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Double Biproduct Hom-Bialgebra and Related Quasitriangular Structures*

Tianshui MA[1] Haiying LI[2] Linlin LIU[2]

Abstract Let (H,β) be a Hom-bialgebra such that $\beta^2 = \mathrm{id}_H$. (A,α_A) is a Hom-bialgebra in the left-left Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ and (B,α_B) is a Hom-bialgebra in the right-right Hom-Yetter-Drinfeld category $\mathbb{YD}^H_H$. The authors define the two-sided smash product Hom-algebra $(A\natural H\natural B, \alpha_A\otimes\beta\otimes\alpha_B)$ and the two-sided smash coproduct Hom-coalgebra $(A\diamond H\diamond B, \alpha_A\otimes\beta\otimes\alpha_B)$. Then the necessary and sufficient conditions for $(A\natural H\natural B, \alpha_A\otimes\beta\otimes\alpha_B)$ and $(A\diamond H\diamond B, \alpha_A\otimes\beta\otimes\alpha_B)$ to be a Hom-bialgebra (called the double biproduct Hom-bialgebra and denoted by $(A^{\natural}_{\diamond}H^{\natural}_{\diamond}B, \alpha_A\otimes\beta\otimes\alpha_B)$) are derived. On the other hand, the necessary and sufficient conditions for the smash coproduct Hom-Hopf algebra $(A\diamond H, \alpha_A\otimes\beta)$ to be quasitriangular are given.

Keywords Double biproduct, Hom-Yetter-Drinfeld category, Radford's biproduct, Hom-Yang-Baxter equation
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1 Introduction

Hom-structures (Lie algebras, algebras, coalgebras and Hopf algebras) have been intensively investigated in the literature recently (see [2, 4, 6, 9, 12–15, 21–26]). Hom-algebras are generalizations of algebras obtained by a twisting map, which were introduced for the first time in [14] by Makhlouf and Silvestrov. The associativity is replaced by Hom-associativity, and Hom-coassociativity for a Hom-coalgebra can be considered in a similar way.

In [21, 25], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras, and the dual version (i.e., comodule Hom-coalgebras) was studied by Zhang in [27]. Based on Yau's definition of module Hom-algebras, the first two authors and Yang in [9] constructed the smash product Hom-Hopf algebra $(A\natural H, \alpha\otimes\beta)$ generalizing the Molnar's smash product (see [16]), gave the cobraided structure (in the sense of Yau's definition in [24]) on $(A\natural H, \alpha\otimes\beta)$, and also considered the case of twist tensor product Hom-Hopf algebra. Makhlouf and Panaite defined and studied a class of Yetter-Drinfeld modules over Hom-bialgebras in [12] and derived the constructions of twistors, pseudotwistors, twisted tensor product and smash product in the setting of Hom-case. Especially, in [6], we

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[1]Corresponding author. School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, China. E-mail: matianshui@yahoo.com
[2]School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, China. E-mail: haiyingli2012@yahoo.com liulinlin2016@163.com
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obtained the following result: Let (H, β) be a Hom-bialgebra such that $\beta^2 = \mathrm{id}_H$, and (A, α) be a left (H, β)-module Hom-algebra and a left (H, β)-comodule Hom-coalgebra. $(A\natural_{\diamond} H, \alpha \otimes \beta)$ is a Radford's biproduct Hom-bialgebra if and only if (A, α) is a Hom-bialgebra in the left-left Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$. In [23], Yau introduced a twisted generalization of quantum groups, called quasitriangular Hom-bialgebras. They are non-associative and non-coassociative analogues of Drinfeld's quasitriangular bialgebras. Each quasitriangular Hom-bialgebra comes with a solution of the quantum Hom-Yang-Baxter equation, which is a non-associative version of the quantum Yang-Baxter equation. Solutions of the Hom-Yang-Baxter equation can be obtained from modules of suitable quasitriangular Hom-bialgebras.

As we all know, the Radford biproduct plays an important role in the lifting method for the classification of finite dimensional pointed Hopf algebras (see [1]). Some related results about Radford's biproduct have recently been given in [3, 7–8, 10, 18]. Let H be a bialgebra. A is a bialgebra in the left-left Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ and B is a bialgebra in the right-right Yetter-Drinfeld category $\mathcal{YD}^H_H$. In [11], Majid gave a construction of bialgebra $A^{\#}_{\times} H^{\#}_{\times} B$ by combining the two-sided smash product algebra $A\#H\#B$ with the two-sided smash coproduct coalgebra $A \times H \times B$, which generalizes the Radford biproduct bialgebra.

In this paper, we generalize the Majid's double biproduct to the Hom-setting, and on the other hand, quasitriangular smash coproduct Hom-Hopf algebras are constructed. This is dual to the results in [9].

This article is organized as follows. In Section 2, we recall some definitions and results which will be used later. In Section 3, we give the right version of Radford's biproduct Hom-bialgebra $(A\natural_{\diamond} H, \alpha_A \otimes \beta)$ and Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ in [6]. We also introduce the notions of two-sided smash product Hom-algebra $(A\natural H\natural B, \alpha_A \otimes \beta \otimes \alpha_B)$ and two-sided smash coproduct Hom-coalgebra $(A \diamond H \diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$. Then we derive the necessary and sufficient conditions for $(A\natural H\natural B, \alpha_A \otimes \beta \otimes \alpha_B)$ and $(A \diamond H \diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$ to be a Hom-bialgebra, which is called double biproduct Hom-bialgebra and denoted by $(A\natural_{\diamond} H\natural_{\diamond} B, \alpha_A \otimes \beta \otimes \alpha_B)$, generalizing the Majid's double biproduct bialgebra. Note that the construction of $(A\natural H\natural B, \alpha_A \otimes \beta \otimes \alpha_B)$ here is different from that defined by Makhlouf and Panaite in [13]. Section 4 is devoted to deriving the necessary and sufficient conditions for the smash coproduct Hom-Hopf algebra $(A \diamond H, \alpha_A \otimes \beta)$ to be quasitriangular. A concrete example for quasitriangular smash coproduct Hom-Hopf algebra is given in Section 5.

2 Preliminaries

Throughout this paper, we follow the definitions and terminologies in [9, 21, 23, 27], with all algebraic systems assumed to be over the field K. Given a K-space M, we write id_M for the identity map on M.

We now recall some useful definitions.

Hom-algebra A Hom-algebra is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a K-linear space, $\mu : A \otimes A \longrightarrow A$ is a K-linear map, $1_A \in A$, and α is an automorphism of A, such that

$$\text{(A1)} \quad \alpha(aa') = \alpha(a)\alpha(a'), \quad \alpha(1_A) = 1_A \quad \text{and}$$

$$\text{(A2)}\quad \alpha(a)(a'a'') = (aa')\alpha(a''), \quad a1_A = 1_A a = \alpha(a)$$

are satisfied for $a, a', a'' \in A$. Here we use the notation $\mu(a \otimes a') = aa'$.

Hom-coalgebra A Hom-coalgebra is a quadruple $(C, \Delta, \varepsilon_C, \beta)$ (abbr. (C, β)), where C is a K-linear space, $\Delta : C \longrightarrow C \otimes C$, $\varepsilon_C : C \longrightarrow K$ are K-linear maps, and β is an automorphism of C, such that

$$\text{(C1)}\quad \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2), \quad \varepsilon_C \circ \beta = \varepsilon_C \quad \text{and}$$
$$\text{(C2)}\quad \beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2), \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c)$$

are satisfied for $c \in A$. Here we use the notation $\Delta(c) = c_1 \otimes c_2$ (summation implicitly understood).

Hom-bialgebra A Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e.,

$$\Delta(hh') = \Delta(h)\Delta(h'), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \quad \varepsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \longrightarrow H$ such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \quad \text{and} \quad S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$ (abbr. (H, γ, S)) a Hom-Hopf algebra.

Let (H, γ) and (H', γ') be two Hom-bialgebras. The linear map $f : H \longrightarrow H'$ is called a Hom-bialgebra map if $f \circ \gamma = \gamma' \circ f$ and at the same time f is a bialgebra map in the usual sense.

Left Hom-module (see [21, 25]) Let (A, β) be a Hom-algebra. A left (A, β)-Hom-module is a triple $(M, \triangleright, \alpha)$, where M is a linear space, $\triangleright : A \otimes M \longrightarrow M$ is a linear map, and α is an automorphism of M, such that

$$\text{(LM1)}\quad \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m) \quad \text{and}$$
$$\text{(LM2)}\quad \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m), \quad 1_A \triangleright m = \alpha(m)$$

are satisfied for $a, a' \in A$ and $m \in M$.

Remark 2.1 (1) It is obvious that (A, μ, β) is a left (A, β)-Hom-module.

(2) When $\beta = \mathrm{id}_A$ and $\alpha = \mathrm{id}_M$, a left (A, β)-Hom-module is the usual left A-module.

Right Hom-module (see [13]) Let (A, β) be a Hom-algebra. A right (A, β)-Hom-module is a triple $(M, \triangleleft, \alpha)$, where M is a linear space, $\triangleleft : M \otimes A \longrightarrow M$ is a linear map, and α is an automorphism of M, such that

$$\text{(RM1)}\quad \alpha(m \triangleleft a) = \alpha(m) \triangleleft \beta(a) \quad \text{and}$$
$$\text{(RM2)}\quad (m \triangleleft a) \triangleleft \beta(a') = \alpha(m) \triangleleft (aa'), \quad m \triangleleft 1_A = \alpha(m)$$

are satisfied for $a, a' \in A$ and $m \in M$.

Left module Hom-algebra (see [21, 25]) Let (H,β) be a Hom-bialgebra and (A,α) be a Hom-algebra. If $(A,\triangleright,\alpha)$ is a left (H,β)-Hom-module and for all $h\in H$ and $a,a'\in A$,

$$\text{(LMA1)}\quad \beta^2(h)\triangleright(aa') = (h_1\triangleright a)(h_2\triangleright a'),$$
$$\text{(LMA2)}\quad h\triangleright 1_A = \varepsilon_H(h)1_A,$$

then $(A,\triangleright,\alpha)$ is called a left (H,β)-module Hom-algebra.

Remark 2.2 (1) When $\alpha=\mathrm{id}_A$ and $\beta=\mathrm{id}_H$, a left (H,β)-module Hom-algebra is the usual left H-module algebra.

(2) In a way similar to the case of Hopf algebras, in [21, 25], Yau concluded that the equation (LMA1) is satisfied if and only if μ_A is a morphism of H-modules for suitable H-module structures on $A\otimes A$ and A, respectively.

Right module Hom-algebra (see [13]) Let (H,β) be a Hom-bialgebra and (A,α) be a Hom-algebra. If $(A,\triangleleft,\alpha)$ is a right (H,β)-Hom-module and for all $h\in H$ and $a,a'\in A$,

$$\text{(RMA1)}\quad (aa')\triangleleft\beta^2(h) = (a\triangleleft h_1)(a'\triangleleft h_2),$$
$$\text{(RMA2)}\quad 1_A\triangleleft h = \varepsilon_H(h)1_A,$$

then $(A,\triangleleft,\alpha)$ is called a right (H,β)-module Hom-algebra.

Left Hom-comodule (see [27]) Let (C,β) be a Hom-coalgebra. A left (C,β)-Hom-comodule is a triple (M,ρ,α), where M is a linear space, $\rho: M\longrightarrow C\otimes M$ (write $\rho(m)=m_{(-1)}\otimes m_{(0)}$, $\forall m\in M$) is a linear map, and α is an automorphism of M, such that

$$\text{(LCM1)}\quad \alpha(m)_{(-1)}\otimes\alpha(m)_{(0)} = \beta(m_{(-1)})\otimes\alpha(m_{(0)})\quad\text{and}$$
$$\text{(LCM2)}\quad \beta(m_{(-1)})\otimes m_{(0)(-1)}\otimes m_{(0)(0)} = m_{(-1)1}\otimes m_{(-1)2}\otimes\alpha(m_{(0)}),$$
$$\varepsilon_C(m_{(-1)})m_{(0)} = \alpha(m)$$

are satisfied for all $m\in M$.

Remark 2.3 (1) It is obvious that (C,Δ_C,β) is a left (C,β)-Hom-comodule.

(2) When $\beta=\mathrm{id}_A$ and $\alpha=\mathrm{id}_M$, a left (C,β)-Hom-comodule is the usual left C-comodule.

