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Modular Invariants and Singularity Indices of Hyperelliptic Fibrations

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Abstract The modular invariants of a family of curves are the degrees of the pullback of the corresponding divisors by the moduli map. The singularity indices were introduced by Xiao (1991) to classify singular fibers of hyperelliptic fibrations and to compute global invariants locally. In semistable case, the author shows that the modular invariants corresponding to the boundary divisor classes are just the singularity indices. As an application, the author shows that the formula of Xiao for relative Chern numbers is the same as that of Cornalba-Harris in semistable case.

Keywords Modular invariants, Singularity indices, Moduli space of curves
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1 Introduction

The modular invariants of a family of curves were introduced by Tan [10]. They are the degrees of the pullback of the corresponding divisors by the moduli map. In the language of arithmetic algebraic geometry, a modular invariant is a certain height of arithmetic curves, for example, Faltings height is the modular invariant corresponding to Hodge class. Modular invariants can be used to describe the lower bound for effective Bogomolov conjecture which is about the finiteness of algebraic points of small height (see [15–16]). More recently, Tan found that the modular invariants are invariants of differential equations, which were expected by mathematicians in 19th century to study the qualitative properties of differential equations (see [11]).

Historically, the study of fibred surfaces is started by Kodaira [6], who gave a complete classification theory for elliptic fibrations. This combinatoric classification of elliptic fibers is used in the computation of the modular invariants. But such a classification is too complicate for the case when the genus $g \geq 2$. There are more than one hundred classes of singular fibers of genus 2 (see [8–9]), and the number of classes of singular fibers increases quickly as the genus becomes bigger. Horikawa [5] classified the singular fibers of genus $g = 2$ into 5 classes from a different point of view. Based on Horikawa's work, Xiao [13–14] introduced the singularity indices (see Definition 2.6) to classify singular fibers of hyperelliptic fibrations. Furthermore, he obtained the local-global formulas, and determined the fundamental group from his classification.

In what follows, we will prove that the two basic invariants, the modular invariants corresponding to boundary divisor classes and the singularity indices, coincide with each other for semistable fibrations.

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Before starting this result, we explain our notations and assumptions.

A family of curves of genus g is a fibration $f : S \to C$ whose general fiber F is a smooth curve of genus g, where S is a complex smooth projective surface, and C is a smooth curve of genus b. The family is called semistable if all the singular fibers are semistable curves (recall that a semistable curve F is a reduced connected curve that has only nodes as singularities and every smooth rational component of F meets the other components at no less than 2 points). If all the smooth fibers are hyperelliptic, we say that the family is hyperelliptic. We always assume that f is relatively minimal, i.e., there is no (-1)-curve in any singular fiber.

If r is a non-negative real number, we denote by $[r]$ the integral part of r. Hence when m is a positive integer, $m - 2\left[\frac{m}{2}\right]$ is zero if m is even, or 1 otherwise.

For a fibration $f : S \to C$, we have three fundamental relative invariants which are non-negative,

$$\begin{aligned} K_f^2 &= K^2_{\frac{S}{C}} = K_S^2 - 8(g-1)(b-1), \\ e_f &= \chi_{\rm top}(S) - 4(g-1)(b-1), \\ \chi_f &= \deg f_*\omega_{\frac{S}{C}} = \chi(\mathcal{O}_S) - (g-1)(b-1). \end{aligned} \tag{1.1}$$

Let f be a locally non-trivial fibration, the slope of f is defined as $\lambda_f = \frac{K_f^2}{\chi_f}$.

For $g \geq 2$, the moduli map $J : C \to \overline{\mathcal{M}}_g$ induced by f is a holomorphic map from C to the moduli space $\overline{\mathcal{M}}_g$ of semistable curves of genus g. For each $\mathbb{Q}$-divisor class η of $\overline{\mathcal{M}}_g$, we can define an invariant $\eta(f) = \deg J^*\eta$ which satisfies the base change property, i.e., if $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ is the pullback fibration of f under a base change $\pi : \widetilde{C} \to C$ of degree d, then $\eta(\widetilde{f}) = d \cdot \eta(f)$ (see [10]). Consequently, for a non-semistable family f, we have $\eta(f) = \frac{\eta(\widetilde{f})}{d}$, where $\widetilde{f}$ is the semistable model of f corresponding to a base change of degree d. We call $\eta(f)$ the modular invariant of f corresponding to η.

Let $\Delta_0, \cdots, \Delta_{\left[\frac{g}{2}\right]}$ be the boundary divisors of $\overline{\mathcal{M}}_g$, and $\delta_i(f)$ be the modular invariant corresponding to the divisor class $\delta_i = [\Delta_i]$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$, $i = 0, 1, \cdots, \left[\frac{g}{2}\right]$. Let $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ be the Hodge class, $\delta = \delta_0 + \cdots + \delta_{\left[\frac{g}{2}\right]}$, and $\kappa = 12\lambda - \delta$. For these classes, we have modular invariants $\lambda(f)$, $\delta(f)$ and $\kappa(f)$ of f. If f is semistable, then

$$\lambda(f) = \chi_f, \quad \delta(f) = e_f, \quad \kappa(f) = K_f^2. \tag{1.2}$$

We say that a singularity p in a semistable curve F is a node of type i if its partial normalization at p consists of two connected components of arithmetic genera i and $g - i \geq i$, for $i > 0$, and is connected for $i = 0$. The node of the semistable curve corresponding to a general point of Δ_0 is α-type, i.e., an ordinary double point of an irreducible curve, hence it is a node of type 0. For a general point in Δ_i, the corresponding node is of type i $(i \geq 1)$ (see Figure 1). Denote by $\delta_i(F)$ the number of nodes of type i $(i \geq 0)$ in F.

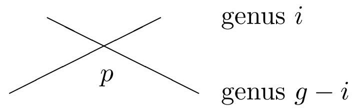

Figure 1 Node of type i $(i \geq 1)$

The general point in the intersection $\Delta_{i_1} \cap \cdots \cap \Delta_{i_k}$ of k distinct boundary divisors corresponds to a semistable curve with k nodes which are of types $i_1, \cdots, i_k$ respectively.

Let $\overline{\mathcal{H}}_g$ be the moduli space of semistable hyperelliptic curves, the restriction of Δ_0 on $\overline{\mathcal{H}}_g$ breaks up into $\Xi_0, \Xi_1, \cdots, \Xi_{\left[\frac{g-1}{2}\right]}$. We denote by Θ_i the restriction of Δ_i $(i \geq 1)$ on $\overline{\mathcal{H}}_g$. Suppose that F is a semistable hyperelliptic curve with hyperelliptic involution σ, and $p \in F$ is a node of type 0. If $p = \sigma(p)$, then we set $k = 0$; if $p \neq \sigma(p)$, and the partial normalization of F at p and $\sigma(p)$ consists of two connected components of arithmetic genera k and $g-k-1 \geq k$, then the node p (resp. nodal pair $\{p, \sigma(p)\}$) is called a node (resp. nodal pair) of type $(0,k)$. Then the nodal pair of the semistable curve corresponding to a general point of Ξ_k is of type $(0,k)$ (see Figure 2).

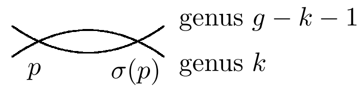

Figure 2 Nodes of type $(0,k)$ $(k \geq 0)$

A semistable hyperelliptic curve is a double cover of a tree of rational curves branched over $2g+2$ points (see [1, X.3]), which is induced by the involution map. Since the points p and $\sigma(p)$ map to the same point in some $\mathbb{P}^1$, we treat them together as a nodal pair $\{p, \sigma(p)\}$.

