Chin. Ann. Math.
37B(6), 2016, 915–928
DOI: 10.1007/s11401-016-1039-4

Chinese Annals of
Mathematics, Series B


# The 3D Non-isentropic Compressible Euler Equations with Damping in a Bounded Domain*

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**Abstract** The authors investigate the global existence and asymptotic behavior of classical solutions to the 3D non-isentropic compressible Euler equations with damping on a bounded domain with slip boundary condition. The global existence and uniqueness of classical solutions are obtained when the initial data are near an equilibrium. Furthermore, the exponential convergence rates of the pressure and velocity are also proved by delicate energy methods.

**Keywords** Non-isentropic, Euler equations, Damping, Exponential convergence
**2000 MR Subject Classification** 35Q30, 76N15

## 1 Introduction

With damping, the three-dimensional compressible Euler equations for non-isentropic flows have the following form:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho u)=0,\\ (\rho u)_t+\nabla\cdot(\rho u\otimes u)+\nabla P=-\alpha\rho u,\\ (\rho\mathcal{E})_t+\nabla\cdot(\rho u\mathcal{E}+uP)=-\alpha\rho u^2.\end{cases}\tag{1.1}$$

Such a system occurs in the mathematical modeling of compressible flow through a porous medium. Here $\rho$, $u=(u_1,u_2,u_3)^t$ and $P$ represent the density, the velocity and the pressure respectively. The total energy $\mathcal{E}=\frac{|u|^2}{2}+e$, where $e$ is the internal energy. The constant $\alpha>0$ models friction. In this paper, we will consider only polytropic fluids, so that the equations of state for the fluid are given by

$$P=R\rho\theta,\quad e=\frac{R}{\gamma-1}\theta,\tag{1.2}$$

where $\theta$ is the absolute temperature. The constants $R>0$ and $\gamma>1$ denote the gas constant and the adiabatic exponent, respectively.

For the isentropic flow, namely $S=$const., (1.1) takes the form

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho u)=0,\\ \partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla P=-\alpha\rho u.\end{cases}\tag{1.3}$$

Manuscript received August 23, 2014. Revised October 10, 2015.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11301172, 11226170, 11571280) and the Scientific Research Fund of Hunan Provincial Education Department (No. 14B077).

The 1D version of (1.3) with various initial and initial-boundary conditions has been studied intensively during the past decades, both classical and weak solutions have been constructed, and the long time behaviors of different solutions have been investigated. There are extensive literatures for both the Cauchy problem and the initial-boundary value problem, and the readers are referred to [2, 8–10, 13, 15-24, 26–27, 36, 39, 41–42] and references therein. For the multi-dimension problem to the isentropic system (1.3), Wang and Yang [34] proved the global existence and asymptotic behavior to the Cauchy problem for the isentropic system (1.3) by the Green function method. Sideris, Thomases and Wang [31] showed that the damping term prevented the development of singularities for small amplitude classical solutions in three-dimensional space, using an equivalent reformulation of the Cauchy problem to obtain effective energy estimates. Pan and Zhao [29] investigated the global existence and asymptotic behavior to the initial boundary value problem for the isentropic system (1.3) by energy method. Fang and Xu [7] studied the existence and asymptotic behavior of $C^1$ solutions on the framework of Besov space. The optimal convergence rates were recently obtained by Tan and Wu [32].

For the adiabatic flow, namely $S \neq$ const., much less is known even for the one-dimensional case. The global existence of smooth solution to the Cauchy problem for 1D version of (1.1) has been proved in [14, 40] for small initial data. The large time behavior of these solutions is known only for some particular initial data (see [11, 25]). For the initial boundary value problem, the readers are referred for instance to [12, 28] and references therein.

From the physical point of view, the 3D model (1.1) describes more realistic phenomena. Also the 3D compressible non-isentropic Euler equations carry some unique features, such as the effect of vorticity, which are totally absent in the 1D case and make the problem more challenging in mathematics. The system (1.1) and the time-asymptotic behavior of the solution are of great importance and are much less understood than its 1D companion. Recently, the authors in [35, 37] studied the global existence and asymptotic behavior of classic solutions to the Cauchy problem and the period boundary problem to the system (1.1) respectively. To our knowledge, there is no work on the global existence and asymptotic behavior of classical solution for the initial boundary value problem to the system (1.1). The main motivation of this article is to give a positive answer to this problem.

To begin with, we note the fact that all thermodynamics variables $\rho, \theta, e, P$ as well as the entropy $S$ can be represented by functions of any two of them. To overcome the difficulties arising from non-isentropic, we rewrite the system (1.1). We take the two variables to be $P$ and $S$, then the equation of state is replaced by

$$\rho = aP^{\frac{1}{\gamma}} \exp\Big\{ - \frac{(\gamma-1)S}{\gamma R} \Big\}, \tag{1.4}$$

where $a > 0$ is a constant. Under the aforementioned assumptions, we can rewrite the system (1.1) in terms of $(P, u, S)$ as follows:

$$\begin{cases} \partial_t P + \gamma P \nabla \cdot u + u \cdot \nabla P = 0, \\ \partial_t u + (u \cdot \nabla) u + \dfrac{\nabla P}{\rho} = -\alpha u, \\ \partial_t S + (u \cdot \nabla) S = 0, \end{cases} \tag{1.5}$$

where $\rho = \rho(P, S)$ is given by (1.4). It should be mentioned that (1.5) is a hyperbolic system, while the dissipation property comes from the damping term. In this paper, we consider the initial boundary value problem for (1.5) with the following initial and boundary conditions:

$$\begin{cases}(P,u,S)(x,0)=(P_0,u_0,S_0)(x), \quad x\in\Omega,\\ u\cdot n|_{\partial\Omega}=0, \quad t\geq 0,\\ \Big(\dfrac{1}{|\Omega|}\displaystyle\int_\Omega P_0^{\frac{1}{\gamma}}\mathrm{d}x\Big)^\gamma=\overline{P}>0,\end{cases} \tag{1.6}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $n$ is the unit outward normal vector on the boundary $\Omega$ and the last condition is imposed to avoid the trivial case, $\rho\equiv 0$.

