Chin. Ann. Math.
37B(6), 2016, 951–962
DOI: 10.1007/s11401-016-1037-6

Chinese Annals of Mathematics, Series B

On the Same n-Types for the Wedges of the Eilenberg-Maclane Spaces*

Dae-Woong LEE[1]

Abstract Given a connected CW-space X, $SNT(X)$ denotes the set of all homotopy types $[X']$ such that the Postnikov approximations $X^{(n)}$ and $X'^{(n)}$ are homotopy equivalent for all n. The main purpose of this paper is to show that the set of all the same homotopy n-types of the suspension of the wedges of the Eilenberg-MacLane spaces is the one element set consisting of a single homotopy type of itself, i.e., $SNT(\Sigma(K(\mathbb{Z}, 2a_1) \vee K(\mathbb{Z}, 2a_2) \vee \cdots \vee K(\mathbb{Z}, 2a_k))) = *$ for $a_1 < a_2 < \cdots < a_k$, as a far more general conjecture than the original one of the same n-type posed by McGibbon and Møller (in [McGibbon, C. A. and Møller, J. M., On infinite dimensional spaces that are rationally equivalent to a bouquet of spheres, Proceedings of the 1990 Barcelona Conference on Algebraic Topology, Lecture Notes in Math., **1509**, 1992, 285–293].)

Keywords Same n-type, Aut, Basic Whitehead product, Samelson product, Bott-Samelson theorem, Tensor algebra, Cartan-Serre theorem, Hopf-Thom theorem

2000 MR Subject Classification 55P15, 55S37, 55P62

1 Introduction

Let us call $X^{(n)}$ the n-th Postnikov approximation of a connected CW-space X. $X^{(n)}$ is a CW-complex obtained from X by adjoining cells of dimension $\geq n+2$ such that $\pi_i(X^{(n)}) = 0$ for $i \geq n+1$ and $\pi_i(X^{(n)}) = \pi_i(X)$ for $i \leq n$. The Postnikov k-invariants $k^{n+1}(X)$ of X are maps $X^{(n-1)} \to K(\pi_n(X), n+1)$ and thus cohomology classes in $H^{n+1}(X^{(n-1)}; \pi_n(X))$ for $n \geq 2$. We say that two connected CW-spaces X and X' have the same n-type if the n-th Postnikov approximations $X^{(n)}$ and $X'^{(n)}$ are homotopy equivalent for all $n \geq 1$.

An interesting question raised by J. H. C. Whitehead is this: Suppose that X and X' are two spaces whose Postnikov approximations, $X^{(n)}$ and $X'^{(n)}$, are homotopy equivalent for each integer n. Does it follow that X and X' have the same homotopy type? It is well known that either if X is finite dimensional (use the cellular approximation theorem) or if X has only a finite number of nonzero homotopy groups, then the answer to Whitehead's question is yes! However, in general, there are examples, founded by Adams [1] and Gray [6] independently, saying that the answer to this question is no! It is also shown that in [16] if the base space of

Manuscript received January 5, 2013. Revised December 21, 2015.
[1]Department of Mathematics, Institute of Pure and Applied Mathematics, Chonbuk National University, 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896, Republic of Korea.
E-mail: dwlee@jbnu.ac.kr
*This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF, in short) funded by the Ministry of Education (No. NRF-2015R1D1A1A09057449).

a sphere fibration $\xi : E \xrightarrow{\pi} B$ is a topological manifold, then a Hopf index theorem can be obtained.

Let $\mathbb{Z}$ be the ring of integers and let Σ denote the suspension functor. For a connected CW-space X, we let $SNT(X)$ denote the set of all homotopy types $[X']$ such that the Postnikov approximations $X^{(n)}$ and $X'^{(n)}$ are homotopy equivalent for all n. This is a pointed set with base point $* = [X]$. It is well known in [11] that the set of all the same homotopy n-types for the k-th iterated suspension of the Eilenberg-MacLane space $K(\mathbb{Z}, 2b+1)$ is trivial for $k \geq 0$; that is, $SNT(\Sigma^k K(\mathbb{Z}, 2b+1)) = *$. One reason of this fact is that $\Sigma^k K(\mathbb{Z}, 2b+1)$ has a rational homotopy type of a single sphere of dimension $k+2b+1$. As we can see, the even dimensional case is much more complicated because $\Sigma K(\mathbb{Z}, 2a)$ has a rational homotopy type of a bouquet of infinitely many spheres of dimensions $2a+1, 4a+1, \cdots, 2na+1, \cdots$. So it is natural to ask in the case of even integers. The first interesting case ($a = 1$) is the following conjecture.

Conjecture 1.1 (see [11, p. 287]) $SNT(\Sigma K(\mathbb{Z}, 2)) = *$.

The positive answer to this conjecture was given in [8]. More generally, what will happen in the case of the suspension of the wedge products of the Eilenberg-MacLane spaces of various types? After suspensions or wedge products of the Eilenberg-MacLane spaces $K(\mathbb{Z}, 2a)$ and $K(\mathbb{Z}, 2b+1)$ for $a, b \geq 1$ as the infinite loop spaces, they become much more intractable, and they are worth mentioning what it is in the SNT-sense. The purpose of this paper is to provide an answer to the above query as a general version of the original same n-type conjecture.

Theorem 1.1 *Let $Y := K(\mathbb{Z}, 2a_1) \vee K(\mathbb{Z}, 2a_2) \vee \cdots \vee K(\mathbb{Z}, 2a_k)$ be the wedge products of the Eilenberg-MacLane spaces, where a_i is the positive integer for $i = 1, 2, \cdots, k$ with $a_1 < a_2 < \cdots < a_k$. Then $SNT(\Sigma Y) = *$.*

In this paper we often do not distinguish notationally between a base point preserving map and its homotopy class. We denote $\mathbb{Q}$ by the set of all rational numbers. As an adjointness, we will make use of the notations Σ and Ω for the suspension and loop functors in the based homotopy category, respectively.

2 Homotopy Self-Equivalences of CW-Spaces

Let $\mathrm{Aut(X)}$ be the group of homotopy classes of homotopy self-equivalences of a space X and let $\mathrm{Aut}(\pi_{\leq n}(\mathrm{X}))$ denote the group of automorphisms of the graded $\mathbb{Z}$-module, $\pi_{\leq n}(X)$, preserving the Whitehead product pairings. McGibbon and Møller (see [11, Theorem 1]) proved the following theorem.