Let

$$\begin{aligned}
&\mathcal{N}_{2,1}(F) = \{p \in F: \ p \text{ is a node of type } (0,0), \ p = \sigma(p)\},\\
&\mathcal{N}_{2,2}(F) = \{\{p, \sigma(p)\} \subset F: \ \{p, \sigma(p)\} \text{ is a nodal pair of type } (0,0), \ p \neq \sigma(p)\}.
\end{aligned}$$

Denote by $\mathcal{N}_{2k+2}(F)$ (resp. $\mathcal{N}_{2k+1}(F)$) the set of all the nodal pairs $\{p, \sigma(p)\}$ of type $(0,k)$ (resp. nodes p of type k) $(k > 0)$. Then we define

$$\begin{aligned}
&\xi_0(F) := |\mathcal{N}_{2,1}(F)| + 2|\mathcal{N}_{2,2}(F)|,\\
&\xi_k(F) := |\mathcal{N}_{2k+2}(F)|, \quad \delta_k(F) := |\mathcal{N}_{2k+1}(F)|, \ k \geq 1.
\end{aligned} \tag{1.3}$$

From now on, we assume that f is hyperelliptic and semistable. Let $\delta_k(f)$ (resp. $\xi_k(f)$) be the modular invariants corresponding to Θ_k (resp. Ξ_k). Then (see [4])

$$\delta_k(f) = \sum_{i=1}^{s} \delta_k(F_i) \quad (k \geq 1), \quad \xi_k(f) = \sum_{i=1}^{s} \xi_k(F_i) \ (k \geq 0), \tag{1.4}$$

where $F_1, \cdots, F_s$ are all singular fibers of f, and

$$\delta_0(f) = \xi_0(f) + \sum_{k \geq 1} 2\xi_k(f). \tag{1.5}$$

It is proved that in [4], if f is semistable, then

$$\begin{aligned}
&(8g+4)\lambda(f) = g\xi_0(f) + \sum_{k=1}^{\left[\frac{g-1}{2}\right]} 2(k+1)(g-k)\xi_k(f) + \sum_{k=1}^{\left[\frac{g}{2}\right]} 4k(g-k)\delta_k(f),\\
&\delta(f) = \xi_0(f) + \sum_{k=1}^{\left[\frac{g-1}{2}\right]} 2\xi_k(f) + \sum_{k=1}^{\left[\frac{g}{2}\right]} \delta_k(f).
\end{aligned} \tag{1.6}$$

On the other hand, for a hyperelliptic fibration $f: S \to C$, Xiao introduced the singularity indices $s_2(f), s_3(f), \cdots, s_{g+2}(f)$ (see Definition 2.6), and he obtained the following formulas (see Theorem 2.1):

$$
\begin{aligned}
(8g+4)\chi_f &= g(s_2(f) - 2s_{g+2}(f)) + \sum_{k=2}^{\left[\frac{g-1}{2}\right]} 2(k+1)(g-k)s_{2k+2}(f) \\
&\quad + \sum_{k=1}^{\left[\frac{g+1}{2}\right]} 4k(g-k)s_{2k+1}(f), \\
e_f &= s_2(f) - 3s_{g+2}(f) + \sum_{k=1}^{\left[\frac{g-1}{2}\right]} 2s_{2k+2}(f) + \sum_{k=1}^{\left[\frac{g+1}{2}\right]} s_{2k+1}(f).
\end{aligned}
\tag{1.7}
$$

Note that Xiao's equations do not need the semistable condition and $s_{g+2}(f) = 0$ if f is semistable (see Corollary 3.1).

Comparing (1.6) with (1.7), it is natural to build up the relation between modular invariants with singularity indices.

A double point p of a semistable curve F is called separable if F becomes disconnected when we normalize F locally at p; otherwise, p is called inseparable. Xiao showed that for each semistable fibration f of genus 2, $s_2(f)$ (resp. $s_3(f)$) is the number of inseparable (resp. separable) double points of all singular fibers of f (see [14]), i.e., $\xi_0(f) = s_2(f),\ \delta_1(f) = s_3(f)$.

If we subdivide the inseparable nodal points into nodes of type $(0, k)$ $(k \geq 0)$, and subdivide the separable nodes into nodes of type i $(i \geq 1)$, then we can get that the modular invariants $\delta_i(f)$, $\xi_j(f)$ are the same as the singularity indices $s_k(f)$.

Theorem 1.1 *If f is a semistable hyperelliptic fibration of genus $g \geq 2$, then*

$$
\delta_k(f) = s_{2k+1}(f)\ (k \geq 1), \quad \xi_k(f) = s_{2k+2}(f)\ (k \geq 0) \tag{1.8}
$$

and

$$
\delta_0(f) = s_2(f) + 2s_4(f) + \cdots + 2s_{2\left[\frac{g-1}{2}\right]+2}(f). \tag{1.9}
$$

Considering the equations in (1.2) and (1.8), it is likely that there exists a more general correspondence between modular invariants and relative invariants. Precisely, we expect that if $\mathcal{M}$ is any kind of moduli space of curves, and η is a divisor class of $\mathcal{M}$, especially the generator of $\mathrm{Pic}(\mathcal{M})$, then there is a reasonable relative invariant which coincides with the modular invariant $\eta(f)$ corresponding to η for each semistable family f of curves in $\mathcal{M}$. Recently, there is another such corresponding showed in [3].

In §2, we recall Xiao's study of hyperelliptic fibration. In §3, we repeat the work (see [12]) of Yuping Tu on semistable criterion, and then we prove our result locally by constructing bijective maps between sets of singularities $\mathcal{R}_*$ with sets of nodes (or nodal pairs) $\mathcal{N}_*$.

2 Singularity Indices

2.1 Genus g data

Let P be a smooth surface, and R be a reduced even divisor (the image of R in $\mathrm{Pic}(P)$ is divisible by 2) on P. Let δ be an invertible sheaf such that $\mathcal{O}_P(R) = \delta^{\otimes 2}$, and we call δ the square root of R for convenience. In fact, a reduced even divisor R on P and an invertible sheaf

δ with $\mathcal{O}_P(R) = \delta^{\otimes 2}$ determine a unique double cover $\pi : S \to P$ branched along R (see [2, I.7]). Thus (R, δ) is called a double cover datum. If R is reduced smooth, then S is smooth.

If $\psi_1 : P_1 \to P$ is a blow-up of P at a singular point x of R of order m, set

$$R_1 := \psi_1^*(R) - 2\Big[\frac{m}{2}\Big]E, \quad \delta_1 := \psi_1^*\delta - \Big[\frac{m}{2}\Big]E, \tag{2.1}$$

where E is the exceptional (-1)-curve of ψ_1. Then (R_1, δ_1) is called a reduced even inverse image of (R, δ) under ψ_1. In what follows, we call R_1 a reduced even inverse image of R briefly, since δ_1 is determined by (R, δ) and R_1.

Definition 2.1 *An even resolution of R is a sequence of blow-ups $\widetilde{\psi} = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_r$,*

$$\widetilde{\psi} : (\widetilde{P}, \widetilde{R}) = (P_r, R_r) \overset{\psi_r}{\to} \cdots \to (P_2, R_2) \overset{\psi_2}{\to} (P_1, R_1) \overset{\psi_1}{\to} (P_0, R_0) = (P, R) \tag{2.2}$$

satisfying the following conditions:

(i) *$\widetilde{R}$ is a smooth reduced even divisor,*

(ii) *R_i is the reduced even inverse image of R_{i-1} under ψ_i.*

Furthermore, $\widetilde{\psi}$ is called the minimal even resolution of the singularities of R if

(iii) *ψ_i is the blow-up of P_{i-1} at a singular point x_i of R_{i-1} for any $1 \le i \le r$.*

If the even resolution of $\widetilde{\psi} : \widetilde{P} \to P$ of R is minimal, then for any even resolution $\psi' : P' \to P$, there exists a morphism $\alpha : P' \to \widetilde{P}$ such that $\alpha(R') = \widetilde{R}, \alpha(\delta') = \widetilde{\delta}$. Here $\alpha(\delta') = \widetilde{\delta}$ means that there exists a divisor $D' \in \mathrm{Pic}(P')$ with $\delta' \cong \mathcal{O}_{P'}(D')$ such that $\widetilde{\delta} \cong \mathcal{O}_{\widetilde{P}}(\alpha(D'))$. Note that the minimal even resolution is unique.

