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Catenoidal Layers for the Allen-Cahn Equation in Bounded Domains*

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(*Dedicated to Haim Brezis on the occasion of his* 70*th birthday*)

Abstract This paper presents a new family of solutions to the singularly perturbed Allen-Cahn equation $\alpha^2\Delta u + u(1-u^2) = 0$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, with Neumann boundary condition and $\alpha > 0$ a small parameter. These solutions have the property that as $\alpha \to 0$, their level sets collapse onto a bounded portion of a complete embedded minimal surface with finite total curvature intersecting $\partial\Omega$ orthogonally and that is non-degenerate respect to $\partial\Omega$. The authors provide explicit examples of surfaces to which the result applies.

Keywords Allen-Cahn equation, Critical minimal surfaces, Critical catenoid, Infinite dimensional gluing method, Neumann boundary condition
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1 Introduction

1.1 Preliminary discussion

In this paper, we study the singularly perturbed boundary value problem

$$\alpha^2\Delta u + u(1-u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\alpha > 0$ is a small parameter, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and n is the inner unit normal vector to $\partial\Omega$.

Solutions to (1.1) correspond exactly to the critical points of the Allen-Cahn energy

$$J_\alpha(u) := \int_\Omega \frac{\alpha}{2}|\nabla u|^2 + \frac{1}{4\alpha}(1-u^2)^2, \quad u \in H^1(\Omega).$$

Equation (1.1) arises for instance in the gradient theory of phase transition when modelling the phase of a material placed in Ω or when studying stationary solutions for bistable reaction kinetics (see [2]).

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Observe that $u = \pm 1$ are global minimizers of J_α, representing stable phases of two different materials in Ω.

We are interested in solutions u connecting the stable phases ± 1. As described in [29], solutions of this type are expected to have a narrow transtition layer from -1 to $+1$ with a nodal set that is asymptotically locally stationary for the perimeter functional. To be more precise, in [29] the author showed that a family of local minimizers $\{u_\alpha\}_\alpha$ of J_α with uniformly bounded energy must converge in $L^1(\Omega)$, up to a subsequence, to a function u^* of the form

$$u^* = \chi_\Lambda - \chi_{\Omega - \Lambda},$$

where χ_E is the characteristic function of a set E and $\Lambda \subset \Omega$ minimizes perimeter. In this case, as $\alpha \to 0$

$$J_\alpha(u_\alpha) \to \mathrm{Per}(\Lambda) \Big(\int_{\mathbb{R}} \frac{1}{2}|w'|^2 + \frac{1}{4}(1 - w^2)^2 \mathrm{d}t \Big), \tag{1.2}$$

where $w(t) = \tanh(\frac{t}{\sqrt{2}})$ is the solution of

$$w'' + w(1 - w^2) = 0 \quad \text{in } \mathbb{R}, \quad w(0) = 0, \quad w' > 0, \quad w(\pm\infty) = \pm 1. \tag{1.3}$$

The above assertion means that for any $c \in (-1, 1)$, the level sets $\{u_\alpha = c\}$ converge to $\partial\Lambda$ as $\alpha \to 0$. This result provided the intuition that ultimately led to important developments in the theory of Γ-convergence and put into light a deep connection between the Allen-Cahn equation and the theory of minimal surfaces. We refer reader to [4–5, 23, 31, 33] for related results and stronger notions of convergence.

The connection between the Allen-Cahn equation and the theory of minimal surfaces has been explored in order to produce nontrivial solutions of equation (1.1), but the general understanding of solutions to this equation is far from being complete. In this regard, it is natural to ask for existence and asymptotic behavior of solutions to (1.1) in general smooth domains. For the case of minimizers, we refer the reader to [3, 18, 31, 35] and references therein. We also refer the reader to [6, 28], where it is established that the only local minimizers in convex domains are the constants ± 1.

The authors in [25], used a measure theoretical approach and the aforementioned intuition in dimension $N = 2$, to construct local minimizers u_α to (1.1) with interfaces collapsing onto a fixed minimizing segment Γ_0 inside Ω that cuts $\partial\Omega$ perpendicularly.

In [26], the author considers a situation similar to that one described in [25], but where Γ_0 is a non-degenerate segment, instead of a stable one. Non-degeneracy of Γ_0 respect to Ω is stated as

$$K_0 + K_1 - |\Gamma_0| K_0 K_1 \neq 0,$$

where K_0, K_1 are the curvatures of $\partial\Omega$ at the points where Γ_0 cuts $\partial\Omega$ orthogonally and $|\Gamma_0|$ corresponds to the length of the segment.

This geometric condition is equivalent to the fact that the eigenvalue problem

$$h'' = \lambda h \quad \text{in } (0, l), \quad K_0 h(0) + h'(0) = 0, \quad K_1 h(l) - h'(l) = 0, \quad l = |\Gamma_0|$$

does not have $\lambda = 0$ as an eigenvalue. The author also provides information about the Morse index of these solutions, which is either one or two, depending on the sign of K_0 and K_1.

The previous construction was generalized in [9] under the same geometrical setting described in [26]. Solutions in [9] have multiple interfaces that in the limit collapse onto the segment Γ_0. Also, at main order, the transition layers interact exponentially respect to their mutual distances giving rise to the Toda system of ODEs.

In dimension $N = 3$, Sakamoto [34] constructed solutions to (1.1) having a narrow transition through a planar disk orthogonal to the boundary of the domain and being non-degenerate in a suitable sense. The author also provides a characterization for this non-degeneracy in terms of the spectrum of the Dirichlet to Neumann map of the planar disk. As for higher dimensions in the setting of manifolds, Pacard and Ritoré [30] constructed solutions having one transition along a codimension one non-degenerate minimal submanifold.

In the spirit of the results mentioned above, we also want to refer the reader to [11–14] dealing with similar results for the inhomogeneous Allen-Cahn equation and [19, 36–37] for semilinear elliptic problems, where resonance phenomena are present.

The underlying geometric problem when constructing solutions to (1.1) with narrow interfaces is the existence of minimal surfaces inside the domain Ω intersecting the boundary $\partial\Omega$ orthogonally. This problem, in a general three dimensional compact Riemannian manifold, has been completely settled in a recent paper by Li [27]. For earlier results in this direction, we refer to [15–16]. One instance is the critical catenoid in a ball whose uniqueness was established by Fraser and Schoen [17].

1.2 Main result

Our goal in this paper is to generalize the results in [26, 34] by taking $N = 3$ and a more general class of minimal surfaces for limiting nodal set.

Let M be a complete embedded minimal surface of finite total curvature in $\mathbb{R}^3$. For over a century, there were known only two examples of such surfaces, namely the plane and the catenoid. In [7–8], Costa gave the first nontrivial example of such a surface with genus one, being properly embedded and having two catenoidal connected components outside a large ball sharing an axis of symmetry and another planar component perpendicular to this axis. Later this construction was generalized in [21–22] to surfaces having the same look as the Costa's surface far away but with arbitrary genus. We refer the interested reader to [24, 32] and references therein, for related results and further generalizations.

For $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, let us denote

$$y' = (y_1, y_2), \quad r = r(y) := \sqrt{y_1^2 + y_2^2}.$$

It is known that for some large but fixed $R_0 > 0$, outside the cylinder $\{y \in \mathbb{R}^3 : r(y) > R_0\}$, M decomposes into finite connected components, say $M_1, \cdots, M_m$, which from now on we will refer to as the ends of M.

For every $k = 1, \cdots, m$, there exists a smooth function $F_k = F_k(y')$ with

$$M_k = \{(y', y_3) \in \mathbb{R}^3 \,:\, r(y', y_3) > R_0,\ y_3 = F_k(y')\},$$

and there exist constants a_k, b_k, b_{ik} satisfying

$$a_1 \le a_2 \le \cdots \le a_k, \qquad \sum_{k=1}^{m} a_k = 0,$$

such that

$$F_k(y') = a_k \log(r) + b_k + b_{ik} \frac{y_i}{r^2} + \mathcal{O}(r^{-3}), \quad \text{as } r \to \infty, \tag{1.4}$$

and relation (1.4) can be differentiated.

It is also known that M is orientable and $\mathbb{R}^3 - M$ has exactly two connected components namely S^+ and S^- (see [20]). Let S^+ be the connected component of $\mathbb{R}^3 - M$ containing the axis x_3 which corresponds to the axis of symmetry of the ends $M_1, \cdots, M_k$.

Let $\nu : M \to S^2$ be the unit normal vector to M pointing towards S^+ and consider Fermi coordinates near M

$$x = y + z\nu(y) \quad \text{for } y \in M,\ |z| < \eta + \delta \log(2 + r(y)),$$

where $\eta, \delta > 0$ are small but fixed. Observe that z corresponds to the signed distance to M, i.e.,

$$|z| = \text{dist}(x, M) \quad \text{for } x = y + z\nu(y)$$

for every $y \in M$ and z small.

Next, consider a smooth bounded domain Ω, such that

(i) Ω contains a portion of the surface M, denoted by $\mathcal{M}$, i.e., $\mathcal{M} := \Omega \cap M$ is non-empty.

(ii) $\Omega - \overline{\mathcal{M}}$ has two connected components which, abusing the notation, we denote by S^+, S^- with the same convention as above.

(iii) For every $k = 1, \cdots, m$, $C_k := M_k \cap \partial\Omega$ is a smooth simple closed curve and

$$\partial\Omega \cap M = \bigcup_{k=1}^{m} C_k.$$

Observe that $\partial\Omega \cap M$ consists of m non-intersecting closed curves.

Following [9, 25, 26], in order to produce solutions to (1.1), $\mathcal{M}$ must be critical and non-degenerate in a suitable sense respect to $\partial\Omega$. To make this concepts precise, let us introduce $\Delta_{\mathcal{M}}$ and $|A_{\mathcal{M}}|$ the Laplace-Beltrami operator and the norm of the second fundamental form of $\mathcal{M}$, respectively. The fact that $\mathcal{M}$ is a minimal surface is equivalent to saying that its mean curvature $H_{\mathcal{M}} = 0$. This implies that $|A_{\mathcal{M}}|^2 = -2K_{\mathcal{M}}$, where $K_{\mathcal{M}}$ is the Gaussian curvature of $\mathcal{M}$.

Recall that n is the inner unit normal vector to $\partial\Omega$ and consider the eigenvalue problem

$$\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h = \lambda h \quad \text{in } \mathcal{M}, \quad \frac{\partial h}{\partial \tau} + I(y)h = 0 \quad \text{on } \partial\mathcal{M}, \tag{1.5}$$

where τ represents the inward unit normal direction to $\partial\mathcal{M}$ respect to $\mathcal{M}$ and $I(y)$ is given by

$$I(y) := \Big\langle \frac{\partial n}{\partial \nu}(y); \nu(y) \Big\rangle \quad \text{for } y \in \partial\mathcal{M}.$$

Our crucial assumptions on $\mathcal{M}$ are the following:

(I) $\mathcal{M}$ cuts orthogonally $\partial\Omega$ along the curve C_k for every $k = 1, \cdots, m$.

(II) $\lambda = 0$ is not an eigenvalue for the problem (1.5) in $H^1(\mathcal{M})$.

As stated in [19], assumption (I) implies that τ and n must coincide along every curve C_k and consequently these curves are geodesics in $\partial\Omega$ in the direction of ν since their normal vectors in $\partial\Omega$ are parallel to n. Therefore, the quantity $I(y)$ corresponds to the geodesic curvature of $\partial\Omega$ in the direction $\nu(y)$ for $y \in C_k$.

Our main result is the following.

