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Revuz Measures, Energy Functionals and Capacities Under Girsanov Transform Induced by α-Excessive Function*

Ruili SONG[1]

Abstract The author investigates the relationships of some potential objects for a right Markov process and the same objects for the Girsanov transformed process induced by α-excessive function including Revuz measures, energy functionals, capacities and Lévy systems in this paper.

Keywords Girsanov transform, Revuz measure, Energy functional, Capacity, Lévy system
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1 Introduction and Preliminaries

In the theory of Markov processes, it is always interesting and important to explore various kinds of relationships between X and the transformed process of X under some kind of transform. Getoor and Steffens [1] studied the relationships among the energy functional, the balayage operators and the capacities for a Markov process and the same objects for its q-subprocesses and u-transform; Ying [2] studied the homogeneous multiplicative functionals of Lévy processes and gave their characterizations; Ying [3] studied the additive and multiplicative functionals of a right Markov process systematically in the setting of weak duality and proved that any subprocess of a nearly symmetric Markov process was also nearly symmetric and gave a generalized Feynman-Kac formula; Ying [4] proved that the killing transform of Markov processes was equivalent to strong subordination of the respective Dirichlet forms and gave a characterization of so-called bivariate smooth measures; Ying [5] studied the h-transform of symmetric Markov processes and corresponding Dirichlet spaces and discussed the drift transformation of Fukushima and Takeda's type; Ying [6] studied the relationships of Revuz measures, several formulas on the energy functional and capacity between any process and its subprocesses. Chen et al. [7] investigated the most general Girsanov transformation leading to another symmetric Markov process. Fukushima et al. [8] extended the classical Douglas integral by using the approach of time change of Markov processes. Song and Ying [9] established a representation formula for transition density function of a certain type of right

Manuscript received April 13, 2015. Revised September 2, 2015.
[1]School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210023, China.
E-mail: 022018031@fudan.edu.cn
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processes under Girsanov transform and gave the infinitesimal generator of the transformed process. He and Ying [10] studied how energy functionals and Revuz measures change under time change of Markov processes and provided an intuitive and direct approach of the Lévy system and jumping measure of time changed process; Song [11] studied how Revuz measures, energy functional, capacity and Lévy system change under some Girsanov transform of Hunt processes.

Ying [5] considered a drift transformation which was in the setting of symmetric right process. In this paper, we consider a Girsanov transformation of a right Markov process which is formally like the drift transformation in [5]. However, the right Markov process in our paper is not symmetric and the transformation is not the same. We will extend the excessive function in [11] to α-excessive function and generalize the results of [11]. In the following, we will give the specific explanations.

We start from a right Markov process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, P^x)_{t\in(0,+\infty)}$ on the state space $(E, \mathscr{E})$ with transition semigroup $(P_t)_{\{t>0\}}$ and resolvent $(U^q)_{\{q\geq 0\}}$. E is, at least, a separable Radon space and $\mathscr{E}$ is the Borel σ-field of E. A cemetery point Δ is adjoined to E as an isolated point of E and $E_\Delta := E \cup \{\Delta\}$, $\mathscr{E}_\Delta := \sigma(\mathscr{E} \cup \{\Delta\})$ (the symbol ':=' is always read as 'is defined to be'). Let $\zeta := \inf\{t : X_t = \Delta\}$ be the lifetime of X. The filtration $(\mathscr{F}, \mathscr{F}_t)$ is the augmented natural filtration of X. Denote by $\mathrm{Exc}^q(X)$ and $S^q(X)$ $(q \geq 0)$ the cones of q-excessive measures and q-excessive functions of X, respectively. As usual we write $S := S^0$, $\mathrm{Exc} := \mathrm{Exc}^0$.

For $\alpha > 0$, let $h \in S^\alpha(X)$ and $0 < h < \infty$. Then $\mathrm{e}^{-\alpha t} h(X_t)$ is a right continuous nonnegative $\mathscr{F}_t$-supermartingale. By Doob-Meyer decomposition, we have

$$\mathrm{e}^{-\alpha t} h(X_t) - h(X_0) = M_t^{[h]} - N_t^{[h]},$$

where $M_t^{[h]}$ is a local martingale, $N_t^{[h]}$ is a continuous increasing additive functional. Refer to [12] for the theory of additive functionals.