If $x_i \in P_{i-1}$ lies in E_j $(j < i)$, that is, $\psi_j \circ \cdots \circ \psi_{i-1}(x_i) = x_j$, then we say that x_i is infinitely near x_j. Let x_i be a singularity of order $\mathrm{ord}_{x_i}(R) = m_i$. If $m_i \le 3$ and for any x_j infinitely near x_i $(j > i)$ we have $m_j \le 3$, then x_i is called a negligible singularity, since such a singularity does not change the invariants K_f^2, χ_f (see [13, (2)]).

Unless stated otherwise, the singularities (resp. smooth points) of R include all the infinitely near singularities (resp. smooth points) of R_i in P_i for $1 \le i \le r$. If we want to specify a singularity (resp. smooth point) p of R, we will point out the surface which p lies in.

Now we want to introduce the genus g datum associated to a hyperelliptic fibration $f : S \to C$, according to Xiao's approach in [13–14].

Since the generic fiber F of f is hyperelliptic, we glue the involution σ_F of F together, and then we get a rational map $\sigma : S \to S$. The map σ is in fact a morphism, because f is assumed to be relatively minimal. Let $\rho : \widetilde{S} \to S$ be the minimal composition of blow-ups of S at all the isolated fixed points of σ, and $\widetilde{\sigma} : \widetilde{S} \to \widetilde{S}$ be the induced map of σ on $\widetilde{S}$. Then $\widetilde{P} = \frac{\widetilde{S}}{\langle\widetilde{\sigma}\rangle}$ is smooth. Let $\widetilde{\theta} : \widetilde{S} \to \widetilde{P}$ be the corresponding double cover branched along a smooth reduced divisor $\widetilde{R}$ in $\widetilde{P}$. Then $\widetilde{\theta}_*(\mathcal{O}_{\widetilde{S}}) \cong \mathcal{O}_{\widetilde{P}} \oplus \widetilde{\delta}^\vee$, where $\widetilde{\delta}^\vee$ is an invertible sheaf with $\widetilde{\delta}^{\otimes 2} \cong \mathcal{O}_{\widetilde{P}}(\widetilde{R})$.

Let $\Phi_K : S \dashrightarrow \mathrm{Proj}(f_*\omega_{\frac{S}{C}})$ be the relative canonical map, then Φ_K is a generic double cover, for its restriction on a generic fiber F is the double cover induced by σ_F. Let $\widehat{\rho} : \widehat{S} \to S$ be the minimal composition of blow-ups at all base points of Φ_K and all isolated fixed points. Then the birational morphism $\widehat{S} \to \widetilde{S}$ is an isomorphism because of the minimality of ρ. Hence $\widehat{\rho} = \rho$ and $\widehat{S} \cong \widetilde{S}$. This gives another process to get the double cover $\widetilde{\theta} : \widetilde{S} \to \widetilde{P}$ and the branch locus $\widetilde{R}$.

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\theta}"] \arrow[d, "\rho"'] & \widetilde{P} \arrow[d, "\widetilde{\psi}"] \\ S \arrow[r, dashed] \arrow[dr, "f"'] & P \arrow[d, "\varphi"] \\ & C \end{tikzcd}$$

The morphism $\widetilde{\varphi}:\widetilde{P}\to C$ induced by f is a birational ruling (a fibration whose general fibers are rational curves). There are many choices to give a birational morphism $\widetilde{\psi}:\widetilde{P}\to P$ mapping to a geometric ruled surface P. The morphism $\widetilde{\psi}$ induces a reduced divisor $R=\widetilde{\psi}(\widetilde{R})$ in P. All such geometric ruled surfaces differ by elementary transforms. We want to choose one such that R^2 is the smallest.

In the rest, a curve D means a nonzero effective divisor.

Definition 2.2 *Let D be an irreducible curve on a fibred surface S with fibration $f:S\to C$. If $f(D)$ is a point, we call D a vertical curve.*

Lemma 2.1 (see [13, Lemma 6]) *There is a birational morphism $\widetilde{\psi}:\widetilde{P}\to P$ over C, where every fiber of the induced morphism $\varphi:P\to C$ is $\mathbb{P}^1$, such that, letting δ be the image of $\widetilde{\delta}$ in P, and R_h be the sum of the non-vertical irreducible components of R. Then R^2 is the smallest among all such choices, and the singularities of R_h are at most of order $g+1$. Therefore as R is reduced, the singularities of R are of order at most $g+2$, and if p is a singular point of order $g+2$, R contains the fiber of φ passing through p.*

Definition 2.3 *Let P be a geometric ruled surface over C, and (R,δ) be a double cover datum on P. If (R,δ) satisfies that $R\Gamma=2g+2$, where Γ is a generic fiber of $\varphi:P\to C$, and the order of any singularity of the non-vertical part R_h of R is at most $g+1$, then we call (P,R,δ) a genus g datum.*

We have shown that there is a genus g datum (P,R,δ) (Lemma 2.1) associated to a given hyperelliptic fibration f. On the other hand, let (P,R,δ) be a genus g datum over a smooth curve C, $\widetilde{\psi}:\widetilde{P}\to P$ be the minimal even resolution of (P,R), and $\widetilde{\theta}:\widetilde{S}\to\widetilde{P}$ be the double cover determined by $(\widetilde{R},\widetilde{\delta})$. Let $\rho:\widetilde{S}\to S$ be the morphism of contracting all the vertical (-1)-curves, then we get a hyperelliptic fibration $f:S\to C$. Hence we need to study the vertical (-1)-curves in $\widetilde{S}$.

Lemma 2.2 (see [14]) *Let (P,R,δ) be a genus g datum, and Γ be any fiber of $P\to C$, whose inverse image in $\widetilde{S}$ is a (-1)-curve. In other words, the strict transform of Γ in $\widetilde{P}$ is a (-2)-curve contained in $\widetilde{R}$. If g is even, then one of the following two cases is satisfied:*

(1) *R_h intersects with Γ at two distinct points x,y, $m_x(R_h)=m_y(R_h)=g+1$; or*

(2) *R_h intersects with Γ at one point, and the point is a singularity of type $(g+1\to g+1)$ (see Definition 2.4), which is tangent to Γ.*

If g is odd, then R_h intersects with Γ at one point, and it is a singularity of type $(g+2\to g+2)$, which is tangent to Γ.

Lemma 2.3 (see [14]) *Suppose that E is a vertical (-1)-curve in $\widetilde{S}$, then the image $\widetilde{E}$ of E in $\widetilde{P}$ is an isolated (-2)-curve contained in $\widetilde{R}$, and $\widetilde{E}$ either comes from a blow-up of a singularity of R with odd order, or is a strict transform of a fiber in Lemma 2.2. Conversely, for any singularity of R with odd order or any fiber in Lemma 2.2, there is a corresponding vertical (-2)-curve.*

Remark 2.1 As stated in [13], if we start from a hyperelliptic fibration $f : S \to C$, we can choose a genus g datum (P, R, δ) such that R^2 is the smallest, and then the case (1) in Lemma 2.2 does not occur. Accordingly, Lemma 2.3 turns to be Lemma 7 in [13]. In what follows, we always assume that the genus g datum associated with f satisfies that R^2 is the smallest.