Before stating the main results, let us introduce some notations for the use throughout this paper. $C$ denotes some positive constant. The norms in the Sobolev spaces $H^m(\Omega)$ and $W^{m,q}(\Omega)$ are denoted respectively by $\|\cdot\|_m$ and $\|\cdot\|_{m,q}$ for $m\geq 0$ and $q\geq 1$. In particular, for $m=0$ we will simply use $\|\cdot\|$ and $\|\cdot\|_{L^q}$. $\|(a,b,c)\|_m$ denotes $\|a\|_m+\|b\|_m+\|c\|_m$. The energy space under consideration is

$$X_k([0,T],\Omega)\equiv\{F:\Omega\times[0,T]\to\mathbb{R}\ (or\ \mathbb{R}^3)\mid \partial_t F\in L^\infty([0,T];H^{3-l}(\Omega)),\ l=0,1,\cdots,k\},$$

equipped with norm

$$|||F(\cdot,t)|||_k\equiv\Big[\sum_{l=0}^k\|\partial_t^l F(\cdot,s)\|_{k-l}^2\Big]^{\frac{1}{2}}$$

for any $F\in X_k([0,T],\Omega)$ and $t\in[0,T]$. Moreover, we use $\langle\cdot,\cdot\rangle$ to denote the inner product in $L^2(\Omega)$. Finally,

$$\nabla=(\partial_1,\partial_2,\partial_3), \quad \partial_i=\partial_{x_i},\ i=1,2,3,$$

and for any integer $l\geq 0$, $\nabla^l f$ denotes all derivatives of order $l$ of the function $f$. And for multi-indices $\alpha$ and $\beta$

$$\alpha=(\alpha_1,\alpha_2,\alpha_3), \quad \beta=(\beta_1,\beta_2,\beta_3),$$

we use

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, \quad |\alpha|=\sum_{i=1}^3\alpha_i,$$

and $C_\alpha^\beta=\frac{\alpha!}{\beta!(\alpha-\beta)!}$, where $\beta\leq\alpha$.

Now, we are ready to state the main results.

**Theorem 1.1** *Assume that the initial data satisfy the compatibility condition, i.e.,* $\partial_t^l u(0)\cdot n|_\Omega=0,\ 0\leq l\leq 3$, *where* $\partial_t^l u(0)\cdot n|_\Omega=0$ *is the* $l$*-th time derivative at* $t=0$ *of any solution of* (1.5)–(1.6), *as calculated from* (1.5) *to yield an expression in terms of* $P_0,u_0$ *and* $S_0$, *and* $\|(P_0^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u_0,S_0-\overline{S})\|_3$ *is sufficiently small. Then the initial boundary value problem* (1.5)–(1.6) *admits a unique solution* $(P,u,S)$ *globally in time with* $P>0$, *satisfying*

$$P^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u,S-\overline{S}\in C^0([0,\infty),H^3(\Omega))\cap C^1([0,\infty),H^2(\Omega))\cap X_3([0,\infty),\Omega).$$

*Moreover, there exist positive constants* $C_0$ *and* $\eta_0$, *which are independent of* $t$, *such that for any* $t\geq 0$, *it holds*

$$|||(P^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}})(\cdot,t)|||_3+|||u(\cdot,t)|||_3\leq C_0\|(P_0^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u_0)\|_3\exp\{-\eta_0 t\}, \tag{1.7}$$

$$\|(S-\overline{S})(\cdot,t)\|\leq C_0\|(P_0^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u_0,S_0-\overline{S})\|_3\exp\{C_0\|(P_0^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u_0)\|_3\}, \tag{1.8}$$

$$|||\partial_t S(\cdot,t)|||_2\leq C_0\|(P_0^{\frac{1}{\gamma}}-\overline{P}^{\frac{1}{\gamma}},u_0,S_0-\overline{S})\|_3\exp\{-\eta_0 t\}. \tag{1.9}$$

**Remark 1.1** The methods of this paper can be applied to study the global existence and asymptotic behavior for the initial boundary value problem to the 3D compressible non-isentropic Navier-Stokes equations without heat conductivity (see [3]) and the 3D viscous liquid-gas two phase flow model (see [38]).

**Remark 1.2** By applying the similar idea of [38], we can also prove that Theorem 1.1 still holds under only the smallness assumption on $H^2$-norm of the initial data.

Now, we sketch the main idea of the proof and explain some of the main difficulties and techniques involved in the process. First, due to non-isentropic, we can not use the methods of [29, 30–32, 34] where the isentropic system (1.2) has been studied. To overcome the difficulties for the appearance of the non-isentropic term, as in [3, 38], we first rewrite the system (1.1) into (1.5). However, we can not work directly on the system of the variables $(P, u, S)$ as in [3, 38]. Indeed, on one hand, the integral $\int_\Omega (P(x,t) - P_0(x))\mathrm{d}x$ is in general not zero since the variable $P$ is not conservative. On the other hand, by noting the dissipation structure of (1.5), it is clear that there is no dissipation estimate for $L^2$-norm of the variable $P$. Therefore, it seems impossible to get the exponential decay estimate on $\|P\|$ by the Poincaré inequality and the Gronwall's inequality as in [29]. The key idea here is that instead of the variables $(P, u, S)$, we study the system of the variables $(\omega, u, S)$ with $w = P^{\frac{\gamma-1}{2\gamma}} - \overline{P}^{\frac{\gamma-1}{2\gamma}}$ (see (2.2)–(2.3) for details). One of main observations in this article is that the dissipative variables $\omega$ and $u$ satisfy the first and second equations of (2.2) whose linear parts possess the same structure as that of the compressible isentropic Euler equations with damping (1.2), while the non-dissipative variable $S$ satisfies the homogeneous transport equation the third equation of (2.2). Then, in order to obtain a priori estimates of solutions to (2.2)–(2.3), we can apply the similar energy method as in [3–5, 29, 33, 37–38] to the first two equations of (2.2) to obtain the uniform bound of $(\omega, u)$ under the assumption that $\|(\omega, u, S)\|_3$ is sufficiently small, see Lemmas 3.1–3.2 in Section 3. With these in hand, the variables $(w, u)$ can be shown to converge exponentially to zero from the Poincaré inequality and Gronwall's inequality. It is worth mentioning that the crucial part of the proof is to obtain a Lyapunov-type energy inequality (see (3.37)). Then, the bound of $S$ will be derived by the exponential decay estimates on $(w, u)$ and the Gronwall's inequality. Second, due to the slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives. As in [29], the main idea is to get the key estimates of $\nabla u$ by $\nabla \times u$ and $\nabla \cdot u$, see Lemma 3.2 below. Using the special structure of (2.2) together with an induction on the number of spatial derivatives, the estimate of total energy is reduced to those for the vorticity and temporal derivatives. And the proof is completed by showing that (1.7) is true for the vorticity and temporal derivatives.