Let

$$Z_t^{[h]} := \int_0^t \frac{\mathrm{d}M_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_{s-})},$$

which is a martingale additive functional of X. Note that for $t < \zeta$, we have

$$\begin{aligned}\Delta Z_t^{[h]} &= \frac{1}{\mathrm{e}^{-\alpha t} h(X_{t-})}(M_t^{[h]} - M_{t-}^{[h]}) \\ &= \frac{1}{\mathrm{e}^{-\alpha t} h(X_{t-})}(\mathrm{e}^{-\alpha t} h(X_t) - \mathrm{e}^{-\alpha t} h(X_{t-})) \\ &= \frac{h(X_t)}{h(X_{t-})} - 1.\end{aligned}$$

Let $Z_t^{[h]} = Z_t^{[h],c} + Z_t^{[h],d}$ be the decomposition as continuous and purely discontinuous parts. Denote by L_t the Doléan-Dade exponential martingale of $Z_t^{[h]}$. Then it admits a representation as the following:

$$L_t^{[h]} = \exp\left(Z_t^{[h],c} - \frac{1}{2}\langle Z^{[h],c}\rangle_t\right) \mathrm{e}^{Z_t^{[h],d}} \prod_{s\leq t} \frac{h(X_s)}{h(X_{s-})} \mathrm{e}^{-\left(\frac{h(X_s)}{h(X_{s-})} - 1\right)} 1_{\{t<\zeta\}},$$

which is a martingale of X. Consequently, the formula

$$\frac{\mathrm{d}Q^x}{\mathrm{d}P^x}\Big|_{\mathscr{F}_t} = L_t^{[h]}$$

uniquely determines a family of probability measure on $(\Omega, \mathscr{F}_\infty)$. It is known that X is a right Markov process on E under these new measures (cf. [13, Section 62]). We will use $(Y, \mathscr{F}, \mathscr{F}_t, Q^x, x \in E)$ to denote the transformed process. Here $Y_t(\omega) = X_t(\omega)$ but we use Y_t for emphasis when working with Q^x. By Itô's formula, we have

$$L_t^{[h]} = \frac{\mathrm{e}^{-\alpha t} h(X_t)}{h(X_0)} \ \exp\Big(\int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)} \Big).$$

Let Q_t and V_t be the semigroup and resolvent of Y, respectively, that is, for any $f \in \mathscr{E}$, we have

$$\begin{aligned} Q_t f(x) &= P^x(f(X_t) L_t^{[h]}) = P^x \Big(f(X_t) \frac{\mathrm{e}^{-\alpha t} h(X_t)}{h(X_0)} \ \exp\Big(\int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)} \Big)\Big), \qquad (1.1) \\ V^q f(x) &= \int_0^\infty \mathrm{e}^{-qt} Q_t f(x) \mathrm{d}t \\ &= P^x \Big(\int_0^\infty \mathrm{e}^{-qt} f(X_t) \frac{\mathrm{e}^{-\alpha t} h(X_t)}{h(X_0)} \ \exp\Big(\int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)} \Big) \mathrm{d}t \Big) \quad (q \geq 0). \end{aligned}$$

The process Y is called the Girsanov transform of X by $(L_t)_{\{t>0\}}$ or induced by α-excessive function h.

In this article, we will get some potential objects under the transform, such as Revuz measures, energy functionals, capacities and Lévy systems.

2 Revuz Measures, Energy Functionals, Capacities

In this section, we will get the relationships of some potential objects between the process X and the transformed process Y which is defined in Section 1. In the following, we denote by $S^q(Y)$ and $\mathrm{Exc}^q(Y)$ $(q \geq 0)$ the cones of q-excessive measures and q-excessive functions of Y, respectively. Particularly, denote by $S(Y)$ and $\mathrm{Exc}(Y)$ the cones of excessive measures and excessive functions of Y, respectively.