Theorem 1.1 *Assume conditions* (i)–(iii) *and* (I)–(II). *Then for every $\alpha > 0$ small enough, there exists a solution u_α to* (1.1), *such that for every $x \in \{\text{dist}(\cdot, \mathcal{M}) < \eta\} \cap \Omega$,*

$$u_\alpha(x) = w\Big(\frac{z - h(y)}{\alpha}\Big) + \mathcal{O}_{H^1(\Omega)}(\alpha), \quad x = y + z\nu(y),$$

where $w(t)$ is determined by (1.3) *and at main order h solves the boundary value problem*

$$\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h = \mathcal{O}_{L^p(\mathcal{M})}(\alpha) \quad in\ \mathcal{M}, \quad \frac{\partial h}{\partial \tau} + I(y)h = \mathcal{O}_{L^p(\partial\mathcal{M})}(\alpha) \quad on\ \partial\mathcal{M},\ p > 3. \tag{1.6}$$

Even more, in the set $\Omega - \{\mathrm{dist}(\cdot, \mathcal{M}) < \eta\}$, as $\alpha \to 0$,

$$u_\alpha(x) \to \begin{cases} +1, & x \in S^+, \\ -1, & x \in S^-. \end{cases} \tag{1.7}$$

Theorem 1.1 provides us with solutions having limiting nodal with multiple catenoidal ends intersecting $\partial\Omega$ orthogonally. We also remark that in the case that the surface M and the domain Ω enjoy axial symmetry, our developments can be carry out in this setting and condition (II) for problem (1.5) is required only in the space in $H^1_{\mathrm{axial}}(\mathcal{M})$.

The paper is organized as follows. In Section 2, we present the invertibility theory of the operator described in (1.6) with Robin boundary conditions and discuss some examples, where our result applies. In Section 3, we present the geometric framework, which we will use to set up the proof of Theorem 1.1. In Section 4, we construct an accurate approximation of the solution to problem (1.1) and Section 5 presents the proof of our main result. The final sections are devoted to provide detailed proofs of lemmas and propositions used in Section 5.

2 Jacobi Operator with Robin Boundary Conditions and Examples

In this part, we consider the equation

$$\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h = f, \quad \frac{\partial h}{\partial \tau} + I(y)h = g \quad \text{on } \partial\mathcal{M} \tag{2.1}$$

for given functions $f \in L^p(\mathcal{M})$ and $g \in L^p(\partial\mathcal{M})$.

Let us assume hypothesis (II) from the introduction. In the case $g = 0$, using the Fourier decomposition method, it is straight-forward to verify that for any $f \in L^2(\mathcal{M})$, there exists a unique solution $h \in W^{2,2}(\mathcal{M})$ satisfying

$$\|h\|_{W^{2,2}(\mathcal{M})} \le C\|f\|_{L^2(\mathcal{M})}.$$

By standard regularity theory, still assuming that $g = 0$, if $p > 2$ and $f \in L^p(\mathcal{M})$, then problem (2.1) has a unique solution $h \in W^{2,p}(\mathcal{M})$ satisfying

$$\|h\|_* := \|D^2 h\|_{L^p(\mathcal{M})} + \|\nabla h\|_{L^\infty(\mathcal{M})} + \|h\|_{L^\infty(\mathcal{M})} \le C\|f\|_{L^p(\mathcal{M})}.$$

From the previous discussion, it follows that for any $p > 2$, there exists $C > 0$ such that given arbitrary functions $f \in L^p(\mathcal{M})$, $g \in L^p(\partial M)$, problem (2.1) has a unique solution $h \in W^{2,p}(\mathcal{M}) \cap C^{1,1-\frac{2}{p}}(\mathcal{M})$ satisfying

$$\|h\|_* \le C(\|f\|_{L^p(\mathcal{M})} + \|g\|_{L^p(\partial\mathcal{M})}).$$

2.1 Comments about conditions (I)–(II)

We comment first on conditions (I)–(II) in a particular case.

Let M be the catenoid in $\mathbb{R}^3$ parameterized by the mapping

$$Y(\mathrm{y}, \theta) := (\sqrt{1+\mathrm{y}^2}\cos\theta, \sqrt{1+\mathrm{y}^2}\sin\theta, \log(\mathrm{y} + \sqrt{1+\mathrm{y}^2})), \quad \mathrm{y} \in \mathbb{R},\ \theta \in (0, 2\pi),$$

which provides coordinates on M in terms of the signed arch-length of the profile curve and the rotation around the x_3-axis, which in our setting corresponds to the axis of symmetry of M.

Since S^+ is the connected component of $\mathbb{R}^3 - M$ containing the x_3-axis, the unit normal vector to M pointing towards S^+ is given by

$$\nu(\mathrm{y},\theta) = \frac{1}{\sqrt{1+\mathrm{y}^2}}\,(-\cos\theta, -\sin\theta, \mathrm{y}), \quad \mathrm{y}\in\mathbb{R},\ \theta\in(0,2\pi).$$

Consider the Fermi coordinates

$$\widetilde{X}(\mathrm{y},\theta,z) = Y(\mathrm{y},\theta) + z\nu(\mathrm{y},\theta), \tag{2.2}$$

which define a change of variables for instance on the neighborhood of M

$$\{Y(\mathrm{y},\theta) + z\nu(\mathrm{y},\theta) : |z| < \eta\} \tag{2.3}$$

for some fixed and small $\eta > 0$.

Let Ω be an axially symmetric domain and recall that $\mathcal{M} = \Omega\cap M$. Since M has two ends and $\partial\Omega\cap M$ is axially symmetric, $\partial\Omega\cap M = C_1\cup C_2$, where C_1, C_2 are parallel, non-intersecting circles. C_1, C_2 are parameterized respectively by

$$Y_i(\theta) := Y(\mathrm{y}_i,\theta), \quad \theta\in(0,2\pi), \quad i = 1,2$$

for some fixed $\mathrm{y}_1 < \mathrm{y}_2$.

To describe $\partial\Omega$ close to the circles $Y_i(\theta)$, we assume the existence of two smooth functions

$$G_1, G_2 : (-\eta,\eta)\to\mathbb{R}, \quad G_i(0) := \mathrm{y}_i, \quad i = 1,2$$

and also that the systems of coordinates

$$X_i(\theta,z) := Y(G_i(z),\theta) + z\nu(G_i(z),\theta), \quad \theta\in(0,2\pi),\ |z|<\eta,\ i=1,2 \tag{2.4}$$

describe the set

$$\{x\in\partial\Omega : x = X_i(\theta,z),\ |z|<\eta,\ \theta\in(0,2\pi),\ i=0,1\}\subset\partial\Omega.$$

Normal deformations of $\mathcal{M}$ within Ω can be described by

$$\widetilde{Y}_h(y,\theta) \;:=\; Y(\mathrm{y}(y,\theta),\theta) + h(y,\theta)\,\nu(\mathrm{y}(y,\theta),\theta), \tag{2.5}$$

where $\|h\|_{C^2(\mathcal{M})} < \eta$ and

$$\mathrm{y}(y,\theta) := \frac{G_2(h(y,\theta)) - G_1(h(y,\theta))}{\mathrm{y}_2 - \mathrm{y}_1}(y - \mathrm{y}_1) + G_1(h(y,\theta))$$

for $\mathrm{y}_1 < y < \mathrm{y}_2$, $\theta\in(0,2\pi)$.

Take any arbitrary $h\in C^2(\mathcal{M})\cap C^1(\overline{\mathcal{M}})$ with $\|h\|_{C^2\mathcal{M}} < \eta$. Denote

$$\mathcal{M}_h := \widetilde{Y}_h([\mathrm{y}_1,\mathrm{y}_2]\times(0,2\pi)),$$

and let g_h be its respective induced metric, with the convention that g_0 is the induced metric of $\mathcal{M}$.

The area functional of $\mathcal{M}_h$ is computed as

$$\mathcal{A}(\mathcal{M}_h) := \int_{\mathcal{M}_h} 1 \, \mathrm{d}A_{g_h} = \int_0^{2\pi} \int_{\mathrm{y}_1}^{\mathrm{y}_2} \sqrt{\det g_h} \mathrm{d}y \mathrm{d}\theta. \tag{2.6}$$

This area functional is of class C^2 and its first variation around $\mathcal{M}$ is given by

$$\begin{aligned} D\mathcal{A}(\mathcal{M})[h] = &- \sum_{i=1}^{2} \int_{C_i} \Big(\frac{\partial_z G_2(0) - \partial_z G_1(0)}{\mathrm{y}_2 - \mathrm{y}_1} (\mathrm{y}_i - \mathrm{y}_1) + \partial_z G_1(0) \Big) h(\mathrm{y}_i, \theta) \mathrm{d}s_{g_0} \\ &+ \int_{\mathcal{M}} H_{\mathcal{M}} \, h \, \mathrm{d}A_{g_0}, \end{aligned} \tag{2.7}$$

where we recall that $H_{\mathcal{M}}$ is the mean curvature of $\mathcal{M}$.

From (2.7), we conclude that $\mathcal{M}$ is critical for the area functional (2.6) if and only if

$$H_{\mathcal{M}} = 0, \quad \partial_z G_1(0) = \partial_z G_2(0) = 0. \tag{2.8}$$

Since $\mathcal{M}$ is a minimal surface, automatically $H_{\mathcal{M}} = 0$. Therefore, (2.8) states that condition (I) is equivalent to the fact that $\mathcal{M}$ is critical for the functional (2.6) respect to normal perturbations of $\mathcal{M}$.

Assuming condition (I), the second variation of the area functional around $\mathcal{M}$ is given by the quadratic form

$$D^2\mathcal{A}(\mathcal{M})[h,h] := (-1)^{i+1} \int_{Ci} \partial_{zz} G_i(0) h^2(y) \mathrm{d}s_{g_0} + \int_{\mathcal{M}} (|\nabla_{\mathcal{M}} h|^2 - |A_{\mathcal{M}}|^2 h) \mathrm{d}A_{g_0}$$

and stability properties of $\mathcal{M}$ respect to Ω are analyzed through the linear eigenvalue problem

$$\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h = \lambda \, h \quad \text{in } \mathcal{M}, \quad \frac{\partial h}{\partial \tau_i} + \partial_{zz} G_i(0) h = 0 \quad \text{on } \partial\mathcal{M}.$$

A similar analysis, but with more careful computations, can be carry out for a more general geometric setting, leading to the same interpretation.

2.2 Examples

In the entire catenoid, the linear equation,

$$\Delta_M Z + |A_M|^2 Z = 0 \quad \text{in } M \tag{2.9}$$

has two axially symmetric entire solutions, namely

$$Z_1(y) = y \cdot e_3, \quad Z_2(y) = y \cdot \nu(y), \quad y \in M$$

corresponding respectively to the invariances of M under translations along the vertical axis and dilations. We refer the reader to Section 4 in [1, 10] for full details.

In the coordinates $y = Y(\mathrm{y}, \theta) \in M$, we have that

$$Z_1(\mathrm{y}) = \frac{\mathrm{y}}{\sqrt{1+\mathrm{y}^2}}, \qquad Z_2(\mathrm{y}) = \frac{\mathrm{y}}{\sqrt{1+\mathrm{y}^2}} \log(\mathrm{y} + \sqrt{1+\mathrm{y}^2}) - 1, \quad \mathrm{y} \in \mathbb{R},$$

from where we observe that Z_1 is odd and Z_2 is even.

Observe also that in $(0, \infty)$, Z_1 is positive and $Z_1(0) = 0$, while Z_2 changes sign only at one point $\mathrm{y} = \mathrm{y}_0 > 0$ and $-1 < Z_2(\mathrm{y}) < 0$ for $0 < \mathrm{y} < \mathrm{y}_0$.

Since

$$\partial_{\mathrm{y}} Z_1(\mathrm{y}) = \frac{1}{(1+\mathrm{y}^2)^{\frac{3}{2}}}, \quad \partial_{\mathrm{y}} Z_2(\mathrm{y}) = \frac{\mathrm{y}}{1+\mathrm{y}^2} + \frac{\log(\mathrm{y}+\sqrt{1+\mathrm{y}^2})}{(1+\mathrm{y}^2)^{\frac{3}{2}}}, \quad \mathrm{y} \in \mathbb{R},$$

we notice that $\partial_{\mathrm{y}} Z_i > 0$ in $(0,\infty)$ for $i = 1, 2$. Therefore, Z_1, Z_2 are strictly increasing in $(0,\infty)$.