Left comodule Hom-coalgebra (see [27]) Let (H,β) be a Hom-bialgebra and (C,α) be a Hom-coalgebra. If (C,ρ,α) is a left (H,β)-Hom-comodule and for all $c\in C$,

$$\text{(LCMC1)}\quad \beta^2(c_{(-1)})\otimes c_{(0)1}\otimes c_{(0)2} = c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)},$$
$$\text{(LCMC2)}\quad c_{(-1)}\varepsilon_C(c_{(0)}) = 1_H\varepsilon_C(c),$$

then (C,ρ,α) is called a left (H,β)-comodule Hom-coalgebra.

Remark 2.4 (1) It is obvious that (H,Δ_H,β) is a left (H,β)-comodule Hom-coalgebra.

(2) When $\alpha=\mathrm{id}_A$ and $\beta=\mathrm{id}_H$, a left (H,β)-comodule Hom-coalgebra is the usual left H-comodule coalgebra.

(3) In a way similar to the case of Hopf algebras, in [27], Zhang and Li concluded that the equation (LCMC1) is satisfied if and only if Δ_C is a morphism of H-comodules for suitable H-comodule structures on $C \otimes C$ and C, respectively.

Left module Hom-coalgebra (see [9]) Let (H, β) be a Hom-bialgebra and (C, α) be a Hom-coalgebra. If $(C, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $c \in A$,

$$\text{(LMC1)} \quad (h \triangleright c)_1 \otimes (h \triangleright c)_2 = (h_1 \triangleright c_1) \otimes (h_2 \triangleright c_2),$$
$$\text{(LMC2)} \quad \varepsilon_C(h \triangleright c) = \varepsilon_H(h)\varepsilon_C(c),$$

then $(C, \triangleright, \alpha)$ is called a left (H, β)-module Hom-coalgebra.

Remark 2.5 When $\alpha = \mathrm{id}_C$ and $\beta = \mathrm{id}_H$, a left (H, β)-module Hom-coalgebra is the usual left H-module coalgebra.

Left comodule Hom-algebra (see [22]) Let (H, β) be a Hom-bialgebra and (A, α) be a Hom-algebra. If (A, ρ, α) is a left (H, β)-Hom-comodule and for all $a, a' \in A$,

$$\text{(LCMA1)} \quad \rho(aa') = a_{(-1)}a'_{(-1)} \otimes a_{(0)}a'_{(0)},$$
$$\text{(LCMA2)} \quad \rho(1_A) = 1_H \otimes 1_A,$$

then (A, ρ, α) is called a left (H, β)-comodule Hom-algebra.

Remark 2.6 When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, a left (H, β)-comodule Hom-algebra is the usual left H-comodule algebra.

Left smash product Hom-algebra (see [6, 9]) Let (H, β) be a Hom-bialgebra and $(A, \triangleright, \alpha)$ be a left (H, β)-module Hom-algebra. Then $(A\natural H, \alpha \otimes \beta)$ ($A\natural H = A \otimes H$ as a linear space) and unit $1_A \otimes 1_H$ is a Hom-algebra with the multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1 \triangleright \alpha^{-1}(a')) \otimes \beta^{-1}(h_2)h',$$

where $a, a' \in A$, $h, h' \in H$, and we call it a left smash product Hom-algebra denoted by $(A\natural H, \alpha \otimes \beta)$.

Remark 2.7 (1) Here the multiplication of smash product Hom-algebra is different from that defined by Makhlouf and Panaite in [13, Theorem 3.1].

(2) When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, we can get the usual smash product algebra $A\#H$ (see [16–17]).

Left smash coproduct Hom-coalgebra (see [6]) Let (H, β) be a Hom-bialgebra and (C, ρ, α) be a left (H, β)-comodule Hom-coalgebra. Then $(C \diamond H, \alpha \otimes \beta)$ ($C \diamond H = C \otimes H$ as a linear space) and counit $\varepsilon_C \otimes \varepsilon_H$ is a Hom-coalgebra with the comultiplication

$$\Delta_{C\diamond H}(c \otimes h) = c_1 \otimes c_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(c_{20}) \otimes h_2,$$

where $c \in C$, $h \in H$, and we call it a left smash coproduct Hom-coalgebra denoted by $(C \diamond H, \alpha \otimes \beta)$.

Left Radford biproduct (see [6]) Let (H,β) be a Hom-bialgebra, and (A,α) be a left (H,β)-module Hom-algebra with module structure $\triangleright: H\otimes A\longrightarrow A$ and a left (H,β)-comodule Hom-coalgebra with comodule structure $\rho: A\longrightarrow H\otimes A$. Then the following are equivalent:

(i) $(A^{\natural}_{\diamond}H, \mu_{A\natural H}, 1_A\otimes 1_H, \Delta_{A\diamond H}, \varepsilon_A\otimes\varepsilon_H, \alpha\otimes\beta)$ is a Hom-bialgebra, where $A\natural H$ is a left smash product Hom-algebra and $A\diamond H$ is a left smash coproduct Hom-coalgebra.

(ii) The following conditions hold ($\forall a,b\in A$ and $h\in H$):

(LR1) (A,ρ,α) is a left (H,β)-comodule Hom-algebra,

(LR2) $(A,\triangleright,\alpha)$ is a left (H,β)-module Hom-coalgebra,

(LR3) ε_A is a Hom-algebra map and $\Delta_A(1_A)=1_A\otimes 1_A$,

(LR4) $\Delta_A(ab)=a_1(\beta^2(a_{2(-1)})\triangleright\alpha^{-1}(b_1))\otimes\alpha^{-1}(a_{2(0)})b_2$, and

(LR5) $h_1\beta(a_{(-1)})\otimes(\beta^3(h_2)\triangleright a_{(0)})=(\beta^2(h_1)\triangleright a)_{(-1)}h_2\otimes(\beta^2(h_1)\triangleright a)_{(0)}$.

Left-left Hom-Yetter-Drinfeld module (see [6]) Let (H,β) be a Hom-bialgebra, $(M,\triangleright_M,\alpha_M)$ be a left (H,β)-module with action $\triangleright_M: H\otimes M\longrightarrow M,\ h\otimes m\mapsto h\triangleright_M m$, and (M,ρ^M,α_M) be a left (H,β)-comodule with coaction $\rho^M: M\longrightarrow H\otimes M,\ m\mapsto m_{(-1)}\otimes m_{(0)}$. Then we call $(M,\triangleright_M,\rho^M,\alpha_M)$ a left-left Hom-Yetter-Drinfeld module over (H,β) if the following condition holds:

$$\text{(LYD)}\quad h_1\beta(m_{(-1)})\otimes(\beta^3(h_2)\triangleright_M m_{(0)})=(\beta^2(h_1)\triangleright_M m)_{(-1)}h_2\otimes(\beta^2(h_1)\triangleright_M m)_{(0)},$$

where $h\in H$ and $m\in M$.

Left-left Hom-Yetter-Drinfeld category (see [6]) Let (H,β) be a Hom-bialgebra. Then the left-left Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ is a braided tensor category (see [5]), with tensor product $(M\otimes N,\alpha_M\otimes\alpha_N)$ and associativity constraints, and the braiding is defined by

$$\triangleright_{M\otimes N}: H\otimes M\otimes N\longrightarrow M\otimes N,\ h\otimes m\otimes n\mapsto(h_1\triangleright_M m)\otimes(h_2\triangleright_N n)$$

and

$$\rho^{M\otimes N}: M\otimes N\longrightarrow H\otimes M\otimes N,\ m\otimes n\mapsto\beta^{-2}(m_{-1}n_{-1})\otimes m_0\otimes n_0,$$

where $h\in H$, $m\in M$ and $n\in N$,

$$a_{M,N,P}:(M\otimes N)\otimes P\longrightarrow M\otimes(N\otimes P),\quad(m\otimes n)\otimes p\mapsto\alpha_M^{-1}(m)\otimes(n\otimes\alpha_P(p))$$

and

$$c_{M,N}: M\otimes N\longrightarrow N\otimes M,\quad m\otimes n\mapsto(\beta^2(m_{(-1)})\triangleright_N\alpha_N^{-1}(n))\otimes\alpha_M^{-1}(m_{(0)}),$$

respectively, as well as unit (K,id_K).

Left Radford biproduct and left-left Yetter-Drinfeld category (see [6]) Let (H,β) be a Hom-bialgebra such that $\beta^2=\mathrm{id}_H$, and (A,α) be a left (H,β)-module Hom-algebra and a left (H,β)-comodule Hom-coalgebra. Then $(A^{\natural}_{\diamond}H,\mu_{A\natural H},1_A\otimes 1_H,\Delta_{A\diamond H},\ \varepsilon_A\otimes\varepsilon_H,\alpha\otimes\beta)$ is a left Radford biproduct Hom-bialgebra if and only if (A,α) is a Hom-bialgebra in the left-left Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$.

Quasitriangular Hom-Hopf algebra (see [23]) A quasitriangular Hom-Hopf algebra is a octuple $(H,\mu,1_H,\Delta,\ \varepsilon,S,\beta,R)$ (abbr. (H,β,R)) in which $(H,\mu,1_H,\Delta,\varepsilon,S,\beta)$ is a Hom-Hopf algebra and $R = R^1 \otimes R^2 \in H \otimes H$, satisfying the following axioms (for all $h \in H$ and $R = r$):

$$\text{(QT1)} \quad \varepsilon(R^1)R^2 = R^1\varepsilon(R^2) = 1,$$
$$\text{(QT2)} \quad {R^1}_1 \otimes {R^1}_2 \otimes \beta(R^2) = \beta(R^1) \otimes \beta(r^1) \otimes R^2 r^2,$$
$$\text{(QT3)} \quad \beta(R^1) \otimes {R^2}_1 \otimes {R^2}_2 = R^1 r^1 \otimes \beta(r^2) \otimes \beta(R^2),$$
$$\text{(QT4)} \quad h_2 R^1 \otimes h_1 R^2 = R^1 h_1 \otimes R^2 h_2,$$
$$\text{(QT5)} \quad \beta(R^1) \otimes \beta(R^2) = R^1 \otimes R^2.$$

Remark 2.8 (1) When $\alpha = \mathrm{id}_H$, a quasitriangular Hom-Hopf algebra is exactly the usual quasitriangular Hopf algebra.

(2) It is slightly different from the definition in [23]. Here we replace the Hom-bialgebra with the Hom-Hopf algebra and also add another two conditions (QT1) and (QT5). Similar to the Hopf algebra setting, the quasitriangular structure R is invertible.

(3) Based on Yau's results in [23], each quasitriangular Hom-Hopf algebra comes with solutions of the quantum Hom-Yang-Baxter equations.

3 Double Biproduct Hom-Bialgebra

In this section, we mainly generalize the double biproduct bialgebra to the Hom-setting. In order to define double biproduct Hom-bialgebra, we need first the right-handed versions of some concepts and results. The proofs are similar to the left-handed versions, so we omit them.

Definition 3.1 *Let (C, β) be a Hom-coalgebra. A right (C, β)-Hom-comodule is a triple (M, δ, α), where M is a linear space, $\delta : M \longrightarrow M \otimes C$ (write $\delta(m) = m_{[0]} \otimes m_{[1]}$, $\forall m \in M$) is a linear map, and α is an automorphism of M, such that*

$$\text{(RCM1)} \quad \alpha(m)_{[0]} \otimes \alpha(m)_{[1]} = \alpha(m_{[0]}) \otimes \beta(m_{[1]}) \quad \textit{and}$$
$$\text{(RCM2)} \quad m_{[0][0]} \otimes m_{[0][1]} \otimes \beta(m_{[1]}) = \alpha(m_{[0]}) \otimes m_{[1]1} \otimes m_{[1]2}, \quad m_{[0]}\varepsilon_C(m_{[1]}) = \alpha(m)$$

are satisfied for all $m \in M$.

Definition 3.2 *Let (H, β) be a Hom-bialgebra and (C, α) be a Hom-coalgebra. If (C, δ, α) is a right (H, β)-Hom-comodule and for all $c \in C$,*

$$\text{(RCMC1)} \quad c_{[0]1} \otimes c_{[0]2} \otimes \beta^2(c_{[1]}) = c_{1[0]} \otimes c_{2[0]} \otimes c_{1[1]}c_{2[1]},$$
$$\text{(RCMC2)} \quad \varepsilon_C(c_{[0]})c_{[1]} = 1_H \varepsilon_C(c),$$

then (C, δ, α) is called a right (H, β)-comodule Hom-coalgebra.