Lemma 2.1 *For any $q \geq 0$, if $u \in S^q(Y)$, then $hu \in S^{q+\alpha}(X)$. Particularly, if $u \in S(Y)$, then $hu \in S^\alpha(X)$.*

Proof By the definition of q-excessive function, we have

$$u \in S^q(Y) \Leftrightarrow \mathrm{e}^{-qt} Q_t u(x) \leq u(x), \quad \lim_{t\to 0} \mathrm{e}^{-qt} Q_t u(x) = u(x).$$

By (1.1), we have

$$P_t f(x) = Q^x \Big(f(X_t) \frac{h(X_0)}{\mathrm{e}^{-\alpha t} h(X_t)} \ \exp\Big(-\int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)} \Big)\Big).$$

Thus it follows that

$$\begin{aligned}
&\mathrm{e}^{-(q+\alpha)t}P_t(hu)(x)\\
&= P^x(\mathrm{e}^{-qt}\mathrm{e}^{-\alpha t}h(X_t)u(X_t))\\
&= Q^x\Big(\mathrm{e}^{-qt}\mathrm{e}^{-\alpha t}h(X_t)u(X_t)\frac{h(X_0)}{\mathrm{e}^{-\alpha t}h(X_t)}\ \exp\Big(-\int_0^t\frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s}h(X_s)}\Big)\Big)\\
&= h(x)\mathrm{e}^{-qt}Q^x\Big(u(X_t)\ \exp\Big(-\int_0^t\frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s}h(X_s)}\Big)\Big)\\
&\le h(x)\mathrm{e}^{-qt}Q_tu(x)\\
&\le h(x)u(x)
\end{aligned}$$

and

$$\lim_{t\to 0}\mathrm{e}^{-(q+\alpha)t}P_t(hu)(x) = h(x)\lim_{t\to 0}\mathrm{e}^{-qt}Q^x(u(X_t)\ \mathrm{e}^{-\int_0^t\frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s}h(X_s)}}) = h(x)u(x).$$

Then also by the definition of q-excessive function, we have $hu\in S^q(X)$.

Lemma 2.2 *For any $q\ge 0$, if $\mu\in \mathrm{Exc}^q(Y)$, then $\frac{1}{h}\mu\in\mathrm{Exc}^{q+\alpha}(X)$. Particularly, if $\mu\in\mathrm{Exc}(Y)$, then $\frac{1}{h}\mu\in\mathrm{Exc}^{\alpha}(X)$.*

Proof We assume that $f\in p\mathscr{E}$ (positive measurable function with respect to $\mathscr{E}$) in the following. By the definition of q-excessive measure, we have $\mu\in\mathrm{Exc}^q(Y)$ if and only if $\mu\mathrm{e}^{-qt}Q_t\le\mu$.

Thus it follows that

$$\begin{aligned}
&\frac{1}{h}\mu\mathrm{e}^{-(q+\alpha)t}P_tf(x)\\
&=\int_E\mathrm{e}^{-(q+\alpha)t}P^x(f(X_t))\frac{1}{h(x)}\mu(\mathrm{d}x)\\
&=\int_E\mathrm{e}^{-qt}\frac{1}{h(x)}Q^x\Big(\mathrm{e}^{-\alpha t}f(X_t)\frac{h(X_0)}{\mathrm{e}^{-\alpha t}h(X_t)}\exp\Big(-\int_0^t\frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s}h(X_s)}\Big)\Big)\mu(\mathrm{d}x)\\
&\le\int_E\mathrm{e}^{-qt}Q^x\Big(\frac{f(X_t)}{h(X_t)}\Big)\mu(\mathrm{d}x)\\
&\le\mu\Big(\frac{f}{h}\Big)\\
&=\frac{1}{h}\mu(f).
\end{aligned}$$

Thus we get the desired result.

The above two lemmas give the relationships of excessive functions and excessive measures of X and Y.

Revuz measures were first introduced for ordinary additive functionals in [14]. Let $m\in\mathrm{Exc}(X)$ and $A\in\mathrm{RAF}$ (raw additive functional). Then the Revuz measure of A relative to m is defined by

$$\rho_A^{X,m}(f):=\uparrow\lim_{t\downarrow 0}\frac{1}{t}P^m\Big(\int_0^t f(X_s)\mathrm{d}A_s\Big),$$

where $\uparrow$ ($\downarrow$) means increasing (decreasing). Let X^α denote the α-subprocess of X, that is, the semigroup of X^α is defined by $P_t^\alpha := \mathrm{e}^{-\alpha t}P_t$. We have

$$\mathrm{Exc}(X) \subset \mathrm{Exc}^\alpha(X) = \mathrm{Exc}(X^\alpha).$$

Let ρ^X, ρ^{X^α} and ρ^Y denote the Revuz measures of X, X^α and Y, respectively.