Theorem 2.1 *Let X be a 1-connected space with finite type over some subring of the rationals. Assume that X has the rational homotopy type of a bouquet of spheres. Then the following three conditions are equivalent:*

(a) $SNT(X) = *$;

(b) *the map* $\mathrm{Aut(X)} \xrightarrow{f \mapsto f^{(n)}} \mathrm{Aut(X^{(n)})}$ *has a finite cokernel for each n;*

(c) *the map* $\mathrm{Aut(X)} \xrightarrow{f \mapsto f_\sharp} \mathrm{Aut}(\pi_{\leq n}(\mathrm{X}))$ *has a finite cokernel for each n.*

In 1976, Wilkerson (see [21, Theorem I]) classified CW-spaces having the same n-type up to homotopy, and proved that for a connected CW-complex X, there is a bijection of pointed sets

$$SNT(X) \approx \lim{}^1 \mathrm{Aut}(\mathrm{X}^{(n)}),$$

where $\lim^1$ is the first derived limit of groups (not necessarily abelian) in the sense of Bousfield and Kan [4]. Thus, if X is a space of finite type, then the torsion subgroup of $\pi_*(X^{(n)})$ can be ignored in the $\lim^1$-calculation (see [12]).

We note that Y has a CW-decomposition of wedges based on the Eilenberg-MacLane spaces $K(\mathbb{Z}, 2a_s)$ as follows:

$$K(\mathbb{Z}, 2a_s) = S^{2a_s} \cup T_1^s \cup_{\gamma_1} \mathrm{e}^{4a_s} \cup T_2^s \cup_{\gamma_2} \mathrm{e}^{6a_s} \cup \cdots \cup T_{n-1}^s \cup_{\gamma_{n-1}} \mathrm{e}^{2na_s} \cup T_n^s \cup_{\gamma_n} \mathrm{e}^{2(n+1)a_s} \cdots$$

for $s = 1, 2, \cdots, k$, where γ_n is an attaching map, and T_n^s denotes the other cells or the Moore spaces for torsions of the reduced homology groups for $n = 1, 2, 3, \cdots$.

In order to define the homotopy self-maps of the suspension of wedges of the Eilenberg-MacLane spaces $K(\mathbb{Z}, 2a_s)$, $s = 1, 2, \cdots, k$, we first define maps $\widehat{\varphi}_n^{a_s} : Y \to \Omega\Sigma Y$ for $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$ as follows.

Definition 2.1 *Let*

$$K(\mathbb{Z}, 2a_s)^c = K(\mathbb{Z}, 2a_1) \vee \cdots \vee K(\mathbb{Z}, 2a_{s-1}) \vee K(\mathbb{Z}, 2a_{s+1}) \vee \cdots \vee K(\mathbb{Z}, 2a_k)$$

for each $s = 1, 2, \cdots, k$, and let Y_t denote the t-skeleton of $Y := K(\mathbb{Z}, 2a_1) \vee K(\mathbb{Z}, 2a_2) \vee \cdots \vee K(\mathbb{Z}, 2a_k)$. Then the cofibration sequences

$$K(\mathbb{Z}, 2a_s)^c \xhookrightarrow{i_{1,a_s}} Y \xrightarrow{p_{1,a_s}} Y/K(\mathbb{Z}, 2a_s)^c$$

and

$$\mathrm{Y}_{2na_s-1} \xhookrightarrow{i_{n,a_s}} Y \xrightarrow{p_{n,a_s}} Y/Y_{2na_s-1}$$

induce the exact sequences of groups

$$[Y/K(\mathbb{Z}, 2a_s)^c, \Omega\Sigma Y] \xrightarrow{p_{1,a_s}^\sharp} [Y, \Omega\Sigma Y] \xrightarrow{i_{1,a_s}^\sharp} [K(\mathbb{Z}, 2a_s)^c, \Omega\Sigma Y]$$

and

$$[Y/Y_{2na_s-1}, \Omega\Sigma Y] \xrightarrow{p_{n,a_s}^\sharp} [Y, \Omega\Sigma Y] \xrightarrow{i_{n,a_s}^\sharp} [Y_{2na_s-1}, \Omega\Sigma Y]$$

for $n \geq 2$ and $s = 1, 2, \cdots, k$. We now take essential maps

$$\widehat{\varphi}_1^{a_s} \in {i_{1,a_s}^\sharp}^{-1}(*) = \ker(i_{1,a_s}^\sharp) \subset [Y, \Omega\Sigma Y]$$

and

$$\widehat{\varphi}_n^{a_s} \in {i_{n,a_s}^\sharp}^{-1}(*) = \ker(i_{n,a_s}^\sharp) \subset [Y, \Omega\Sigma Y]$$

for $n \geq 2$ and $s = 1, 2, \cdots, k$. Similarly, we can choose maps

$$\widehat{\psi}_1^{a_s} : Y/K(\mathbb{Z}, 2a_s)^c \longrightarrow \Omega\Sigma Y$$

and

$$\widehat{\psi}_n^{a_s} : Y/Y_{2na_s-1} \longrightarrow \Omega\Sigma Y$$

with $p_{1,a_s}^\sharp(\widehat{\psi}_1^{a_s}) = \widehat{\varphi}_1^{a_s}$ and $p_{n,a_s}^\sharp(\widehat{\psi}_n^{a_s}) = \widehat{\varphi}_n^{a_s}$ for $n \geq 2$ and $s = 1, 2, \cdots, k$, respectively, by using the above exact sequences.

In the above definition, we note that

$$Y/K(\mathbb{Z},2a_s)^c = S^{2a_s} \cup \text{higher cells}$$

and

$$Y/Y_{2na_s-1} = S^{2na_s} \cup \text{the other sphere(s) and higher cells.}$$

We now have the following definition.