Definition 3.3 *Let (H, β) be a Hom-bialgebra and (C, α) be a Hom-coalgebra. If $(C, \triangleleft, \alpha)$ is a right (H, β)-Hom-module and for all $h \in H$ and $c \in A$,*

$$\text{(RMC1)} \quad (c \triangleleft h)_1 \otimes (c \triangleleft h)_2 = (c_1 \triangleleft h_1) \otimes (c_2 \triangleleft h_2),$$
$$\text{(RMC2)} \quad \varepsilon_C(c \triangleleft h) = \varepsilon_H(h)\varepsilon_C(c),$$

then $(C, \triangleleft, \alpha)$ is called a right (H, β)-module Hom-coalgebra.

Definition 3.4 *Let (H,β) be a Hom-bialgebra and (A,α) be a Hom-algebra. If (A,δ,α) is a right (H,β)-Hom-comodule and for all $a,a'\in A$,*

$$\text{(RCMA1)}\quad \delta(aa')=a_{[0]}a'_{[0]}\otimes a_{[1]}a'_{[1]},$$
$$\text{(RCMA2)}\quad \delta(1_A)=1_A\otimes 1_H,$$

then (A,δ,α) is called a right (H,β)-comodule Hom-algebra.

Definition 3.5 *Let (H,β) be a Hom-bialgebra and $(A,\triangleleft,\alpha)$ be a right (H,β)-module Hom-algebra. Then $(H\natural A,\beta\otimes\alpha)$ ($H\natural A=H\otimes A$ as a linear space) and unit $1_H\otimes 1_A$ is a Hom-algebra with the multiplication*

$$(h\otimes a)(h'\otimes a')=h\beta^{-1}(h'_1)\otimes(\alpha^{-1}(a)\triangleleft h'_2)a',$$

where $a,a'\in A$, $h,h'\in H$, and we call it a right smash product Hom-algebra denoted by $(H\natural A,\beta\otimes\alpha)$.

Proposition 3.1 *Let (H,β) be a Hom-bialgebra and (C,δ,α) be a right (H,β)-comodule Hom-coalgebra. Then $(H\diamond C,\beta\otimes\alpha)$ ($H\diamond C=H\otimes C$ as a linear space) and counit $\varepsilon_H\otimes\varepsilon_C$ is a Hom-coalgebra with the comultiplication*

$$\Delta_{H\diamond C}(h\otimes c)=h_1\otimes\alpha^{-1}(c_{1[0]})\otimes\beta^{-1}(h_2)c_{1[1]}\otimes c_2,$$

where $c\in C$, $h\in H$, and we call it a right smash coproduct Hom-coalgebra denoted by $(H\diamond C,\beta\otimes\alpha)$.

Theorem 3.1 *Let (H,β) be a Hom-bialgebra, and (A,α) be a right (H,β)-module Hom-algebra with module structure $\triangleleft: A\otimes H\longrightarrow A$ and a right (H,β)-comodule Hom-coalgebra with comodule structure $\delta: A\longrightarrow A\otimes H$. Then the following are equivalent:*

(i) *$(H^{\natural}_{\diamond}A,\mu_{H\natural A},1_H\otimes 1_A,\Delta_{H\diamond A},\varepsilon_H\otimes\varepsilon_A,\beta\otimes\alpha)$ is a Hom-bialgebra, where $H\natural A$ is a right smash product Hom-algebra and $H\diamond A$ is a right smash coproduct Hom-coalgebra.*

(ii) *The following conditions hold ($\forall a,b\in A$ and $h\in H$):*

(RR1) *(A,δ,α) is a right (H,β)-comodule Hom-algebra,*

(RR2) *$(A,\triangleleft,\alpha)$ is a right (H,β)-module Hom-coalgebra,*

(RR3) *ε_A is a Hom-algebra map and $\Delta_A(1_A)=1_A\otimes 1_A$,*

(RR4) *$\Delta_A(ab)=a_1\alpha^{-1}(b_{1[0]})\otimes(\alpha^{-1}(a_2)\triangleleft\beta^2(b_{1[1]}))b_2$, and*

(RR5) *$(a_{[0]}\triangleleft\beta^3(h_1))\otimes\beta(a_{[1]})h_2=(a\triangleleft\beta^2(h_2))_{[0]}\otimes h_1(a\triangleleft\beta^2(h_2))_{[1]}$.*

Definition 3.6 *Let (H,β) be a Hom-bialgebra, $(M,\triangleleft_M,\alpha_M)$ be a right (H,β)-module with action $\triangleleft_M: M\otimes H\longrightarrow M$, $m\otimes h\mapsto m\triangleleft_M h$ and (M,δ^M,α_M) be a right (H,β)-comodule with coaction $\delta^M: M\longrightarrow M\otimes H$, $m\mapsto m_{[0]}\otimes m_{[1]}$. Then we call $(M,\triangleleft_M,\delta^M,\alpha_M)$ a right-right Hom-Yetter-Drinfeld module over (H,β) if the following condition holds:*

$$\text{(RYD)}\quad (m_{[0]}\triangleleft\beta^3(h_1))\otimes\beta(m_{[1]})h_2=(m\triangleleft\beta^2(h_2))_{[0]}\otimes h_1(m\triangleleft\beta^2(h_2))_{[1]},$$

where $h\in H$ and $m\in M$.

Definition 3.7 *Let (H,β) be a Hom-bialgebra. Then the right-right Hom-Yetter-Drinfeld category $\mathbb{YD}_H^H$ is a braided tensor category, with tensor product $(M\otimes N,\alpha_M\otimes\alpha_N)$ and associativity constraints, and the braiding is defined by*

$$a_{M,N,P}:(M\otimes N)\otimes P\longrightarrow M\otimes(N\otimes P),\quad (m\otimes n)\otimes p\mapsto \alpha_M^{-1}(m)\otimes(n\otimes\alpha_P(p))$$

and

$$c_{M,N}:M\otimes N\longrightarrow N\otimes M,\quad m\otimes n\mapsto \alpha_N^{-1}(n_{[0]})\otimes(\alpha_M^{-1}(m)\triangleleft_M\beta^2(n_{[1]})),$$

respectively, as well as unit (K,id_K).

Theorem 3.2 *Let (H,β) be a Hom-bialgebra such that $\beta^2=\mathrm{id}_H$, and (A,α) be a right (H,β)-module Hom-algebra and a right (H,β)-comodule Hom-coalgebra. Then $(H\overset{\natural}{\diamond}A,\mu_{H\natural A},1_H\otimes 1_A,\Delta_{H\diamond A},\varepsilon_H\otimes\varepsilon_A,\beta\otimes\alpha)$ is a right Radford biproduct Hom-bialgebra if and only if (A,α) is a Hom-bialgebra in the right-right Hom-Yetter-Drinfeld category $\mathbb{YD}_H^H$.*

Next we introduce the two-sided smash product Hom-algebra, the two-sided smash coproduct Hom-coalgebra and the double biproduct Hom-bialgebra.

Proposition 3.2 *Let (H,β) be a Hom-bialgebra, $(A,\triangleright,\alpha_A)$ be a left (H,β)-module Hom-algebra and $(B,\triangleleft,\alpha_B)$ be a right (H,β)-module Hom-algebra. Then $(A\natural H\natural B,\alpha_A\otimes\beta\otimes\alpha_B)$ ($A\natural H\natural B=A\otimes H\otimes B$ as a linear space) and unit $1_A\otimes 1_H\otimes 1_B$ is a Hom-algebra with the multiplication*

$$(a\otimes h\otimes b)(a'\otimes h'\otimes b')=a(h_1\triangleright\alpha_A^{-1}(a'))\otimes\beta^{-1}(h_2h'_1)\otimes(\alpha_B^{-1}(b)\triangleleft h'_2)b',$$

where $a,a'\in A,\ h,h'\in H,\ b,b'\in B$, and we call it a two-sided smash product Hom-algebra denoted by $(A\natural H\natural B,\alpha_A\otimes\beta\otimes\alpha_B)$.

Proof It is direct to prove that

$$(a\otimes h\otimes b)(1_A\otimes 1_H\otimes 1_B)=(1_A\otimes 1_H\otimes 1_B)(a\otimes h\otimes b)=\alpha_A(a)\otimes\beta(h)\otimes\alpha_B(b).$$

On the other hand, for all $a,a',a''\in A,\ h,h',h''\in H$ and $b,b',b''\in B$, we have

$$\begin{aligned}
&(\alpha_A(a)\otimes\beta(h)\otimes\alpha_B(b))((a'\otimes h'\otimes b')(a''\otimes h''\otimes b''))\\
&=\alpha_A(a)(\beta(h)_1\triangleright\alpha_A^{-1}(a'(h'_1\triangleright\alpha_A^{-1}(a''))))\otimes\beta^{-1}(\beta(h)_2\beta^{-1}(h'_2h''_1)_1)\\
&\quad\otimes(b\triangleleft\beta^{-1}(h'_2h''_1)_2)((\alpha_B^{-1}(b')\triangleleft h''_2)b'')\\
&\overset{\text{(A1)(C1)}}{=}\alpha_A(a)(\beta(h_1)\triangleright(\alpha_A^{-1}(a')\alpha_A^{-1}(h'_1\triangleright\alpha_A^{-1}(a''))))\otimes\beta^{-1}(\beta(h_2)(\beta^{-1}(h'_{21})\beta^{-1}(h''_{11})))\\
&\quad\otimes(b\triangleleft(\beta^{-1}(h'_{22})\beta^{-1}(h''_{12})))((\alpha_B^{-1}(b')\triangleleft h''_2)b'')\\
&\overset{\text{(LMA1)(A2)(C2)}}{=}\alpha_A(a)((\beta^{-1}(h_{11})\triangleright\alpha_A^{-1}(a'))(\beta^{-1}(h_{12})\triangleright\alpha_A^{-1}(h'_1\triangleright\alpha_A^{-1}(a''))))\\
&\quad\otimes\beta^{-1}((h_2\beta^{-1}(h'_{21}))h''_{11})\otimes(\alpha_B^{-1}(b\triangleleft(\beta^{-1}(h'_{22})\beta^{-1}(h''_{12})))(\alpha_B^{-1}(b')\triangleleft h''_2))\alpha_B(b'')\\
&\overset{\text{(LM1)(RM2)}}{=}\alpha_A(a)((\beta^{-1}(h_{11})\triangleright\alpha_A^{-1}(a'))(\beta^{-1}(h_{12})\triangleright(\beta^{-1}(h'_1)\triangleright\alpha_A^{-2}(a''))))\\
&\quad\otimes\beta^{-1}((h_2\beta^{-1}(h'_{21}))h''_{11})\otimes(\alpha_B^{-1}((\alpha_B^{-1}(b)\triangleleft\beta^{-1}(h'_{22}))\triangleleft h''_{12})(\alpha_B^{-1}(b')\triangleleft h''_2))\alpha_B(b'')\\
&\overset{\text{(LM2)}}{=}\alpha_A(a)((\beta^{-1}(h_{11})\triangleright\alpha_A^{-1}(a'))((\beta^{-2}(h_{12})\beta^{-1}(h'_1))\triangleright\alpha_A^{-1}(a'')))
\end{aligned}$$

$$
\begin{aligned}
&\otimes \beta^{-1}((h_2\beta^{-1}(h'_{21}))h''_{11}) \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \lhd \beta^{-1}(h'_{22})) \lhd h''_{12})(\alpha_B^{-1}(b') \lhd h''_2))\alpha_B(b'')\\
&\overset{(C2)}{=} \alpha_A(a)((h_1 \rhd \alpha_A^{-1}(a'))((\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \rhd \alpha_A^{-1}(a''))) \otimes \beta^{-1}((\beta^{-1}(h_{22})\beta^{-1}(h'_{12}))\\
&\quad \times \beta(h''_1)) \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \lhd h'_2) \lhd h''_{21})(\alpha_B^{-1}(b') \lhd \beta^{-1}(h''_2)))\alpha_B(b'')\\
&\overset{(RM1)}{=} \alpha_A(a)((h_1 \rhd \alpha_A^{-1}(a'))((\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \rhd \alpha_A^{-1}(a''))) \otimes \beta^{-1}((\beta^{-1}(h_{22})\\
&\quad \times \beta^{-1}(h'_{12}))\beta(h''_1)) \otimes ((\alpha_B^{-1}(\alpha_B^{-1}(b) \lhd h'_2) \lhd \beta^{-1}(h''_{21}))(\alpha_B^{-1}(b') \lhd \beta^{-1}(h''_2)))\alpha_B(b'')\\
&\overset{(RMA1)(C1)}{=} \alpha_A(a)((h_1 \rhd \alpha_A^{-1}(a'))((\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \rhd \alpha_A^{-1}(a'')))\\
&\quad \otimes \beta^{-1}((\beta^{-1}(h_{22})\beta^{-1}(h'_{12}))\beta(h''_1)) \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b) \lhd h'_2)\alpha_B^{-1}(b') \lhd \beta(h''_2))\alpha_B(b'')\\
&\overset{(C1)(A1)}{=} \alpha_A(a)((h_1 \rhd \alpha_A^{-1}(a'))(\beta^{-1}(\beta^{-1}(h_2h'_1)_1) \rhd \alpha_A^{-1}(a'')))\\
&\quad \otimes \beta^{-1}(\beta^{-1}(h_2h'_1)_2\beta(h'')_1) \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \lhd h'_2)b') \lhd \beta(h'')_2)\alpha_B(b'')\\
&\overset{(LM1)(A2)}{=} (a(h_1 \rhd \alpha_A^{-1}(a')))(\beta^{-1}(h_2h'_1)_1 \rhd a'') \otimes \beta^{-1}(\beta^{-1}(h_2h'_1)_2\beta(h'')_1)\\
&\quad \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \lhd h'_2)b') \lhd \beta(h'')_2)\alpha_B(b'')\\
&= ((a \otimes h \otimes b)(a' \otimes h' \otimes b'))(\alpha_A(a'') \otimes \beta(h'') \otimes \alpha_B(b'')),
\end{aligned}
$$

which finishes the proof.