Consequently, we only need to consider genus g datum for hyperelliptic fibrations.

2.2 Singularity indices

Based on the above preparation, we are able to define the singularity indices.

Let (P, R, δ) be a genus g datum over a smooth curve C, and $\widetilde{\psi}$ in (2.2) be the minimal even resolution of (P, R). We decompose $\widetilde{\psi}$ into $\psi' : \widetilde{P} \to \widehat{P}$ followed by $\widehat{\psi} : \widehat{P} \to P$, where ψ' and $\widehat{\psi}$ are composed respectively of negligible and non-negligible blow-ups. We may assume $\widehat{\psi} = \psi_1 \circ \cdots \circ \psi_t$ for $t \le r$. Denote by $(\widehat{R}, \widehat{\delta})$ the reduced even inverse image of (R, δ) in $\widehat{P}$.

Definition 2.4 *Let x_i be a singularity of R_{i-1} of order $2k+1$ $\left(1 \le k \le \left[\frac{g+1}{2}\right]\right)$. If R_i has a unique singularity on the inverse image of x_i, say x_{i+1}, with order $2k+2$, then we call x_i a singularity of type $(2k+1 \to 2k+1)$.*

Definition 2.5 *Let $f : S \to C$ be a fibration and D be a reduced curve on S. Let $\phi : D \to C$ be the natural morphism induced by f. Let $\nu : \widetilde{D} \to D$ be the normalization of D, D_h be the union of all the irreducible components of $\widetilde{D}$ which maps projectively onto C, and $\nu_h : D_h \to D$ be the induced map. The ramification index $r(D)$ of ϕ is defined as follows:*

If $q \in D_h$ is a ramification point of $\phi \circ \nu_h$, then the ramification index $r_q(D)$ is defined as usual.

If p is a singularity of D of order m_p, then the ramification index is $r_p(D) = m_p(m_p - 1)$.

If E is an isolated vertical curve of $\widetilde{D}$, then the ramification index is $r_E(D) = \chi_{\rm top}(E)$.

Furthermore, we define

$$r(D) := \sum_{q \in D_h} r_q(D) + \sum_{p \in D} m_p(m_p - 1) - \sum_{\substack{E \subset \widetilde{D} \text{ isolated} \\ \text{vertical curve}}} \chi_{\rm top}(E). \tag{2.3}$$

Remark 2.2 It is easy to see that $r(D) = D^2 + DK_{\frac{S}{C}}$, from the adjoint formula $K_S D + D^2 = -2\chi(\mathcal{O}(D))$ (see [14]).

When we consider a singular fiber F of f, the singularities and ramification points of branch locus are those over $f(F)$ if there is no confusion.

Definition 2.6 (see [13–14]) *Let $f : S \to C$ be a hyperelliptic fibration, and (P, R, δ) be the corresponding genus g datum. Suppose that F is a singular fiber of f. We denote by Γ the fiber of $P \to C$ over $f(F)$. The singularity indices $s_k(F)$ $(2 \le k \le g+2)$ are defined as follows.*

(1) *Let $E_1, \cdots, E_k$ be all the isolated vertical (-2)-curves in $\widehat{R}$. Letting $\widehat{R}_p = \widehat{R} - E_1 - \cdots - E_k$, then $s_2(F)$ is defined to be the ramification index of $\widehat{R}_p$ over the point $f(F)$. Concisely, if we denote by $\mathcal{R}_{2,1}(F)$ the set of all ramification points of $\widetilde{R}$ over $f(F)$, by $\mathcal{R}_{2,2}(F)$ the set of all singularities of $\widehat{R}_p$, and by $\mathcal{R}_{2,-}(F)$ the set of all vertical components in $\widehat{R}_p$, then*

$$s_2(F) = \sum_{q \in \mathcal{R}_{2,1}(F)} \left((\Gamma, \widetilde{R})_q - 1\right) + \sum_{q \in \mathcal{R}_{2,2}(F)} m_q(m_q - 1) - 2|\mathcal{R}_{2,-}(F)|. \tag{2.4}$$

(2) *If k is odd, denote by $\mathcal{R}_k(F)$ the set of all singularities of R of type $(k \to k)$, then $s_k(F) := |\mathcal{R}_k(F)|$.*

(3) *If $k \geq 4$ is even, denote by $\mathcal{R}_k(F)$ the set of all singularities of R of order k, not belonging to a singularity of type $(k+1 \to k+1)$ or $(k-1 \to k-1)$, then $s_k(F) := |\mathcal{R}_k(F)|$.*

Define

$$s_k(f) = \sum_{i=1}^{s} s_k(F_i), \tag{2.5}$$

where $F_1, \cdots, F_s$ are all the singular fibers of f.

Remark 2.3 Xiao introduced the singularity indices in order to compute the contribution of singular fibers to the invariants K_f^2, χ_f. It is convenient to put x_i, x_{i+1} in Definition 2.4 together, and regard the pair $\{x_i, x_{i+1}\}$ of points as one singularity of type $(2k+1 \to 2k+1)$, that is, the total contribution of x_i and x_{i+1} to singularity indices adds one to s_{2k+1} only.

Example 2.1 Let (x,t) be the local coordinate of $\mathbb{P}^1 \times \Delta$, where Δ is the open unit disc of $\mathbb{C}$. Let

$$\begin{aligned} h(x,t) = {} & (x+t)((x-a_0)^2+t)((x-a_1)^2+t^2) \\ & \cdot ((x-a_2+t)^2+t^3)((x-a_2-t)^2+t^3)((x-a_3)^3+t^6), \end{aligned} \tag{2.6}$$

where a_i's are distinct nonzero complex numbers. Let $f: S_\Delta \to \Delta$ be the local fibration of genus g defined by $y^2 = h(x,t)$. Let $F = f^{-1}(0)$ be the fiber of f over the origin, and Γ be the fiber of $\mathbb{P}^1 \times \Delta \to \Delta$ over the origin.

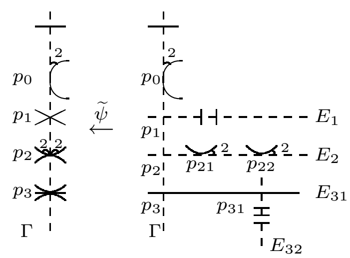

Figure 3 The minimal even resolution

The branch locus is $R = \{(x,t) \in \mathbb{P}^1 \times \Delta : h(x,t) = 0\}$, and $R\Gamma' = 12$, where Γ' is the generic fiber of $\mathbb{P}^1 \times \Delta \to \Delta$. Hence $g = 5$ by Riemann-Hurwitz formula. Let $p_i = (a_i, 0)$ $(i = 0,1,2,3)$, then p_2 and p_3 are non-negligible.

Let p_{21} and p_{22} be the infinitely near points of p_2, which are smooth points of $\widehat{R}$. Let p_{31} be the infinitely near singularity of p_3, then $\{p_3, p_{31}\}$ is a singularity of type $(3 \to 3)$. Therefore $\widehat{R}_p = R$ which is the strict transform in $\widehat{P}$, and

$$\begin{aligned} & \mathcal{R}_{2,1}(F) = \{p_0, p_{21}, p_{22}\}, \quad \mathcal{R}_{2,2}(F) = \{p_1\}, \quad \mathcal{R}_{2,-}(F) = \emptyset, \\ & \mathcal{R}_3(F) = \{\{p_3, p_{31}\}\}, \quad \mathcal{R}_4(F) = \{p_2\}. \end{aligned} \tag{2.7}$$

Furthermore, the singularity indices are

$$(s_2(F), s_3(F), \cdots, s_7(F)) = (5,1,1,0,0,0). \tag{2.8}$$

Using the singularity indices, Xiao obtained the following formulas.