Definition 2.2 *We define the rationally non-trivial homotopy elements $\widehat{x}_1^{a_s}$ and $\widehat{x}_n^{a_s}$ of the homotopy groups modulo torsions $\pi_{2a_s}(\Omega\Sigma Y)/\text{torsion}$ and $\pi_{2na_s}(\Omega\Sigma Y)/\text{torsion}$ by $\widehat{x}_1^{a_s} = \widehat{\psi}_1^{a_s}|_{S^{2a_s}}$ and $\widehat{x}_n^{a_s} = \widehat{\psi}_n^{a_s}|_{S^{2na_s}}$, respectively, for $s = 1, 2, \cdots, k$ and $n \geq 2$.*

We now take the self-maps $\varphi_n^{a_s} : \Sigma Y \to \Sigma Y$ and maps $x_n^{a_s} : S^{2na_s+1} \to \Sigma Y$ as the adjointness of $\widehat{\varphi}_n^{a_s} : Y \to \Omega\Sigma Y$ and $\widehat{x}_n^{a_s} : S^{2na_s} \to \Omega\Sigma Y$, respectively, for $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$. We then order the basic Whitehead products (see [7]) of weight 1 on the graded homotopy groups modulo torsion, $\pi_*(\Sigma Y)/\text{torsion}$, as follows: We order the rationally non-trivial elements $x_m^{a_s}$ and $x_n^{a_t}$ of $\pi_*(\Sigma Y)/\text{torsion}$ as $x_m^{a_s} < x_n^{a_t}$ either if $\dim(x_m^{a_s}) < \dim(x_n^{a_t})$, or if $\dim(x_m^{a_s}) = \dim(x_n^{a_t})$ and $a_s < a_t$ for $s, t = 1, 2, \cdots, k$ and $m, n = 1, 2, 3, \cdots$.

Let $[\varphi_m^{a_s}, \varphi_n^{a_t}] : \Sigma Y \to \Sigma Y$ be the commutator of self-maps $\varphi_m^{a_s}$ and $\varphi_n^{a_t}$; that is

$$[\varphi_m^{a_s}, \varphi_n^{a_t}] = \varphi_m^{a_s} + \varphi_n^{a_t} - \varphi_m^{a_s} - \varphi_n^{a_t},$$

where the operations are the suspension additions on ΣY. By using this suspension structure, we construct self-maps of ΣY by $\mathrm{I} + [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$, where I is the identity map of ΣY and $[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$ is the l-th iterated commutator of self-maps $\varphi_{n_i}^{a_{s_i}} : \Sigma Y \to \Sigma Y$, $i = 1, 2, \cdots, l$ on the suspension structure for $s_i = 1, 2, \cdots, k$, and $n_i = 1, 2, 3, \cdots$. The Whitehead theorem asserts that the above self-maps $\mathrm{I} + [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$ of ΣY are actually homotopy self-equivalences.

We note that the above iterated commutator maps

$$[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]] : \Sigma Y \longrightarrow \Sigma Y$$

do make sense because there are infinitely many non-zero cohomology cup products in Y so that it has the infinite Lusternik-Schnirelmann category (see [20, Chapter X] and [18]). Moreover, Arkowitz and Curjel (see [2, Theorem 5]) showed that the n-fold commutator is of finite order if and only if all n-fold cup products of any positive dimensional rational cohomology classes of a space vanish.

Remark 2.1 (a) Let x be a rationally non-trivial indecomposable element of the homotopy groups $\pi_{2(n_1 a_{s_1} + n_2 a_{s_2} + \cdots + n_l a_{s_l})+1}(\Sigma Y)$. Then

$$(\mathrm{I} + [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]])_\sharp(x) \;=\; x + [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]_\sharp(x),$$

where the first addition is the one of suspension structure on ΣY, while the second addition refers to the one of homotopy groups (see [8, Lemma 3.2]).

(b) Let $J : Y \to \Omega\Sigma Y$ be the James map. Then we have

$$\Omega[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]] \circ J = [\widehat{\varphi}_{n_l}^{a_{s_l}}, [\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]]$$

in the group $[Y, \Omega\Sigma Y]$ (see also [9, Lemma 4]).

By using the Serre spectral sequence of a path space fibration

$$K(\mathbb{Z}, 2a_s - 1) \longrightarrow PK(\mathbb{Z}, 2a_s) \longrightarrow K(\mathbb{Z}, 2a_s)$$

for each $s = 1, 2, \cdots, k$, we have an algebra isomorphism $H^*(K(\mathbb{Z}, 2a_s); \mathbb{Q}) \cong \mathbb{Q}[\alpha_s]$. Here $\mathbb{Q}[\alpha_s]$ is the polynomial algebra over $\mathbb{Q}$ generated by α_s of dimension $2a_s$; that is, α_s is a generator of $H^{2a_s}(K(\mathbb{Z}, 2a_s); \mathbb{Q})$ with $\langle \alpha_s^m, \alpha_n' \rangle = \delta_{mn}$, where α_n' is a rational homology generator of dimension $2na_s$.

3 Proof of Theorem 1.1

We point out that the proof of Theorem 1.1 depends highly on Theorem 2.1. We remark that the total rational homotopy group $\widehat{\mathcal{L}} = \pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}$ of $\Omega\Sigma Y$ is a graded Lie algebra over $\mathbb{Q}$ with Lie bracket $\langle\ ,\ \rangle$ given by the Samelson product which is called the rational homotopy Lie algebra of ΣY (see [14] for the de Rham homotopy theory). For $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$, we let $\widehat{\mathcal{L}}_{\le a_s, n}$ denote the subalgebra of $\widehat{\mathcal{L}}$ generated by all free algebra generators of degree less than or equal to $2na_s$, that is

$$\widehat{\mathcal{L}}_{\le a_s, n} = \pi_{\le 2na_s}(\Omega\Sigma Y) \otimes \mathbb{Q}$$

with generators $\widehat{\chi}_{n_j}^{a_{s_i}} \in \pi_{2n_j a_{s_i}}(\Omega\Sigma Y_{\mathbb{Q}})$ so that $n_j a_{s_i} \le na_s$, where $\widehat{\chi}_{n_j}^{a_{s_i}} : S^{2n_j a_{s_i}} \to \Omega\Sigma Y_{\mathbb{Q}}$ is the composition $r \circ \widehat{x}_{n_j}^{a_{s_i}}$ of the rationally non-trivial indecomposable element $\widehat{x}_{n_j}^{a_{s_i}} : S^{2n_j a_{s_i}} \to \Omega\Sigma Y$ of $\pi_{2n_j a_{s_i}}(\Omega\Sigma Y)/\text{torsion}$ for $s_i = 1, 2, \cdots, k$ and $n_j = 1, 2, 3, \cdots$ with the rationalization $r : \Omega\Sigma Y \to \Omega\Sigma Y_{\mathbb{Q}}$. As an adjointness,

$$\mathcal{L}_{\le a_s, n} = \pi_{\le 2na_s+1}(\Sigma Y) \otimes \mathbb{Q}$$

with the Whitehead product $[\ ,\ ]_W$ has the graded quasi-Lie algebra structure which is called the Whitehead algebra with generators $\chi_{n_j}^{a_{s_i}} \in \pi_{2n_j a_{s_i}+1}(\Sigma Y_{\mathbb{Q}})$.