Observe that assumption (I) implies that along the circles C_i, $n = \tau_i$, where τ_i is the inward unit tangent vector of the profile curve of the catenoid M along the circles C_i, $i = 1, 2$.

Also $K_i := (-1)^{i+1} \partial_{zz} G_i(0)$ corresponds to the curvature of the integral curve of $\partial\Omega$ in the direction of ν along the circle C_i.

The axially symmetric framework and computations similar to those carried out in [1], yield that in the coordinates $y = Y(\mathrm{y}, \theta)$,

$$\Delta_{\mathcal{M}} = \partial_{\mathrm{yy}} + \frac{\mathrm{y}}{1+\mathrm{y}^2}\partial_{\mathrm{y}}, \qquad |A_{\mathcal{M}}|^2 = \frac{2}{(1+\mathrm{y}^2)^2}.$$

Hence, non-degeneracy of $\mathcal{M}$ respect to $\partial\Omega$ is equivalent to saying that the only solution to

$$\partial_{\mathrm{yy}} Z + \frac{\mathrm{y}}{1+\mathrm{y}^2}\partial_{\mathrm{y}} Z + \frac{2}{(1+\mathrm{y}^2)^2} Z = 0, \quad \mathrm{y}_1 < \mathrm{y} < \mathrm{y}_2, \quad \frac{\partial Z}{\partial \mathrm{y}}(\mathrm{y}_i) + (-1)^i K_i\, Z(\mathrm{y}_i) = 0 \tag{2.10}$$

is the trivial one.

Basic theory of ODEs and the developments from Section 4 in [1] imply that $\lambda = 0$ is not an eigenvalue of (2.10) if and only if

$$\det \begin{bmatrix} \partial_{\mathrm{y}} Z_1(\mathrm{y}_1) + K_1 Z_1(\mathrm{y}_1) & \partial_{\mathrm{y}} Z_2(\mathrm{y}_1) + K_1 Z_2(\mathrm{y}_1) \\ \partial_{\mathrm{y}} Z_1(\mathrm{y}_2) - K_2 Z_1(\mathrm{y}_2) & \partial_{\mathrm{y}} Z_2(\mathrm{y}_2) - K_2 Z_2(\mathrm{y}_2) \end{bmatrix} \neq 0. \tag{2.11}$$

Observe that condition (2.11) is clearly invariant under dilations.

2.2.1 Example 1

If in addition to the axial symmetry of Ω, we assume that $\partial\Omega$ is almost flat along the circles C_1, C_2 in the direction of the normal ν, i.e., $K_1 = K_2 = 0$, then $\mathcal{M}$ is non-degenerate respect to Ω.

To verify this claim, we notice that in this case, condition (2.11) is equivalent to

$$\frac{\partial_{\mathrm{y}} Z_2(\mathrm{y}_1)}{\partial_{\mathrm{y}} Z_1(\mathrm{y}_1)} \neq \frac{\partial_{\mathrm{y}} Z_2(\mathrm{y}_2)}{\partial_{\mathrm{y}} Z_1(\mathrm{y}_2)},$$

which holds true since the function $\mathrm{y} \mapsto \frac{\partial_{\mathrm{y}} Z_2(\mathrm{y})}{\partial_{\mathrm{y}} Z_1(\mathrm{y})}$ is strictly increasing and $\mathrm{y}_1 < \mathrm{y}_2$.

2.2.2 Example 2

Assume now that the catenoidal portion $\mathcal{M}$ is even respect to the vertical axis, i.e., $-\mathrm{y}_1 = \mathrm{y}_2 =: \overline{\mathrm{y}} > 0$. Let

$$\Omega := \left\{ x = (x_1, x_2, x_3) \,:\, \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \right\}$$

be an ellipsoid of revolution, where $a, b > 0$.

Using the coordinates $X_i(\theta, z)$ from (2.4) with $G = G_2(z) = -G_1(z)$ such that $G(0) = \overline{\mathrm{y}}$, $\partial\Omega$ near $\mathcal{M}$ is described by the implicit relation

$$\frac{1}{a^2}\left(\sqrt{1+G^2(z)} - \frac{z}{\sqrt{1+G^2(z)}}\right)^2 + \frac{1}{b^2}\left(\log(G(z)+\sqrt{1+G^2(z)}) + z\,\frac{G(z)}{\sqrt{1+G^2(z)}}\right)^2 = 1.$$

Implicit function theorem yields that

$$\partial_z G_2(0) = -\partial_z G_1(0) = \frac{-\frac{1}{a^2} + \frac{1}{b^2}\frac{\overline{\mathrm{y}}}{\sqrt{1+\overline{\mathrm{y}}^2}}\log(\overline{\mathrm{y}} + \sqrt{1+\overline{\mathrm{y}}^2})}{\frac{\overline{\mathrm{y}}}{a^2} + \frac{1}{b^2}\frac{1}{\sqrt{1+\overline{\mathrm{y}}^2}}\log(\overline{\mathrm{y}} + \sqrt{1+\overline{\mathrm{y}}^2})},$$

so that $\mathcal{M}$ is critical respect to the ellipsoid Ω if a, b satisfy

$$Z_2(\overline{\mathrm{y}}) = \frac{1}{a^2} - \frac{1}{b^2}, \tag{2.12}$$

and since $\min\{Z_2(\mathrm{y}) : \mathrm{y} \in \mathbb{R}\} = -1$, this imposes a restriction on a and b that

$$\frac{1}{a^2} > \frac{1}{b^2} - 1. \tag{2.13}$$

The monotonicity of $Z_2(\mathrm{y})$ allows the following interpretation of the criticality of $\mathcal{M}$: Once the ellipsoid has been fixed satisfying (2.13), there is exactly one catenoid that cuts the boundary of the ellipsoid perpendicularly.

Next, assume that $R = a = b$, so that $\Omega = B_R(0)$. In this case, $\mathcal{M}$ corresponds to the so-called critical catenoid. This situation was treated in [16], where $\mathcal{M}$ is the solution of a maximization problem for the first Steklov eigenvalue of the Dirichlet to Neumann mapping in bounded domains.

Using the same notation as above, one can verify that for the critical catenoid $\mathcal{M}$, $K_1 = K_2 = \frac{\overline{\mathrm{y}}}{1+\overline{\mathrm{y}}^2} = \frac{1}{R}$. From the determinant in (2.11) and relation (2.12), the non-degeneracy of the critical catenoid respect to the sphere $\partial B_R(0)$ is equivalent to the expression

$$2\partial_{\mathrm{y}} Z_2(\overline{\mathrm{y}})(\partial_{\mathrm{y}} Z_1(\overline{\mathrm{y}}) + K Z_1(\overline{\mathrm{y}})) \neq 0,$$

which holds true since all the quantities involved are positive.

2.2.3 Example 3

Concerning stability issues let us consider the quadratic form in

$$\mathrm{Q}(h,h) := -\int_{\partial\mathcal{M}} I(y) h^2 \mathrm{d}s_{g_0} + \int_{\mathcal{M}} (|\nabla h|^2 - |A_{\mathcal{M}}|^2 h^2) \mathrm{d}A_{g_0}, \quad h \in H^1(\mathcal{M}).$$

We first establish conditions on $\mathcal{M}$ to be minimizer of the area functional.

Proposition 2.1 *Assume that Z is a smooth positive solution to the linear equation* (2.9) *in an open set of the catenoid M, containing the portion $\mathcal{M} = \Omega \cap M$. For every smooth function φ in $\mathcal{M}$ it holds that*

$$Q(\varphi,\varphi) = -\int_{\partial\mathcal{M}} \Big(\frac{\partial \log(Z)}{\partial\tau} + I(y)\Big)\varphi^2 \mathrm{d}s_{g_0} + \int_{\mathcal{M}} \Big|\nabla\varphi - \frac{\varphi}{Z}\nabla Z\Big|^2 \mathrm{d}A_{g_0},$$

where τ is the inner normal vector to $\partial\mathcal{M}$. Consequently, if

$$I(y) < \frac{\partial \log(Z)}{\partial(-\tau)}, \quad y \in \partial\mathcal{M},$$

then $\mathcal{M}$ is minimizer for the area functional.

Proof The proof follows directly testing equation (2.9) against $\psi = \frac{\varphi^2}{Z}$ and integrating by parts.

Assume that $K_1, K_2 < 0$, so that Ω is a non-convex domain. If in addition $\mathcal{M}$ is an even catenoidal portion with small area, then $\mathcal{M}$ is a local minimizer for the area functional (2.6). To see this, it suffices to consider $\mathcal{M}$ an even piece of catenoid contained in an open set of M where $Z = -Z_2 > 0$, where we also have that

$$\frac{\partial \log(Z)}{\partial(-\tau)} > 0.$$

The claim follows by a direct application of the previous proposition.

On the other hand, if the area of $\mathcal{M}$ is large enough, $\mathcal{M}$ resembles the entire catenoid M, which has Morse index 1. Cutting off an eigenfunction of $\Delta_M + |A_M|^2$ associated to its positive eigenvalue, in a way that the boundary condition plays no role at infinity, we obtain a direction in $H^1(\mathcal{M})$ where the second variation of surface area is negative and therefore a catenoidal portion $\mathcal{M}$ with large area would be unstable.

The former situation ocurrs also in a general complete embedded minimal surface with finite total curvature for which the Morse index is also finite.

3 Geometric Computations

Let M be a complete embedded minimal surface with finite total curvature. In this part, we compute the Euclidean Laplacian in an open neighborhood of $\mathcal{M}$ inside Ω and the normal derivate $\frac{\partial}{\partial n}$ in $\partial\Omega$ near $\partial\mathcal{M}$ in suitable systems of coordinates.

First we compute the Euclidean Laplacian well inside the set Ω close to $\mathcal{M}$. Following the developments from [10], denote by M_0 the part of M inside the cylinder $\{\mathrm{y} : r(\mathrm{y}) < R_0 + 1\}$. To parameterize M_0, we take an open set $\mathcal{U} \subset \mathbb{R}^2$ and a mapping $\mathrm{y} \in \mathcal{U} \to y := Y_0(\mathrm{y})$ with associated induced metric given by $g := (g_{ij})_{2\times 2}$. The Laplace-Beltrami operator of M inside the cylinder can be computed as the elliptic operator

$$\Delta_M = \frac{1}{\sqrt{\det(g)}} \partial_i (\sqrt{\det(g)} g^{ij}\ \partial_j) = a^0_{ij}\partial_{ij} + b^0_i \partial_i, \tag{3.1}$$

where $g^{-1} = (g^{ij})_{2\times 2}$ is the inverse of the metric g, and the coefficients a^0_{ij}, b^0_i are smooth.

Next, consider the set

$$D := \{\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2) \in \mathbb{R}^2 \ :\ r(\mathrm{y}) > R_0\}.$$

For $k = 1, \cdots, m$, we parameterize M_k, the k-th end of M with the mapping

$$\mathrm{y} \in D \mapsto Y_k(\mathrm{y}) := \mathrm{y}_i e_i + F_k(\mathrm{y}) e_3.$$

Notice that the unit normal vector to M at a point $y \in M_k$ has the expression in coordinates

$$\begin{aligned}\nu(y) &:= \frac{(-1)^k}{\sqrt{1 + |\nabla F_k(\mathrm{y})|^2}} (\partial_i F_k e_i - e_3) \\ &= (-1)^k e_3 + a_k \frac{\mathrm{y}_i}{r^2} + \mathcal{O}(r^{-2}), \quad y = Y_k(\mathrm{y}),\end{aligned}$$

so that $\partial_i \nu = \mathcal{O}(r^{-2})$ and $|A_M|^2 = \mathcal{O}(r^{-4})$ as $r \to \infty$.