Theorem 2.1 (see [14, Theorem 5.1.7]) *Let $f : S \to C$ be a hyperelliptic fibration of genus g, then*

$$(8g+4)\chi_f = g(s_2(f) - 2s_{g+2}(f)) + \sum_{k=2}^{\left[\frac{g-1}{2}\right]} 2(k+1)(g-k)s_{2k+2}(f) + \sum_{k=1}^{\left[\frac{g+1}{2}\right]} 4k(g-k)s_{2k+1}(f),$$

$$e_f = s_2(f) - 3s_{g+2}(f) + \sum_{k=1}^{\left[\frac{g-1}{2}\right]} 2s_{2k+2}(f) + \sum_{k=1}^{\left[\frac{g+1}{2}\right]} s_{2k+1}(f),$$

$$(2g+1)K_f^2 = (g-1)s_2(f) + 3s_{g+2}(f) + \sum_{k=1}^{\left[\frac{g-1}{2}\right]} a_k s_{2k+2}(f) + \sum_{k=1}^{\left[\frac{g+1}{2}\right]} b_k s_{2k+1}(f),$$

where $a_k = 6((k+1)(g-k) - 4g - 2)$ and $b_k = 12k(g-k) - 2g - 1$.

Corollary 2.1 (see [14]) *If f is hyperelliptic, then the slope of f*

$$\frac{4g-4}{g} \leqslant \lambda_f \leqslant \begin{cases} 12 - \dfrac{8g+4}{g^2}, & \text{if } g \text{ is even,} \\ 12 - \dfrac{8g+4}{g^2-1}, & \text{if } g \text{ is odd.} \end{cases}$$

Moreover, the left equality holds if and only if $s_2(f) \neq 0, s_k = 0$ $(k > 2)$, and the right equality holds if and only if $s_{2\left[\frac{g}{2}\right]+1} \neq 0$ and the rest singularity indices are all zero.

3 Modular Invariants in Semistable Case

At the beginning of this section, we fix notations firstly.

Let (P, R, δ) be a genus g datum over a smooth curve C, and $\widetilde{\psi}$ in (2.2) be the minimal even resolution. Let $f : S \to C$ be the fibration determined by the datum, and F be a singular fiber of f. Denote by $\widetilde{F}$ the total transform of F by $\rho : \widetilde{S} \to S$, which is a birational morphism contracting all the vertical (-1)-curves. Let Γ be the fiber of $\varphi : P \to C$ over $t = f(F)$, and we call Γ the image of F in P briefly. Let $\widetilde{\Gamma} = \widetilde{\psi}^*(\Gamma)$ be the total transform of Γ by the minimal even resolution $\widetilde{\psi} : \widetilde{P} \to P$ of R. To keep it simple, we also denote by R (resp. Γ) the strict transform of R (resp. Γ) under the even resolution $\widetilde{\psi}$.

$$\begin{tikzcd}
\widetilde{F} \subset \widetilde{S} \arrow[rr, "\widetilde{\theta}"] \arrow[d, "\rho"'] & & \widetilde{P} \supset \widetilde{\Gamma} \arrow[d, "\widetilde{\psi}"] \\
F \subset S \arrow[rr, dashed] \arrow[dr, "f"'] & & P \supset \Gamma \arrow[dl, "\varphi"] \\
& t \in C &
\end{tikzcd}$$

Denote by $B = \widetilde{\theta}^{-1}(\Gamma)$ the inverse image of Γ in $\widetilde{F}$, and by $B_i = \widetilde{\theta}^{-1}(E_i)$ the inverse image of the exceptional curve E_i. Then B (resp. B_i) may be composed by two irreducible curves B' and B'' (resp. B_i' and B_i''). Letting

$$\widetilde{\Gamma} = \Gamma + \sum_{i=1}^{r} m_i E_i, \tag{3.1}$$

then

$$\widetilde{F}=\widetilde{\theta}^*(\widetilde{\Gamma})=\widetilde{\theta}^*(\Gamma)+\sum_{i=1}^{r}m_i\widetilde{\theta}^*(E_i)=nB+\sum_{i=1}^{r}n_iB_i, \tag{3.2}$$

where $n=1,2$ and $n_i=m_i$ or $n_i=2m_i$. Therefore, $F=\rho(\widetilde{F})$ is obtained by contracting (-1)-curves in $\widetilde{F}$.

Definition 3.1 *An even resolution at point p of R is a sequence of blow-ups $\check{\psi}_p=\psi_1\circ\psi_2\circ\cdots\circ\psi_l$*

$$(\check{P},\check{R})=(P_l,R_l)\stackrel{\psi_l}{\to}\cdots\to(P_2,R_2)\stackrel{\psi_2}{\to}(P_1,R_1)\stackrel{\psi_1}{\to}(P_0,R_0)=(P,R) \tag{3.3}$$

satisfying the following conditions:

(i) *All the points of $\check{R}$ infinitely near p, including p, are smooth.*

(ii) *R_i is the reduced even inverse image of R_{i-1} under ψ_i.*

Furthermore, $\check{\psi}_p$ is called the minimal even resolution at p if

(iii) *ψ_i is the blow-up of P_{i-1} at a singular point p_i of R_{i-1}, which is infinitely near p, for any $1\le i\le l$.*

If the resolution $\check{\psi}_p$ is minimal, we call the desired number l_p of blow-ups the length of the minimal even resolution $\check{\psi}_p$ at p. The exceptional curves E_i's $(1\le i\le l)$ in P_l are called exceptional curves from p briefly. For example, if p is an ordinary singularity of even order, then $l_p=1$; if p is a singularity of type $(3\to3)$, then $l_p\ge 2$.

Let p be a singularity of R, and $E_1,\cdots,E_{l_p}$ be all the exceptional curves from p in P_{l_p}. Set $\mathcal{E}_p:=m_1E_1+\cdots+m_{l_p}E_{l_p}$, $\mathcal{B}_p:=\widetilde{\theta}^*(\mathcal{E}_p)$, where $m_i=\mathrm{mult}_{E_i}(\mathcal{E}_p)=\mathrm{mult}_{E_i}(\widetilde{\Gamma})$. Then we call $\mathcal{E}_p$ the block of $\widetilde{\Gamma}$ from p, and call $F_p:=\rho(\mathcal{B}_p)$ the block of F from p. Assume that Γ is not contained in R. Let $p_1,\cdots,p_e$ be all the singularities of R on Γ in P, and $\mathcal{B}_{p_0}=\widetilde{\theta}^*(\Gamma)$. Then we can decompose F into finite blocks $F=F_{p_0}+F_{p_1}+\cdots+F_{p_e}$, and we call this decomposition the modular decomposition of F.

Example 3.1 (Continuation of Example 2.1) Let $f:S_\Delta\to\Delta$ be the local fibration in Example 2.1. Then $l_{p_1}=1,\ l_{p_2}=1,\ l_{p_3}=2$. The blocks of $\widetilde{\Gamma}$ are $\mathcal{E}_{p_1}=E_1,\ \ \mathcal{E}_{p_2}=E_2,\ \ \mathcal{E}_{p_3}=E_{31}+E_{32}$.