Remark 3.1 We consider the following cofibration sequence:

$$S^{2(n_1 a_{s_1} + n_2 a_{s_2})+1} \xrightarrow{[x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W} S^{2n_1 a_{s_1}+1} \vee S^{2n_2 a_{s_2}+1} \longrightarrow S^{2n_1 a_{s_1}+1} \times S^{2n_2 a_{s_2}+1},$$

where

$$x_{n_j}^{a_{s_i}} \in \pi_{2n_j a_{s_i}+1}(S^{2n_j a_{s_i}+1}) \cong \mathbb{Z} \subset \pi_{2n_j a_{s_i}+1}(\Sigma Y)/\text{torsion}$$

are the rationally non-trivial homotopy elements. By considering the homotopy cofibre of the above Whitehead product map and the cohomology cup product argument on it, we can see that $[x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W$ is rationally non-trivial, and that by induction on l the iterated basic Whitehead products $[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W$ in the graded homotopy group $\pi_*(\Sigma Y)/\text{torsion}$ are also rationally non-trivial (see [10, Lemma 3.5] for details).

Thus we can define the following.

Definition 3.1 *The basic Whitehead product* $[\chi_{n_l}^{a_{s_l}}, [\chi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\chi_{n_1}^{a_{s_1}}, \chi_{n_2}^{a_{s_2}}]_W \cdots]_W]_W$ *is said to be a purely decomposable generator of the rational homotopy group in dimension* $2(n_1 a_1 + n_2 a_2 + \cdots + n_l a_l) + 1$ *if* $s_1 = s_2 = \cdots = s_l$, *and it is said to be a hybrid decomposable generator if there is at least one* s_i *which differs from one of those* s_j, *where* $i \in \{1, 2, \cdots, l\}$ *and* $j = 1, 2, \cdots, l$.

Recall that

$$\widetilde{H}_*(Y;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}\{\beta_n^{a_s} \mid n=1,2,3,\cdots \text{ and } s=1,2,\cdots,k\}$$

as a graded $\mathbb{Z}$-module and

$$\widetilde{H}_*(Y;\mathbb{Q}) \cong \mathbb{Q}\{b_n^{a_s} \mid n=1,2,3,\cdots \text{ and } s=1,2,\cdots,k\}$$

as a graded $\mathbb{Q}$-module, where $\beta_n^{a_s}$ and $b_n^{a_s}$ are the standard generators of the homology groups $H_{2na_s}(Y;\mathbb{Z})/\text{torsion}$ and $H_{2na_s}(Y;\mathbb{Q})$, respectively for $n=1,2,3,\cdots$ and $s=1,2,\cdots,k$. The Bott-Samelson theorem (see [3]) says that the Pontryagin algebra $H_*(\Omega\Sigma Y;\mathbb{Q})$ is isomorphic to the tensor algebra $TH_*(Y;\mathbb{Q})$ generated by $\{b_n^{a_s} \mid n=1,2,3,\cdots \text{ and } s=1,2,\cdots,k\}$.

Let $\mathrm{ad}[\varphi_{n_l}^{a_{s_l}},[\varphi_{n_{l-1}}^{a_{s_{l-1}}},\cdots,[\varphi_{n_1}^{a_{s_1}},\varphi_{n_2}^{a_{s_2}}]\cdots]]: Y \longrightarrow \Omega\Sigma Y$ be the adjoint of the iterated commutator map $[\varphi_{n_l}^{a_{s_l}},[\varphi_{n_{l-1}}^{a_{s_{l-1}}},\cdots,[\varphi_{n_1}^{a_{s_1}},\varphi_{n_2}^{a_{s_2}}]\cdots]]: \Sigma Y \longrightarrow \Sigma Y$. Then we have

$$\mathrm{ad}[\varphi_{n_l}^{a_{s_l}},[\varphi_{n_{l-1}}^{a_{s_{l-1}}},\cdots,[\varphi_{n_1}^{a_{s_1}},\varphi_{n_2}^{a_{s_2}}]\cdots]] = [\widehat{\varphi}_{n_l}^{a_{s_l}},[\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}},\cdots,[\widehat{\varphi}_{n_1}^{a_{s_1}},\widehat{\varphi}_{n_2}^{a_{s_2}}]\cdots]],$$

since the map $\mathrm{ad} = \widehat{\ } : [\Sigma Y, \Sigma Y] \longrightarrow [Y, \Omega\Sigma Y]$ defined by

$$(\mathrm{ad}\varphi)(y)(t) = \widehat{\varphi}(y)(t) = \varphi\langle y,t\rangle$$

is an isomorphism of groups, where $\varphi \in [\Sigma Y, \Sigma Y],\ y \in Y,\ t \in I$ and $\langle y,t\rangle \in \Sigma Y$. Moreover, we have the following lemma.