In coordinates $Y_k(\mathrm{y})$ on M_k, the metric $g := (g_{ij})_{2\times 2}$ satistisfies

$$g_{ij} = \delta_{ij} + \mathcal{O}(r^{-2}), \quad i,j = 1,2, \text{ as } r \to \infty,$$

and this relations can be differentiated.

We compute the Laplace-Beltrami operator on M_k, using again formula (3.1) to find that

$$\Delta_M = \Delta_{\mathrm{y}} + \mathcal{O}(r^{-2})\partial_{ij} + \mathcal{O}(r^{-3})\partial_i, \quad \text{on } M_k. \tag{3.2}$$

The surface M is parameterized completely by the $m+1$ local coordinates described above and we observe that expression (3.1) holds in the entire M, where for $k = 1, \cdots, m$ the coefficients on M_k satisfy

$$a^0_{ij}(y) = \delta_{ij} + \mathcal{O}(r^{-2}), \quad b_i(y) = \mathcal{O}(r^{-3}) \quad \text{for } k = 1, \cdots, m, \text{ as } r \to \infty.$$

Fermi coordinates given by the mapping $X(y,z) := y + z\nu(y)$ provide a change of variables in the neighborhood of M

$$\mathcal{N} := \{x = y + z\nu(y) \,:\, |z| < \eta\}.$$

In the set $\mathcal{N}$, the formula

$$\Delta_X = \partial_{zz} + \Delta_M \,-\, z|A_M|^2\,\partial_z \,+\, D \tag{3.3}$$

is valid, where Δ_M is computed in (3.1) for $\mathcal{N} \cap M_0$ and (3.2) for $\mathcal{N} \cap M_k$ and

$$D \,=\, z\,a^1_{ij}(y,z)\,\partial_{ij} \,+\, z\,b^1_i(y,z)\,\partial_i \,+\, z^3\,b^1_3(y,z)\,\partial_z.$$

On the ends of M, the smooth functions $a_{ij}(y,z)$, $b_i(y,z)$ satisfy

$$\begin{aligned} &|a^1_{ij}| \,+\, |r\,\nabla a^1_{ij}| \,=\, O(r^{-2}), \quad |b^1_i| \,+\, |r\,\nabla b^1_i| \,=\, O(r^{-3}), \\ &|b^1_3| \,+\, |r\,\nabla b^1_3| \,=\, O(r^{-8}), \end{aligned} \tag{3.4}$$

as $r \to \infty$, uniformly on z in the neighborhood $\mathcal{N}$ of M (see [10, Lemma 2.1]).

Let us comment further on expression (3.3). For fixed and small z, the mean curvature of the normally translated surface

$$M_z := \{y + z\nu(y) \,:\, y \in M\} \subset \mathcal{N}$$

is given by

$$H_{M_z} := H_M - z|A_M|^2 + z^2(k_1^3 + k_2^3) + z^3(k_1^4 + k_2^4) + \mathcal{O}(z^4\,r^{-10}),$$

where k_1, k_2 are the principal curvatures of M. Since M is a minimal surface, $H_M = k_1 + k_2 = 0$. It follows that $k_1^3 + k_2^3 = 0$ and so

$$H_{M_z} := -z|A_M|^2 + z^3 b^1_3(y,z).$$

From the asymptotics of $\nabla_M \nu$, we have that $k_i = \mathcal{O}(r^{-2})$, and thus we obtain the expansion for b^1_3 in (3.4) follows.

In what follows, we consider a large dilation of $\mathcal{M}$, denoted by $\mathcal{M}_\alpha := \alpha^{-1}\mathcal{M}$ for $\alpha > 0$ small. Let us denote the dilated ends of M by $M_{k,\alpha} := \alpha^{-1} M_k$ for $k = 1, \cdots, m$.

For a smooth function h defined in $\mathcal{M}$, we consider dilated and translated Fermi coordinates

$$X_{\alpha,h}(y,t) := X(\widetilde{y}, z), \quad \widetilde{y} = \alpha y, \quad z = \alpha(t + h(\alpha y))$$

for $y \in \mathcal{M}_\alpha$ and $|t + h(\alpha y)| < \frac{\eta}{\alpha}$.

Scaling and translating expression (3.3), we obtain

$$\alpha^2 \Delta_X = \Delta_{X_{\alpha,h}} = \partial_{tt} + \Delta_{\mathcal{M}_\alpha} - \alpha^2\{\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h\}\partial_t - \alpha^2 |A_{\mathcal{M}}|^2 t \partial_t \\ - 2\alpha\, a^0_{ij} \partial_i h\, \partial_{jt} + \alpha^2\, a^0_{ij} \partial_i h \partial_j\, h \partial_{tt} + D_{\alpha,h}, \tag{3.5}$$

where

$$D_{\alpha,h} = \alpha(t+h) a^1_{ij}(\alpha y, \alpha(t+h)) \left(\partial_{ij} - 2\alpha \partial_i h\, \partial_{it} - \alpha^2 \partial_{ij} h \partial_t + \alpha^2 \partial_i h\, \partial_j h\, \partial_{tt}\right) \\ + \alpha^2 (t+h) b^1_i(\alpha y, \alpha(t+h)) \left(\partial_i - \alpha \partial_i h\, \partial_t\right) \\ + \alpha^4 (t+h)^3 b^1_3(\alpha y, \alpha(t+h))\, \partial_t. \tag{3.6}$$

Expression (3.5) holds true in the region

$$\mathcal{N}_{\alpha,h} := \left\{ y + (t + h(\alpha y))\nu(\alpha y) \,:\, y \in \mathcal{M}_\alpha,\ |t + h| < \frac{\eta}{\alpha} + \right\}, \tag{3.7}$$

and we will use it to handle equation (1.1) well inside the region $\alpha^{-1}(\Omega \cap \mathcal{N})$.

The previous geometric considerations above do not take into account the effect of $\partial\Omega \cap \mathcal{N}$.

To handle boundary computations, we use that the surface $\mathcal{M}$ is orthogonal to $\partial\Omega$. Let us fix $k = 1, \cdots, m$. From assumption (iii) in the introduction, we may assume that the closed simple curve $C_k := M_k \cap \Omega$ is parameterized by

$$\upsilon \in (0, l_k) \mapsto \gamma_k := \gamma_k(\upsilon).$$

The mapping γ_k has a smooth orthogonal extension to an open neighborhood of C_k in M_k. Abusing the notation, we write this extension as

$$(\rho, \upsilon) \mapsto \gamma_k = \gamma_k(\rho, \upsilon), \quad \rho \in (-\delta, \delta),\ \upsilon \in (0, l_k), \tag{3.8}$$

which satisfies

$$\gamma_k(0, \upsilon) = \gamma_k(\upsilon), \quad \partial_\upsilon \gamma_k(0, \upsilon) = \partial_\upsilon \gamma_k(\upsilon), \quad \partial_\rho \gamma_k \perp \partial_\upsilon \gamma_k$$

and $\gamma_k([0,\delta) \times (0, l_k)) \subset M_k \cap \overline{\Omega}$. The coordinates (ρ, υ) can be thought as polar coordinates in M_k near C_k.

Using formula (3.1) and coordinates $\gamma_k(\rho, \upsilon)$ and omitting the depedence on k, the Laplace-Beltrami operator of $\mathcal{M}$ close to C_k, takes the form

$$\Delta_{\mathcal{M}} = a^0_{ij} \partial_{ij} + b^0_i \partial_j, \quad i, j = \rho, \upsilon, \tag{3.9}$$

where

$$a^0_{\rho\rho}(\rho, \upsilon) = |\partial_\rho \gamma_k|^{-2}, \quad a^0_{\upsilon\upsilon}(\rho, \upsilon) = |\partial_\upsilon \gamma_k|^{-2}, \quad a^0_{\rho\upsilon} = a^0_{\upsilon\rho} = 0, \\ b^0_\rho(\rho, \upsilon) = |\partial_\rho \gamma_k|^{-2} |\partial_\upsilon \gamma_k|^{-2} \langle \partial_{\rho\upsilon} \gamma_k; \partial_\upsilon \gamma_k \rangle + |\partial_\rho \gamma_k|^{-4} \langle \partial_{\rho\rho} \gamma_k; \partial_\rho \gamma_k \rangle, \\ b^0_\upsilon(\rho, \upsilon) = |\partial_\rho \gamma_k|^{-2} |\partial_\upsilon \gamma_k|^{-2} \langle \partial_{\rho\upsilon} \gamma_k; \partial_\rho \gamma_k \rangle + |\partial_\rho \gamma_k|^{-4} \langle \partial_{\upsilon\upsilon} \gamma_k; \partial_\upsilon \gamma_k \rangle.$$

Associated to the coordinate system $y = \gamma_k(\rho, \upsilon)$, we consider Fermi coordinates

$$X_k(\rho, \upsilon, z) := \gamma_k(\rho, \upsilon) + z\nu(\rho, \upsilon)$$

in the neighborhood of M_k

$$\mathcal{N}_k := \{\gamma_k(\rho,\upsilon) + z\nu(\rho,\upsilon) \,:\, |z|<\eta,\ |\rho|<\delta,\ \upsilon\in(0,l_k)\}.$$

To described $\Omega\cap\mathcal{N}_k$ near C_k, we assume the existence of a smooth function $G_k = G_k(\upsilon,z)$ with $G_k(\upsilon,0)=0$ and such that

$$\partial\Omega\cap\mathcal{N}_k := \{\gamma_k(0,G_k(\upsilon,z)) + z\nu_k(0,G_k(\upsilon,z)) \,:\, \upsilon\in(0,l_k),\ |z|<\eta\}.$$

A translation along the integral lines of M_k associated to the parameterization $\gamma_k(\rho,\upsilon)$, is given by

$$\rho(s,\upsilon,z) := s + G_k(\upsilon,z), \quad |s|<\delta,\ \upsilon\in(0,l_k),\ |z|<\eta$$

taking $\delta>0$ smaller if necessary.

Modified Fermi coordinates

$$\widetilde{X}(s,\upsilon,z) := \gamma_k(\rho(s,\upsilon,z),\upsilon) + z\nu(\rho(s,\upsilon,z),\upsilon)$$

describe also the set

$$\Omega\cap\mathcal{N}_k := \{x=\widetilde{X}(s,\upsilon,z)\in\Omega \,:\, |z|<\eta,\ |s|<\delta,\ \upsilon\in(0,l_k)\}.$$

Observe that

$$\partial_z\widetilde{X}(0,\upsilon,0) = \nu(0,\upsilon) + \partial_\rho\gamma_k(0,\upsilon)\cdot\partial_z G_k(\upsilon,0)$$

and

$$\partial_\upsilon\widetilde{X}(0,\upsilon,0) = \partial_\upsilon\gamma_k(0,\upsilon), \quad \partial_s\widetilde{X}(0,\upsilon,0) = \partial_\rho\gamma_k(0,\upsilon),$$

so that, assumption (I) is equivalent to saying that $\partial_z\widetilde{X}(0,\upsilon,0)$ and $\partial_s\widetilde{X}(0,\upsilon,0)$ are orthogonal vectors, and hence $\partial_z G_k(\upsilon,0)=0$.