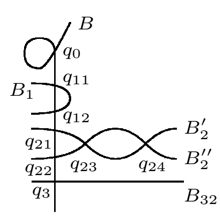

Figure 4 Modular decomposition of F

Here E_1,E_2,E_{32} are not contained in $\widetilde{R}$, and E_{31} is contained in $\widetilde{R}$. Then the blocks of F are $F_{p_0}=B,\ F_{p_1}=B_1,\ F_{p_2}=B_2'+B_2'',\ F_{p_3}=B_{32}$. In these equations, B is a rational curve with a node q_0; B_1 is $\mathbb{P}^1$ meeting B at two points q_{11},q_{12}; B_2' and B_2'' are both $\mathbb{P}^1$ meeting with B at q_{21},q_{22} respectively and meeting with each other at two points q_{23},q_{24}; and B_{32} is a smooth elliptic curve meeting with B at q_3. Then F is semistable, and the modular decomposition of F is $F=\sum\limits_{i=0}^{2}F_{p_i}=B+B_1+(B_2'+B_2'')+B_{32}$.

3.1 Semistable criterion

There is a criterion for semistable hyperelliptic fiber given by Tu [12]. We rewrite the result and its proof here, because the reference is in Chinese.

Lemma 3.1 (see [12]) *If F is a semistable singular fiber of a hyperelliptic fibration $f : S \to C$, then we have the following results:*

(1) *If g is odd, then Γ is not contained in R; if g is even and Γ is contained in R, then Γ is the fiber in Lemma 2.2.*

(2) *If p is a smooth point of R in P, then the intersection number of R with Γ at p is $(R, \Gamma)_p \leq 2$.*

(3) *If p is a singularity of R in P, then we have $(R, \Gamma)_p = \mathrm{ord}_p(R)$.*

(4) *Let $q \in R_i$ $(i \geq 1)$ be an infinitely near singularity, then there is exactly one exceptional curve E_q passing through q, and $(R, E_q)_q = \mathrm{ord}_q(R)$.*

(5) *Let $q \in R_i$ and E_q be the same as (4). If $\mathrm{ord}_q(R) = l$ is even, then E_q is not contained in $\widetilde{R}$. If $\mathrm{ord}_q(R) = l$ is odd, then either E_q is contained in $\widetilde{R}$, and thus E_q is from a singularity of type $(k \to k)$ (k is odd, $k \geq l$); or E_q is not contained in $\widetilde{R}$, and thus q is a singularity of type $(l \to l)$.*

(6) *Let q be an infinitely near smooth point, and E_q be the irreducible component passing through q, then $(R, E_q)_q \leq 2$ and E_q is not contained in $\widetilde{R}$.*

Proof (1) Suppose $\Gamma \subseteq \widetilde{R}$. Then B is a component of $\widetilde{F}$ with multiplicity 2, for $\pi^*(\Gamma) = 2B$, furthermore, $B^2 = \frac{\Gamma^2}{2}$. If $\Gamma^2 \leq -4$, then $B^2 \leq -2$. Hence B is a multiple component in $\widetilde{F}$ which can not be contracted, contradicting with the assumption that F is semistable. Thus we get that if $\Gamma \subseteq \widetilde{R}$, then Γ is a (-2)-curve in $\widetilde{P}$. If g is odd, then any singularity of R is of type $(g + 2 \to g + 2)$, and we need twice blow-up so that the intersection point of Γ with the exceptional curve is a smooth point of R. Hence there is a (-1)-curve, say E_2, with multiplicity 2 in $\widetilde{\Gamma}$. It is easy to see that E_2 is not contained in $\widetilde{R}$, and $\pi^*(2E_2) = 2B_2$ in F is irreducible with $B_2^2 \leq -2$. Therefore B_2 is an un-contractible multiple component in semistable curve $\widetilde{F}$, which is impossible. So when g is odd, Γ is not contained in R.

(2) In what follows, we may assume that Γ is not contained in R since (1). Let $n = (R, \Gamma)_p$, we take the local coordinate (x, t) of p such that the local equations of Γ and R near p are $t = 0$ and $t + x^n = 0$ respectively. Then the local equation of F in S is $y^2 - x^n = 0$. If $n \geq 3$, it is a singularity of type A_{n-1} on F, and thus F is not semistable.

(3) Suppose not, then $(R, \Gamma)_p > \mathrm{ord}_p(R)$. Let ψ_1 be the blow-up at p, and E_1 be the exceptional curve. Then the intersection point p' of Γ with E_1 is still on R. Let ψ_2 be the successive blow-up at p_1 and E_2 be the exceptional curve. Then the total transform of Γ by $\psi_1 \circ \psi_2$ is $\widetilde{\Gamma}_2 = \Gamma + 2E_2 + E_1$, and B_2 is with multiplicity at least 2 in F. Hence B_2 is a (-1)-curve in $\widetilde{S}$, E_2 is a (-2)-curve in $\widetilde{R}$, and p_1 is a singularity of type $(k \to k)$ (k is odd) (see Lemma 2.2). Furthermore, there is a singularity p_2 on E_2 of order $k + 1$. Let ψ_3 be the blow-up at p_2 with exceptional curve E_3. Then $\widetilde{\Gamma}_3 = \Gamma + 2E_3 + 2E_2 + E_1$, E_3 is not contained in $\widetilde{R}$, and B_3 is an un-contractile multiple component in $\widetilde{F}$.

(4) Suppose that E_1 and E_2 are both through q. When $\mathrm{ord}_q(R)$ is even, then the exceptional curve E_3 of the blow-up at q is of multiplicity at least 2, and E_3 is not contained in $\widetilde{R}$. So B_3 is an un-contractile multiple component in F. When $\mathrm{ord}_q(R) = k$ is odd, then q should be of type $(k \to k)$, and E_3 is contained in $\widetilde{R}$ with multiplicity at least 2. Blowing up the infinitely near singularity q' of q, then the exceptional curve E_4 is not contained in $\widetilde{R}$ of multiplicity at least 2, which is impossible. The proof of the second part of (5) is analogous to that of (3).

(5) Suppose that $\mathrm{ord}_q(R)$ is even and E_q is contained in $\widetilde{R}$, then $\mathrm{ord}_{\underset{\sim}{q}}(R_i)$ is odd. The exceptional curve E' of the blow-up $\psi : P_{i+1} \to P_i$ at q is contained in $\widetilde{R}$. Thus $E_q^2 \leq -2$ in $\widetilde{P}$, and $E_q^2 \leq -4$ in $\widetilde{S}$. So E_q corresponds to an un-contractile multiple component in F. Consequently, we proved the first part of (4). The second part of (4) is a direct corollary of Lemma 2.3.

(6) The proof of the second part is the same as that of (1), and the rest is the same as that of (2). We omit the detail.

Remark 3.1 Let F be a semistable fiber of f, and p be a singularity of R. Then there is exactly one irreducible component E_p passing through p. We call E_p the exceptional curve through p. Note that E_p is either Γ or an exceptional curve.

Corollary 3.1 *If F is a semistable hyperelliptic fiber of genus g, then $s_{g+2}(F) = 0$.*

Proof By Lemma 3.1 (1), we know that if g is odd, then $s_{g+2}(F) = 0$; if g is even, then Γ is the fiber in Lemma 2.2, and we can check the result directly.

3.2 Proof of Theorem 1.1

We first consider the effect of the smooth points of R to the arithmetic genus.