The plan of the rest of this paper is as follows. In Section 2, we reformulate the original system to get a quasi-linear symmetric hyperbolic system and give some basic facts that will be used in this paper together with the local existence result. In Section 3, we prove Theorem 1.1 by delicate energy estimates.

## 2 Reformulation and Local Existence

In this section, we are going to reformulate the initial-boundary value problem (1.5)–(1.6). First we reformulate (1.5) to get a symmetric hyperbolic system. Introducing the nonlinear transformation $\widetilde{\omega} = P^{\frac{\gamma-1}{2\gamma}}$, we get from the original system (1.5) that

$$\begin{cases} \widetilde{\omega}_t + u\cdot\nabla\widetilde{\omega} + \frac{\gamma-1}{2}\widetilde{\omega}\nabla\cdot u = 0, \\ \dfrac{a(\gamma-1)^2}{4\gamma}\exp\Big\{-\dfrac{(\gamma-1)S}{\gamma R}\Big\}u_t + \dfrac{\gamma-1}{2}\widetilde{\omega}\nabla\widetilde{\omega} + \dfrac{a(\gamma-1)^2}{4\gamma}\exp\Big\{-\dfrac{(\gamma-1)S}{\gamma R}\Big\}u\cdot\nabla u \\ = -\alpha a\dfrac{(\gamma-1)^2}{4\gamma}\exp\Big\{-\dfrac{(\gamma-1)S}{\gamma R}\Big\}u, \\ S_t + u\cdot\nabla S = 0. \end{cases} \tag{2.1}$$

Denoting $\omega = \frac{1}{\kappa}(\widetilde{\omega} - \overline{\omega})$ with $\kappa = \frac{\gamma-1}{2}$, $\overline{\omega} = \overline{P}^{\frac{\gamma-1}{2\gamma}}$, $s = \dfrac{a\exp\Big\{-\frac{(\gamma-1)S}{\gamma R}\Big\}}{\gamma}$, we get the desired symmetric system for the perturbation $(w, u, s)$

$$\begin{cases} \omega_t + u\cdot\nabla\omega + \kappa\omega\nabla\cdot u + \overline{\omega}\nabla\cdot u = 0, \\ su_t + \kappa\omega\nabla\omega + \overline{\omega}\nabla\omega + su\cdot\nabla u = -\alpha su, \\ s_t + u\cdot\nabla s = 0. \end{cases} \tag{2.2}$$

The initial and boundary conditions become

$$\begin{cases} (\omega, u, s)(x,0) = (\omega_0, u_0, s_0), \\ u\cdot n|_{\partial\Omega} = 0, \quad t \ge 0, \end{cases} \tag{2.3}$$

with

$$\omega_0 = \frac{1}{\kappa}(P_0^{\frac{\gamma-1}{2\gamma}} - \overline{P}^{\frac{\gamma-1}{2\gamma}}), \quad s_0 = \frac{a\exp\{e - \frac{(\gamma-1)S_0}{\gamma R}\}}{\gamma}.$$

Before giving the proof of Theorem 1.1, we state the local existence result for the system (2.2)–(2.3), which can be established using the arguments in [3, 26–27].

**Proposition 2.1** (Local Existence) *Let* $\overline{s} = \frac{a\exp\{-\frac{(\gamma-1)\overline{S}}{\gamma R}\}}{\gamma} > 0$ *be fixed and suppose that* $(\omega_0, u_0, s_0 - \overline{s}) \in H^3(\Omega)$ *are such that* $\inf_{x\in\Omega}\{P_0(x) + \overline{P}\} > 0$, *and satisfy the compatibility condition, i.e.,* $\partial_t^l u(0)\cdot n|_\Omega = 0$, $0 \le l \le 3$. *Then there exists a positive constant* $\varepsilon_0$ *such that if* $\|(\omega_0, u_0, s_0 - \overline{s})\|_3 \le \varepsilon_0$, *then there exists a positive constant* $T_0$ *depending on* $\varepsilon_0$ *such that the initial-boundary value problem* (2.2)–(2.3) *admits a unique solution* $(\omega, u, s - \overline{u}) \in C^1(\overline{\Omega}\times[0,T_0]) \cap X_3([0,T_0],\Omega)$ *which satisfies*

$$\inf_{\substack{x\in\Omega \\ 0\le t\le T_0}}\{P(x,t) + \overline{P}\} > 0$$

*and*

$$\sup_{0\le t\le T_0}\|(\omega, u, s-\overline{s})(\cdot,t)\|_3 \le 2\|(\omega_0, u_0, s_0 - \overline{s})\|_3.$$

To prove global existence of a smooth solution with small initial data, it suffices to establish global a priori estimate of the solution.

**Proposition 2.2** (A Priori Estimate) *Let* $(\omega_0, u_0, s_0 - \overline{s}) \in H^3(\Omega)$ *and suppose that the initial-boundary value problem* (2.2)–(2.3) *has a solution* $(\omega, u, s - \overline{s}) \in C^1(\overline{\Omega}\times[0,T]) \cap X_3([0,T],\Omega)$ *for given* $T > 0$. *Then there exist a small positive constant* $\varepsilon_1(\le \varepsilon_0)$ *and two positive constants* $C_1$ *and* $\eta_1$, *which are independent of* $T$, *such that if*

$$\sup_{0\le t\le T}\|(\omega, u, s-\overline{s})(\cdot,t)\|_3 \le \varepsilon_1, \tag{2.4}$$

*then for any* $t \in [0,T]$, *it holds that*

$$|||(\omega(\cdot,t)|||_3 + |||u(\cdot,t)|||_3 \le C_1\|(\omega_0,u_0)\|_3 \exp\{-\eta_1 t\}, \tag{2.5}$$
$$\|(s-\overline{s})(\cdot,t)\| \le C_1\|s_0-\overline{s}\| \exp\{C_1\|(\omega_0,u_0)\|_3\}, \tag{2.6}$$
$$|||\partial_t s(\cdot,t)|||_2 \le C_1\|(\omega_0,u_0,s_0-\overline{s})\|_3 \exp\{-\eta_1 t\}. \tag{2.7}$$