Summarizing, the function $G_k(\upsilon,z)$ satisfies

$$G_k(\upsilon,0)=0, \quad \partial_z G_k(\upsilon,0)=0, \quad \upsilon\in(0,l_k). \tag{3.10}$$

From (3.10), the asymptotic expansion in powers of z of $\widetilde{X}(s,\upsilon,z)$ reads as

$$\widetilde{X}(s,\upsilon,z) := \gamma_k(s,\upsilon) + z\nu(s,\upsilon) + \frac{z^2}{2}q_1(s,\upsilon) + \frac{z^3}{6}q_2(s,\upsilon) + \mathcal{O}(z^4), \tag{3.11}$$

where $q_1, q_2 \perp \nu$ with expressions given by

$$q_1(s,\upsilon) := \partial_\rho\gamma_k(s,\upsilon)\cdot\partial_{zz}G_k(\upsilon,0), \quad q_2(s,\upsilon) = \partial_\rho\gamma_k\cdot\partial_z^{(3)}G_k(\upsilon,0) + 3\partial_\rho\nu\cdot\partial_{zz}G_k(\upsilon,0).$$

Taking derivatives in expression (3.11) and omitting the dependence on k, we compute the induced metric on $\Omega\cap\mathcal{N}_k$, which takes the form

$$\widetilde{g} = \begin{pmatrix} |\partial_\rho\gamma|^2 & 0 & 0 \\ 0 & |\partial_\upsilon\gamma|^2| & 0 \\ 0 & 0 & 1 \end{pmatrix} + z\begin{pmatrix} -2\widetilde{M} & -2\widetilde{N} & \langle\partial_\rho\gamma;q_1\rangle \\ -2\widetilde{N} & -2\widetilde{R} & \langle\partial_\upsilon\gamma;q_1\rangle \\ \langle\partial_\rho\gamma;q_1\rangle & \langle\partial_\upsilon\gamma;q_1\rangle & 0 \end{pmatrix} + \mathcal{O}(z^2), \tag{3.12}$$

where

$$-\widetilde{M} = \langle\partial_\rho\gamma;\partial_\rho\nu\rangle, \quad -\widetilde{R} = \langle\partial_\upsilon\gamma;\partial_\upsilon\nu\rangle, \quad -2\widetilde{N} = \langle\partial_\rho\gamma;\partial_\upsilon\nu\rangle + \langle\partial_\upsilon\gamma;\partial_\rho\nu\rangle,$$

where all the entries of the matrices above are evaluated at (s,υ).

The inverse of the metric $\widetilde{g}$, $\widetilde{g}^{-1} = (\widetilde{g}^{ij})_{3\times 3}$, has the asymptotic expression

$$\widetilde{g}^{-1} = \begin{pmatrix} |\partial_\rho \gamma|^{-2} & 0 & 0 \\ 0 & |\partial_\upsilon \gamma|^{-2}| & 0 \\ 0 & 0 & 1 \end{pmatrix} + z \begin{pmatrix} 2|\partial_\rho\gamma|^{-4}\widetilde{M} & 2|\partial_\rho\gamma|^{-2}|\partial_\upsilon\gamma|^{-2}\widetilde{N} & -|\partial_\rho\gamma|^{-2}\langle\partial_\rho\gamma; q_1\rangle \\ 2|\partial_\rho\gamma|^{-2}|\partial_\upsilon\gamma|^{-2}\widetilde{N} & 2|\partial_\upsilon\gamma|^{-4}\widetilde{R} & -|\partial_\upsilon\gamma|^{-2}\langle\partial_\upsilon\gamma; q_1\rangle \\ -|\partial_\rho\gamma|^{-2}\langle\partial_\rho\gamma; q_1\rangle & -|\partial_\upsilon\gamma|^{-2}\langle\partial_\upsilon\gamma; q_1\rangle & 0 \end{pmatrix} + \mathcal{O}(z^2). \quad (3.13)$$

Following [19], we denote

$$\begin{aligned} &l_1(\upsilon) = |\partial_\upsilon \gamma(0,\upsilon)| > 0, \quad l_2(\upsilon) = |\partial_\rho\gamma(0,\upsilon)| > 0, \quad I(\upsilon) = l_2(\upsilon)^2\,\partial_{zz}G_k(\upsilon, 0), \\ &A(\upsilon) = \langle\partial_{\rho\rho}\gamma(0,\upsilon); \partial_\rho\gamma(0,\upsilon)\rangle, \quad C(\upsilon) = \langle\partial_{\rho\upsilon}\gamma(0,\upsilon); \partial_\upsilon\gamma(0,\upsilon)\rangle, \\ &R(\upsilon) = \langle\partial_{\upsilon\upsilon}\gamma(0,\upsilon); \partial_\upsilon\gamma(0,\upsilon)\rangle, \quad E(\upsilon) = \langle\partial_{\rho\upsilon}\gamma(0,\upsilon); \partial_\rho\gamma(0,\upsilon)\rangle. \end{aligned}$$

Consider again the function $h \in C^2(\mathcal{M})$, written in the coordinates (3.8) as $h = h(\rho, \upsilon)$. Taking dilated and translated modified Fermi coordinates

$$\widetilde{X}_{\alpha,h}(s,\theta,t) = \alpha^{-1}\widetilde{X}(\alpha s, \alpha\theta, \alpha(t + h(\alpha s, \alpha\theta)))$$

for

$$0 < s < \frac{\delta}{\alpha}, \quad \theta \in \Big(0, \frac{l_k}{\alpha}\Big), \quad |z| < \frac{\eta}{\alpha},$$

and after a series of lengthy, but necessary computations, we arrive to the expressions

$$\begin{aligned} \Delta_{\mathcal{M}_\alpha} = {}& \frac{1}{l_1^2(\alpha\theta)}\partial_{\theta\theta} + \frac{1}{l_2^2(\alpha\theta)}\partial_{ss} - 2\alpha s\frac{A(\alpha\theta)}{l_2^4(\alpha\theta)}\partial_{ss} + \alpha\Big(\frac{C(\alpha\theta)}{l_1^2(\alpha\theta)l_2^2(\alpha\theta)} - \frac{A(\alpha\theta)}{l_2^4(\alpha\theta)}\Big)\partial_s \\ &+ \alpha\Big(\frac{E(\alpha\theta)}{l_1^2(\alpha\theta)l_2^2(\alpha\theta)} - \frac{R(\alpha\theta)}{l_1^4(\alpha\theta)}\Big)\partial_\theta + \mathcal{O}(\alpha^2) \quad \text{in } \alpha^{-1}\mathcal{N}_k \cap M_{k,\alpha}, \end{aligned} \quad (3.14)$$

and in the coordinates $\widetilde{X}_{\alpha,h}$ in the set $\alpha^{-1}(\Omega\cap\mathcal{N}_k)$,

$$\alpha^2\Delta_{\widetilde{X}} = \Delta_{\widetilde{X}_{\alpha,h}} = \partial_{tt} + \Delta_{\mathcal{M}_\alpha} - \alpha^2|A_{\mathcal{M}}|^2\,t\partial_t - \alpha^2\{\Delta_{\mathcal{M}}h + |A^2_{\mathcal{M}}h\}\partial_t + D_0 + \widetilde{D}_{\alpha,h}. \quad (3.15)$$

The asymptotic expressions for $D_0, \widetilde{D}_{\alpha,h}$ read as follows:

$$\begin{aligned} \widetilde{D}_0 = {}& -2\alpha\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}(t+h)(\partial_{st} - \alpha\partial_\rho h\partial_{tt}) - \alpha l_1^{-2}(\alpha\theta)\partial_{\upsilon\upsilon}h\partial_{\theta t} - \alpha l_2^{-2}(\alpha\theta)\partial_{\rho\rho}h\partial_{st} \\ &+ \alpha^2 l_1^{-2}(\alpha\theta)|\partial_\upsilon h|^2\partial_{tt} + \alpha^2 l_2^{-2}(\alpha\theta)|\partial_\rho h|^2\partial_{tt} - 2\alpha^3 s l_2^{-4}(\alpha\theta)A(\alpha\theta)|\partial_\rho h|^2\partial_{tt} \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \widetilde{D}_{\alpha,h} = {}& \alpha^2\widetilde{a}_1(\partial_{\theta t} - \alpha\partial_\upsilon h\partial_{tt}) + \alpha^2\widetilde{a}_2(\partial_{st} - \alpha\partial_\rho h\partial_{tt}) \\ &+ \alpha^2\widetilde{b}_1(\partial_\theta - \alpha\partial_\upsilon h\partial_t) + \alpha^2\widetilde{b}_2(\partial_s - \alpha\partial_\rho h\partial_t) \\ &+ \alpha^3(t+h)^2\widetilde{b}_3(\alpha s, \alpha\theta, \alpha(t+h))\partial_t + \alpha^4\widetilde{R}_\alpha, \end{aligned} \quad (3.17)$$

where the functions $\widetilde{a}_1, \widetilde{a}_2, \widetilde{b}_1, \widetilde{b}_2, \widetilde{b}_3$ are smooth with bounded derivatives, and $\widetilde{R}_\alpha$ is a differential operator having C^1 dependence on h and its derivatives.

Next, we turn our attention to the boundary condition. Since at $\partial\Omega \cap \mathcal{N}_k$, the vectors $\frac{\partial \widetilde{X}}{\partial \upsilon}$, $\frac{\partial \widetilde{X}}{\partial z}$ span the tangent space of $\partial\Omega$ and

$$\Big\langle \frac{\partial \widetilde{X}}{\partial \upsilon}; n \Big\rangle = 0, \quad \Big\langle \frac{\partial \widetilde{X}}{\partial z}; n \Big\rangle = 0, \quad \langle n; n\rangle = 1$$

along the curve C_k, we can write

$$n = \sqrt{\widetilde{g}^{11}} \frac{\partial \widetilde{X}}{\partial s} + \frac{\widetilde{g}^{12}}{\sqrt{\widetilde{g}^{11}}} \frac{\partial \widetilde{X}}{\partial \upsilon} + \frac{\widetilde{g}^{13}}{\sqrt{\widetilde{g}^{11}}} \frac{\partial \widetilde{X}}{\partial z}.$$

It can be check directly from (3.12)–(3.13) that the boundary condition reads as

$$\frac{\partial}{\partial n} = \sqrt{\widetilde{g}^{11}} \partial_s + \frac{\widetilde{g}^{12}}{\sqrt{\widetilde{g}^{11}}} \partial_\upsilon + \frac{\widetilde{g}^{13}}{\sqrt{\widetilde{g}^{11}}} \partial_z,$$

so that, after dilating and translating with the coordinates $\widetilde{X}_{\alpha,h}(s,\theta,t)$, we find that the boundary condition becomes

$$\begin{aligned} \frac{\partial}{\partial n} = &-\partial_s + \alpha I(\alpha\theta) t\partial_t + \alpha\{\partial_\rho h + I(\alpha\theta)h\}\partial_t - [2\alpha(t+h)m_1(\alpha\theta) + \alpha^2(t+h)^2 \widetilde{d}_1(\alpha\theta)]\partial_s \\ &+ [\alpha(t+h)m_2(\alpha\theta) + \alpha^2 \widetilde{d}_2(\alpha\theta, \alpha(t+h))]\partial_\theta [\alpha^2(t+h)^2 I(\alpha\theta) m_1(\alpha\theta) \\ &+ 2\alpha^2 m_1(\alpha\theta)\partial_\rho h(t+h) - \alpha^2 m_2(\theta)\partial_\upsilon h(t+h) + \alpha^3 \widetilde{B}_\alpha]\partial_t, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} m_1(\upsilon) &= |l_2(\upsilon)|^{-2} < \langle \partial_\rho \nu(0,\upsilon); \partial_\rho \gamma(0,\upsilon)\rangle, \\ m_2(\upsilon) &= |l_1(\upsilon)|^{-2} (\langle \partial_\upsilon \nu(0,\upsilon); \partial_\rho \gamma(0,\upsilon)\rangle + \langle \partial_\rho \nu(0,\upsilon); \partial_\upsilon \gamma(0,\upsilon)\rangle) \end{aligned}$$

and $\widetilde{d}_1(\upsilon)$, $\widetilde{d}_2(\upsilon, z)$, $\widetilde{B}_\alpha = \widetilde{B}_\alpha(\theta, t, h, \nabla_{\mathcal{M}} h)$ has C^1 dependence on its variables.

We remark that in the case that M is the catenoid and Ω is an axially symmetric domain containing a catenoidal portion, following the scheme in [9, 26], one can parameterize with only one set of coordinates and the calculations reduce considerably.

4 Approximation of the Solution

The proof of our main result relies on a Lyapunov-Schmidt procedure near an almost solution of equation (1.1). This section is devoted to find an accurate global approximation to perform this reduction.

