text{as } \varepsilon\to 0^+.
\end{aligned}
$$

To the last term, using Lemma 2.2 and Lemma 2.4, we have

$$
\begin{aligned}
\Big|\int_0^T\int_\Omega w^\varepsilon\times u^\varepsilon\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t\Big|&\le C\|u^\varepsilon\|_{L^2(0,T;L^4(\Omega))}\|\omega_i^\varepsilon\|_{L^\infty(0,T;L^4(\Omega))}\\
&\le C\varepsilon^{\frac12}\to 0\quad\text{as } \varepsilon\to 0^+.
\end{aligned}
$$

It is obvious that

$$
\int_0^T\int_\Omega f\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t\to\int_0^T\int_\Omega \mathcal{A}f\cdot\varphi\mathrm{d}x\mathrm{d}t\quad\text{as } \varepsilon\to 0^+ \tag{4.1}
$$

and

$$
\begin{aligned}
\int_0^T\int_\Omega \nabla p^\varepsilon\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t&=-\int_0^T\int_\Omega p^\varepsilon\omega_i^\varepsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\
&\to-\int_0^T\int_\Omega \mathcal{A}p\nabla\cdot\varphi\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega \mathcal{A}\cdot\nabla p\varphi\mathrm{d}x\mathrm{d}t
\end{aligned}\tag{4.2}
$$

as ε tends to zero.

Finally, we consider the limit of $-\mu\int_0^T\int_\Omega\triangle u^\varepsilon\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t$. We have

$$
\begin{aligned}
&-\mu\int_0^T\int_\Omega \triangle u^\varepsilon\cdot\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t\\
=&\mu\int_0^T\int_\Omega\nabla u^\varepsilon:\nabla\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t+\mu\int_0^T\int_\Omega\nabla u^\varepsilon:\omega_i^\varepsilon\otimes\nabla\varphi\mathrm{d}x\mathrm{d}t\\
=&\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}
$$

Due to (3.2), we have

$$
|\mathrm{I}_2|\le C\varepsilon(\varepsilon^{-1}\|\nabla u^\varepsilon\|_{L^2(\Omega\times(0,T))})\|w_i^\varepsilon\|_{L^2(\Omega\times(0,T))}\to 0
$$

as ε tends to zero.

Integrating by parts in I_1, we have

$$
\begin{aligned}
\mathrm{I}_1&=-\mu\int_0^T\int_\Omega u^\varepsilon\cdot\triangle\omega_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t-\mu\int_0^T\int_\Omega u^\varepsilon\otimes\nabla\varphi:\nabla\omega_i^\varepsilon\mathrm{d}x\mathrm{d}t\\
&=\mathrm{I}_{11}+\mathrm{I}_{12}.
\end{aligned}
$$

Due to (3.3), we have

$$|\mathrm{I}_{12}| \leq C\varepsilon(\varepsilon^{-2}\|u^\varepsilon\|_{L^2(\Omega\times(0,T))})(\varepsilon\|\nabla\omega_i^\varepsilon\|_{L^2(\Omega\times(0,T))}) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

By Lemma 2.4, we write I_{11} in the following way:

$$\begin{aligned}
\mathrm{I}_{11} &= \mu\int_0^T\int_\Omega \varepsilon^{-2}u^\varepsilon\cdot(e_i-\varepsilon\nabla\pi_i^\varepsilon)\varphi\mathrm{d}x\mathrm{d}t \\
&= \mu\int_0^T\int_\Omega \varepsilon^{-2}u^\varepsilon\cdot e_i\varphi\mathrm{d}x\mathrm{d}t - \mu\int_0^T\int_\Omega \varepsilon^{-1}u^\varepsilon\cdot\nabla\pi_i^\varepsilon\varphi\mathrm{d}x\mathrm{d}t \\
&= \mathrm{I}_{111}+\mathrm{I}_{112}.
\end{aligned}$$

It is obvious

$$|\mathrm{I}_{112}| = \Big|\mu\int_0^T\int_\Omega \varepsilon^{-1}u^\varepsilon\ \pi_i^\varepsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\Big| \leq C\varepsilon \to 0 \quad \text{as } \varepsilon\to 0^+$$

and

$$\mathrm{I}_{111} \to \mu\int_0^T\int_\Omega u\cdot e_i\varphi\mathrm{d}x\mathrm{d}t. \tag{4.3}$$

Combining (4.1)–(4.3), we obtain

$$\mu u = \mathcal{A}(-\nabla p + f) \quad \text{in } \mathcal{D}'(\Omega\times(0,T)). \tag{4.4}$$

To pass the limit to (1.2), we take $\varphi \in C_0^\infty(\Omega\times[0,T))$ as a test function and we have

$$\begin{aligned}
&-\int_0^T\int_\Omega w^\varepsilon\varphi_t\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \mathrm{div}\,(u^\varepsilon w^\varepsilon)\varphi\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \kappa w^\varepsilon\varphi\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\int_\Omega m\varphi\mathrm{d}x\mathrm{d}t + \int_\Omega w_0^\varepsilon\varphi(x,0)\mathrm{d}x.
\end{aligned}$$