**Proof of Theorem 1.1** Choose $\varepsilon_2$, $C_1$ and $\eta_1$ such that $\varepsilon_2 = \min\{1, \frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2C_1}\exp\{C_1\}\}$, $C_1 = C_0$ and $\eta_1 = \eta_0$. Then the local solution of (2.2)–(2.3) can be continued globally in time, provided that the smallness condition $\|(\omega_0,u_0,s_0-\overline{s})\|_3 \le \varepsilon_2$ is satisfied. In fact, we have $\|(\omega_0,u_0,s_0-\overline{s})\|_3 \le \varepsilon_2 \le \varepsilon_1$. Therefore, by Proposition 2.1, there is a positive constant $T_1 = T_1(\varepsilon_1)$ such that a solution exists on $[0,T_1]$ and satisfies $\|(\omega,u,s-\overline{s})(\cdot,t)\|_3 \le 2\|(\omega_0,u_0,s_0-\overline{s})\|_3 \le \varepsilon_1$ for $t \in [0,T_1]$. Hence we can apply Proposition 2.2 with $T = T_1$ to get $\|(\omega,u,s-\overline{s})(\cdot,T_1)\|_3 \le C_1\exp\{C^1\}\varepsilon_2 \le \frac{\varepsilon_1}{2} \le \varepsilon_1$. Therefore, we can apply Proposition 2.1 by taking $t = T_1$ as the new initial time. Then we have a solution on $[T_1,2T_1]$ with the estimate $\|(\omega,u,s-\overline{s})(\cdot,t)\|_3 \le 2\|(\omega,u,s-\overline{s})(\cdot,T_1)\|_3 \le \varepsilon_1$ for $t \in [T_1,2T_1]$. Therefore $\|(\omega,u,s-\overline{s})(\cdot,t)\|_3 \le \varepsilon_1$ holds on $[0,2T_1]$. Hence Proposition 2.2 again gives the estimates (2.5)–(2.7) for $t \in [0,2T_1]$. In the same way we can extend the solution to the interval $[0,nT_1]$ successively, $n = 1,2,\cdots$, and get a global solution. The estimates (1.7)–(1.9) is a consequence of (2.5)–(2.7). This completes the proof of Theorem 1.1.

The proof of Proposition 2.2 is based on several steps of careful energy estimates which are stated as a sequence of lemmas in Section 3.

## 3 Global Existence and Large Time Behavior

In this section, we devote ourselves to prove Proposition 2.2. For convenience, we let

$$W_1(t) \equiv |||\omega(\cdot,t)|||_3^2 + |||u(\cdot,t)|||_3^2 = \sum_{l=0}^{3}(\|\partial_t^l\omega(\cdot,t)\|_{3-l}^2 + \|\partial_t^l u(\cdot,t)\|_{3-l}^2), \tag{3.1}$$

$$\begin{aligned} W_2(t) &\equiv W_1(t) + |||s(\cdot,t)-\overline{s}|||_3^2 \\ &= \sum_{l=0}^{3}(\|\partial_t^l\omega(\cdot,t)\|_{3-l}^2 + \|\partial_t^l u(\cdot,t)\|_{3-l}^2 + \|\partial_t^l(s(\cdot,t)-\overline{s})\|_{3-l}^2). \end{aligned} \tag{3.2}$$

Throughout this section, we suppose that the initial-boundary value problem (2.2)–(2.3) has a solution $(\omega,u,s-\overline{s})$ in the space $C^1(\overline{\Omega}\times[0,T]) \cap X_3([0,T],\Omega)$ with some $T \in (0,+\infty]$, and the inequality (2.4) holds. We also omit the variable $t$ of all functions in the proof of different lemmas in this section for simplicity.

In what follows, a series of lemmas on the energy estimates are given. First we recall some inequalities of Sobolev type (see [6]).

**Lemma 3.1** *Let* $\Omega$ *be any bounded domain in* $\mathbb{R}^3$ *with smooth boundary. Then it holds:*
(i) $\|f\|_{L^\infty(\Omega)} \le C\|f\|_{H^2(\Omega)}$,
(ii) $\|f\|_{L^q(\Omega)} \le C\|f\|_{H^1(\Omega)}$, $2 \le q \le 6$
*for some constant* $C > 0$ *depending only on* $\Omega$.

As in [29], the following lemma (see [1]) plays an important role in our proofs, which gives the estimate of $\nabla u$ by $\nabla\cdot u$ and $\nabla\times u$.

**Lemma 3.2** *Let $u \in H^k(\Omega)$ be a vector-valued function satisfying $u \cdot n|_\Omega = 0$, where $n$ is the unit outer normal of $\partial\Omega$. Then*

$$\|u\|_k \le C(\|\nabla \cdot u\|_{k-1} + \|\nabla \times u\|_{k-1} + \|u\|_{k-1}) \tag{3.3}$$

*for $k \ge 1$, where constant $C$ depends only on $k$ and $\Omega$.*

The next lemma is an application of Lemma 3.2, which is crucial to complete the proof of Proposition 2.2. Indeed, the lemma states that the bounds of spatial derivatives can be controlled by those of the temporal derivatives and the vorticity. Let $\upsilon = \nabla \times u$ and define

$$U(t) \equiv \sum_{l=0}^{3}(\|\partial_t^l \omega\|^2 + \|\partial_t^l u\|^2), \quad V(t) \equiv \sum_{l=0}^{2}\|\partial_t^l \upsilon\|^2. \tag{3.4}$$

**Lemma 3.3** *Under the assumptions of Proposition 2.2, there exists a constant $C_2 > 0$ which is independent of $\varepsilon$ such that*

$$W_1(t) \le C_2(U(t) + V(t)). \tag{3.5}$$

**Proof** From the equation $(2.2)_2$, we have

$$\nabla\omega = -\frac{1}{\kappa\omega + \overline{\omega}}(\alpha s u + s u_t + s u \cdot \nabla u). \tag{3.6}$$