For this, denote $f(u) := u(1-u^2)$ and we consider

$$w(t) = \tanh\Big(\frac{t}{\sqrt{2}}\Big), \quad s \in \mathbb{R},$$

the solution of the ordinary differential equation (ODE for short)

$$w''(s) + w(1-w^2) = 0, \quad s \in \mathbb{R}, \quad w(0) = 0, \quad w' > 0, \quad w(\pm\infty) = \pm 1,$$

which has the asymptotics

$$\begin{aligned} w(t) &= 1 - 2\,e^{-\sqrt{2}\,t} + \mathcal{O}(e^{-2\sqrt{2}|t|}), \quad t > 1, \\ w(t) &= -1 + 2\,e^{\sqrt{2}\,t} + \mathcal{O}(e^{-2\sqrt{2}|t|}), \quad t < -1, \\ w'(t) &= 2\sqrt{2}\,e^{-\sqrt{2}\,|t|} + \mathcal{O}(e^{-2\sqrt{2}|t|}), \quad |t| > 1. \end{aligned} \tag{4.1}$$

Set $\Omega_\alpha := \alpha^{-1}\Omega$, $\partial\Omega_\alpha = \alpha^{-1}\partial\Omega$ and recall that $\mathcal{M}_\alpha = \alpha^{-1}\mathcal{M}$. After a rescaling, we are led to consider the problem

$$\Delta u + f(u) = 0 \quad \text{in } \Omega_\alpha, \quad \frac{\partial u}{\partial n_\alpha} = 0 \quad \text{on } \partial\Omega_\alpha, \tag{4.2}$$

where n_α stands for the inward unit normal vector to $\partial\Omega_\alpha$.

In what follows for a function $U = U(x)$, we denote

$$S(U) := \Delta U + f(U).$$

4.1 The inner approximation

Let $p > 2$ and take $h \in W^{2,p}(\mathcal{M})$ satisfying the a priori estimate

$$\|D^2 h\|_{L^p(\mathcal{M})} + \|\nabla h\|_{L^\infty(\mathcal{M})} + \|h\|_{L^\infty(\mathcal{M})} \le \mathcal{K}\alpha, \tag{4.3}$$

where the constant $\mathcal{K}$ is going to be chosen large but independent of $\alpha > 0$.

Using the coordinates $X_{\alpha,h}$ in the set $\mathcal{N}_{\alpha,h}$ described in (3.7), we set as first local approximation

$$u_0(x) = w(t), \quad x = X_{\alpha,h}(y,\theta,t) \in \alpha^{-1}(\Omega \cap \mathcal{N}).$$

When computing the error created by u_0 using (3.5)–(3.6), in $\alpha^{-1}(\Omega \cap \mathcal{N})$, we find that

$$\begin{aligned} S(u_0) = &-\alpha^2\{\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h\} w'(t) - \alpha^2 |A_{\mathcal{M}}|^2 t w'(t) + \alpha^2 \partial_i h \partial_j h w''(t) \\ &- \alpha^3 (t+h) a_{ij}^1(\alpha y, \alpha(t+h))(\partial_{ij} h w'(t) - \partial_i h \partial_j h w''(t)) \\ &- \alpha^3 b_i^1(\alpha y, \alpha(t+h)) \partial_i h w'(t) - \alpha^4 (t+h) b_3^1(\alpha y, \alpha(t+h)) w'(t), \end{aligned} \tag{4.4}$$

where $|A_{\mathcal{M}}|, h, \partial_i h, \partial_{ij} h$ are evaluated at αy.

Observe that if we take $h = 0$, the size and behavior of the error in expression (4.4) is given by

$$-\alpha^2 |A_{\mathcal{M}}|^2 t w'(t) + \alpha^4 b_3^1(\alpha y, \alpha t) t^3 w'(t).$$

As in [10], due to the presence of the $\mathcal{O}(\alpha^2)$ term, we need to improve this approximation. Hence we consider the function $\psi_1(t)$ solving the ODE

$$\partial_{tt}\psi_1(t) + F'(w(t))\psi_1(t) = t w'(t), \quad t \in \mathbb{R}. \tag{4.5}$$

Using variations of parameters formula and the fact that

$$\int_{\mathbb{R}} t (w'(t))^2 \mathrm{d}t = 0,$$

we obtain that $\psi_1(t)$ given by the formula

$$\psi_1(t) = -w(t) \int_0^t w'(s)^{-2} \int_s^\infty \xi w'(\xi)^2 \, \mathrm{d}\xi \mathrm{d}s,$$

from where it follows that for any $\sigma \in (0, \sqrt{2})$,

$$\|\mathrm{e}^{\sigma|t|} \partial_t^{(j)} \psi_1\|_{L^\infty(\mathbb{R})} \le C_j, \quad j \in \mathbb{N}.$$

So, we consider as a second approximation in the region $\alpha^{-1}(\Omega \cap \mathcal{N})$ the function

$$u_1(x) \;=\; w(t) \;+\; \phi_1(y,t), \tag{4.6}$$

where in the coordinates $X_{\alpha,h}$

$$\phi_1(y,t) \;=\; \alpha^2|A_{\mathcal{M}}(\alpha y)|^2\,\psi_1(t).$$

Computing the inner error of this new approximation, we find that

$$\begin{aligned} S(u_1) &= \Delta\phi_1 + f(w(t))\phi_1 + S(u_0) + f(w(t)+\phi_1) - f(w(t)) - f'(w(t))\phi_1 \\ &= -\alpha^2\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}w'(t) + \alpha^2\partial_i h\partial_j h w''(t) \\ &\quad - \alpha^3(t+h)a^1_{ij}(\alpha y, \alpha(t+h))(\partial_{ij}hw'(t) - \partial_i h\partial_j h w''(t)) \\ &\quad - \alpha^3 b^1_i(\alpha y, \alpha(t+h))\partial_i h w'(t) - \alpha^4(t+h)b^1_3(\alpha y, \alpha(t+h))w'(t) \\ &\quad + \alpha^4\Delta_{\mathcal{M}}(|A_{\mathcal{M}}|^2)\psi_1(t) - \alpha^4\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}|A_{\mathcal{M}}|^2\psi(t) + \alpha^4|A_{\mathcal{M}}|^4 t\partial_t\psi_1(t) \\ &\quad - 2\alpha^4 a^0_{ij}(\alpha y)\partial_i h\partial_j(|A_{\mathcal{M}}|^2)\partial_t\psi_1(t) + \alpha^4 a^0_{ij}(\alpha y)\partial_i h\partial_j h|A_{\mathcal{M}}|^2\partial_{tt}\psi_1(t)N(\phi_1) \\ &\quad + \alpha^5 R_{1,\alpha}(\alpha y, t, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h), \end{aligned} \tag{4.7}$$

where

$$N(\phi_1) = f(w(t))\phi_1 + S(u_0) + f(w(t)+\phi_1) - f(w(t)) - f'(w(t))\phi_1 \sim \mathcal{O}(\alpha^4 \mathrm{e}^{-\sigma|t|}),$$

and the differential operator $R_{1,\alpha}$ has C^1 dependence on all of its variables with

$$|\nabla R_{1,\alpha}| + |R_{1,\alpha}| \le C\mathrm{e}^{-\sigma|t|}, \quad \text{for } 0<\sigma<\sqrt{2}.$$

From the error (4.7), we see that in the open neighborhood of $\mathcal{M}$, $\alpha^{-1}(\Omega\cap\mathcal{N})$

$$|S(u_1) + \alpha^2\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}w'(t)| \le C\alpha^4\mathrm{e}^{-\sigma|t|}. \tag{4.8}$$

4.2 Boundary correction

It is clear that our approximation u_1 can be defined in the set $\alpha^{-1}(\Omega\cap\mathcal{N})$, but u_1 does not satisfy in general the boundary condition. In this regard, we need to make a further improvement of the approximation u_1 by adding boundary correction terms.

Let us consider a smooth cut-off function β such that

$$\beta(\rho) = \begin{cases} 1, & 0\le\rho<\dfrac{\delta}{2}, \\ 0, & \rho>\delta, \end{cases}$$

where $\delta>0$ is the constant in (3.8).

For the k-th end of $\alpha^{-1}M$, $M_{k,\alpha}$, we consider a cut-off function $\beta_\alpha = \beta_{k,\alpha}(x) = \beta(\alpha s)$ for $x = \widetilde{X}_{\alpha,h}(s,\theta,t)$.

Near the boundary, we consider an approximation of the form

$$u_2(x) = u_1(x) + \sum_{k=1}^{m}(\beta_{\alpha,k}\phi_{2,k}(x) + \beta_{\alpha,k}\phi_{3,k}(x)),$$

where $\phi_{2,k}$, $\phi_{3,k}(x)$ are defined in $\alpha^{-1}(\Omega\cap\mathcal{N}_k)$ and will be chosen of order $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ respectively.

We first compute the error of the boundary condition created by the approximation u_2 using the coordinates $\widetilde{X}_{\alpha,h}$ and expression (3.18) for a fixed end M_k. We again omit the explicit dependence of k, but noticing that the developments in this part hold true regardless of the end we are working with, since the supports of the cut-off functions $\beta_{\alpha,k}$ within the region $\mathcal{N}_{\alpha,h}$ close to every end $M_{k,\alpha}$ are far away from each other.

From (4.3), we know that $h = \mathcal{O}_{W^{2,p}(\mathcal{M})}(\alpha)$. Splitting the boundary error in powers of α, we find from (3.18),

$$\begin{aligned}\widetilde{B}(u_2) = & -\partial_s\phi_2 + \alpha I(\alpha\theta)tw'(t) - \partial_s\phi_3 + \alpha(\partial_\rho h_1 + I(\alpha\theta)h_1)w'(t)\\ & - \alpha^2 I(\alpha\theta)m_1(\alpha\theta)t^2 w'(t) + \alpha I(\alpha\theta)t\partial_t\phi_2 - 2\alpha m_1(\alpha\theta)t\partial_s\phi_2\\ & - \alpha^3\partial_\rho(|A_{\mathcal{M}}|^2)\psi_1(t) + \alpha^3 I(\alpha\theta)|A_{\mathcal{M}}|^2\partial_t\psi_1(t)\\ & + \alpha\{\partial_\rho h + I(\alpha\theta)h\}\partial_t\phi_2 - 2m_1(\alpha\theta)h\partial_s\phi_2 - \alpha^2(t+h)^2\widetilde{d}_1(\alpha\theta)\partial_s\phi_2\\ & - 2\alpha^2 I(\alpha\theta)m_1(\alpha\theta)hw'(t) - \alpha^2 I(\alpha\theta)m_1(\alpha\theta)t^2\partial_t\phi_2 + \alpha^4\widetilde{B}_{0,\alpha},\end{aligned} \tag{4.9}$$

where the term $\widetilde{B}_{0,\alpha}$ satisfies that

$$|\nabla\widetilde{B}_{0,\alpha}| + |\widetilde{B}_{0,\alpha}| \le C\mathrm{e}^{-\sigma|t|}.$$

Our goal is to get a boundary error of order $\mathcal{O}(\alpha^3\mathrm{e}^{-\sigma}|t|)$ by choosing h, ϕ_2, ϕ_3 satisfying

$$\begin{aligned}& -\partial_s\phi_2 + \alpha I(\alpha\theta)tw'(t) + \alpha(\partial_\rho h_1 + I(\alpha\theta)h_1)w'(t) - \partial_s\phi_3\\ & -\alpha^2 I(\alpha\theta)m_1(\alpha\theta)t^2 w'(t) + \alpha I(\alpha\theta)t\partial_t\phi_2 - 2\alpha m_1(\alpha\theta)t\partial_s\phi_2 = 0.\end{aligned}$$