Dually, we have the following proposition.

Proposition 3.3 *Let (H,β) be a Hom-bialgebra, (A,ρ,α_A) be a left (H,β)-comodule Hom-coalgebra and (B,δ,α_B) be a right (H,β)-comodule Hom-coalgebra. Then $(A \diamond H \diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$ ($A \diamond H \diamond B = A \otimes H \otimes B$ as a linear space) and counit $\varepsilon_A \otimes \varepsilon_H \otimes \varepsilon_B$ is a Hom-coalgebra with comultiplication*

$$\Delta(a \otimes h \otimes b) = a_1 \otimes a_{2(-1)}\beta^{-1}(h_1) \otimes \alpha_B^{-1}(b_{1[0]}) \otimes \alpha_A^{-1}(a_{2(0)}) \otimes \beta^{-1}(h_2)b_{1[1]} \otimes b_2,$$

where $a \in A,\ h \in H,\ b \in B$, and we call it a two-sided smash coproduct Hom-coalgebra denoted by $(A \diamond H \diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$.

Theorem 3.3 *Let (H,β) be a Hom-bialgebra such that $\beta^2 = \mathrm{id}_H$, (A,α_A) be a Hom-bialgebra in the left-left Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ and (B,α_B) be a Hom-bialgebra in the right-right Hom-Yetter-Drinfeld category $\mathbb{YD}^H_H$. Then the two-sided smash product Hom-algebra $(A\natural H\natural B, \alpha_A \otimes \beta \otimes \alpha_B)$ equipped with the two-sided smash coproduct Hom-coalgebra $(A \diamond H \diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$ becomes a Hom-bialgebra if and only if*

$$
\begin{aligned}
\text{(DB)}\qquad &\beta(a_{(-1)}) \otimes b_{[0]} \otimes a_{(0)} \otimes \beta(b_{[1]})\\
&= a_{(-1)1} \otimes (\alpha^{-1}(b_{[0]}) \lhd \beta^2(a_{(-1)2})) \otimes (\beta^2(b_{[1]1}) \rhd \alpha^{-1}(a_{(0)})) \otimes b_{[1]2},
\end{aligned}
$$

where $a \in A$ and $b \in B$.

In this case, we call this Hom-bialgebra a double biproduct Hom-bialgebra and denote it by $(A^\natural_\diamond H^\natural_\diamond B, \alpha_A \otimes \beta \otimes \alpha_B)$.

Proof ($\Leftarrow$) We only need to check that $\Delta_{A\diamond H\diamond B}$ is a Hom-algebra map. For all $a, a' \in A, h, h' \in H$ and $b, b' \in B$, we have

$$\Delta_{A\diamond H\diamond B}((a \otimes h \otimes b)(a' \otimes h' \otimes b'))$$

$$
\begin{aligned}
&= (a(h_1 \triangleright \alpha_A^{-1}(a')))_1 \otimes (a(h_1 \triangleright \alpha_A^{-1}(a')))_{2(-1)}\beta^{-1}(\beta^{-1}(h_2h'_1)_1)\\
&\quad \otimes \alpha_B^{-1}(((\alpha_B^{-1}(b) \triangleleft h'_2)b')_{1[0]}) \otimes \alpha_A^{-1}((a(h_1 \triangleright \alpha_A^{-1}(a')))_{2(0)})\\
&\quad \otimes \beta^{-1}(\beta^{-1}(h_2h'_1)_2)((\alpha_B^{-1}(b) \triangleleft h'_2)b')_{1[1]} \otimes ((\alpha_B^{-1}(b) \triangleleft h'_2)b')_2\\
&\overset{(A1)(C1)}{=} (a(h_1 \triangleright \alpha_A^{-1}(a')))_1 \otimes (a(h_1 \triangleright \alpha_A^{-1}(a')))_{2(-1)}(\beta^{-2}(h_{21})\beta^{-2}(h'_{11}))\\
&\quad \otimes \alpha_B^{-1}(((\alpha_B^{-1}(b) \triangleleft h'_2)b')_{1[0]}) \otimes \alpha_A^{-1}((a(h_1 \triangleright \alpha_A^{-1}(a')))_{2(0)})\\
&\quad \otimes (\beta^{-2}(h_{22})\beta^{-2}(h'_{12}))((\alpha_B^{-1}(b) \triangleleft h'_2)b')_{1[1]} \otimes ((\alpha_B^{-1}(b) \triangleleft h'_2)b')_2\\
&\overset{(LR4)(RR4)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}((h_1 \triangleright \alpha_A^{-1}(a'))_1)) \otimes (\alpha_A^{-1}(a_{2(0)})(h_1 \triangleright \alpha_A^{-1}(a'))_2)_{(-1)}\\
&\quad \times (\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \otimes \alpha_B^{-1}(((\alpha_B^{-1}(b) \triangleleft h'_2)_1\alpha_B^{-1}(b'_{1[0]}))_{[0]}) \otimes \alpha_A^{-1}((\alpha_A^{-1}(a_{2(0)})\\
&\quad \times (h_1 \triangleright \alpha_A^{-1}(a'))_2)_{(0)}) \otimes (\beta^{-2}(h_{22})\beta^{-2}(h'_{12}))((\alpha_B^{-1}(b) \triangleleft h'_2)_1\alpha_B^{-1}(b'_{1[0]}))_{[1]}\\
&\quad \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \triangleleft h'_2)_2) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{(LCA1)(RCA1)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}((h_1 \triangleright \alpha_A^{-1}(a'))_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\\
&\quad \times (h_1 \triangleright \alpha_A^{-1}(a'))_{2(-1)})(\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \otimes \alpha_B^{-1}((\alpha_B^{-1}(b) \triangleleft h'_2)_{1[0]}\alpha_B^{-1}(b'_{1[0]})_{[0]})\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(h_1 \triangleright \alpha_A^{-1}(a'))_{2(0)}) \otimes (\beta^{-2}(h_{22})\beta^{-2}(h'_{12}))((\alpha_B^{-1}(b) \triangleleft h'_2)_{1[1]}\\
&\quad \times \alpha_B^{-1}(b'_{1[0]})_{[1]}) \otimes (\alpha_B^{-1}((\alpha_B^{-1}(b) \triangleleft h'_2)_2) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{(LMC1)(RMC1)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(h_{11} \triangleright \alpha_A^{-1}(a')_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\\
&\quad \times (h_{12} \triangleright \alpha_A^{-1}(a')_2)_{(-1)})(\beta^{-2}(h_{21})\beta^{-2}(h'_{11})) \otimes \alpha_B^{-1}((\alpha_B^{-1}(b)_1 \triangleleft h'_{21})_{[0]}\alpha_B^{-1}(b'_{1[0]})_{[0]})\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(h_{12} \triangleright \alpha_A^{-1}(a')_{(2)})_{(0)}) \otimes (\beta^{-2}(h_{22})\beta^{-2}(h'_{12}))((\alpha_B^{-1}(b)_1 \triangleleft h'_{21})_{[1]}\\
&\quad \times \alpha_B^{-1}(b'_{1[0]})_{[1]}) \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b)_2 \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{(A2)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(h_{11} \triangleright \alpha_A^{-1}(a')_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\beta^{-1}((h_{12} \triangleright \alpha_A^{-1}(a')_2)_{(-1)})\\
&\quad \times (\beta^{-2}(h_{21})))\beta^{-1}(h'_{11})) \otimes \alpha_B^{-1}((\alpha_B^{-1}(b)_1 \triangleleft h'_{21})_{[0]}\alpha_B^{-1}(b'_{1[0]})_{[0]})\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(h_{12} \triangleright \alpha_A^{-1}(a')_{(2)})_{(0)}) \otimes (\beta^{-2}(h_{22})\beta^{-1}(\beta^{-2}(h'_{12})\\
&\quad \times (\alpha_B^{-1}(b)_1 \triangleleft h'_{21})_{[1]}))\beta(\alpha_B^{-1}(b'_{1[0]})_{[1]}) \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b)_2 \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{(C2)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a')_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\\
&\quad \times \beta^{-1}((\beta^{-1}(h_{211}) \triangleright \alpha_A^{-1}(a')_2)_{(-1)})(\beta^{-3}(h_{212})))h'_1 \otimes \alpha_B^{-1}((\alpha_B^{-1}(b)_1 \triangleleft \beta^{-1}(h'_{212}))_{[0]}\\
&\quad \times \alpha_B^{-1}(b'_{1[0]})_{[0]}) \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(\beta^{-1}(h_{211}) \triangleright \alpha_A^{-1}(a')_{(2)})_{(0)})\\
&\quad \otimes (\beta^{-2}(h_{22})\beta^{-1}(\beta^{-3}(h'_{211})(\alpha_B^{-1}(b)_1 \triangleleft \beta^{-1}(h'_{212}))_{[1]}))\beta(\alpha_B^{-1}(b'_{1[0]})_{[1]})\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b)_2 \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{(C1)}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a')_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\\
&\quad \times \beta^{-1}((\beta^2(\beta^{-3}(h_{21})_1) \triangleright \alpha_A^{-1}(a')_2)_{(-1)})(\beta^{-3}(h_{21})_2))h'_1\\
&\quad \otimes \alpha_B^{-1}((\alpha_B^{-1}(b)_1 \triangleleft \beta^2(\beta^{-3}(h'_{21})_2))_{[0]}\alpha_B^{-1}(b'_{1[0]})_{[0]})\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(\beta^{-3}(h_{21})_1 \triangleright \alpha_A^{-1}(a')_{(2)})_{(0)})\\
&\quad \otimes (\beta^{-2}(h_{22})\beta^{-1}(\beta^{-3}(h'_{21})_1(\alpha_B^{-1}(b)_1 \triangleleft \beta^2(\beta^{-3}(h'_{21})_2)))_{[1]}))\beta(\alpha_B^{-1}(b'_{1[0]})_{[1]})\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b)_2 \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(LR5)(RR5)}}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a')_1)) \otimes (\alpha_A^{-1}(a_{2(0)})_{(-1)}\beta^{-1}(\beta^{-3}(h_{21})_1\\
&\quad \times \beta(\alpha_A^{-1}(a')_{2(-1)})))h'_1 \otimes \alpha_B^{-1}((\alpha_B^{-1}(b)_{1[0]} \triangleleft \beta^3(\beta^{-3}(h'_{21})_1))\alpha_B^{-1}(b'_{1[0]})_{[0]})\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)})_{(0)}(\beta^3(\beta^{-3}(h_{21})_2) \triangleright \alpha_A^{-1}(a')_{(2)(0)})\\
&\quad \otimes (\beta^{-2}(h_{22})\beta^{-1}(\beta(\alpha_B^{-1}(b)_{1[1]})\beta^{-3}(h'_{21})_2))\beta(\alpha_B^{-1}(b'_{1[0]})_{[1]})\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b)_2 \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{\text{(LCM1)(RCM1)(C1)}}{=} a_1(\beta^2(a_{2(-1)}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a'_1)))\\
&\quad \otimes (\alpha_A^{-1}(a_{2(0)(-1)})\beta^{-1}(\beta^{-3}(h_{211})a'_{2(-1)}))h'_1 \otimes \alpha_B^{-1}((\alpha_B^{-1}(b_{1[0]}) \triangleleft h'_{211})\alpha_B^{-1}(b'_{1[0][0]}))\\
&\quad \otimes \alpha_A^{-1}(\alpha_A^{-1}(a_{2(0)(0)})(h_{212} \triangleright \alpha_A^{-1}(a'_{(2)(0)}))) \otimes (\beta^{-2}(h_{22})\beta^{-1}(b_{1[1]}\beta^{-3}(h'_{212})))b'_{1[0][1]}\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b_2) \triangleleft h'_{22}) \triangleleft \beta^2(b'_{1[1]}))b'_2\\
&\overset{\text{(LCM2)(RCM2)}}{=} a_1(\beta(a_{2(-1)1}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a'_1)))\\
&\quad \otimes (\beta^{-1}(a_{2(-1)2})\beta^{-1}(\beta^{-3}(h_{211})a'_{2(-1)}))h'_1 \otimes \alpha_B^{-1}((\alpha_B^{-1}(b_{1[0]}) \triangleleft h'_{211})b'_{1[0]}\\
&\quad \otimes \alpha_A^{-1}(a_{2(0)}(h_{212} \triangleright \alpha_A^{-1}(a'_{(2)(0)}))) \otimes (\beta^{-2}(h_{22})\beta^{-1}(b_{1[1]}\beta^{-3}(h'_{212})))b'_{1[1]1}\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b_2) \triangleleft h'_{22}) \triangleleft \beta(b'_{1[1]2}))b'_2\\
&\overset{\text{(A2)}}{=} a_1(\beta(a_{2(-1)1}) \triangleright \alpha_A^{-1}(\beta(h_1) \triangleright \alpha_A^{-1}(a'_1)))\\
&\quad \otimes (\beta^{-1}(a_{2(-1)2})\beta^{-3}(h_{211}))(a'_{2(-1)}\beta^{-1}(h'_1) \otimes \alpha_B^{-1}((\alpha_B^{-1}(b_{1[0]}) \triangleleft h'_{211})b'_{1[0]}\\
&\quad \otimes \alpha_A^{-1}(a_{2(0)}(h_{212} \triangleright \alpha_A^{-1}(a'_{(2)(0)}))) \otimes (\beta^{-2}(h_{22})b_{1[1]})(\beta^{-3}(h'_{212})\beta^{-1}(b'_{1[1]1}))\\
&\quad \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b_2) \triangleleft h'_{22}) \triangleleft \beta(b'_{1[1]2}))b'_2\\
&\overset{\text{(C2)}}{=} a_1(\beta(a_{2(-1)1}) \triangleright \alpha_A^{-1}(h_{11} \triangleright \alpha_A^{-1}(a'_1))) \otimes (\beta^{-1}(a_{2(-1)2})\beta^{-2}(h_{12}))(a'_{2(-1)}\beta^{-2}(h'_{11})\\
&\quad \otimes \alpha_B^{-1}((\alpha_B^{-1}(b_{1[0]}) \triangleleft \beta(h'_{12}))b'_{1[0]} \otimes \alpha_A^{-1}(a_{2(0)}(\beta(h_{21}) \triangleright \alpha_A^{-1}(a'_{(2)(0)})))\\
&\quad \otimes (\beta^{-2}(h_{22})b_{1[1]})(\beta^{-2}(h'_{21})\beta^{-1}(b'_{1[1]1})) \otimes (\alpha_B^{-1}(\alpha_B^{-1}(b_2) \triangleleft h'_{22}) \triangleleft \beta(b'_{1[1]2}))b'_2\\
&\overset{\text{(LM1)(RM1)}}{=} a_1(\beta(a_{2(-1)1}) \triangleright (\beta^{-1}(h_{11}) \triangleright \alpha_A^{-2}(a'_1)))\\
&\quad \otimes (\beta^{-1}(a_{2(-1)2})\beta^{-2}(h_{12}))(a'_{2(-1)}\beta^{-2}(h'_{11}) \otimes (\alpha_B^{-2}(b_{1[0]}) \triangleleft h'_{12})\alpha_B^{-1}(b'_{1[0]})\\
&\quad \otimes \alpha_A^{-1}(a_{2(0)})(h_{21} \triangleright \alpha_A^{-2}(a'_{(2)(0)})) \otimes (\beta^{-2}(h_{22})b_{1[1]})(\beta^{-2}(h'_{21})\beta^{-1}(b'_{1[1]1}))\\
&\quad \otimes ((\alpha_B^{-2}(b_2) \triangleleft \beta^{-1}(h'_{22})) \triangleleft \beta(b'_{1[1]2}))b'_2\\
&\overset{\text{(LM2)(RM2)}}{=} a_1((a_{2(-1)1}\beta^{-1}(h_{11})) \triangleright \alpha_A^{-1}(a'_1))\\
&\quad \otimes (\beta^{-1}(a_{2(-1)2})\beta^{-2}(h_{12}))(a'_{2(-1)}\beta^{-2}(h'_{11}) \otimes (\alpha_B^{-2}(b_{1[0]}) \triangleleft h'_{12})\alpha_B^{-1}(b'_{1[0]})\\
&\quad \otimes \alpha_A^{-1}(a_{2(0)})(h_{21} \triangleright \alpha_A^{-2}(a'_{(2)(0)})) \otimes (\beta^{-2}(h_{22})b_{1[1]})(\beta^{-2}(h'_{21})\beta^{-1}(b'_{1[1]1}))\\
&\quad \otimes (\alpha_B^{-1}(b_2) \triangleleft (\beta^{-1}(h'_{22})b'_{1[1]2}))b'_2\\
&\overset{\text{(C1)(A1)}}{=} a_1((a_{2(-1)}\beta^{-1}(h_1))_1 \triangleright \alpha_A^{-1}(a'_1)) \otimes \beta^{-1}((a_{2(-1)}\beta^{-1}(h_1))_2)(a'_{2(-1)}\beta^{-2}(h'_{11})\\
&\quad \otimes (\alpha_B^{-2}(b_{1[0]}) \triangleleft h'_{12})\alpha_B^{-1}(b'_{1[0]} \otimes \alpha_A^{-1}(a_{2(0)})(h_{21} \triangleright \alpha_A^{-2}(a'_{(2)(0)}))\\
&\quad \otimes (\beta^{-2}(h_{22})b_{1[1]})\beta^{-1}((\beta^{-1}(h'_2)b'_{1[1]})_1) \otimes (\alpha_B^{-1}(b_2) \triangleleft (\beta^{-1}(h'_2)b'_{1[1]})_2)b'_2\\
&= a_1((a_{2(-1)}\beta^{-1}(h_1))_1 \triangleright \alpha_A^{-1}(a'_1)) \otimes \beta^{-1}((a_{2(-1)}\beta^{-1}(h_1))_2)(a'_{2(-1)}\beta^{-2}(h'_{11})\\
&\quad \otimes (\alpha_B^{-2}(b_{1[0]}) \triangleleft h'_{12})\alpha_B^{-1}(b'_{1[0]} \otimes \alpha_A^{-1}(a_{2(0)})(h_{21} \triangleright \alpha_A^{-2}(a'_{(2)(0)}))
\end{aligned}
$$