Using the smallness of $W_2(t)$, Lemma 3.1 and Cauchy-Schwarz inequality, we easily get

$$\begin{aligned}
\|\nabla\omega\|^2 &\le \Big\|\frac{1}{\kappa\omega + \overline{\omega}}(\alpha s u + s u_t + s u \cdot \nabla u)\Big\|^2 \\
&\le C\Big\|\frac{1}{\kappa\omega + \overline{\omega}}\Big\|_{L^\infty}^2 \|s\|_{L^\infty}^2(\|u\|^2 + \|u_t\|^2 + \|u\|_{L^\infty}^2\|\nabla u\|^2) \\
&\le C(\|u\|^2 + \|u_t\|^2) + CW_1^2(t)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\|\nabla\omega_t\|^2 &\le \Big\|\partial_t\Big[\frac{1}{\kappa\omega + \overline{\omega}}(\alpha s u + s u_t + s u \cdot \nabla u)\Big]\Big\|^2 \\
&\le C(\|\omega_t u\|^2 + \|\omega_t u_t\|^2 + \|\omega_t u \cdot \nabla u\|^2 + \|s_t u\|^2 + \|s_t u_t\|^2 + \|s_t u \cdot \nabla u\|^2 \\
&\quad + \|u_t\|^2 + \|u_{tt}\|^2 + \|u_t \cdot \nabla u\|^2 + \|u \cdot \nabla u_t\|^2) \\
&\le C(\|u\|_{L^\infty}^2\|\omega_t\|^2 + \|\omega_t\|_{L^4}^2\|u_t\|_{L^4}^2 + \|\omega_t\|_{L^4}^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^4}^2 \\
&\quad + \|u\|_{L^\infty}^2\|s_t\|^2 + \|s_t\|_{L^4}^2\|u_t\|_{L^4}^2 + \|s_t\|_{L^4}^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^4}^2 \\
&\quad + \|u_t\|^2 + \|u_{tt}\|^2 + \|\nabla u\|_{L^\infty}^2\|u_t\|^2 + \|\nabla u\|_{L^\infty}^2\|\nabla u_t\|^2) \\
&\le C(\|u_t\|^2 + \|u_{tt}\|^2) + CW_1(t)W_2(t).
\end{aligned} \tag{3.8}$$

Taking time derivatives of (3.6) twice, after a tedious but direct computation, we also have

$$\|\nabla\omega_{tt}\|^2 \le C(\|u_{tt}\|^2 + \|u_{ttt}\|^2) + CW_1(t)W_2(t). \tag{3.9}$$

By using the first equation of (2.2), we have

$$\nabla \cdot u = -\frac{1}{\kappa\omega + \overline{\omega}}(\omega_t + u \cdot \nabla\omega). \tag{3.10}$$

So, we can easily get

$$\|\nabla\cdot u\|^2\le C(\|\omega_t\|^2+W_1^2(t)). \tag{3.11}$$

Using Lemma 3.2 with $k=1$ and (3.11), we obtain

$$\begin{aligned}\|u\|_1^2&\le C(\|\nabla\cdot u\|^2+\|v\|^2+\|u\|^2)\\&\le C(\|\omega_t\|^2+\|v\|^2+\|u\|^2)+CW_1^2(t).\end{aligned} \tag{3.12}$$

Next, we take time derivatives of (3.10). It is easy to see that every time derivative up to order two of $\nabla\cdot u$ can be bounded by $U(t)+W_1(t)W_2(t)$. Furthermore, together with an induction on the number of spatial derivatives, the same is true for any derivative up to order two of $\nabla\cdot\omega$ and $\nabla\cdot u$. By applying Lemma 3.2 with $k=1,2,3$ respectively, we can deduce that

$$\begin{aligned}&\sum_{l=0}^{3}(\|\partial_t^l\omega(\cdot,t)\|_{3-l}^2+\|\partial_t^l u(\cdot,t)\|_{3-l}^2)\\&\le C\sum_{l=0}^{3}(\|\partial_t^l\omega\|^2+\|\partial_t^l u\|^2)+C\sum_{l=0}^{2}\|\partial_t^l v\|^2+CW_2(t)W_1(t).\end{aligned} \tag{3.13}$$

Since $W_2(t)$ is small, we prove (3.5). Therefore, the proof of Lemma 3.3 is completed.

Lemma 3.3 reduces the estimate of $W(t)$ to those for $U(t)$ and $V(t)$. In the following, we will devote ourselves to deduce the estimates of $U(t)$ and $V(t)$.

**Lemma 3.4** *Under the assumptions of Proposition* 2.2*, there exists a constant* $C_3>0$ *which is independent of* $\varepsilon$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^{3}\|\partial_t\omega\|^2+s\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^{3}\|\partial_t u\|^2+2\alpha s\sum_{l=0}^{3}\|\partial_t^l u\|^2\le C_3W_2^{\frac12}(t)W_1(t). \tag{3.14}$$

**Proof** In the following, we will prove Lemma 3.4 by five steps.

**Step 1** Zero order estimate Multiplying the first and second equations of (2.2) by $\omega$, $u$ respectively and then integrating them over $\Omega$, using the boundary condition $u\cdot n|_{\partial\Omega}=0$, we have

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\alpha s\|u\|^2=-\langle u\cdot\nabla\omega+\kappa\omega\nabla\cdot u,\omega\rangle-\langle\kappa\omega\nabla\omega+su\cdot\nabla u,u\rangle. \tag{3.15}$$

From Lemma 3.1, Hölder's inequality and Cauchy-Schwarz inequality, we have

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\alpha s\|u\|^2\le C(\|\nabla\omega\|_{L^\infty}+\|\nabla u\|_{L^\infty})(\|\omega\|^2+\|u\|^2)\le CW_1^{\frac32}(t). \tag{3.16}$$