Theorem 2.1 *Let $\mu \in \mathrm{Exc}(Y)$ and A be an increasing continuous additive functional which is finite on $[0,\zeta[$. Then*

$$\rho_A^{Y,\mu} = h \cdot \rho_A^{X^\alpha,\frac{1}{h}\mu}.$$

Particularly, if $\frac{1}{h}\mu \in \mathrm{Exc}(X)$, then

$$\rho_A^{Y,\mu} = h \cdot \rho_A^{X\frac{1}{h}\mu}.$$

Proof If $A_t = \int_0^t a(X_s)\mathrm{d}s$, where a is a bounded positive Borel function, then for any $\mu \in \mathrm{Exc}(X)$, we have $\rho_A^{X,\mu} = a\cdot\mu$. Since $\mu \in \mathrm{Exc}(Y)$, by Lemma 2.2, we have $\frac{1}{h}\mu \in \mathrm{Exc}(X^\alpha)$. Thus

$$\rho_A^{Y,\mu} = a\cdot\mu, \quad \rho_A^{X^\alpha,\frac{1}{h}\mu} = a\cdot\frac{1}{h}\mu.$$

Then

$$\rho_A^{Y,\mu} = h \cdot \rho_A^{X^\alpha,\frac{1}{h}\mu}.$$

For a general A which is an increasing continuous additive functional, define a strictly increasing continuous additive functional H by $H_t := A_t + t\wedge\zeta$, then

$$\rho_H^{Y,\mu} = \rho_A^{Y,\mu} + \mu := \widetilde{\mu}, \quad \rho_H^{X^\alpha,\frac{1}{h}\mu} = \rho_A^{X^\alpha,\frac{1}{h}\mu} + \frac{1}{h}\mu := \widetilde{\frac{1}{h}}\mu.$$

By time-change, let

$$\widetilde{X}_t^\alpha := X^\alpha_{H_t^{-1}}, \quad \widetilde{Y}_t := Y_{H_t^{-1}}, \quad \widetilde{A_t} := A_{H_t^{-1}}.$$

Since $A_t \ll H_t$, then $\widetilde{A_t} \ll \mathrm{d}t$. By Motoo's theorem (cf. [13, (66.2)]), there is a bounded positive Borel function $\widetilde{a}$, such that $\widetilde{A_t} = \int_0^t \widetilde{a}(X_s)\mathrm{d}s$ for all $t \geq 0$ a.s. $P^{\frac{1}{h}\mu}$. By the discussion in the first part of the proof, we have

$$\begin{aligned}
&\rho_{\widetilde{A_t}}^{\widetilde{X}^\alpha,\widetilde{\frac{1}{h}}\mu} = \widetilde{a}\widetilde{\frac{1}{h}}\mu = \widetilde{a}\Big(\rho_A^{X^\alpha,\frac{1}{h}\mu} + \frac{1}{h}\mu\Big),\\
&\rho_{\widetilde{A_t}}^{\widetilde{Y},\widetilde{\mu}} = \widetilde{a}\widetilde{\mu} = \widetilde{a}(\rho_A^{Y,\mu} + \mu).
\end{aligned}$$

And by time-change (cf. [10]), we have

$$\rho_A^{X^\alpha,\frac{1}{h}\mu} = \rho_{\widetilde{A_t}}^{\widetilde{X}^\alpha,\widetilde{\frac{1}{h}}\mu}, \quad \rho_A^{Y,\mu} = \widetilde{\rho}_{\widetilde{A_t}}^{\widetilde{Y},\widetilde{\mu}}.$$

Thus it follows that

$$\rho_A^{Y,\mu} = h \cdot \rho_A^{X^\alpha,\frac{1}{h}\mu}.$$

If $\frac{1}{h}\mu \in \mathrm{Exc}(X)$, by [15, (8.10)], we have

$$\rho_A^{X^\alpha,\frac{1}{h}\mu} = \rho_A^{X,\frac{1}{h}\mu},$$

then

$$\rho_A^{Y,\mu} = h \cdot \rho_A^{X,\frac{1}{h}\mu}.$$

This theorem gives the relationship of Revuz measures between X and Y. In the proof, we use the method in [16]. We see that time-change plays an important role in the above proof.

The energy functional L (of X) is defined on $\mathrm{Exc}(X) \times S(X)$ by

$$L^X(m,u) = \sup\{\mu(u) : \mu U \le m\}.$$

Refer to [15] for the basic properties of energy functional L.