Lemma 3.1 *Let $j : Y_t \hookrightarrow Y$ and $q : Y_t \to S^t$ be the inclusion map and the projection to the top cell of Y_t, respectively. Then the following diagram*

$$\begin{tikzcd} Y_t \arrow[r, hook, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\simeq"] & Y \arrow[d, "{[\widehat{\varphi}_{n_l}^{a_{s_l}},[\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}},\cdots,[\widehat{\varphi}_{n_1}^{a_{s_1}},\widehat{\varphi}_{n_2}^{a_{s_2}}]\cdots]]}"] \\ S^t \arrow[r, "{\langle\widehat{x}_{n_l}^{a_{s_l}},\langle\widehat{x}_{n_{l-1}}^{a_{s_{l-1}}},\cdots,\langle\widehat{x}_{n_1}^{a_{s_1}},\widehat{x}_{n_2}^{a_{s_2}}\rangle\cdots\rangle\rangle}"'] & \Omega\Sigma Y \end{tikzcd} \qquad (3.1)$$

is commutative up to homotopy, where $t = 2(n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l})$ and $\langle\widehat{x}_{n_l}^{a_{s_l}},\langle\widehat{x}_{n_{l-1}}^{a_{s_{l-1}}},\cdots,\langle\widehat{x}_{n_1}^{a_{s_1}},\widehat{x}_{n_2}^{a_{s_2}}\rangle\cdots\rangle\rangle$ is the iterated Samelson product.

Proof We first consider the exact sequence

$$[Y/Y_{t-1}, Y\wedge Y] \xrightarrow{p^*} [Y, Y\wedge Y] \xrightarrow{i^*} [Y_{t-1}, Y\wedge Y],$$

induced by a cofibration sequence

$$Y_{t-1} \overset{i}{\hookrightarrow} Y \xrightarrow{p} Y/Y_{t-1}.$$

Let $\overline{\Delta} : Y \to Y\wedge Y$ be the reduced diagonal map (i.e., the composite of the diagonal $\Delta : Y \to Y\times Y$ with the projection $\pi : Y\times Y \to Y\wedge Y$ onto the smash product) and let $p_{n_i,a_{s_i}} : Y \to Y/Y_{2n_ia_{s_i}-1}$ be the projection for $i=1,2$. Then by using the cellular approximation theorem, and considering the cell structure of $Y\wedge Y$ and the composition with

$$p_{n_1,a_{s_1}} \wedge p_{n_2,a_{s_2}} : Y\wedge Y \longrightarrow Y/Y_{2n_1a_{s_1}-1} \wedge Y/Y_{2n_2a_{s_2}-1},$$

we have $(p_{n_1,a_{s_1}} \wedge p_{n_2,a_{s_2}}) \circ \overline{\Delta} \circ i = *$. From the above exact sequence, there exists a map

$$\nabla : Y/Y_{t-1} \longrightarrow Y/Y_{2n_1a_{s_1}-1} \wedge Y/Y_{2n_2a_{s_2}-1}$$

such that $\nabla \circ p = (p_{n_1,a_{s_1}} \wedge p_{n_2,a_{s_2}}) \circ \overline{\Delta}$.

By using this fact, we now consider the following commutative diagram up to homotopy (see also [13] in the case of the infinite complex projective space):

$$\begin{tikzcd}
 & Y_{t-1} \arrow[d, hook, "i"] & & \\
 & Y \arrow[r, "\overline{\Delta}"] \arrow[d, "p"] & Y \wedge Y \arrow[r, "\widehat{\varphi}_{n_1}^{a_{s_1}} \wedge \widehat{\varphi}_{n_2}^{a_{s_2}}"] \arrow[d, "p_{n_1,a_{s_1}} \wedge p_{n_2,a_{s_2}}"] & \Omega\Sigma Y \wedge \Omega\Sigma Y \arrow[d, "\mathrm{id}"] \\
Y_t \arrow[ur, hook, "j"] \arrow[r] \arrow[dr, "q"'] & Y/Y_{t-1} \arrow[r, "\nabla"] & Y/Y_{2n_1a_{s_1}-1} \wedge Y/Y_{2n_2a_{s_2}-1} \arrow[r, "\widehat{\psi}_{n_1}^{a_{s_1}} \wedge \widehat{\psi}_{n_2}^{a_{s_2}}"] & \Omega\Sigma Y \wedge \Omega\Sigma Y \arrow[d, "C"] \\
 & S^t \arrow[u] \arrow[r, "="] & S^{2n_1a_{s_1}} \wedge S^{2n_2a_{s_2}} \arrow[u] \arrow[ur, "\widehat{x}_{n_1}^{a_{s_1}} \wedge \widehat{x}_{n_2}^{a_{s_2}}"] \arrow[r, "\langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle"] & \Omega\Sigma Y
\end{tikzcd} \tag{3.2}$$

where $t = 2(n_1a_{s_1} + n_2a_{s_2})$ and $C : \Omega\Sigma Y \wedge \Omega\Sigma Y \to \Omega\Sigma Y$ is the commutator map with respect to the loop operation, that is

$$C(\widehat{\varphi}_{n_1}^{a_{s_1}}(y), \widehat{\varphi}_{n_2}^{a_{s_2}}(y)) = \widehat{\varphi}_{n_1}^{a_{s_1}}(y) \cdot \widehat{\varphi}_{n_2}^{a_{s_2}}(y) \cdot (\widehat{\varphi}_{n_1}^{a_{s_1}}(y))^{-1} \cdot (\widehat{\varphi}_{n_2}^{a_{s_2}}(y))^{-1}.$$

Here the multiplication is the loop multiplication and the inverse means the loop inverse $\nu : \Omega\Sigma Y \to \Omega\Sigma Y$ defined by $\nu(\omega) = \omega^{-1}$, where $\omega^{-1}(t) = \omega(1-t),\ t \in [0,1]$. It shows that

$$[\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \circ j = \langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle \circ\ q.$$

The proof in case of the l-fold iterated commutators and the Samelson products goes to the same way by substituting $\widehat{\varphi}_{n_l}^{a_{s_l}}$ and $[\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]$ for $\widehat{\varphi}_{n_1}^{a_{s_1}}$ and $\widehat{\varphi}_{n_2}^{a_{s_2}}$, respectively (similarly for the iterated Samelson products of homotopy classes).

Lemma 3.2 *Let $h : \pi_*(\Omega\Sigma Y) \to H_*(\Omega\Sigma Y; \mathbb{Q})$ be the Hurewicz homomorphism. Then*

$$h(\langle \widehat{x}_{n_l}^{a_{s_l}}, \langle \widehat{x}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, \langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle \cdots \rangle\rangle) = [\widehat{\varphi}_{n_l}^{a_{s_l}}, [\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]]_*(b_n^{a_s}),$$

where $b_n^{a_s}$ is the standard generator of rational homology in dimension $2(n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l})$.