Lemma 3.2 *Let F be a semistable fiber of f. Assume that the image Γ of F in P is not contained in R. Suppose that all the intersection points $p_1, \cdots, p_{k_1}$, $q_1, \cdots, q_{k_2}$ of R with Γ are smooth, where $(\Gamma, R)_{p_i} = 2$ and $(\Gamma, R)_{q_j} = 1$.*

(1) *If $k_2 \neq 0$, then F is an irreducible curve with k_1 nodes corresponding to p_i's, the geometric genus of F is $\big[\frac{k_2-1}{2}\big]$, and*

$$p_a(F) = \Big[\frac{\Gamma R - 1}{2}\Big] = \Big[\frac{2k_1 + k_2 - 1}{2}\Big] = \Big[k_1 + \frac{k_2 - 1}{2}\Big].$$

(2) *If $k_2 = 0$, then F is composed of two smooth rational curves which meet with each other at k_1 distinct points. Thus*

$$F = \widetilde{\Theta}^*(\Gamma) + \sum_{i=1}^{k_1} \widetilde{\Theta}^*(E_i) = (B' + B'') + \sum_{i=1}^{k_1} B_i,$$

where every irreducible component is a smooth rational curve, B_i meets B' and B'' normally at one point respectively for each $1 \leq i \leq k_1$, and there is no other intersection. Hence

$$p_a(F) = \Big[\frac{\Gamma R - 1}{2}\Big] = k_1 - 1.$$

Proof The proof is obvious, and we omit it.

Then we consider the effect of singularities.

Lemma 3.3 *Suppose that F is a semistable fiber of f, and the image Γ of F in P is not contained in R. If p is a singularity of R such that the exceptional curve E_p through p is not contained in the branch locus, then the arithmetic genus of the block F_p of F from p is*

$$p_a(F_p) = \Big[\frac{(E_p, R)_p - 1}{2}\Big]. \tag{3.4}$$

Proof We use induction on the length l_p of the minimal even resolution $\widetilde{\psi}_p$ of R at p. Note that $\big[\frac{(2k+2)-1}{2}\big] = \big[\frac{(2k+1)-1}{2}\big] = k$, and $\mathrm{ord}_p(R) = (R, E_p)_p$ for any singularity of R on E_p from Lemma 3.1. We may assume that E_p is Γ, since the proof for exceptional curves is similar.

If $l_p = 1$, then $\mathrm{ord}_p(R) = 2k+2$ is even and p is an ordinary singularity. The exceptional curve E_1 from p is not contained in R, and E_1 meets R in P_1 transversely at $2k+2$ distinct points. Hence $\mathcal{E}_p = E_1$, and $\mathcal{B}_p = B_1$ with $p_a(B_1) = k$.

If $l_p = 2$ and $\mathrm{ord}_p(R) = 2k+2$ is even, then there is exactly one infinitely near singularity p_1 of R in P_1, which is an ordinary singularity of even order, say $2k_2$. Hence E_1, E_2 are not contained in R, $\mathcal{E}_p = E_1 + E_2$, and $\mathcal{E}_p$ meets R in P_2 transversely at $2k+2$ distinct points. Let $E_1R = 2k_1 + 2$, then $k_1 + k_2 = k$. Thus $p_a(B_1) = k_1$, $p_a(B_2) = k_2 - 1$, and B_1 intersects with B_2 at two points transversely. So $p_a(F_p) = p_a(\mathcal{B}_p) = p_a(B_1) + p_a(B_2) + 1 = k$.

If $l_p = 2$ and $\mathrm{ord}_p(R) = 2k+1$ is odd, then p is a singularity of $(2k+1 \to 2k+1)$. So E_1 is contained in R, E_2 is not contained in R, and $\mathcal{E}_p = E_1 + E_2$, where E_2 meets R_2 in P_2 transversely at $2k+2$ distinct points. It is easy to see that B_1 is a (-1)-curve and B_2 is a smooth curve with genus k. Hence $p_a(F_p) = p_a(B_2) = k$.

Assume that (3.4) holds for any positive integer $l < l_p$. We want to prove that (3.4) holds for l_p.

If $\mathrm{ord}_p(R) = 2k+1$ is odd, let $\psi_1 : P_1 \to P$ be the blow-up at p. Then there is exactly one infinitely near singularity q of R in P_1, and $(R, E_1)_q = 2k+1$, $\mathrm{ord}_q(R_1) = 2k+2$. Let $\psi_2 : P_2 \to P_1$ be the successive blow-up at q. It is clear that E_1 is contained in R, but E_2 is not.

Let $q_1, \cdots, q_\alpha$ be all the infinitely near singularities of q in P_2. Hence $l_{q_i} < l_p$ for $1 \le i \le \alpha$. Suppose that $q_1, \cdots, q_\beta$ ($\beta \le \alpha$) are all the singularities with even order. Let $(R, E_2)_{q_i} = 2k_i + 2$ for $1 \le i \le \beta$, and let $(R, E_2)_{q_j} = 2k_j + 1$ for $\beta + 1 \le j \le \alpha$. Let the total intersection number of R with E_2 at all the smooth points of R in P_2 be $(R, E_2)_{\mathrm{sm}}$. Then

$$\begin{aligned}
2k+2 &= \sum_{i+1}^{\beta}(2k_i+2) + \sum_{j=\beta+1}^{\alpha}(2k_j+1) + (R, E_2)_{\mathrm{sm}} + 1 \\
&= 2(k_1 + \cdots + k_\beta) + 2\beta + 2(k_{\beta+1} + \cdots + k_\alpha) + (\alpha - \beta) + (R, E_2)_{\mathrm{sm}} + 1 \\
&= 2(k_1 + \cdots + k_\alpha) + (R, E_2)_{\mathrm{sm}} + (\alpha + \beta) + 1.
\end{aligned}$$

Hence

$$k = \Big(\sum_{i=1}^{\alpha} k_i\Big) + \frac{(R, E_2)_{\mathrm{sm}} + \alpha + \beta - 1}{2}. \tag{3.5}$$

It is easy to see that in $\widetilde{P}$, $\widetilde{R}E_2 = (R, E_2)_{\mathrm{sm}} + (\alpha - \beta) + 1$. By Lemma 3.2,

$$p_a(B_2) = \Big[\frac{((R, E_2)_{\mathrm{sm}} + (\alpha - \beta) + 1) - 1}{2}\Big] = \frac{(R, E_2)_{\mathrm{sm}} + \alpha - \beta - 1}{2}. \tag{3.6}$$

The block of $\widetilde{\Gamma}$ from p is $\mathcal{E}_p = E_1 + E_2 + \sum\limits_{i=1}^{\alpha} \mathcal{E}_{q_i}$. Combining (3.5) and (3.6), then

$$\begin{aligned}
p_a(F_p) = p_a(\mathcal{B}_p - 2B_1) &= p_a(B_2) + \Big(\sum_{i=1}^{\alpha} p_a(F_{q_i})\Big) + \beta \\
&= \frac{(R, E_2)_{\mathrm{sm}} + \alpha - \beta - 1}{2} + \Big(\sum_{i=1}^{\alpha} k_i\Big) + \beta \\
&= k,
\end{aligned} \tag{3.7}$$

where the block F_{q_i} intersects with B_2 at two points, and adds one to the arithmetic genus for each $1 \le i \le \beta$. Here we used the induction assumption.

If $\mathrm{ord}_p(R) = 2k+2$ is even, take $\psi_1 : P_1 \to P$, the blow-up at p. Let $q_1, \cdots, q_\alpha$ be all the infinitely near singularities of p on p_1. Then the rest of the proof is the same as the odd case above.

Now we can prove the identities between singularity indices (Definition 2.6) with modular invariants $\delta_i(F),\ \xi_j(F)$ (see (1.3)–(1.5)).

Theorem 3.1 *Let $f : S \to C$ be a semistable hyperelliptic fibration of genus g, and F be a singular fiber of f, then*

$$s_{2k+1}(F) = \delta_k(F)\ (k \ge 1), \quad s_{2k+2}(F) = \xi_k(F)\ (k \ge 0). \tag{3.8}$$

Proof (1) Proof of $s_{2k+1}(F) = \delta_k(F),\ k \ge 1$.