Let $L^X,\ L^\alpha$ and L^Y denote the energy functional of $X,\ X^\alpha$ and Y, respectively.

Before we get our results, we will explain our ideas as follows. Since $h \in \mathrm{Exc}^\alpha(X)$ and $\mathrm{Exc}^\alpha(X) = \mathrm{Exc}(X^\alpha)$, then $h \in \mathrm{Exc}(X^\alpha)$. Since $0 < h < \infty$, let $X^{\alpha,h}$ denote the h-transform of X^α, $P_t^{\alpha,h}$ denote the semigroup of $X^{\alpha,h}$, then for $t > 0$,

$$P_t^{\alpha,h} f(x) = P^x\Big(\mathrm{e}^{-\alpha t} f(X_t)\frac{h(X_t)}{h(X_0)}\Big).$$

From (1.1), we see that the process $X^{\alpha,h}$ can be seen as the killing process of Y by the decreasing multiplication functional M, where

$$M := \exp\Big(-\int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)}\Big).$$

We will use the process $X^{\alpha,h}$ as a bridge to get the relationships of energy functional and capacity between the process X and the process Y. We will get the relationship between $L^\alpha(L^X)$ and L^Y in the following Theorem 2.2.

Theorem 2.2 *Let* $\mu \in \mathrm{Exc}(Y)$, $u \in S(Y)$, *then*

$$L^\alpha\Big(\frac{1}{h}\mu, hu\Big) = L^Y(\mu,u) + \rho_{N^{[h]}}^{X^\alpha,\frac{1}{h}\mu}(u).$$

Particularly, if $\frac{1}{h}\mu \in \mathrm{Exc}(X)$, $hu \in S(X)$, *then*

$$L^X\Big(\frac{1}{h}\mu, hu\Big) + \alpha\mu(u) = L^Y(\mu,u) + \rho_{N^{[h]}}^{X,\frac{1}{h}\mu}(u).$$

Proof By Lemmas 2.1–2.2, we have

$$\frac{1}{h}\mu \in \mathrm{Exc}(X^\alpha), \quad hu \in S(X^\alpha).$$

Due to [1, (4.8)], we have

$$L^{X^{\alpha,h}}(\mu,u) = L^\alpha\Big(\frac{1}{h}\mu, hu\Big). \tag{2.1}$$

The process $X^{\alpha,h}$ can be seen as the M-subprocess of Y, where

$$M = \exp\Big(- \int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)} \Big).$$

Due to Theorem 3.3 in [6], we have

$$L^{X^{\alpha,h}}(\mu, u) = L^Y(\mu, u) + \rho_M^{Y,\mu}(u). \tag{2.2}$$

By (2.1)–(2.2), we have

$$L^\alpha\Big(\frac{1}{h}\mu, hu\Big) = L^Y(\mu, u) + \rho_M^{Y,\mu}(u). \tag{2.3}$$

By Theorem 2.1, we have

$$\rho_M^{Y,\mu} = h \cdot \rho_M^{X^\alpha, \frac{1}{h}\mu}. \tag{2.4}$$

The Stieltjes logarithm of M is

$$[M]_t := \int_0^t \frac{\mathrm{d}(-M_s)}{M_s} = \int_0^t \frac{\mathrm{d}N_s^{[h]}}{\mathrm{e}^{-\alpha s} h(X_s)}.$$

Due to [6, Corollary 2.8], we have

$$\rho_M^{X^\alpha, \frac{1}{h}\mu} = \rho_{[M]}^{X^\alpha, \frac{1}{h}\mu} = \frac{1}{h} \cdot \rho_{N^{[h]}}^{X^\alpha, \frac{1}{h}\mu}. \tag{2.5}$$

Using (2.3)–(2.5), it follows that

$$L^\alpha\Big(\frac{1}{h}\mu, hu\Big) = L^Y(\mu, u) + \rho_{N^{[h]}}^{X^\alpha, \frac{1}{h}\mu}(u).$$