**Step 2** First order estimate Differentiating the first and second equations of (2.2) with respect to $t$ once, multiplying the resultant equations by $\omega_t$, $u_t$ respectively, integrating over $\Omega$ and using the boundary conditions $\partial_t^l u\cdot n|_{\partial\Omega}=0$ with $l=0,1$, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_t\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2+s\|u_t\|^2\\&=-\langle(u\cdot\nabla\omega+\kappa\omega\nabla\cdot u)_t,\omega_t\rangle-\langle(\kappa\omega\nabla\omega+su\cdot\nabla u)_t,u_t\rangle-\langle s_t(u_t+au),u_t\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\langle u_t\cdot\nabla\omega,\omega_t\rangle+\frac{1}{2}\langle\nabla\cdot u,|\omega_t^2|\rangle-\kappa\langle\omega_t\nabla\cdot u,\omega_t\rangle+\kappa\langle u_t\omega_t,\nabla\omega\rangle-\kappa\langle\omega_t\nabla\omega,u_t\rangle\\
&\quad-\langle s_tu\cdot\nabla u,u_t\rangle-\langle su_t\cdot\nabla u,u_t\rangle+\frac{1}{2}\langle\nabla su+s\nabla\cdot u,|u_t|^2\rangle-\langle s_t(u_t+au),u_t\rangle\\
&\le C(\|\nabla\omega\|_{L^\infty}+\|\nabla u\|_{L^\infty}+\|s_t\|_{L^\infty}\|\nabla u\|_{L^\infty}+\|\nabla s\|_{L^\infty}\|u\|_{L^\infty}+\|s_t\|_{L^\infty})\\
&\quad\times(\|\omega_t\|^2+\|u\|^2+\|u_t\|^2)
\end{aligned}
\tag{3.17}
$$

for some constant $C>0$. From Lemma 3.1, Hölder inequality and Cauchy-Schwarz inequality, we deduce that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_t\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2+s\|u_t\|^2\le CW_2^{\frac{1}{2}}(t)W_1(t). \tag{3.18}
$$

**Step 3** Second order estimate Repeating the above procedure again for 2nd order time derivatives, we can get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{tt}\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{tt}\|^2+\alpha s\|u_{tt}\|^2\\
&= -\langle(u\cdot\nabla\omega+\kappa\omega\nabla\cdot u)_{tt},\omega_{tt}\rangle-\langle(\kappa\omega\nabla\omega+su\cdot\nabla u)_{tt},u_{tt}\rangle\\
&\quad-\langle s_{tt}(u_t+au),u_{tt}\rangle-\langle s_t(u_{tt}+au_t),u_{tt}\rangle\\
&= -\langle u_{tt}\cdot\nabla\omega+2u_t\cdot\nabla\omega_t,\omega_{tt}\rangle+\frac{1}{2}\langle\nabla\cdot u,|\omega_{tt}^2|\rangle-\kappa\langle\omega_{tt}\nabla\cdot u+2\omega_t\nabla\cdot u_t,\omega_{tt}\rangle\\
&\quad+\kappa\langle u_{tt}\omega_{tt},\nabla\omega\rangle-\kappa\langle\omega_{tt}\nabla\omega+\omega_t\nabla\omega_t,u_{tt}\rangle-\langle(s_{tt}u+2s_tu_t+su_{tt})\cdot\nabla u,u_{tt}\rangle\\
&\quad-2\langle(s_tu+su_t)\cdot\nabla u_t,u_{tt}\rangle+\frac{1}{2}\langle\nabla su+s\nabla\cdot u,|u_{tt}|^2\rangle-\langle s_{tt}(u_t+au),u_{tt}\rangle\\
&\quad-2\langle s_t(u_{tt}+au_t),u_{tt}\rangle\\
&\le C(\|\nabla(\omega,u)\|_{L^\infty}+\|(\omega_t,u_t)\|_{L^\infty})(\|\nabla(\omega_t,u_t)\|^2+\|(\omega_{tt},u_{tt})\|^2)\\
&\quad+C(\|u\|_{L^\infty}\|\nabla u\|_{L^\infty}+\|s_t,\nabla s\|_{L^\infty}\|(u,\nabla u)\|_\infty)(\|s_{tt}\|^2+\|u_t\|^2+\|u_{tt}\|^2)\\
&\quad+C\|s_{tt}\|_{L^3}\|u_t\|_{L^3}\|u_{tt}\|_{L^3}+C\|s_{tt}\|_{L^3}\|u\|_{L^3}\|u_{tt}\|_{L^3}+\|s_t\|_{L^\infty}\|(u_t,u_{tt})\|^2
\end{aligned}
\tag{3.19}
$$

for some constant $C>0$. By virtue of Lemma 3.1, Hölder inequality and Cauchy-Schwarz inequality, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{tt}\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{tt}\|^2+s\|u_{tt}\|^2\le CW_2^{\frac{1}{2}}(t)W_1(t). \tag{3.20}
$$

**Step 4** Third order estimate Repeating the above procedure again for 3rd order time derivatives, we get the following

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{ttt}\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ttt}\|^2+as\|u_{ttt}\|^2\\
&= -\langle(u\cdot\nabla\omega+\kappa\omega\nabla\cdot u)_{ttt},\omega_{ttt}\rangle-\langle(\kappa\omega\nabla\omega+su\cdot\nabla u)_{ttt},u_{ttt}\rangle\\
&\quad-\langle s_{ttt}(u_t+au),u_{ttt}\rangle-3\langle s_{tt}(u_{tt}+au_t),u_{ttt}\rangle-3\langle s_t(u_{ttt}+au_{tt}),u_{ttt}\rangle\\
&= -\langle u_{ttt}\cdot\nabla\omega+3u_{tt}\cdot\nabla\omega_t+3u_t\cdot\nabla\omega_{tt},\omega_{ttt}\rangle+\frac{1}{2}\langle\nabla\cdot u,|\omega_{ttt}^2|\rangle\\
&\quad-\kappa\langle\omega_{ttt}\nabla\cdot u+3\omega_{tt}\nabla\cdot u_t+3\omega_t\nabla\cdot u_{tt},\omega_{ttt}\rangle+\kappa\langle u_{ttt}\omega_{ttt},\nabla\omega\rangle\\
&\quad-\kappa\langle\omega_{ttt}\nabla\omega+3\omega_{tt}\nabla\omega_t+3\omega_t\nabla\omega_{tt},u_{ttt}\rangle\\
&\quad-\langle(s_{ttt}u+3s_{tt}u_t+3s_tu_{tt}+su_{ttt})\cdot\nabla u,u_{ttt}\rangle\\
&\quad-3\langle(s_{tt}u+2s_tu+2su_t+su_{tt})\cdot\nabla u_t+(s_tu+su_t)\cdot\nabla u_{tt},u_{ttt}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\langle \nabla s u+s\nabla\cdot u,|u_{ttt}|^2\rangle-\langle s_{ttt}(u_t+au),u_{ttt}\rangle\\
&-3\langle s_{tt}(u_{tt}+au_t),u_{ttt}\rangle-3\langle s_t(u_{ttt}+au_{tt}),u_{ttt}\rangle\\
=&\sum_{i=1}^{11}I_i. \qquad (3.21)
\end{aligned}
$$