Proof By applying homology to the above homotopy commutative diagram (3.2) in the case of the two-fold commutators and the Samelson products, we obtain

$$h(\langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle) = [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]_*(b_n^{a_s})$$

in rational homology of $\Omega\Sigma Y$. Here $na_s = n_1a_{s_1} + n_2a_{s_2}$ and $b_n^{a_s}$ is the standard generator of $H_{2(n_1a_{s_1}+n_2a_{s_2})}(Y;\mathbb{Q})$. The homotopy commutative diagram (3.1) in Lemma 3.1 shows that this lemma is still true for the l-th iterated commutators and the iterated Samelson products, as required.

By considering the cell structure of the product of CW-spaces (this works for countable CW-complexes or when one factor is locally finite), we have the following lemma.

Lemma 3.3 *If X is a CW-complex of finite type with base point x_0 as the zero skeleton and if f and $g : X \to \Omega X'$ are the base point preserving maps with $f|_{X_p} \simeq *$ and $g|_{X_q} \simeq *$, respectively, then the restriction of the commutator $[f, g] : X \to \Omega X'$ to the $(p + q)$-skeleton of X is inessential.*

Proof For details, see [10, Lemma 2.3].

Lemma 3.4 *Let $t = 2(n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l})$. Then*

$$[\widehat{\varphi}_{n_l}^{a_{s_l}}, [\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]]|_{Y_{t-1}} : Y_{t-1} \longrightarrow \Omega\Sigma Y$$

is inessential, where $s_i = 1, 2, \cdots, k$ and $n_i = 1, 2, 3, \cdots$ for $i \in \{1, 2, \cdots, l\}$.

Proof We prove this lemma by induction on l. Since $\mathrm{ad}[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] = \widehat{[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]} = [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]$ and $\widehat{\varphi}_{n_i}^{a_{s_i}}|_{Y_{2n_ia_{s_i}-1}} \simeq *$ for $s_i = 1, 2, \cdots, k$, and $n_i = 1, 2, 3, \cdots$, by Lemma 3.3, we see that the commutator $[\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]$ restricts to the trivial map on the skeleton $Y_{2(n_1a_{s_1}+n_2a_{s_2})-2}$. By considering the cell structures of the Eilenberg-MacLane spaces described above, we see that Y has no cells in some ranges of dimensions, more precisely, between dimensions $2n_1a_{s_1} + 2n_2a_{s_2} - 2$ and $2n_1a_{s_1} + 2n_2a_{s_2} - 1$, that is

$$Y_{2(n_1a_{s_1}+n_2a_{s_2})-2} = Y_{2(n_1a_{s_1}+n_2a_{s_2})-1}.$$

The cellular approximation theorem shows that the restriction $[\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]|_{Y_{2(n_1a_{s_1}+n_2a_{s_2})-1}}$ to the skeleton is null homotopic.

We now suppose that $[\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]|_{Y_{2(n_1a_{s_1}+n_2a_{s_2}+\cdots+n_{n-1}a_{s_{l-1}})-1}}$ is inessential. Since $\widehat{\varphi}_{n_l}^{a_{s_l}}|_{Y_{2n_la_{s_l}-1}} \simeq *$, the similar argument as described above shows that

$$[\widehat{\varphi}_{n_l}^{a_{s_l}}, [\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}] \cdots]]|_{Y_{t-1}} \simeq *.$$

By induction on l, we complete the proof of this lemma.

Lemma 3.5 *For each basic Whitehead product $[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W$ of the graded homotopy group $\pi_*(\Sigma Y)$, we can construct the corresponding iterated commutator $[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$ in the group $[\Sigma Y, \Sigma Y]$ such that*

$$(I+[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]])_\sharp(x_n^{a_s}) = x_n^{a_s} + \lambda[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W,$$

where $\lambda \neq 0$, and $x_n^{a_s}$ and $x_{n_i}^{a_{s_i}}$ are rationally non-trivial indecomposable elements, and $na_s = n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l}$.

Proof We argue about a matter with induction on l again. We first show that

$$[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_\sharp(x_n^{a_s}) \;=\; \lambda[x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W,$$

where $\lambda \neq 0$, and $na_s = n_1 a_{s_1} + n_2 a_{s_2}$. To do this, we consider the following commutative diagram:

$$\begin{tikzcd}
\pi_*(\Omega\Sigma Y) \arrow[r, "{\Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_\sharp}"] \arrow[d, "h"'] & \pi_*(\Omega\Sigma Y) \arrow[d, "h"] \\
H_*(\Omega\Sigma Y;\mathbb{Q}) \arrow[r, "{\Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_*}"] & H_*(\Omega\Sigma Y;\mathbb{Q})
\end{tikzcd}$$

The Cartan-Serre theorem (see [5, Theorem 16.10]) asserts that the Hurewicz homomorphism $h : \pi_*(\Omega\Sigma Y) \to H_*(\Omega\Sigma Y;\mathbb{Q})$ becomes an isomorphism

$$\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q} \cong PH_*(\Omega\Sigma Y;\mathbb{Q}),$$

where the latter is a primitive subspace of $H_*(\Omega\Sigma Y;\mathbb{Q})$. Thus we observe that

$$h(\widehat{x}_n^{a_s}) = \lambda b_n^{a_s} + \text{decomposables} \quad (\lambda \neq 0)$$

for each $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$ (compare with the Hurewicz map of the Brown-Peterson spectra in [15, p. 166]). Here $\widehat{x}_n^{a_s}$ is the rationally non-trivial indecomposable element of the homotopy groups, and $b_n^{a_s}(= E_*(b_n^{a_s}))$ is the rational homology generator in dimension $2na_s$, where $E : Y \to \Omega\Sigma Y$ is the canonical inclusion. We now have

$$\begin{aligned}
h\Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_\sharp(\widehat{x}_n^{a_s}) &= \Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_* h(\widehat{x}_n^{a_s}) && \text{(by commutativity)}\\
&= [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]_*(\lambda b_n^{a_s} + \text{decomposables}) && \text{(by Remark 2.1 (b))}\\
&= [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]_*(\lambda b_n^{a_s}) + 0 && \text{(by Lemma 3.4)}\\
&= \lambda h(\langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle) && \text{(by Lemma 3.2)}\\
&= h(\lambda \langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle). && (3.3)
\end{aligned}$$