$$\otimes\beta^{-1}((\beta^{-1}(h_{22})\beta(b_{1[1]}))(\beta^{-1}(h'_2)b'_{1[1]})_1)\otimes(\alpha_B^{-1}(b_2)\lhd(\beta^{-1}(h'_2)b'_{1[1]})_2)b'_2$$

$$\overset{(DB)}{=} a_1((a_{2(-1)}\beta^{-1}(h_1))_1\rhd\alpha_A^{-1}(a'_1))\otimes\beta^{-1}((a_{2(-1)}\beta^{-1}(h_1))_2)(\beta^{-1}(a'_{2(-1)1})\beta^{-2}(h'_{11})$$
$$\otimes((\alpha_B^{-3}(b_{1[0]})\lhd a'_{2(-1)2})\lhd h'_{12})\alpha_B^{-1}(b'_{1[0]})\otimes\alpha_A^{-1}(a_{2(0)})$$
$$\times(h_{21}\rhd(b_{1[1]1}\rhd\alpha_A^{-3}(a'_{(2)(0)})))\otimes\beta^{-1}((\beta^{-1}(h_{22})b_{1[1]2})(\beta^{-1}(h'_2)b'_{1[1]})_1)$$
$$\otimes(\alpha_B^{-1}(b_2)\lhd(\beta^{-1}(h'_2)b'_{1[1]})_2)b'_2$$

$$\overset{(LM2)(RM2)}{=} a_1((a_{2(-1)}\beta^{-1}(h_1))_1\rhd\alpha_A^{-1}(a'_1))\otimes\beta^{-1}((a_{2(-1)}\beta^{-1}(h_1))_2)$$
$$\times(\beta^{-1}(a'_{2(-1)1})\beta^{-2}(h'_{11}))\otimes(\alpha_B^{-2}(b_{1[0]})\lhd(a'_{2(-1)2}\beta^{-1}(h'_{12})))\alpha_B^{-1}(b'_{1[0]})$$
$$\otimes\alpha_A^{-1}(a_{2(0)})((\beta^{-1}(h_{21})b_{1[1]1})\rhd\alpha_A^{-2}(a'_{(2)(0)}))$$
$$\otimes\beta^{-1}((\beta^{-1}(h_{22})b_{1[1]2})(\beta^{-1}(h'_2)b'_{1[1]})_1)\otimes(\alpha_B^{-1}(b_2)\lhd(\beta^{-1}(h'_2)b'_{1[1]})_2)b'_2$$

$$\overset{(A1)(C1)}{=} a_1((a_{2(-1)}\beta^{-1}(h_1))_1\rhd\alpha_A^{-1}(a'_1))\otimes\beta^{-1}((a_{2(-1)}\beta^{-1}(h_1))_2(a'_{2(-1)}\beta^{-1}(h'_1))_1)$$
$$\otimes(\alpha_B^{-2}(b_{1[0]})\lhd(a'_{2(-1)}\beta^{-1}(h'_1))_2)\alpha_B^{-1}(b'_{1[0]})\otimes\alpha_A^{-1}(a_{2(0)})$$
$$\times((\beta^{-1}(h_2)b_{1[1]})_1\rhd\alpha_A^{-2}(a'_{(2)(0)}))\otimes\beta^{-1}((\beta^{-1}(h_2)b_{1[1]})_2(\beta^{-1}(h'_2)b'_{1[1]})_1)$$
$$\otimes(\alpha_B^{-1}(b_2)\lhd(\beta^{-1}(h'_2)b'_{1[1]})_2)b'_2$$
$$=\Delta_{A\diamond H\diamond B}(a\otimes h\otimes b)\Delta_{A\diamond H\diamond B}(a'\otimes h'\otimes b'),$$

and $\Delta(1_A\otimes 1_H\otimes 1_B)=1_A\otimes 1_H\otimes 1_B\otimes 1_A\otimes 1_H\otimes 1_B$ is easy.

($\Rightarrow$) Set $a=1_A,\ h=h'=1_H$, and $b'=1_B$ in

$$\Delta_{A\diamond H\diamond B}((a\otimes h\otimes b)(a'\otimes h'\otimes b'))=\Delta_{A\diamond H\diamond B}(a\otimes h\otimes b)\Delta_{A\diamond H\diamond B}(a'\otimes h'\otimes b'),$$

and we have

$$\alpha_A(a')_1\otimes\beta(\alpha_A(a')_{2(-1)})\otimes\alpha_B^{-1}(\alpha_B(b)_{1[0]})\otimes\alpha_A^{-1}(\alpha_A(a')_{2[0]})\otimes\beta(\alpha_B(b)_{1[1]})\otimes\alpha_B(b)_2$$
$$=\alpha_A(a'_1)\otimes\beta(a'_{2(-1)})_1\otimes\alpha_B(\alpha_B^{-2}(b_{1[0]})\lhd\beta(a'_{2(-1)})_2)$$
$$\otimes\alpha_A(\beta(b_{1[1]})_1\rhd\alpha_A^{-2}(a'_{2(0)}))\otimes\beta(b_{1[1]})_2\otimes\alpha_B(b_2).$$

Then, applying $\varepsilon_A\otimes\mathrm{id}_H\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\mathrm{id}_H\otimes\varepsilon_B$ to the above equation, by (C1), we obtain the condition (DB).