Note that

$$\begin{aligned}
\Big|\int_0^T\int_\Omega \mathrm{div}\,(u^\varepsilon w^\varepsilon)\varphi\mathrm{d}x\mathrm{d}t\Big| &= \Big|-\int_0^T\int_\Omega u^\varepsilon w^\varepsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\Big| \\
&\leq C\varepsilon^2 \to 0 \quad \text{as } \varepsilon\to 0^+.
\end{aligned}$$

Passing the limit, we obtain

$$-\int_0^T\int_\Omega w\varphi_t\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \kappa w\varphi\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega m\varphi\mathrm{d}x\mathrm{d}t + \int_\Omega w_0\varphi(x,0)\mathrm{d}x,$$

which implies that

$$\frac{\partial w}{\partial t} + \kappa w = m \quad \text{in } \mathcal{D}'(\Omega\times(0,T)) \tag{4.5}$$

and $w|_{t=0} = w_0(x)$.

The expression of the degree of freedom in (4.6) can be written as

$$w = w_0 \mathrm{e}^{-\kappa t} + \int_0^t m(x,\tau)\mathrm{e}^{-\kappa(t-\tau)}\mathrm{d}\tau, \tag{4.6}$$

where we assume that w vanishes on the boundary of Ω.

At last, using the fact that $\varepsilon^{-2}u^\varepsilon$ is bounded in $L^2(\Omega \times (0,T))$ and $\varepsilon^{-2}\frac{\partial u^\varepsilon}{\partial t}$ is bounded in $H^{-1}(0,T;L^2(\Omega))$, we conclude that $\varepsilon^{-2}u^\varepsilon$ is bounded in $C([0,T];L^2(\Omega))$. Then $\varepsilon^{-2}u^\varepsilon$ make sense at $t=0$. Passing the limit, we have $u|_{t=0} = u_0$. For any $t \in (0,T)$, $\varepsilon^{-1}u^\varepsilon$ is bounded in $H_0^1(\Omega)$. Then the trace of $\varepsilon^{-2}u^\varepsilon$ on $\partial\Omega$ makes sense. Passing the limit, we obtain $u|_{\partial\Omega} = 0$.

Collecting all the information above, Theorem 2.2 is then proved.

Acknowledgements We show our appreciations to the anonymous reviewers for their evaluating on this work. The authors also express their thanks to the editors for their selfless help.

References

[1] Sanchez-Palencia, E., Nonhomogeneous media and vibration theory, Lecture Notes in Physics, Vol. 127, Springer-Verlag, North Holland, Amsterdam, 1979.

[2] Keller, J. B., Darcy's law for flow in porous media and the two-space method, Nonlinear Partial Differential Equations in Engineering and Applied Science, Proc. Conf., Univ. Rhode Island, Kingston. RI, 1979, Lect. Notes Pure Appl. Math., Vol. 54, Dekker, New York, 1980, 429–443.

[3] Bensoussan, A., Lions, J. L. and Papanicolaou, G., Asymptotic analysis for periodic structures, Studies in Mathematics and Its Applications, Vol. 5, Co., Amsterdam, New York, North-Holland, 1978.

[4] Tartar, L., Incompressible fluid flow in a porous medium convergence of the homogenization process, Appendix to Lecture Notes in Physics, Vol. 127, Springer-Velag, Berlin, 1980.

[5] Allaire, G., Homogenization of the Stokes flow in a connected porous medium, *Asymptot. Anal.*, **2**, 1989, 203–222.

[6] Allaire, G., Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I-II, *Arch. Rat. Mech. Anal.*, **113**, 1991, 209–259, 261–298.

[7] Allaire, G., Homogenization of the Navier-Stokes equations with a slip boundary condition, *Comm. Pure. Appl. Math.*, **44**, 1991, 605–641.

[8] Mikelic, A., Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary, *Ann. Mat. Pura et. appl.*, **158**(4), 1991, 167–179.

[9] Mikelic, A. and Aganovic, I., Homogenization of stationary of miscible fluids in a domain boundary, *SIAM J. Math. Anal.*, **19**, 1988, 287–294.

[10] Masmoudi, N., Homogenization of the compressible Navier-Stokes equations in a porous medium, *ESIAM Control Optim. Calc. Var.*, **8**, 2002, 885–906.

[11] Zhao, H. and Zheng-An, Y., Homogenization of the time discretized compressible Navier-Stokes system, *Nonlinear Analysis*, **75**, 2012, 2486–2498.

[12] Brinkman, H. C., A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.*, **A1**, 1947, 27–34.

[13] Lions, P. L., Mathematical Topics in Fluid Mechanics, Vol. 1, The Clarendon Press Oxford University Press, New York, 1996.

[14] Lipton, R. and Avellanda, M., Darcy's law for slow viscous flow past a stationary array of bubbles, *Proc. Roy. Soc. Edinbergh Sect. A*, **114**(1–2), 1990, 71–79.

[15] Hornung, U., Homogenization and porous media, Interdisciplinary Applied Mathematics Series, Springer-Verlag, New York, 1997.

[16] Temam, R., Navier-Stokes Equations, North Holland, Amsterdam, 1979.

[17] Nečas, J., Sur les normes équivalentes dans $W_p^k(\Omega)$ et sur la coercivité des formes formellement positives, Séminaire Equations aux Dérivées partielles, les presses de l'Université de Montréal, 1966, 102–128.