It can be noticed that the above zero term is derived from the fact that the restriction $[\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]|_{Y_{2(n_1 a_{s_1} + n_2 a_{s_2})-1}}$ to the skeleton is inessential by Lemma 3.4; that is

$$[\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]_*(b_{n_j}^{a_{s_i}}) = 0$$

for $\dim(b_{n_j}^{a_{s_i}}) \leq 2(n_1 a_{s_1} + n_2 a_{s_2}) - 1$ in rational homology of $\Omega\Sigma Y$. Moreover, we see that $\widehat{x}_n^{a_s}$ and $\langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle$ are rationally non-trivial indecomposable and decomposable elements, respectively, in $\pi_{2(n_1 a_{s_1} + n_2 a_{s_2})}(\Omega\Sigma Y)/\text{torsion}$, by Remark 3.1 as adjointness for decomposable generators, and that $h(\widehat{x}_n^{a_s})$ is spherical, and thus primitive. Now considering the above equation (3.3), we observe that

$$\Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]_\sharp(\widehat{x}_n^{a_s}) = \lambda \langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}} \rangle.$$

On the other hand, $\Omega[\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}]$ is a loop map, thus it is an H-map. Furthermore, the Scheerer's theorem (see [17, p. 75]) says that there is a bijection between $[\Sigma Y, \Sigma Y]$ and the set $[\Omega\Sigma Y, \Omega\Sigma Y]_H$ of homotopy classes of H-maps $\Omega\Sigma Y \to \Omega\Sigma Y$. Therefore, by taking the adjoint of the Samelson product, we obtain the result.

We now suppose that the result holds for the $(l-1)$-fold Whitehead product. Since

$$[\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]\cdots]|_{Y_{2(n_1 a_{s_1} + n_2 a_{s_2} + \cdots + n_{l-1} a_{s_{l-1}})-1}} \simeq *,$$

and the iterated Samelson product $\langle \widehat{x}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, \langle \widehat{x}_{n_1}^{a_{s_1}}, \widehat{x}_{n_2}^{a_{s_2}}\rangle \cdots\rangle$ is rationally non-trivial, by using the first result above and combining with $\widehat{\varphi}_{n_l}^{a_{s_l}} : Y \to \Omega\Sigma Y$, we can construct an iterated commutator map $[\widehat{\varphi}_{n_l}^{a_{s_l}}, [\widehat{\varphi}_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\widehat{\varphi}_{n_1}^{a_{s_1}}, \widehat{\varphi}_{n_2}^{a_{s_2}}]\cdots]]$ such that, after taking the adjointness, the desired formula of this lemma is obtained.

Remark 3.2 We turn now to the other types of purely decomposable generators, namely $[[\chi_1^{a_s}, \chi_2^{a_s}]_W, [\chi_1^{a_s}, \chi_3^{a_s}]_W]_W$ and $[[\chi_1^{a_s}, \chi_2^{a_s}]_W, [\chi_1^{a_s}, \chi_4^{a_s}]_W]_W$, consisting of the basic Whitehead products of the rational homotopy. It can be shown that we can also consider the iterated commutators $[[\varphi_1^{a_s}, \varphi_2^{a_s}], [\varphi_1^{a_s}, \varphi_3^{a_s}]]$ and $[[\varphi_1^{a_s}, \varphi_2^{a_s}], [\varphi_1^{a_s}, \varphi_4^{a_s}]]$ (corresponding to the basic Whitehead products $[[\chi_1^{a_s}, \chi_2^{a_s}]_W, [\chi_1^{a_s}, \chi_3^{a_s}]_W]_W$ and $[[\chi_1^{a_s}, \chi_2^{a_s}]_W, [\chi_1^{a_s}, \chi_4^{a_s}]_W]_W$, respectively) satisfying Lemma 3.5 whose proof goes to the similar way.

By using the results described above, we now proceed to the proof of Theorem 1.1 as follows.

If X is a connected H-space of finite type, then X has k-invariants of finite order, and $H^*(X;\mathbb{Q})$ becomes a Hopf algebra which is the tensor product of exterior algebras with odd degree generators and polynomial algebras with even degree generators. On the space level, this means that every H-space has a rational homotopy type of a product of rational Eilenberg-MacLane spaces. The Eckmann-Hilton dual of the Hopf-Thom theorem (see [19, p. 263–269] and [20, Chapter III]) says that $\Sigma K(\mathbb{Z}, 2a_s)$ has the rational homotopy type of the wedge products of the infinite number of spheres, that is

$$\Sigma K(\mathbb{Z}, 2a_s) \simeq_{\mathbb{Q}} S^{2a_s+1} \vee S^{4a_s+1} \vee S^{6a_s+1} \vee \cdots \vee S^{2na_s+1} \vee \cdots$$

for each $s = 1, 2, \cdots, k$. By using both the basic Whitehead products and the Hilton's theorem (see [7]), we can find various kinds of rational homotopy indecomposable and purely decomposable generators on $\pi_*(\Sigma Y) \otimes \mathbb{Q}$ as follows:

Table 1 $s = 1, 2, \cdots, k$

n	dimension	indecomposable	purely decomposable
1	$2a_s+1$	$\chi_1^{a_s}$	-
2	$4a_s+1$	$\chi_2^{a_s}$	-
3	$6a_s+1$	$\chi_3^{a_s}$	$[\chi_1^{a_s}, \chi_2^{a_s}]_W$
4	$8a_s+1$	$\chi_4^{a_s}$	$[\chi_1^{a_s}, \chi_3^{a_s}]_W, [\chi_1^{a_s}[\chi_1^{a_s}, \chi_2^{a_s}]_W]_W$
5	$10a_s+1$	$\chi_5^{a_s}$	$[\chi_1^{a_s}, \chi_4^{a_s}]_W, [\chi_1^{a_s}[\chi_1^{a_s}, \chi_3^{a_s}]_W]_W, [\chi_2^{a_s}, \chi_3^{a_s}]_W,$ $[\chi_1^{a_s}, [\chi_1^{a_s}[\chi_1^{a_s}, \chi_2^{a_s}]_W]_W]_W, [\chi_2^{a_s}[\chi_1^{a_s}, \chi_2^{a_s}]_W]_W$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Moreover, we can see that there exist hybrid decomposable generators of the rational homotopy. The hybrid decomposable generator might be occurred firstly in dimension $2a_3 + 1$. For example, if $a_1 = 1,\ a_2 = 3$ and $a_3 = 4$, then $[\chi_1^{a_1}, \chi_1^{a_2}]_W$ and $[\chi_1^{a_2}[\chi_1^{a_1}, \chi_4^{a_1}]_W]_W$ are the hybrid decomposable generators in $\pi_9(\Sigma Y) \otimes \mathbb{Q}$ and $\pi_{17}(\Sigma Y) \otimes \mathbb{Q}$, respectively. The number of purely or hybrid decomposable generators increases dramatically as the homotopy dimensions are on the increase.