Remark 3.1 (1) When $\alpha_A=\mathrm{id}_A,\ \beta=\mathrm{id}_H$, and $\alpha_B=\mathrm{id}_B$, we get Majid's double biproduct bialgebra in [11].

(2) Let $B=K$, and we obtain the left Radford's biproduct Hom-bialgebra. Let $A=K$, and we obtain the following right Radford's biproduct Hom-bialgebra $H^{\natural}_{\diamond}A$.

Corollary 3.1 *Let (H,β) be a Hom-bialgebra, and (A,α) be a right (H,β)-module Hom-algebra with module structure $\lhd: A\otimes H\longrightarrow A$ and a right (H,β)-comodule Hom-coalgebra with comodule structure $\delta: A\longrightarrow A\otimes H$. Then the following are equivalent:*

(i) *$(H^{\natural}_{\diamond}A,\mu_{H\natural A},1_H\otimes 1_A,\Delta_{H\diamond A},\varepsilon_H\otimes\varepsilon_A,\beta\otimes\alpha)$ is a Hom-bialgebra, where $H\natural A$ is a right smash product Hom-algebra and $H\diamond A$ is a right smash coproduct Hom-coalgebra.*

(ii) *The following conditions hold ($\forall a,b\in A$ and $h\in H$):*

(RR1) *(A,δ,α) is a right (H,β)-comodule Hom-algebra,* (RR2) $(A, \triangleleft, \alpha)$ *is a right* (H, β)*-module Hom-coalgebra,*

(RR3) ε_A *is a Hom-algebra map and* $\Delta_A(1_A) = 1_A \otimes 1_A$*,*

(RR4) $\Delta_A(ab) = a_1\alpha^{-1}(b_{1[0]}) \otimes (\alpha^{-1}(a_2) \triangleleft \beta^2(b_{1[1]}))b_2$*, and*

(RR5) $(a_{[0]} \triangleleft \beta^3(h_1)) \otimes \beta(a_{[1]})h_2 = (a \triangleleft \beta^2(h_2))_{[0]} \otimes h_1(a \triangleleft \beta^2(h_2))_{[1]}$.

Also, we have the following corollary.

Corollary 3.2 *Let* (H, β) *be a Hom-bialgebra such that* $\beta^2 = \mathrm{id}_H$*, and* (A, α) *be a right* (H, β)*-module Hom-algebra and a right* (H, β)*-comodule Hom-coalgebra. Then* $(H^{\natural}_{\diamond}A, \mu_{H\natural A}, 1_H \otimes 1_A, \Delta_{H\diamond A}, \varepsilon_H \otimes \varepsilon_A, \beta \otimes \alpha)$ *is a right Radford biproduct Hom-bialgebra if and only if* (A, α) *is a Hom-bialgebra in the right-right Hom-Yetter-Drinfeld category* $\mathbb{YD}^H_H$*.*

4 Quasitriangular Smash Coproduct Hom-Hopf Algebras

In this section, we introduce a class of new Hom-Hopf algebras: The T-smash coproduct $C \diamond_T H$, generalizing the T-smash coproduct studied in [3, 14]. The Hom-smash coproduct Hom-Hopf algebra is a special case. Necessary and sufficient conditions for the smash coproduct Hom-Hopf algebra to be quasitriangular are given.

In a way dual to [9, Theorem 3.1], we have the following proposition.

Proposition 4.1 *Let* $(C, \Delta_C, \varepsilon_C, \alpha)$ *and* $(H, \Delta_H, \varepsilon_H, \beta)$ *be two Hom-coalgebras, and* $T : C \otimes H \longrightarrow H \otimes C$ *(write* $T(c \otimes h) = h_T \otimes c_T,\ \forall c \in C,\ h \in H$*) be a linear map such that for all* $c \in C$ *and* $h \in H$*,*

$$\text{(T)} \quad \alpha(c)_T \otimes \beta(h)_T = \alpha(c_T) \otimes \beta(h_T).$$

Then $(C \diamond_T H, \alpha \otimes \beta)$ *(*$C \diamond_T H = C \otimes H$ *as a linear space) and counit* $\varepsilon_C \otimes \varepsilon_H$ *with the comultiplication*

$$\Delta_{C\diamond_T H}(c \otimes h) = c_1 \otimes \beta^{-1}(h_1)_T \otimes \alpha^{-1}(c_{2T}) \otimes h_2$$

becomes a Hom-coalgebra if and only if the following conditions hold:

$$\text{(TS1)} \quad \varepsilon_H(h_T)c_T = \varepsilon_H(h)\alpha(c); \quad h_T\varepsilon_C(c_T) = \beta(h)\varepsilon_C(c),$$

$$\text{(TS2)} \quad h_{T1} \otimes h_{T2} \otimes \alpha(c_T) = \beta(\beta^{-1}(h_1)_T) \otimes h_{2t} \otimes c_{Tt},$$

$$\text{(TS3)} \quad \beta(h_T) \otimes \alpha(c)_{T1} \otimes \alpha(c)_{T2} = h_{Tt} \otimes \alpha(c_1)_t \otimes \alpha(c_{2T}),$$

where $c \in C,\ h \in H$ *and* t *is a copy of* T*.*

We call this a Hom-coalgebra T*-smash coproduct Hom-coalgebra and denote it by* $(C \diamond_T H, \alpha \otimes \beta)$*.*

Remark 4.1 (1) Let $T(c \otimes h) = c_{-1}h \otimes c_0$ in $C \diamond_T H$, and we can get the smash coproduct Hom-coalgebra $C \diamond H$.

(2) When $\alpha = \mathrm{id}_C$ and $\beta = \mathrm{id}_H$, we can get the usual T-smash coproduct coalgebra (see [3, 10]).

Theorem 4.1 *Let* (C, α, S_C) *and* (H, β, S_H) *be two Hom-Hopf algebras, and* $T : C \otimes H \longrightarrow H \otimes C$ *be a linear map. Then the* T*-smash coproduct Hom-coalgebra* $(C \diamond_T H, \alpha \otimes \beta)$ *equipped with the tensor product Hom-algebra structure becomes a Hom-bialgebra if and only if* T *is a*

Hom-algebra map. Furthermore, the T-smash coproduct Hom-bialgebra $(C \diamond_T H, \alpha\otimes\beta)$ is a Hom-Hopf algebra with antipode $\overline{S}$ defined by

$$\overline{S}(c\otimes h) = S_C(\alpha^{-1}(c_T))\otimes S_H(\beta^{-1}(h)_T).$$

Proof We only prove that $\overline{S}$ is an antipode of $(C \diamond_T H, \alpha\otimes\beta)$. The rest is straightforward by direct computation. For all $c\in C$ and $h\in H$,

$$\begin{aligned}
(\overline{S} * \mathrm{id}_{C\diamond_T H})(c\otimes h) &= S_C(\alpha^{-1}(c_{1t}))\alpha^{-1}(c_{2T})\otimes S_H(\beta^{-1}(\beta^{-1}(h_1)_T)_t)h_2\\
&\overset{(\mathrm{T})}{=} S_C(\alpha^{-1}(c_{1t}))\alpha^{-1}(c_2)_T\otimes S_H(\beta^{-2}(h_1)_{Tt})h_2\\
&\overset{(\mathrm{TS3})}{=} S_C(\alpha^{-1}(c_{T1}))\alpha^{-1}(c_{T2})\otimes S_H(\beta(\beta^{-2}(h_1)_T))h_2\\
&= \alpha^{-1}(S_C(c_{T1}))\alpha^{-1}(c_{T2})\otimes S_H(\beta(\beta^{-2}(h_1)_T))h_2\\
&\overset{(\mathrm{A1})}{=} \alpha^{-1}(S_C(c_{T1})c_{T2})\otimes S_H(\beta(\beta^{-2}(h_1)_T))h_2\\
&\overset{(\mathrm{A1})}{=} 1_C\varepsilon_C(c_T)\otimes S_H(\beta(\beta^{-2}(h_1)_T))h_2\\
&\overset{(\mathrm{TS1})}{=} 1_C\varepsilon_C(c)\otimes S_H(\beta^2(\beta^{-2}(h_1)))h_2\\
&= 1_C\varepsilon_C(c)\otimes S_H(h_1)h_2\\
&= 1_C\otimes 1_H\overline{\varepsilon}(c\otimes h)
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id}_{C\diamond_T H} * \overline{S})(c\otimes h) &= c_1 S_C(\alpha^{-1}(\alpha^{-1}(c_{2T})_t))\otimes\beta^{-1}(h_1)_T S_H(\beta^{-1}(h_2)_t)\\
&\overset{(\mathrm{T})}{=} c_1 S_C(\alpha^{-2}(c_{2Tt}))\otimes\beta^{-1}(h_1)_T S_H(\beta^{-1}(h_{2t}))\\
&\overset{(\mathrm{TS2})}{=} c_1 S_C(\alpha^{-2}(c_{2T}))\otimes\beta^{-1}(h_{T1})S_H(\beta^{-1}(h_{T2}))\\
&\overset{(\mathrm{A1})}{=} c_1 S_C(\alpha^{-2}(c_{2T}))\otimes\beta^{-1}(h_{T1}S_H(h_{T2}))\\
&= c_1 S_C(\alpha^{-2}(c_{2T}))\otimes\beta^{-1}(1_H)\varepsilon_H(h_T)\\
&\overset{(\mathrm{TS1})}{=} c_1 S_C(c_2)\otimes 1_H\varepsilon_H(h)\\
&= 1_C\otimes 1_H\overline{\varepsilon}(c\otimes h),
\end{aligned}$$

while

$$\begin{aligned}
\overline{S}(\alpha(c)\otimes\beta(h)) &= S_C(\alpha^{-1}(\alpha(c)_T))\otimes S_H(h_T)\\
&\overset{(\mathrm{T})}{=} S_C(\alpha^{-1}(\alpha(c_T)))\otimes S_H(\beta(\beta^{-1}(h)_T))\\
&= S_C(\alpha(\alpha^{-1}(c_T)))\otimes\beta(S_H(\beta^{-1}(h)_T))\\
&= \alpha(S_C(\alpha^{-1}(c_T)))\otimes\beta(S_H(\beta^{-1}(h)_T))\\
&= (\alpha\otimes\beta)(\overline{S}(c\otimes h)),
\end{aligned}$$

which finishes the proof.

Theorem 4.2 *Let (C,α,S_C) and (H,β,S_H) be two Hom-Hopf algebras, and (C,ρ,α) be a left (H,β)-comodule Hom-coalgebra. Then the smash coproduct Hom-coalgebra $(C\diamond H,\alpha\otimes\beta)$*

endowed with the tensor product Hom-algebra structure becomes a Hom-bialgebra if and only if (C,ρ,α) *is a left* (H,β)*-comodule Hom-algebra and the following condition holds:*

$$c_{(-1)}h\otimes c_{(0)}=hc_{(-1)}\otimes c_{(0)}.$$

Moreover, the smash coproduct Hom-bialgebra $(C\diamond H,\alpha\otimes\beta)$ *is a Hom-Hopf algebra with the antipode*

$$S_{C\diamond H}(c\otimes h)=S_C(\alpha^{-1}(c_{(0)}))\otimes S_H(c_{(-1)}\beta^{-1}(h)).$$

Proof Let $T(c\otimes h)=c_{(-1)}h\otimes c_{(0)},\ \forall c\in C,\ h\in H$ in Theorem 4.1.

Next, we generalize the concept of compatibility Hopf algebra pairs (see [10]) to the Hom-setting.