We do this step by step. First, let us choose ϕ_2 solving the equation

$$\begin{aligned}&\partial_{tt}\phi_2 + \Delta_{\mathcal{M}_\alpha}\phi_2 + f'(w(t))\phi_2 = 0 \quad \text{in } \mathcal{M}_\alpha\times\mathbb{R},\\ &\partial_s\phi_2 = \alpha I(\alpha\theta)tw'(t),\end{aligned}$$

from where we obtain that $\phi_2(\cdot,t)$ is odd in the variable t, and from Proposition 6.1 it follows that in the norms described in (5.2)–(5.3) and for $p > 3$ and $\sigma \in (0,\sqrt{2})$, there exists $C > 0$ such that

$$\|D^2\phi_2\|_{p,\sigma} + \|\mathrm{e}^{\sigma|t|}\nabla\phi_2\|_{L^\infty(\mathcal{M}_\alpha\times\mathbb{R})} + \|\mathrm{e}^{\sigma|t|}\phi_2\|_{L^\infty(\mathcal{M}_\alpha\times\mathbb{R})} \le C\alpha.$$

To choose ϕ_3, we make the decomposition

$$\begin{aligned}&t^2 w'(t) = c_1\, w'(t) + g_1(t), \quad c_1 = \|w'\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}} t^2(w'(t))^2\mathrm{d}t,\\ &t\partial_s\phi_2(\cdot,t) = c_2(\cdot)w'(t) + g_2(\cdot,t), \quad \int_{\mathbb{R}} g_2(\cdot,t)w'(t)\mathrm{d}t = 0,\end{aligned} \tag{4.10}$$

and we write the second line in (4.9) as

$$\begin{aligned}& + \alpha(\partial_\rho h + I(\alpha\theta)h)w'(t) - \alpha^2 c_1 I(\alpha\theta)m_1(\alpha\theta)w'(t) - 2\alpha m_1(\alpha\theta)c_2(\theta)w'(t)\\ & - \alpha^2 I(\alpha\theta)m_1(\alpha\theta)g_1(t) - 2\alpha m_1(\alpha\theta)g_2(\theta,t) + \alpha I(\alpha\theta)t\partial_t\phi_2.\end{aligned}$$

Hence, let ϕ_3 satisfy the boundary condition on $\partial\mathcal{M}_\alpha\times\mathbb{R}$ written in coordinates $\widetilde{X}_{\alpha,h}(s,\theta,t)$

$$\partial_s\phi_3(0,\theta,t) = -\alpha^2 I(\alpha\theta)m_1(\alpha\theta)g_1(t) - 2\alpha m_1(\alpha\theta)g_2(\theta,t) + \alpha I(\alpha\theta)t\partial_t\phi_2. \tag{4.11}$$

Next we compute the error of the approximation u_2 near the boundary. Denote

$$\begin{aligned}S_{\mathrm{out}}(u_2) := & S(u_1) + \Delta\phi_2 + f'(w(t))\phi_2 + \Delta\phi_3 + f'(w(t))\phi_3 + \frac{1}{2}f''(w(t))(\phi_2+\phi_3)^2\\ & + [f'(u_1) - f'(w(t))](\phi_2+\phi_3) + \frac{1}{2}[f''(u_1) - f''(w(t))](\phi_2+\phi_3)^2\\ & + \Big[f(u_1+\phi_2+\phi_3) - f(u_1) - f'(u_1)(\phi_2+\phi_3) - \frac{1}{2}f''(u_1)(\phi_2+\phi_3)^2\Big].\end{aligned}$$

Using our choice for ϕ_2 and expression (3.15), in coordinates $\widetilde{X}_{\alpha,h}(s,\theta,t)$ this error is written as

$$\begin{aligned}
S_{\text{out}}(u_2) = {} & S(u_1) + \partial_{tt}\phi_3 + \Delta_{\mathcal{M}_\alpha}\phi_3 + f'(w(t))\phi_3 - \alpha^2|A_{\mathcal{M}}|^2 t\partial_t\phi_2 \\
& + \frac{1}{2}f''(w(t))\phi_2^2 - 2\alpha\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}t\partial_{st}\phi_2 \\
& - \alpha^2\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}\partial_t\phi_2 - 2\alpha\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}h\partial_{st}\phi_2 \\
& + 2\alpha^2\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}(t+h)\partial_\rho h\partial_{tt}\phi_2 \\
& - \alpha l_1^{-2}(\alpha\theta)\partial_{vv}h^2\partial_{\theta t}\phi_2 - \alpha l_2^{-2}(\alpha\theta)\partial_{\rho\rho}h\partial_{st}\phi_2 + \alpha^2 l_1^{-2}(\alpha\theta)|\partial_v h|^2\partial_{tt}\phi_2 \\
& + \alpha^2 l_2^{-2}(\alpha\theta)|\partial_{\rho\rho}h|^2\partial_{tt}\phi_2 - 2\alpha^3 s l_2^{-4}(\alpha\theta)A(\alpha\theta)|\partial_\rho h|^2\partial_{tt}\phi_2 \\
& + \alpha^2\widetilde{a}_1(\alpha s,\alpha\theta,\alpha(t+h))\{\partial_{\theta t}\phi_2 - \alpha\partial_v h\partial_{tt}\phi_2\} \\
& + \alpha^2\widetilde{a}_2(\alpha s,\alpha\theta,\alpha(t+h))\{\partial_{st}\phi_2 - \alpha\partial_\rho h\partial_{tt}\phi_2\} \\
& + \alpha^2\widetilde{b}_1(\alpha s,\alpha\theta,\alpha(t+h))\{\partial_\theta\phi_2 - \alpha\partial_v h\partial_t\phi_2\} \\
& + \alpha^2\widetilde{b}_2(\alpha s,\alpha\theta,\alpha(t+h))\{\partial_s\phi_2 - \alpha\partial_\rho h\partial_t\phi_2\} \\
& + f'(w(t))(\phi_2\cdot\phi_3 + \frac{1}{2}\phi_3^2) + [f'(u_1) - f'(w(t))](\phi_2+\phi_3) \\
& + \alpha^3(t+h)\widetilde{b}_3(\alpha s,\alpha\theta,\alpha(t+h))\partial_t\phi_2 + \widetilde{R}_{2,\alpha},
\end{aligned} \tag{4.12}$$

where $\widetilde{R}_{2,\alpha} = \widetilde{R}_{2,\alpha}(\alpha s,\alpha\theta,t,h,\nabla_{\mathcal{M}}h,D^2_{\mathcal{M}}h) = \mathcal{O}(\alpha^4)$ and

$$|D\widetilde{R}_{2,\alpha}| + |\widetilde{R}_{2,\alpha}| \le C\alpha^4.$$

Notice that

$$\begin{aligned}
S(u_2) = {} & (1-\beta_\alpha)S(u_1) + \beta_\alpha S_{out}(u_2) + 2\nabla\beta_\alpha\cdot\nabla\phi_2 + 2\nabla\beta_\alpha\cdot\nabla\phi_3 \\
& + (\phi_2+\phi_3)\Delta\beta_\alpha f(u_1+\beta_\alpha(\phi_2+\phi_3)) - f(u_1) \\
& - \beta_\alpha f(u_1+\phi_2+\phi_3) - (1-\beta_\alpha)f(u_1)
\end{aligned}$$

and

$$\begin{aligned}
& \nabla\beta_\alpha\cdot\nabla\phi_2 = \alpha\partial_s\beta(\alpha s)\partial_s\phi_2 = \mathcal{O}(\alpha^2 \mathrm{e}^{-\sigma|t|}) \\
& f(u_1+\beta_\alpha(\phi_2+\phi_3)) - f(u_1) - \beta_\alpha f(u_1+\phi_2+\phi_3) \\
& - (1-\beta_\alpha)f(u_1) = \mathcal{O}(\alpha^4\mathrm{e}^{-\sigma|t|}).
\end{aligned} \tag{4.13}$$

Hence in order to improve the approximation, we need to get rid of the terms in the first line of (4.12) and the term in (4.13). Let ϕ_3 solve the linear problem

$$\begin{aligned}
& \partial_{tt}\phi_3 + \Delta_{\mathcal{M}_\alpha}\phi_3 + f'(w(t))\phi_3 = \alpha^2|A_{\mathcal{M}}|^2 t\partial_t\phi_2 - \frac{1}{2}f''(w(t))\phi_2^2 + 2\alpha\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}t\partial_{st}\phi_2 \\
& \qquad\qquad - \alpha\partial_s\beta(\alpha s)\partial_s\phi_2 \quad \text{in } \mathcal{M}_\alpha\times\mathbb{R}, \\
& \partial_s\phi_3(0,\theta,t) = -\alpha^2 I(\alpha\theta)m_1(\alpha\theta)g_1(t) - 2\alpha m_1(\alpha\theta)g_2(\theta,t) + \alpha I(\alpha\theta)t\partial_t\phi_2, \\
& \int_{\mathbb{R}}\phi_3(\cdot,t)w'(t)\mathrm{d}t = 0 \quad \text{in } M_\alpha.
\end{aligned}$$

From Proposition 6.1, ϕ_3 satisfies

$$\|D^2\phi_3\|_{p,\sigma} + \|e^{\sigma|t|}\nabla\phi_3\|_{L^\infty(\mathcal{M}_\alpha\times\mathbb{R})} + \|e^{\sigma|t|}\phi_3\|_{L^\infty(\mathcal{M}_\alpha\times\mathbb{R})} \le C\alpha^2.$$

From expression (4.12), we directly check that

$$S(u_2) = S(u_1) + E_{0,\alpha} + E_{1,\alpha}, \tag{4.14}$$

where $E_{i,\alpha} = E_{i,\alpha}(\alpha s, \alpha\theta, t, h, \nabla_{\mathcal{M}} h, D^2_{\mathcal{M}} h) = \mathcal{O}(\alpha^{3+i})$ and

$$|DE_{i,\alpha}| + |E_{i,\alpha}| \le C\alpha^{3+i}, \quad i = 0, 1.$$

From (4.9) and (4.11), the boundary error takes the form

$$\begin{aligned}\widetilde{B}(u_2) &= \alpha(\partial_\rho h_1 + I(\alpha\theta)h_1)w'(t) - \alpha^2 c_1 I(\alpha\theta)m_1(\alpha\theta)w'(t) - 2\alpha m_1(\alpha\theta)c_2(\theta)w'(t)\\ &\quad + \widetilde{B}_{1,\alpha} + \widetilde{B}_{2,\alpha},\end{aligned} \tag{4.15}$$

where c_1, c_2 are described in (4.10), $\|c_2\|_{L^\infty(\partial\mathcal{M}_\alpha)} \le C\alpha$ and

$$\widetilde{B}_{i,\alpha} = \widetilde{B}_{i,\alpha}(\alpha s, \alpha\theta, t, h, \nabla_{\mathcal{M}} h) = \mathcal{O}(\alpha^{2+i}), \quad i = 1, 2.$$

Finally, to get the right size of the boundary error in (4.9), we impose on h the boundary condition

$$\partial_\rho h + I(\upsilon)h = \alpha c_1 I(\upsilon)m_1(\upsilon) + 2m_1(\upsilon)c_2\Big(\frac{\upsilon}{\alpha}\Big),$$

and pulling back the rescaling in $\alpha > 0$,

$$\Big\|\alpha c_1 I m_1 + 2m_1 c_2\Big(\frac{\cdot}{\alpha}\Big)\Big\|_{L^\infty(\partial\mathcal{M})} \le C\alpha.$$

4.3 Global approximation

Observe that so far, our approximation is defined only in the open set $\alpha^{-1}(\Omega\cap\mathcal{N})$.

The idea to get a global approximation is to interpolate the approximation u_2 well inside $\mathcal{N}_{\alpha,h}$, with the function

$$\mathbb{H}(x) := \begin{cases} +1, & x \in \alpha^{-1}S^+,\\ -1, & x \in \alpha^{-1}S^- \end{cases} \tag{4.16}$$

outside $\mathcal{N}_{\alpha,h}$.