Since the ranks between the graded homotopy group modulo torsion and the graded rational homotopy group coincide with each other, we can also find the corresponding indecomposable and decomposable elements on $\pi_*(\Sigma Y)$/torsion. More precisely, it can be seen from the above table that there is only one indecomposable generator, up to sign, of the homotopy group $\pi_{2na_s+1}(\Sigma Y)$/torsion for each $n = 1, 2, 3, \cdots$ and $s = 1, 2, \cdots, k$, while there are various kinds of purely or hybrid decomposable generators in it (possibly) for $n \geq 2$.

We now let $L = (\pi_*(\Sigma Y)/\text{torsion}, \ [\ ,\ ]_W)$ and $L_{\leq a_s,n} = (\pi_{\leq 2na_s+1}(\Sigma Y)/\text{torsion}, \ [\ ,\ ]_W)$ be the Whitehead algebras (corresponding to $\mathcal{L}$ and $\mathcal{L}_{\leq a_s,n}$, respectively) under the Whitehead products. And we denote $I_n^{a_s}L$ and $D_n^{a_s}L$ by the indecomposable and decomposable components, respectively, of the homotopy group modulo torsions, namely, $\pi_{2na_s+1}(\Sigma Y)$/torsion. Then we have that $I_n^{a_s}L \cong \mathbb{Z}$, and thus $\text{Aut}(I_n^{a_s}L) \cong \mathbb{Z}_2$ for each $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$. Moreover, the following sequence

$$0 \longrightarrow \text{Hom}(I_n^{a_s}L, D_n^{a_s}L) \xrightarrow{f} \text{Aut}(L_{\leq a_s,n}) \xrightarrow{g} \text{Aut}(L_{<a_s,n}) \oplus \text{Aut}(I_n^{a_s}L) \longrightarrow 0$$

is exact for each $s = 1, 2, \cdots, k$ and $n = 1, 2, 3, \cdots$ (see [11]). Here the map f sends

$$[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]_\sharp \in \text{Hom}(I_n^{a_s}L, D_n^{a_s}L)$$

to

$$I + j \circ [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]_\sharp \circ q \in \text{Aut}(L_{\leq a_s,n}),$$

and the map g is given by restriction and projection, where $q : L_{\leq a_s,n} \to I_n^{a_s}L$ is the projection and $j : D_n^{a_s}L \hookrightarrow L_{\leq a_s,n}$ is the inclusion. We observe that the above short exact sequence is still valid since we are working on $\pi_{\leq 2na_s+1}(\Sigma Y)$/torsion. Furthermore, we get $\text{Aut}(\pi_{2a_s+1}(\Sigma Y)/\text{torsion}) \cong \mathbb{Z}_2$ for $s = 1, 2, \cdots, k$, and $\text{Aut}(\pi_{\leq 2na_s+1}(\Sigma Y)/\text{torsion})$ is infinite for all $n \geq 3$ and $s = 1, 2, \cdots, k$. Therefore the induction step begins. We now suppose that the map $\text{Aut}(\Sigma Y) \to \text{Aut}(L_{<a_s,n})$ has a finite index. For each basic (iterated) Whitehead product $[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W \in D_n^{a_s}L$ (or other types of iterated Whitehead products) with $na_s = n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l}$ and $x_{n_i}^{a_{s_i}} \in I_{n_i}^{s_i}L$ for $i = 1, 2, \cdots, l$, by Lemma 3.5, we can always establish an iterated commutator $[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$ (or other types of iterated commutators), and thus we have a homotopy self-equivalence

$$I + [\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]] \in \text{Aut}(\Sigma Y)$$

completely depending on the form of $[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W$ such that the restriction $(I+[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]])_\sharp|_{L_{<a_s,n}}$ to the subalgebra $L_{<a_s,n}$ is the identity, and

$$(I+[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]])_\sharp(x_n^{a_s}) = x_n^{a_s} + \lambda[x_{n_l}^{a_{s_l}}, [x_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [x_{n_1}^{a_{s_1}}, x_{n_2}^{a_{s_2}}]_W \cdots]_W]_W,$$

where $\lambda \neq 0$, and $na_s = n_1a_{s_1} + n_2a_{s_2} + \cdots + n_la_{s_l}$. By considering the indecomposable and (purely or hybrid) decomposable generators, induction hypothesis and Theorem 2.1, we finally complete the proof of Theorem 1.1.

Remark 3.3 One may wonder why the k-th suspensions are not mentioned in this paper (or the previous papers [9–10]) for $k \geq 2$. Indeed, the homotopy self-equivalences $I +$

$[\varphi_{n_l}^{a_{s_l}}, [\varphi_{n_{l-1}}^{a_{s_{l-1}}}, \cdots, [\varphi_{n_1}^{a_{s_1}}, \varphi_{n_2}^{a_{s_2}}] \cdots]]$ constructed in our main theorem are not as well behaved as one might wish on the self-maps of the k-th suspension of a given CW-space Y for $k \geq 2$ since the group $[\Sigma^k Y, \Sigma^k Y]$ becomes abelian for $k \geq 2$. However, it is reasonable for us to conjecture that there are lots of self-maps in this abelian group which are nontrivial rationally, but suspend to the trivial self-map of $\Sigma^{k+1} Y$.

Acknowledgement The author is grateful to the anonymous referees for their careful readings and many helpful suggestions that improved the quality of the paper.

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