If $\frac{1}{h}\mu \in \mathrm{Exc}(X),\ hu \in S(X)$, since

$$L^\alpha\Big(\frac{1}{h}\mu, hu\Big) = L^X\Big(\frac{1}{h}\mu, hu\Big) + \alpha\mu(u)$$

and

$$\rho_{N^{[h]}}^{X^\alpha, \frac{1}{h}\mu} = \rho_{N^{[h]}}^{X, \frac{1}{h}\mu},$$

then we have

$$L^X\Big(\frac{1}{h}\mu, hu\Big) + \alpha\mu(u) = L^Y\Big(\mu, u\Big) + \rho_{N^{[h]}}^{X, \frac{1}{h}\mu}(u).$$

The Hunt's balayage operation R_T^q is defined on $\mathrm{Exc}^q (q > 0)$. That is, for any terminal time T,

$$R_T^q\mu(f) := L^q(\mu, P_T^q U^q f) \quad \text{for } \mu \in \mathrm{Exc}^q(X).$$

For $\mu \in \mathrm{Exc}(X)$,

$$R_T\mu := \uparrow \lim_{q \downarrow 0} R_T^q \mu.$$

Let $\mathscr{E}^e := \sigma\big(\bigcup_{q \geq 0} S^q \big)$. If $B \in \mathscr{E}^e$ and $\mu \in \mathrm{Exc}^q$ $(q \geq 0)$, we shall write $R_B^q\mu$ in place of $R_{T_B}^q\mu$.

For $B \in \mathscr{E}^e$ and $q \geq 0$, the q-capacity of B with $m \in \mathrm{Exc}(X)$ is defined by

$$\Gamma^q_m(B) := L^q(m, P^q_B 1) = L^q(R^q_B m, 1),$$
$$\Gamma_m(B) := \Gamma^0(B).$$

That is

$$\Gamma_m(B) = L(m, P_B 1) = L(R_B m, 1).$$

An exact terminal time T is called strict if $T \circ \theta_T = 0$ a.s. In the case that $T = T_B$, the hitting time of $B \in \mathscr{E}^e$, T is strict if and only if $X_T \in B^r$, where B^r is the set of regular points of B. The set B is called strict if T_B is. Let Γ^X and Γ^Y denote the capacity of X and Y, respectively.

Theorem 2.3 *Let $B \in \mathscr{E}$ and be strict, $\mu \in \mathrm{Exc}(Y)$. Then*

$$\Gamma^Y_\mu(B) + \frac{1}{h}\rho^{X^\alpha, \frac{1}{h}R_B\mu}_{N^{[h]}}(P^\alpha_B h) = L^\alpha\Big(\frac{1}{h}\mu, P^\alpha_B h\Big) = L^\alpha\Big(R^\alpha_B\Big(\frac{1}{h}\mu\Big), h\Big). \tag{2.6}$$

Particularly, if $\frac{1}{h}\mu, \frac{1}{h}R_B\mu \in \mathrm{Exc}(X)$ and $P^\alpha_B h \in S(X)$, then

$$\Gamma^Y_\mu(B) + \frac{1}{h}\rho^{X, \frac{1}{h}R_B\mu}_{N^{[h]}}(P^\alpha_B h) = L^X\Big(\frac{1}{h}\mu, P^\alpha_B h\Big) + \frac{\alpha}{h}\mu(P^\alpha_B h). \tag{2.7}$$

Proof Since $\mu \in \mathrm{Exc}(Y)$, $h \in S(X^\alpha)$, then $R_B\mu \in \mathrm{Exc}(Y)$, $P^\alpha_B h \in S(X^\alpha)$. By Lemma 2.2, we have $\frac{1}{h}\mu, \frac{1}{h}R_B\mu \in \mathrm{Exc}(X^\alpha)$.

By the definition of capacity, we have

$$\Gamma^{X^{\alpha,h}}_\mu(B) = L^{X^{\alpha,h}}(\mu, P^{\alpha,h}_B 1) = L^\alpha\Big(\frac{1}{h}\mu, hP^{\alpha,h}_B 1\Big) = L^\alpha\Big(\frac{1}{h}\mu, P^\alpha_B h\Big) = L^\alpha\Big(R^\alpha_B\Big(\frac{1}{h}\mu\Big), h\Big). \tag{2.8}$$

In the above reasoning, we use (2.1) to get the second equality and use [1, (5.4)(i)] to get the third equality.