We define a bijective map

$$\alpha_{2k+1} : \mathcal{R}_{2k+1}(F) \to \mathcal{N}_{2k+1}(F) \tag{3.9}$$

between sets as follows.

If $p \in \mathcal{R}_{2k+1}(F)$, then E_p (the exceptional curve through p) is not contained in $\widetilde{R}$. Let $\widetilde{\Gamma} = \mathcal{E}_p^c + \mathcal{E}_p$, where $\mathcal{E}_p$ is the block of $\widetilde{\Gamma}$ from p. Then the decomposition of F is $F = F_p^c + F_p$, where $p_a(F_p) = \big[\frac{(R,E_p)_p - 1}{2}\big] = k$, and F_p^c intersects with F_p^c at a point, say q, which is a node of type k. We define $\alpha_{2k+1}(p) = q \in \mathcal{N}_{2k+1}(F)$, then α_{2k+1} is well-defined.

On the other hand, if $q \in \mathcal{N}_{2k+1}(F)$, then F consists of a genus k curve F_q and a genus $g-k$ curve F_q^c, and F_q meets F_q^c at q transversely. Then q is an isolated fixed point of the hyperelliptic involution σ. Thus the inverse image of q in $\widetilde{F}$ under $\rho : \widetilde{S} \to S$ is a (-1)-curve B. Hence $\widetilde{\theta}(B)$ is a (-2)-curve contained in $\widetilde{R}$, which is from a singularity, say p, of type $(2k'+1 \to 2k'+1)$ (see Lemma 2.3 and Lemma 3.1 (4)). Since $\widetilde{\theta}^*(\mathcal{E}_p) = \rho^*(F_q)$, the arithmetic genus of the block of F from p is $k' = p_a(\widetilde{\theta}^*(\mathcal{E}_p)) = p_a(F_q) = k$. Thus $p \in \mathcal{R}_{2k+1}(F)$, and $\alpha_{2k+1}(p) = q$. Hence it is clear that α_{2k+1} is surjective and injective.

Therefore, $s_{2k+1}(F) = |\mathcal{R}_{2k+1}(F)| = |\mathcal{N}_{2k+1}(F)| = \delta_k(F)$.

(2) Similar proof of $s_{2k+2}(F) = \xi_k(F),\ k \ge 1$.

We define a bijective map

$$\alpha_{2k+2} : \mathcal{R}_{2k+2}(F) \to \mathcal{N}_{2k+2}(F) \tag{3.10}$$

between sets as follows.

If $p \in \mathcal{R}_{2k+2}(F)$, then E_p is not contained in $\widetilde{R}$, and the exceptional curve E_1 of the blow-up at p is not in $\widetilde{R}$ either. Hence $\widetilde{\theta}^{-1}(p)$ consists of two points q and $\sigma(q)$. Let $\widetilde{\Gamma} = \mathcal{E}_p + \mathcal{E}_p^c$, then $F = F_q + F_q^c$, $p_a(F_q) = k$ and F_q meets F_q^c at q and $\sigma(q)$ transversely. So the nodal pair $\{q, \sigma(q)\} \in \mathcal{N}_{2k+2}(F)$. Hence we are able to define $\alpha_{2k+1}(p) = \{q, \sigma(q)\}$.

On the other hand, if $\{q, \sigma(q)\} \in \mathcal{N}_{2k+2}(F)$, then $F = F_q + F_q^c$, where $p_a(F_q) = k$, and they intersect with each other at two points q and $\sigma(q)$ transversely. We may assume that $\widetilde{F} = F_q + F_q^c$, which meet at q and $\sigma(q)$. Then $\widetilde{\theta}(q) = \widetilde{\theta}(\sigma(q))$, say p, is an intersection point of two curves not in $\widetilde{R}$. Hence we can decompose $\widetilde{\Gamma}$ as $\widetilde{\Gamma} = \mathcal{E}_p + \mathcal{E}_p^c$, where $R\mathcal{E}_p = 2k+2$ and $\mathcal{E}_p$ meets $\mathcal{E}_p^c$ at p only. The curve $\mathcal{E}_p$ is from a singularity of order $\mathrm{ord}_p(R) = R\mathcal{E}_q = 2k+2$. Therefore, p is the inverse image of $\{q, \sigma(q)\}$ under α_{2k+2}, and α_{2k+2} is bijective.

So $s_{2k+2}(F) = |\mathcal{R}_{2k+2}(F)| = |\mathcal{N}_{2k+2}(F)| = \xi_k(F)$.

(3) Proof of $s_2(F)=\xi_0(F)$.

If E is a vertical components of $\widehat{R}$, then $B=\widetilde{\theta}^*(E)$ is a multiple component of $\widetilde{F}$. So B is a (-1)-curve for F is semistable, and then we know that E is a (-2)-curve in $\widehat{R}$. Hence $\widehat{R}_p$ is the strict transform of R in $\widehat{P}$, and $|\mathcal{R}_{2,-}(F)|=0$.

If $p\in\mathcal{R}_{2,1}(F)$, then p is a smooth point of R, $(R,E_p)_p=2$, $r_p(R)=1$, and $\widetilde{\theta}^{-1}(p)$ is an α-type node q. Conversely, each α-type node q is a singularity p of type A_1 whose local equation is $t+x^2=0$. So we get a bijective map

$$\alpha_{2,1}:\mathcal{R}_{2,1}(F)\to\mathcal{N}_{2,1}(F). \tag{3.11}$$

If $p\in\mathcal{R}_{2,2}(F)$, then p is an ordinary double point, and $r_p(R)=2$. By the same discussion in (2), we can obtain a bijective map

$$\alpha_{2,2}:\mathcal{R}_{2,2}(F)\to\mathcal{N}_{2,2}(F). \tag{3.12}$$

Hence $s_2(F)=|\mathcal{R}_{2,1}(F)|+2|\mathcal{R}_{2,2}(F)|=|\mathcal{N}_{2,1}(F)|+2|\mathcal{N}_{2,2}(F)|=\xi_0(F)$.

Proof of Theorem 1.1 It is a corollary of the above theorem.

Remark 3.2 Let $f:S\to C$ be a hyperelliptic fibration of genus $g\geq 2$, and $\widetilde{f}:\widetilde{S}\to\widetilde{C}$ be a semistable model of f. Then by Corollary 2.1 and Theorem 1.1, we know that $\widetilde{f}$ has the lowest slope if and only if the image $[f]$ of f by the moduli map intersects with Ξ_0 only, and $\widetilde{f}$ has the highest slope if and only if $[f]$ intersects with $\Delta_{\left[\frac{g}{2}\right]}$ only. See [7] for families with the highest slope.

Example 3.2 (Continuation of Example 3.1) From the analysis of the blocks of F in Example 3.1, we can easy to know that $p_a(F_{p_1})=1$, $p_a(F_{p_2})=1$, $p_a(F_{p_3})=1$, and the sets of nodes are

$$\begin{aligned}&\mathcal{N}_{2,1}(F)=\{q_0,q_{23},q_{24}\},\quad \mathcal{N}_{2,2}(F)=\{(q_{11},q_{12})\},\\&\mathcal{N}_3(F)=\{q_3\},\quad \mathcal{N}_4(F)=\{(q_{21},q_{22})\}.\end{aligned} \tag{3.13}$$

Hence the numbers of nodes on F are

$$(\xi_0(F),\xi_1(F),\xi_2(F))=(5,1,0),\quad (\delta_1(F),\delta_2(F))=(1,0). \tag{3.14}$$

Comparing these equations with (2.7) and (2.8), we give an example for Theorem 3.1.

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