By virtue of Lemma 3.1, Hölder's inequality and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|I_1|&\le \|(\nabla\omega,u_t)\|_{L^\infty}\|(u_{ttt},\nabla\omega_{tt})\|\|\omega_{ttt}\|+\|u_{tt}\|_{L^4}\|\nabla\omega_t\|_{L^4}\|\omega_{ttt}\|\\
&\le CW_2^{\frac{1}{2}}(t)W_1(t).
\end{aligned}
$$

Bounds for the other terms are obtained in a similar way, we finally deduce

$$
\sum_{i=1}^{11}|I_i|\le CW_2^{\frac{1}{2}}(t)W_1(t)+CW_2(t)W_1(t).
$$

Since $W_2(t)$ is small, substituting the above two inequalities into (3.21), we finally obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{ttt}\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ttt}\|^2+as\|u_{ttt}\|^2\le CW_2^{\frac{1}{2}}(t)W_1(t). \qquad (3.22)
$$

**Step 5** Proof of Lemma 3.4 Putting (3.16), (3.18), (3.20) and (3.22) together gives (3.14). This completes the proof of Lemma 3.4.

Lemma 3.4 contains only the dissipation in velocity. In the next lemma, we will deduce the dissipation in pressure due to nonlinearity.

**Lemma 3.5** *Under the assumptions of Proposition* 2.2*, there exist two positive constants* $C_4, C_5$ *which are independent of* $\varepsilon$ *such that*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{l=1}^{3}\int_\Omega(-\partial_t^{l-1}\omega\partial_t^l\omega)\mathrm{d}x\Big)+\sum_{l=0}^{3}\|\partial_t^l\omega\|^2\le C_4W_2^{\frac{1}{2}}(t)W_1(t)+C_5\sum_{l=0}^{3}\|\partial_t^lu\|^2. \qquad (3.23)
$$

**Proof** First of all, we notice that $P^{\frac{1}{\gamma}}$ satisfies the continuity equation, i.e., $\partial_tP^{\frac{1}{\gamma}}+\mathrm{div}(P^{\frac{1}{\gamma}}u)=0$, which yields $\int_\Omega(P^{\frac{1}{\gamma}}(t)-\overline{P}^{\frac{1}{\gamma}})\mathrm{d}x=0$. This together with Poincaré's inequality implies that $\|P^{\frac{1}{\gamma}}(t)-\overline{P}^{\frac{1}{\gamma}}\|\le C\|\nabla P^{\frac{1}{\gamma}}\|$. Since $W_2(t)$ is small, we can deduce that $\|P^{\frac{1}{\gamma}}(t)-\overline{P}^{\frac{1}{\gamma}}\|$ is equivalent to $\|\omega\|$ and $\|\nabla P^{\frac{1}{\gamma}}\|$ is equivalent to $\|\nabla\omega\|$, thus we have $\|\omega\|\le C\|\nabla\omega\|$. By using (3.7), we have

$$
\|\omega\|^2\le C(\|u\|^2+\|u_t\|^2)+CW_1^2(t). \qquad (3.24)
$$

Differentiating the first equation of (2.2) with respect to $t$, we get

$$
\omega_{tt}=-(u\cdot\nabla\omega)_t-[\kappa\omega(\nabla\cdot u)]_t-\overline{\omega}\nabla\cdot u_t. \qquad (3.25)
$$

Multiplying (3.25) by $\omega$ and integrating the resultant equation over $\Omega$, we obtain

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega\omega\omega_t\mathrm{d}x\Big)+\|\omega_t\|^2=\langle(u\cdot\nabla\omega)_t,\omega\rangle+\langle[(\kappa\omega+\overline{\omega})(\nabla\cdot u)]_t,\omega\rangle. \qquad (3.26)
$$

From the second equation of (2.2), we have

$$(\kappa\omega+\overline{\varpi})\nabla\omega=-\frac{(u_t+u\cdot\nabla u+au)}{s}. \tag{3.27}$$

Substituting (3.27) into (3.26), using the boundary conditions $\partial_t^l u\cdot n|_{\partial\Omega}=0$ with $l=0,1$ and integrating by part, we have

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \omega\omega_t\mathrm{d}x\Big)+\|\omega_t\|^2\\
&=\langle u_t\cdot\nabla\omega,\omega\rangle+\langle u\cdot\nabla\omega_t,\omega\rangle+\kappa\langle\omega_t\nabla\cdot u,\omega\rangle+\langle(\kappa\omega+\overline{\varpi})\nabla\cdot u_t,\omega\rangle\\
&=\langle u_t\cdot\nabla\omega,\omega\rangle-\langle\omega\nabla\cdot u+u\cdot\nabla\omega,\omega_t\rangle+\kappa\langle\omega_t\nabla\cdot u,\omega\rangle\\
&\quad-\langle\kappa\nabla\omega u_t,\omega\rangle-\langle u_t,(\kappa\omega+\overline{\varpi})\nabla\omega\rangle\\
&=\langle u_t\cdot\nabla\omega,\omega\rangle-\langle\omega\nabla\cdot u+u\cdot\nabla\omega,\omega_t\rangle+\kappa\langle\omega_t\nabla\cdot u,\omega\rangle\\
&\quad-\langle\kappa\nabla\omega u_t,\omega\rangle+\Big\langle u_t,\frac{(u_t+u\cdot\nabla u+au)}{s}\Big\rangle.
\end{aligned} \tag{3.28}$$

Using the same idea in proof of Lemma 3.4, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \omega\omega_t\mathrm{d}x\Big)+\|\omega_t\|^2\le C(W_1^{\frac12}(t)W_2(t)+\|u(t)\|^2+\|u_t(t)\|^2). \tag{3.29}$$

Repeating the above procedure again for 2nd and 3rd order time derivatives of (3.25), we have

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \omega_t\omega_{tt}\mathrm{d}x\Big)+\|\omega_{tt}\|^2\le C(W_1^{\frac12}(t)W_2(t)+\|u_t(t)\|^2+\|u_{tt}(t)\|^2),\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \omega_{tt}\omega_{ttt}\mathrm{d}x\Big)+\|\omega_{ttt}\|^2\le C(W_1^{\frac12}(t)W_2(t)+\|u_{tt}(t)\|^2+\|u_{ttt}(t)\|^2),
\end{aligned}$$

which together with (3.24) and (3.29) implies (3.23). This completes the proof of Lemma 3.5.