Since the process $X^{\alpha,h}$ is the M-subprocess of Y, and by using [6, (4.11)], we have

$$\Gamma^{X^{\alpha,h}}_\mu(B) = \Gamma^Y_\mu(B) + \rho^{Y, R_B\mu}_M(Q^M_B 1). \tag{2.9}$$

By (2.4)–(2.5), we get

$$\rho^{Y, R_B\mu}_M = \rho^{X^\alpha, \frac{1}{h}R_B\mu}_{N^{[h]}} \tag{2.10}$$

and

$$Q^M_B 1 = \frac{1}{h}P^\alpha_B h. \tag{2.11}$$

By (2.8)–(2.11), we have the result (2.6).

Particularly, if $\frac{1}{h}\mu, \frac{1}{h}R_B\mu \in \mathrm{Exc}(X)$ and $P^\alpha_B h \in S(X)$, we have

$$\rho^{X^\alpha, \frac{1}{h}R_B\mu}_{N^{[h]}}(P^\alpha_B h) = \rho^{X, \frac{1}{h}R_B\mu}_{N^{[h]}}(P^\alpha_B h), \tag{2.12}$$
$$L^\alpha\Big(\frac{1}{h}\mu, P^\alpha_B h\Big) = L^X\Big(\frac{1}{h}\mu, P^\alpha_B h\Big) + \frac{\alpha}{h}\mu(P^\alpha_B h). \tag{2.13}$$

By (2.6) and (2.12)–(2.13), we have the result (2.7).

3 Lévy System

A Lévy system for X is a pair (N,H), where N is a kernel on $(E,\mathscr{E}^u)$ (where $\mathscr{E}^u$ is the σ-algebra of universally measurable subsets of E) with $N(x,\{x\})=0$ for any $x\in E$ and H is a continuous additive functional of X having bounded 1-potential, such that for any $F\in p\mathscr{E}^u\times\mathscr{E}^u$ vanishing on the diagonal and any predictable process Z, we have

$$P^x\sum_{0<s\le t}Z_sF(X_{s-},X_s)=P^x\int_0^t Z_sNF(X_s)\mathrm{d}H_s,$$

where

$$NF(x):=\int F(x,y)N(x,\mathrm{d}y).$$

Refer to [13, §73] for the existence of Lévy systems. Now we state the relationship of Lévy system of X and Y.

Theorem 3.1 *If (N,H) is a Lévy system of X, then (N',H) is a Lévy system of Y, where*

$$N'(x,\mathrm{d}y):=N(x,\mathrm{d}y)\frac{h(y)}{h(x)}.$$

Proof By the definition of Lévy system, for any $F\in p\mathscr{E}^u\times\mathscr{E}^u$, we have

$$\begin{aligned}
&Q^x\sum_{0<s\le t}Z_sF(X_{s-},X_s)\\
&=P^x\sum_{0<s\le t}Z_sF(X_{s-},X_s)L_s\\
&=P^x\sum_{0<s\le t}Z_sF(X_{s-},X_s)\mathrm{e}^{-\alpha s}\frac{h(X_s)}{h(X_0)}\mathrm{e}^{\int_0^s\frac{\mathrm{d}N_r^{[h]}}{\mathrm{e}^{-\alpha r}h(X_r)}}\\
&=\frac{1}{h(x)}P^x\sum_{0<s\le t}\mathrm{e}^{-\alpha s}Z_sh(X_{s-})F(X_{s-},X_s)\frac{h(X_s)}{h(X_{s-})}\mathrm{e}^{\int_0^s\frac{\mathrm{d}N_r^{[h]}}{\mathrm{e}^{-\alpha r}h(X_r)}}\\
&=\frac{1}{h(x)}P^x\int_0^t\mathrm{e}^{-\alpha s}Z_sh(X_{s-})\mathrm{e}^{\int_0^s\frac{\mathrm{d}N_r^{[h]}}{\mathrm{e}^{-\alpha r}h(X_r)}}N'F(X_s)\mathrm{d}H_s\\
&=P^x\int_0^t\mathrm{e}^{-\alpha s}Z_s\frac{h(X_s)}{h(X_0)}\mathrm{e}^{\int_0^s\frac{\mathrm{d}N_r^{[h]}}{h(X_r)}}N'F(X_s)\mathrm{d}H_s\\
&=Q^x\int_0^tZ_sN'F(X_s)\mathrm{d}H_s.
\end{aligned}$$

The fifth equation is due to the continuity of H. Then also by the definition of Lévy system, we have that (N',H) is a Lévy system of the process Y.

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