Definition 4.1 *Let* (C,α,S_C) *and* (H,β,S_H) *be two Hom-Hopf algebras, and* $\vartheta=\vartheta^1\otimes\vartheta^2\in C\otimes H$*. A Hom-compatibility Hopf algebra triple is a triple* (C,H,ϑ) *such that* $(\vartheta=\overline{\vartheta})$

$$\begin{aligned}
&\text{(CT1)}\quad \varepsilon_C(\vartheta^1)\vartheta^2=1_H,\quad \vartheta^1\varepsilon_H(\vartheta^2)=1_C,\\
&\text{(CT2)}\quad {\vartheta^1}_1\otimes{\vartheta^1}_2\otimes\beta(\vartheta^2)=\alpha(\vartheta^1)\otimes\alpha(\overline{\vartheta}^1)\otimes\vartheta^2\overline{\vartheta}^2,\\
&\text{(CT3)}\quad \alpha(\vartheta^1)\otimes{\vartheta^2}_1\otimes{\vartheta^2}_2=\vartheta^1\overline{\vartheta}^1\otimes\beta(\overline{\vartheta}^2)\otimes\beta(\vartheta^2),\\
&\text{(CT4)}\quad \alpha(\vartheta^1)\otimes\beta(\vartheta^2)=\vartheta^1\otimes\vartheta^2.
\end{aligned}$$

Remark 4.2 (1) When $\alpha=\mathrm{id}_C$ and $\beta=\mathrm{id}_H$, we can get the compatibility Hopf algebra pairs.

(2) If (H,β,R) is a quasitriangular Hom-Hopf algebra, then (H,H,R) is a Hom-compatibility Hopf algebra triple.

(3) ϑ is (convolution) invertible with $\vartheta^{-1}=S_C(\vartheta^1)\otimes\vartheta^2$.

Proposition 4.2 *Let* $(C\diamond_T H,\alpha\otimes\beta)$ *be a* T*-smash coproduct Hom-Hopf algebra. Define*

$$\psi: C\diamond_T H\longrightarrow C,\ \psi(c\otimes h)=c\varepsilon_H(h),\quad \varphi: C\diamond_T H\longrightarrow H,\ \varphi(c\otimes h)=\varepsilon_C(c)h$$

for all $c\in C$ *and* $h\in H$*. Then* ψ *and* φ *are both Hom-bialgebra maps.*

Proof Straightforward.

Let $(C\diamond_T H,\alpha\otimes\beta)$ be a T-smash coproduct Hom-Hopf algebra, and $R\in C\diamond_T H\otimes C\diamond_T H$. Define

$$\begin{aligned}
P&=(\psi\otimes\psi)(R)\in C\otimes C,\quad Q=(\varphi\otimes\varphi)(R)\in H\otimes H,\\
U&=(\psi\otimes\varphi)(R)\in C\otimes H,\quad V=(\varphi\otimes\psi)(R)\in H\otimes C.
\end{aligned}$$

The following two lemmas are obvious.

Lemma 4.1 *Let* $(C\diamond_T H,\alpha\otimes\beta)$ *be a* T*-smash coproduct Hom-Hopf algebra. If* R *satisfies* (QT1)*, then*

$$\varepsilon_C(P^1)P^2=P^1v_C(P^2)=1_C,$$

$$\varepsilon_H(Q^1)Q^2 = Q^1 v_H(Q^2) = 1_H,$$
$$\varepsilon_C(U^1)U^2 = 1_H, \quad U^1\varepsilon_H(U^2) = 1_C,$$
$$\varepsilon_H(V^1)V^2 = 1_C, \quad V^1 v_C(V^2) = 1_H.$$

Lemma 4.2 *Let $(C \diamond_T H, \alpha\otimes\beta)$ be a T-smash coproduct Hom-Hopf algebra. If R satisfies* (QT5) *for $\alpha\otimes\beta$, then*

$$(\alpha\otimes\alpha)(P) = P, \quad (\beta\otimes\beta)(Q) = Q, \quad (\alpha\otimes\beta)(U) = U, \quad (\beta\otimes\alpha)(V) = V.$$

Lemma 4.3 *Let $(C \diamond_T H, \alpha\otimes\beta, R)$ be a quasitriangular T-smash coproduct Hom-Hopf algebra. Then, we have*

$$\text{(QS)} \quad (\alpha\otimes\beta\otimes\alpha\otimes\beta)(R) = U^1P^1\otimes Q^1V^1\otimes P^2V^2\otimes U^2Q^2.$$

Proof By (QT2) and (QT3), we have

$$\begin{aligned} &R^1{}_1\otimes\beta^{-1}(R^2{}_1)_T\otimes\alpha^{-1}(R^1{}_{2T})\otimes R^2{}_2\otimes R^3{}_1\otimes\beta^{-1}(R^4{}_2)_t\otimes\alpha^{-1}(R^3{}_{2t})\otimes R^4{}_2 \\ = &R^1\overline{R}^1\otimes R^2\overline{R}^2\otimes r^1\overline{r}^1\otimes r^2\overline{r}^2\otimes\overline{R}^3\overline{r}^3\otimes\overline{R}^4\overline{r}^4\otimes R^3r^3\otimes R^4r^4. \end{aligned}$$

Applying $\psi\otimes\varphi\otimes\psi\otimes\varphi$ to the above equation, we can get (QS).

Lemma 4.4 *Let $(C \diamond_T H, \alpha\otimes\beta, R)$ be a quasitriangular T-smash coproduct Hom-Hopf algebra. Then, for all $c\in C$, and $h\in H$, we have*

$$\begin{aligned} &\text{(D1)} \quad \beta^{-1}(V^1)_T\otimes P^1{}_T\otimes P^2V^2 = V^1\otimes\alpha(P^1)\otimes V^2P^2, \\ &\text{(D2)} \quad \beta^{-1}(Q^1)_T\otimes U^1{}_T\otimes U^2Q^2 = Q^1\otimes\alpha(U^1)\otimes Q^2U^2, \\ &\text{(D3)} \quad Q^1V^1\otimes\beta^{-1}(Q^2)_T\otimes V^2{}_T = V^1Q^1\otimes Q^2\otimes\alpha(V^2), \\ &\text{(D4)} \quad U^1P^1\otimes\beta^{-1}(U^2)_T\otimes P^2{}_T = P^1U^1\otimes U^2\otimes\alpha(P^2), \\ &\text{(D5)} \quad \beta(h)V^1\otimes\alpha(c)V^2 = V^1h_T\otimes V^2\alpha^{-1}(\alpha(c)_T), \\ &\text{(D6)} \quad \alpha^{-1}(\alpha(c)_T)U^1\otimes h_TU^2 = U^1\alpha(c)\otimes U^2\beta(h). \end{aligned}$$

Proof By (QT2), we can obtain

$$\begin{aligned} &R^1{}_1\otimes\beta^{-1}(R^2{}_1)_T\otimes\alpha^{-1}(R^1{}_{2T})\otimes R^2{}_2\otimes\alpha(R^3)\otimes\beta(R^4) \\ = &\alpha(R^1)\otimes\beta(R^2)\otimes\alpha(r^1)\otimes\beta(r^2)\otimes R^3r^3\otimes R^4r^4. \end{aligned} \tag{4.1}$$

Applying $\varphi\otimes\psi\otimes\psi$ to (4.1), we have that (D1) holds by (QS) and (T). Similarly, applying $\varphi\otimes\psi\otimes\varphi$ to (4.1), we can get (D2) by (QS) and (T).

By (QT3), we have

$$\begin{aligned} &\alpha(R^1)\otimes\beta(R^2)\otimes R^3{}_1\otimes\beta^{-1}(R^4{}_2)_T\otimes\alpha^{-1}(R^3{}_{2T})\otimes R^4{}_2 \\ = &R^1r^1\otimes R^2r^2\otimes\alpha(r^3)\otimes\beta(r^4)\otimes\alpha(R^3)\otimes\beta(R^4). \end{aligned} \tag{4.2}$$

(D3) can be obtained by applying $\varphi\otimes\varphi\otimes\psi$ to (4.2) and by (QS) and (T). Likewise, one gets (D4) by using $\psi\otimes\varphi\otimes\psi$ to (4.2) and by (QS) and (T).

By (QT4), for all $c\in C$ and $h\in H$, we have

$$\begin{aligned}&\alpha^{-1}(c_{2T})R^1\otimes h_2R^2\otimes c_1R^3\otimes\beta^{-1}(h_1)_TR^4\\&=R^1c_1\otimes R^2\beta^{-1}(h_1)_T\otimes R^3\alpha^{-1}(c_{2T})\otimes R^4h_2.\end{aligned}\tag{4.3}$$

Apply $\varphi\otimes\psi$ to (4.3), we get (D5). (D6) is derived by applying $\psi\otimes\varphi$ to (4.3).

Lemma 4.5 *Given a quasitriangular structure R on a T-smash coproduct Hom-Hopf algebra $(C\diamond_T H,\alpha\otimes\beta)$, consider the induced elements P, Q, U and V. Then*

(1) (C,α,P) and (H,β,Q) are quasitriangular Hom-Hopf algebras, and

(2) (C,H,U) and (H,C,V) are Hom-compatibility Hopf algebra triples.

Proof (1) Applying $\varphi\otimes\varphi\otimes\varphi$ to (4.1) and (4.2), we can get (QT2) and (QT3) for P, respectively. (QT4) can be derived by applying $\varphi\otimes\varphi$ to (4.3). Then by Lemmas 4.1–4.2, (C,α,P) is a quasitriangular Hom-Hopf algebra. Similarly, we can prove that (H,β,Q) is a quasitriangular Hom-Hopf algebra.

(2) Apply $\psi\otimes\psi\otimes\varphi$ to (4.1), and $\psi\otimes\varphi\otimes\varphi$ to (4.2). (CT2) and (CT3) can be obtained for U, respectively. Then (C,H,U) is a Hom-compatibility Hopf algebra triple by Lemmas 4.1–4.2. The rest of (4.2) can be similarly demonstrated.

Lemma 4.6 *Let $(C\diamond_T H,\alpha\otimes\beta)$ be a T-smash coproduct Hom-Hopf algebra. If there exist elements $P\in C\otimes C$, $Q\in H\otimes H$, $U\in C\otimes H$ and $V\in H\otimes C$ such that*

(1) (C,α,P) and (H,β,Q) are quasitriangular Hom-Hopf algebras,

(2) (C,H,U) and (H,C,V) are Hom-compatibility Hopf algebra triples, and

(3) the conditions (D1)–(D6) *in Lemma* 4.4 *hold,*

then $(C\diamond_T H,\alpha\otimes\beta,R)$ is a quasitriangular Hom-Hopf algebra with the quasitriangular structure given by

$$(\alpha\otimes\beta\otimes\alpha\otimes\beta)(R)=U^1P^1\otimes Q^1V^1\otimes P^2V^2\otimes U^2Q^2.$$

Proof It is obvious that R satisfies (QT1) and (QT5).

Next, we show that (QT3) holds for R:

$$\begin{aligned}\text{LHS}&=U^1P^1\otimes Q^1V^1\otimes\alpha^{-1}(P^2)_1\alpha^{-1}(V^2)_1\otimes\beta^{-1}(\beta^{-1}(U^2)_1\beta^{-1}(Q^2)_1)_T\\&\quad\otimes\alpha^{-1}((\alpha^{-1}(P^2)_2\alpha^{-1}(V^2)_2)_T)\otimes\beta^{-1}(U^2)_2\beta^{-1}(Q^2)_2\\&\overset{(C1)}{=}U^1P^1\otimes Q^1V^1\otimes\alpha^{-1}(P^2{}_1)\alpha^{-1}(V^2{}_1)\otimes\beta^{-1}(\beta^{-1}(U^2{}_1)\beta^{-1}(Q^2{}_1))_T\\&\quad\otimes\alpha^{-1}((\alpha^{-1}(P^2{}_2)\alpha^{-1}(V^2{}_2))_T)\otimes\beta^{-1}(U^2{}_2)\beta^{-1}(Q^2{}_2)\\&\overset{(QT3)(CT3)}{=}\alpha^{-1}((U^1u^1)(P^1p^1))\otimes\beta^{-1}((Q^1q^1)(V^1v^1))\otimes p^2v^2\otimes\beta^{-1}(u^2q^2)_T\\&\quad\otimes\alpha^{-1}((P^2V^2)_T)\otimes U^2Q^2\\&\overset{(A2)}{=}\alpha^{-1}((U^1\alpha^{-1}(u^1P^1))\alpha(p^1))\otimes\beta^{-1}((Q^1\beta^{-1}(q^1V^1))\beta(v^1))\otimes p^2v^2\otimes\beta^{-1}(u^2q^2)_T\\&\quad\otimes\alpha^{-1}((P^2V^2)_T)\otimes U^2Q^2\\&\overset{(D4)(D3)}{=}\alpha^{-1}((U^1\alpha^{-1}(P^1u^1))\alpha(p^1))\otimes\beta^{-1}((Q^1\beta^{-1}(V^1q^1))\beta(v^1))\otimes p^2v^2\otimes u^2q^2\\&\quad\otimes P^2V^2\otimes U^2Q^2\\&\overset{(A2)}{=}\alpha^{-1}((U^1P^1)(u^1p^1))\otimes\beta^{-1}((Q^1V^1)(q^1v^1))\otimes p^2v^2\otimes u^2q^2\otimes P^2V^2\otimes U^2Q^2\end{aligned}$$