Now, we are ready to combine Lemma 3.4 and Lemma 3.5 to deduce the total dissipation. To do this, we let $D_1>0$ be a suitably large positive constant, and define

$$H(t)\equiv D_1\Big[\sum_{l=0}^3(\|\partial_t^l\omega\|^2+s\|\partial_t^l u\|^2)\Big]-\sum_{l=1}^3\int_\Omega(\partial_t^{l-1}\omega\partial_t^l\omega)\mathrm{d}x. \tag{3.30}$$

Since $D_1>0$ is large enough and $\|s(\cdot,t)-\overline{s}\|_3$ is sufficiently small, the function $H(t)$ is equivalent to $U(t)$.

**Lemma 3.6** *Under the assumptions of Proposition* 2.2, *there exist two positive constants* $C_6,C_7$ *which are independent of* $\varepsilon$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)+C_6H(t)\le C_7W_2^{\frac12}(t)W_1(t). \tag{3.31}$$

**Proof** $D_1\times(3.14)+(3.23)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)+(2D_1\alpha s-C_5)\sum_{l=0}^3\|\partial_t^l u\|^2+\sum_{l=0}^3\|\partial_t^l\omega\|^2\le CW_2^{\frac12}(t)W_1(t). \tag{3.32}$$

Since $D_1>0$ is large and $\|s(\cdot,t)-\overline{s}\|_3$ is small, we deduce (3.31) directly from (3.32). This completes the proof of lemma.

The last lemma is concerned with the dissipation in $V(t)$ defined in (3.4).

**Lemma 3.7** *Under the assumptions of Proposition* 2.2*, there exists a positive constant* $C_8$ *which is independent of* $\varepsilon$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\alpha V(t)\le C_8W_2^{\frac{1}{2}}(t)W_1(t). \tag{3.33}$$

**Proof** Taking the curl of the second equation of (2.2), we get

$$\upsilon_t(t)+a\upsilon(t)=-\nabla\times\Big(\frac{\kappa\omega\nabla\omega+\overline{\omega}\nabla\omega}{s}\Big)-u\cdot\nabla\upsilon+\upsilon\cdot\nabla u+\upsilon(\nabla\cdot u). \tag{3.34}$$

Taking any mixed derivative of the above equation, we obtain

$$\begin{aligned}&\partial_t^{\alpha_1}\partial_x^{\alpha_2}\upsilon_t(t)+\alpha\partial_t^{\alpha_1}\partial_x^{\alpha_2}\upsilon(t)\\ =\ &\partial_t^{\alpha_1}\partial_x^{\alpha_2}\Big[-\nabla\times\Big(\frac{\kappa\omega\nabla\omega+\overline{\omega}\nabla\omega}{s}\Big)-u\cdot\nabla\upsilon+\upsilon\cdot\nabla u+\upsilon(\nabla\cdot u)\Big],\end{aligned} \tag{3.35}$$

where $\alpha_1,\alpha_2$ satisfy $|\alpha_1|+|\alpha_2|\le 2$. Noticing that $\nabla\times(\kappa\omega\nabla\omega+\overline{\omega}\nabla\omega)=0$, multiplying the above equation by $\partial_t^{\alpha_1}\partial_x^{\alpha_2}\upsilon(t)$ and integrating the resulting equation by using the boundary condition, together with the standard energy estimate used in deriving Lemmas 3.4–3.5, we deduce (3.33). This completes the proof of lemma.

Now we are in a position to prove Proposition 2.2.

**Proof of Proposition 2.2** If we define

$$H_1(t)=H(t)+V(t), \tag{3.36}$$

then from Lemmas 3.6–3.7, there exist two positive constants $C_9, C_{10}$ which are independent of $\varepsilon$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+C_9H_1(t)\le C_{10}W_2^{\frac{1}{2}}(t)W_1(t). \tag{3.37}$$

Moreover, since $H_1(t)$ is equivalent to $U(t)+V(t)$, we have from Lemma 3.3 that

$$W_1(t)\le CH_1(t). \tag{3.38}$$

Since $\varepsilon>0$ is small, combining (3.37) and (3.38), we have that there exists a constant $C_{11}>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+C_{11}H_1(t)\le 0, \tag{3.39}$$

which yields the exponential decaying of $H_1(t)$. Since $H_1(t)$ is equivalent to $U(t)+V(t)$, (2.5) follows from Lemma 3.3 immediately. Next, we prove (2.6). To do this, by multiplying the third equation of (2.2) by $s-\overline{s}$ and integrating over $\Omega$, using the boundary condition $u\cdot n|_{\partial\Omega}=0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|s-\overline{s}\|^2\le C\|\nabla u\|_{L^\infty}\|s-\overline{s}\|^2,$$

which yields (2.6). Finally, by symmetry, boundary conditions and some tedious but straightforward calculation, we have the energy estimates on the entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|s-\overline{s}\|_3^2\le C\|u\|_3\|s-\overline{s}\|_3^2,$$

thus we have $\|s(\cdot, x) - \overline{s}\|_3 \leq C\|s_0 - \overline{s}\|_3 \exp\{C\|(\omega_0, u_0)\|_3\}$. Taking the derivatives of the third equation of (2.2), we have

$$\|\partial_t^{\alpha_2}\partial_x^{\alpha_1} s_t\| \leq C\||u|\|_3\|s - \overline{s}\|_3 \leq C\|(\omega_0, u_0, s_0 - \overline{s})\|_3 \exp\{-\eta t\}, \tag{3.40}$$

where $|\alpha_1| + |\alpha_2| \leq 2$. In fact, we first take $|\alpha_1| = 0$, then we can easily prove that (3.40) is right. Taking an induction on $\alpha$, we finally deduce (3.40) which gives (2.7). This completes the proof of Proposition 2.2.

**Acknowledgement** The authors would like to thank the anonymous referees for their valuable suggestions and comments which have helped to improve the manuscript.

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