$= \text{RHS}.$

(QT2) for R can be proved by the similar method. And we check (QT4) as follows:

$$
\begin{aligned}
\text{LHS} &= \alpha^{-1}(c_{2T})R^1 \otimes h_2R^2 \otimes c_1R^3 \otimes \beta^{-1}(h_1)_TR^4 \\
&= \alpha^{-1}(c_{2T})\alpha^{-1}(U^1P^1) \otimes h_2\beta^{-1}(Q^1V^1) \otimes c_1\alpha^{-1}(P^2V^2) \otimes \beta^{-1}(h_1)_T\beta^{-1}(U^2Q^2) \\
&\overset{(A1)}{=} \alpha^{-1}(c_{2T}(U^1P^1)) \otimes \beta^{-1}(\beta(h_2)(Q^1V^1)) \otimes \alpha^{-1}(\alpha(c_1)(P^2V^2)) \\
&\quad \otimes \beta^{-1}(\beta(\beta^{-1}(h_1)_T)(U^2Q^2)) \\
&\overset{(A2)}{=} \alpha^{-1}((\alpha^{-1}(c_{2T})U^1)\alpha(P^1)) \otimes \beta^{-1}((h_2Q^1)\beta(V^1)) \otimes \alpha^{-1}((c_1P^2)\alpha(V^2)) \\
&\quad \otimes \beta^{-1}((\beta^{-1}(h_1)_TU^2)\beta(Q^2)) \\
&\overset{(D6)}{=} \alpha^{-1}((U^1c_2)\alpha(P^1)) \otimes \beta^{-1}((h_2Q^1)\beta(V^1)) \otimes \alpha^{-1}((c_1P^2)\alpha(V^2)) \\
&\quad \otimes \beta^{-1}((U^2h_1)\beta(Q^2)) \\
&\overset{(A2)}{=} \alpha^{-1}(\alpha(U^1)(c_2P^1)) \otimes \beta^{-1}((h_2Q^1)\beta(V^1)) \otimes \alpha^{-1}((c_1P^2)\alpha(V^2)) \\
&\quad \otimes \beta^{-1}(\beta(U^2)(h_1Q^2)) \\
&\overset{(QT4)}{=} \alpha^{-1}(\alpha(U^1)(P^1c_1)) \otimes \beta^{-1}((Q^1h_1)\beta(V^1)) \otimes \alpha^{-1}((P^2c_2)\alpha(V^2)) \\
&\quad \otimes \beta^{-1}(\beta(U^2)(Q^2h_2)) \\
&\overset{(A1)(A2)}{=} \alpha^{-1}(U^1P^1)c_1 \otimes \beta^{-1}(\beta(Q^1)(h_1V^1)) \otimes \alpha^{-1}(\alpha(P^2)(c_2V^2)) \otimes \beta^{-1}(U^2Q^2)h_2 \\
&\overset{(D5)}{=} \alpha^{-1}(U^1P^1)c_1 \otimes \beta^{-1}(\beta(Q^1)(V^1\beta^{-1}(h_1)_T)) \otimes \alpha^{-1}(\alpha(P^2)(V^2\alpha^{-1}(c_{2T}))) \\
&\quad \otimes \beta^{-1}(U^2Q^2)h_2 \\
&\overset{(A1)(A2)}{=} \alpha^{-1}(U^1P^1)c_1 \otimes \beta^{-1}(Q^1V^1)\beta^{-1}(h_1)_T \otimes \alpha^{-1}(P^2V^2)\alpha^{-1}(c_{2T}) \otimes \beta^{-1}(U^2Q^2)h_2 \\
&= R^1c_1 \otimes R^2\beta^{-1}(h_1)_T \otimes R^3\alpha^{-1}(c_{2T}) \otimes R^4h_2 \\
&= \text{RHS}.
\end{aligned}
$$

Therefore, $(C \diamond_T H, \alpha \otimes \beta, R)$ is a quasitriangular Hom-Hopf algebra.

Thus it follows from Lemmas 4.1–4.6 that we have the following theorems.

Theorem 4.3 *The T-smash coproduct Hom-Hopf algebra $(C \diamond_T H, \alpha \otimes \beta)$ is quasitriangular if and only if there exist elements $P \in C \otimes C$, $Q \in H \otimes H$, $U \in C \otimes H$ and $V \in H \otimes C$ such that (C, α, P) and (H, β, Q) are quasitriangular Hom-Hopf algebras, (C, H, U) and (H, C, V) are Hom-compatibility Hopf algebra triples, and the conditions* (D1)–(D6) *in Lemma* 4.4 *hold. Moreover, the quasitriangular structure R on $(C \diamond_T H, \alpha \otimes \beta)$ has a decomposition*

$$(\alpha \otimes \beta \otimes \alpha \otimes \beta)(R) = U^1P^1 \otimes Q^1V^1 \otimes P^2V^2 \otimes U^2Q^2.$$

Theorem 4.4 *The smash coproduct Hom-Hopf algebra $(C \diamond H, \alpha \otimes \beta)$ is quasitriangular if and only if there exist elements $P \in C \otimes C$, $Q \in H \otimes H$, $U \in C \otimes H$ and $V \in H \otimes C$ such that (C, α, P) and (H, β, Q) are quasitriangular Hom-Hopf algebras, (C, H, U) and (H, C, V) are Hom-compatibility Hopf algebra triples, and the conditions* (E1)–(E6) *below hold:*

$$\text{(E1)} \quad {P^1}_{(-1)}\beta^{-1}(V^1) \otimes {P^1}_{(0)} \otimes P^2V^2 = V^1 \otimes \alpha(P^1) \otimes V^2P^2,$$

(E2) $U^1{}_{(-1)}\beta^{-1}(Q^1)\otimes U^1{}_{(0)}\otimes U^2Q^2=Q^1\otimes\alpha(U^1)\otimes Q^2U^2,$

(E3) $Q^1V^1\otimes V^2{}_{(-1)}\beta^{-1}(Q^2)\otimes V^2{}_{(0)}=V^1Q^1\otimes Q^2\otimes\alpha(V^2),$

(E4) $U^1P^1\otimes P^2{}_{(-1)}\beta^{-1}(U^2)\otimes P^2{}_{(0)}=P^1U^1\otimes U^2\otimes\alpha(P^2),$

(E5) $\beta(h)V^1\otimes\alpha(c)V^2=V^1\alpha(c)_{(-1)}h\otimes V^2\alpha^{-1}(\alpha(c)_{(0)}),$

(E6) $\alpha^{-1}(\alpha(c)_{(0)})U^1\otimes\alpha(c)_{(-1)}hU^2=U^1\alpha(c)\otimes U^2\beta(h).$

Moreover, the quasitriangular structure R on $(C\diamond H,\alpha\otimes\beta)$ has a decomposition

$$(\alpha\otimes\beta\otimes\alpha\otimes\beta)(R)=U^1P^1\otimes Q^1V^1\otimes P^2V^2\otimes U^2Q^2.$$

Proof Let $T(c\otimes h)=c_{(-1)}h\otimes c_{(0)},\ \forall a\in A,\ h\in H$ in Theorem 4.3.

5 Applications

In this section, we extend the applications of the main results in Section 4 to a concrete example.

The following result is clear.

Lemma 5.1 *Let $K\mathbb{Z}_2=K\{1,a\}$ be a Hopf group algebra (see [19]). Then $(K\mathbb{Z}_2,\mathrm{id}_{K\mathbb{Z}_2},Q)$ is a quasitriangular Hom-Hopf algebra, where $Q=\frac{1}{2}(1\otimes 1+a\otimes 1+1\otimes a-a\otimes a)$.*

Let $T_{2,-1}=K\{1,g,x,gx\mid g^2=1,x^2=0,xg=-gx\}$ be Taft's Hopf algebra (see [20]), and its coalgebra structure and antipode are given by

$$\Delta(g)=g\otimes g,\quad \Delta(x)=x\otimes g+1\otimes x,\quad \Delta(gx)=gx\otimes 1+g\otimes gx,$$
$$\varepsilon(g)=1,\quad \varepsilon(x)=0,\quad \varepsilon(gx)=0$$

and

$$S(g)=g,\quad S(x)=gx,\quad S(gx)=-x.$$

Define a linear map α: $T_{2,-1}\longrightarrow T_{2,-1}$ by

$$\alpha(1)=1,\quad \alpha(g)=g,\quad \alpha(x)=kx,\quad \alpha(gx)=kgx,$$

where $0\neq k\in K$. Then α is an automorphism of Hopf algebras.

So we can get a Hom-Hopf algebra $H_\alpha=(T_{2,-1},\alpha\circ\mu_{T_{2,-1}},1_{T_{2,-1}},\Delta_{T_{2,-1}}\circ\alpha,\varepsilon_{T_{2,-1}},\alpha)$ (see [15]).

Lemma 5.2 *Let H_α be the Hom-Hopf algebra defined as above. Then (H_α,α,P) is a quasitriangular Hom-Hopf algebra, where $P=\frac{1}{2}(1\otimes 1+g\otimes 1+1\otimes g-g\otimes g)$.*

Proof It is straightforward by a tedious computation.

Theorem 5.1 *Let $K\mathbb{Z}_2$ be the Hopf group algebra and H_α be the Hom-Hopf algebra defined as above. Define the comodule action $\rho: H_\alpha\longrightarrow K\mathbb{Z}_2\otimes H_\alpha$ by*

$$\rho: H_\alpha\longrightarrow K\mathbb{Z}_2\otimes H_\alpha,$$
$$1_{H_\alpha}\mapsto 1_{K\mathbb{Z}_2}\otimes 1_{H_\alpha},$$

$$g \mapsto 1_{K\mathbb{Z}_2} \otimes g,$$
$$x \mapsto ka \otimes x,$$
$$gx \mapsto ka \otimes gx.$$

Then by a routine computation we can get that (H_α, ρ, α) *is a left* $K\mathbb{Z}_2$*-comodule Hom-coalgebra. Therefore,* $(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2})$ *is a smash coproduct Hom-coalgebra.*

Furthermore, $(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2})$ *with the tensor product Hom-algebra becomes a Hom-Hopf algebra, where the antipode* $\overline{S}$ *is given by*

$$\begin{aligned}
&\overline{S}(1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}) = 1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}, && \overline{S}(1_{H_\alpha} \otimes a) = 1_{H_\alpha} \otimes a,\\
&\overline{S}(g \otimes 1_{K\mathbb{Z}_2}) = g \otimes 1_{K\mathbb{Z}_2}, && \overline{S}(g \otimes a) = g \otimes a,\\
&\overline{S}(x \otimes 1_{K\mathbb{Z}_2}) = gx \otimes a, && \overline{S}(x \otimes a) = gx \otimes 1_{K\mathbb{Z}_2},\\
&\overline{S}(gx \otimes 1_{K\mathbb{Z}_2}) = -x \otimes a, && \overline{S}(gx \otimes a) = -x \otimes 1_{K\mathbb{Z}_2}.
\end{aligned}$$

Lemma 5.3 *Let* $K\mathbb{Z}_2$ *be the Hopf group algebra and* H_α *be the Hom-Hopf algebra defined as above. Define*

$$U = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + g \otimes 1 - g \otimes a) \in H_\alpha \otimes K\mathbb{Z}_2,$$
$$V = \frac{1}{2}(1 \otimes 1 + a \otimes 1 + 1 \otimes g - a \otimes g) \in K\mathbb{Z}_2 \otimes H_\alpha.$$

Then $(H_\alpha, K\mathbb{Z}_2, U)$ *and* $(K\mathbb{Z}_2, H_\alpha, V)$ *are two Hom-compatibility Hopf algebra triples.*

Proof Straightforward.

Theorem 5.2 *With the notations as above, the smash coproduct Hom-Hopf algebra*

$$(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2}, R)$$

is a quasitriangular Hom-Hopf algebra, where

$$R = \frac{1}{2}(1 \otimes 1 \otimes 1 \otimes 1 + g \otimes a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g \otimes a - g \otimes a \otimes g \otimes a).$$

Proof It is easy to prove that the conditions (E1)–(E6) hold. And by Lemmas 5.1–5.3 and Theorem 4.4, we can finish the proof.

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