Let us take a non-negative function $\widehat{\beta} \in C^\infty(\mathbb{R})$ such that

$$\widehat{\beta}(s) = \begin{cases} 1, & |s| \le 1,\\ 0, & |s| \ge 2, \end{cases}$$

and consider the following cut-off function in $\mathcal{N}_{\alpha,h}$ given by

$$\beta_\eta(x) = \widehat{\beta}\Big(|t + h(\alpha y)| - \frac{\eta}{\alpha} + 2\Big), \quad x = X_{\alpha,h}(y,t) \in \mathcal{N}_{\alpha,h}.$$

With the aid of this, we set up as approximation in Ω_α the function

$$U(x) = \beta_\eta(x)u_2(x) + (1 - \beta_\eta(x)\mathbb{H}(x), \quad x \in \Omega_\alpha. \tag{4.17}$$

We compute the new error created by this approximation as follows:

$$S(U) \,=\, \Delta U \,+\, f(U) \,=\, \beta_\eta S(u_2) + E,$$

where

$$E := f(\beta_\eta U) - \beta_\eta f(U) + 2\nabla\beta_\eta \cdot \nabla u + u\Delta\beta_\eta.$$

Using $z = |t + h(\alpha y)|$, we see that the derivatives of β_η do not depend on the derivatives of h. On the other hand, due to the choice of β_η and the explicit form of E, the error created only takes into account the values of β_η in the set

$$x = X_{\alpha,h}(y,\theta,t) \in \mathcal{N}_{\alpha,h}, \quad |t + h(\alpha y)| \geq \frac{\eta}{\alpha} + 4\ln(r(\alpha y)) - 2,$$

so we get the following estimate for the error E:

$$|E| \leq C\mathrm{e}^{-\frac{\eta}{\alpha}}.$$

5 The Proof of Theorem 1.1

The proof of Theorem 1.1 is fairly technical. To keep the presentation as clear as possible, we sketch the steps of the proof, and in the next sections, we give the detailed proofs of the lemmas and propositions mentioned here.

We introduce suitable norms to set up an appropriate functional analytic scheme for the proof of Theorem 1.1. For $\alpha > 0$, $1 < p \leq \infty$ and a function $f(x)$, defined in Ω_α, we set

$$\|f\|_{p,\sim} := \sup_{x\in\mathbb{R}^3} \|f\|_{L^p(B_1(x))}. \tag{5.1}$$

We also consider for functions $g = g(y,t)$, $\phi = \phi(y,t)$, defined in the whole $\mathcal{M}_\alpha \times \mathbb{R}$, the norms

$$\|g\|_{p,\sigma} := \sup_{(y,t)\in\mathcal{M}_\alpha\times\mathbb{R}} \mathrm{e}^{\sigma|t|}\|g\|_{L^p(B_1(y,t);\mathrm{d}V_\alpha)}, \tag{5.2}$$

$$\|\phi\|_{2,p,\sigma} := \|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma}, \tag{5.3}$$

where $\mathrm{d}V_\alpha := \mathrm{d}y_{g_{\mathcal{M}_\alpha}}\mathrm{d}t$. In the case $p = +\infty$, we have that $L^\infty(B_1(y,t), \mathrm{d}V_\alpha) = L^\infty(B_1(y,t))$.

For a function G defined in $\partial\mathcal{M}_\alpha \times \mathbb{R}$, we set

$$\|G\|_{p,\sigma} := \|\mathrm{e}^{\sigma|t|}G\|_{L^p(\partial\mathcal{M}_\alpha\times\mathbb{R})},$$

and we recall the norm for the parameter function h

$$\|h\|_* = \|D^2h\|_{L^p(\mathcal{M})} + \|\nabla h\|_{L^\infty(\mathcal{M})} + \|h\|_{L^\infty(\mathcal{M})}. \tag{5.4}$$

We look for a solution to equation (1.1) of the form

$$u_\alpha(x) = U(x) + \varphi(x),$$

where $U(x)$ is the global approximation defined in (4.17) and φ is going to be chosen small in some appropriate sense. Thus, we need to solve the problem

$$\Delta\varphi + f'(U)\varphi + S(U) + N(\varphi) = 0,$$

or equivalently

$$\Delta\varphi + f(U)\varphi = -S(W) - N(\varphi) = -\beta_\eta S(u_2) - E - N(\varphi), \tag{5.5}$$

where

$$N(\varphi) = f(U+\varphi) - f(U) - f'(U)\varphi.$$

5.1 Gluing procedure

To solve problem (5.5), we consider again the cut-off function $\widehat{\beta}$ from Subsection 4.3, to define for every $n \in \mathbb{N}$

$$\zeta_n(x) := \begin{cases} \widehat{\beta}(|t+h(\alpha y)| - \frac{\eta}{\alpha} + n), & \text{if } x = X_{\alpha,h}(y_1,y_2,t) \in \mathcal{N}_{\alpha,h}, \\ 0, & \text{if } x \in \mathcal{N}_{\alpha,h}. \end{cases} \tag{5.6}$$

We look for a solution $\varphi(x)$ to (5.5) of the form

$$\varphi(x) = \zeta_2(x)\phi(y,t) + \psi(x),$$

where $\phi(y,t)$ is defined for every $(y,t) \in M_\alpha \times \mathbb{R}$ and $\psi(x)$ is defined in the whole Ω_α. So, we find from equation (5.5) that

$$\begin{aligned} &\zeta_2[\Delta_{\mathcal{N}_{\alpha,h}}\phi + f'(U)\phi + \zeta_1 U\psi + S(U) + \zeta_2 N(\phi+\psi)] \\ &+ \Delta\psi - [2-(1-\zeta_2)[f'(U)+2]]\psi + (1-\zeta_2)S(U) \\ &+ 2\nabla\zeta_2\cdot\nabla_{\mathcal{N}_{\alpha,h}}\phi + \phi\Delta\zeta_2 + (1-\zeta_2)N[\zeta_2\phi+\psi] = 0. \end{aligned}$$

Hence, we will have constructed a solution to equation (5.5), if solve the system

$$\Delta_{\mathcal{N}_{\alpha,h}}\phi + f'(U)\phi + \zeta_2 U\psi + S(U) + \zeta_2 N(\phi+\psi) = 0 \quad \text{in } |t+h(\alpha y)| < \frac{\eta}{\alpha} - 1, \tag{5.7}$$

$$\begin{aligned} &\Delta\psi - [2-(1-\zeta_2)[f'(U)+2]]\psi + (1-\zeta_2)S(U) \\ &+ 2\nabla\zeta_2\cdot\nabla_{\mathcal{N}_{\alpha,h}}\phi + \phi\Delta\zeta_2 + (1-\zeta_2)N[\zeta_2\phi+\psi] = 0 \quad \text{in } \Omega_\alpha. \end{aligned} \tag{5.8}$$

As for the boundary conditions, we compute

$$\beta_\eta \frac{\partial u_2}{\partial n_\alpha} + \zeta_2\frac{\partial\phi}{\partial n_\alpha} + (u_2 - \mathbb{H}(x))\frac{\partial\beta_\eta}{\partial n_\alpha} + \phi\frac{\partial\zeta_2}{\partial n_\alpha} + \frac{\partial\psi}{\partial n_\alpha} = 0.$$

Therefore, the boundary condition is reduced to the boundary system

$$\beta_\eta \frac{\partial u_2}{\partial n_\alpha} + \zeta_2\frac{\partial\phi}{\partial n_\alpha} = 0, \tag{5.9}$$

$$\frac{\partial\psi}{\partial n_\alpha} + (u_2 - \mathbb{H}(x))\frac{\partial\beta_\eta}{\partial n_\alpha} + \phi\frac{\partial\zeta_2}{\partial n_\alpha} = 0. \tag{5.10}$$

We solve first (5.8)–(5.10), using the fact that the potential $2-(1-\zeta_2)[f'(U)+2]$ is uniformly positive, so that the linear operator behaves like $\Delta - 2$. A solution $\psi = \Psi(\phi)$ is then found from the contraction mapping principle. We collect this discussion in the following lemma, that will be proven in detail in Subsection 7.1.

Proposition 5.1 *Let $3 < p \le \infty$ and $\alpha > 0$ be sufficiently small. For every h satisfying (4.3) and every ϕ such that $\|\phi\|_{2,p,\sigma} \le 1$, problem (5.8)–(5.10) has a unique solution $\psi = \Psi(\phi)$ and the operator $\Psi(\phi)$ is Lipschitz in ϕ. More precisely, $\Psi(\phi)$ satisfies that*

$$\|\psi\|_X := \|D^2\psi\|_{p,\sim} + \|D\psi\|_\infty + \|\psi\|_\infty \le C\mathrm{e}^{-\frac{c\eta}{\alpha}}, \tag{5.11}$$

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le C\mathrm{e}^{-\frac{c\eta}{\alpha}}\|\phi_1 - \phi_2\|_{2,p,\sigma}, \tag{5.12}$$

and the constant $C > 0$ depends only on p.

Next, we extend (5.7)–(5.9) to a qualitatively similar equation in $\mathcal{M}_\alpha \times \mathbb{R}$. Let us set

$$R(\phi) := \zeta_4[\Delta_{\mathcal{N}_{\alpha,h}} - \partial_{tt} - \Delta_{\mathcal{M}_\alpha}].$$

Observe that $R(\phi)$ is understood to be zero for $|t + h(\alpha y)| > \frac{\eta}{\alpha} + 2$, and so we consider the equation

$$\begin{aligned} &\partial_{tt}\phi + \Delta_{\mathcal{M}_\alpha}\phi + f'(w(t))\phi \\ &= -\widetilde{S}(u_2) - R(\phi) - (f'(u_2) - f'(w(t)))\phi - \zeta_2 u_2\psi - \zeta_2 N(\phi + \psi) \quad \text{in } \mathcal{M}_\alpha \times \mathbb{R}, \end{aligned} \tag{5.13}$$

where from expression (4.12) and omitting the depedence on k, we have on the k-th end $M_{k,\alpha}$ that

$$\begin{aligned} \widetilde{S}(u_2) = \widetilde{S}(u_1) &- \alpha^2\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}\partial_t\phi_2 - 2\alpha\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}h\partial_{st}\phi_2 \\ &+ 2\alpha^2\frac{I(\alpha\theta)}{l_1(\alpha\theta)l_2(\alpha\theta)}(t+h)\partial_\rho h\partial_{tt}\phi_2 + 2\alpha\partial_\rho\beta(\alpha s)\partial_s\phi_3 + \alpha^2\partial_{\rho\rho}\beta(\alpha s)(\phi_2 + \phi_3) \\ &- \alpha l_1^{-2}(\alpha\theta)\partial_{\upsilon\upsilon}h^2\partial_{\theta t}\phi_2 - \alpha l_2^{-2}(\alpha\theta)\partial_{\rho\rho}h\partial_{st}\phi_2 + \alpha^2 l_1^{-2}(\alpha\theta)|\partial_\upsilon h|^2\partial_{tt}\phi_2 \\ &+ \alpha^2 l_2^{-2}(\alpha\theta)|\partial_{\rho\rho}h|^2\partial_{tt}\phi_2 - 2\alpha^3 s l_2^{-4}(\alpha\theta)A(\alpha\theta)|\partial_\rho h|^2\partial_{tt}\phi_2 + f'(w(t))\Big(\phi_2\cdot\phi_3 + \frac{1}{2}\phi_3^2\Big) \\ &+ [f'(u_1) - f'(w(t))](\phi_2 + \phi_3) + \alpha^2\zeta_4\widetilde{a}_1(\alpha s, \alpha\theta, \alpha(t+h))\{\partial_{\theta t}\phi_2 - \alpha\partial_\upsilon h\partial_{tt}\phi_2\} \\ &+ \alpha^2\zeta_4\widetilde{a}_2(\alpha s, \alpha\theta, \alpha(t+h))\{\partial_{st}\phi_2 - \alpha\partial_\rho h\partial_{tt}\phi_2\} \\ &+ \alpha^2\zeta_4\widetilde{b}_1(\alpha s, \alpha\theta, \alpha(t+h))\{\partial_\theta\phi_2 - \alpha\partial_\upsilon h\partial_t\phi_2\} \\ &+ \alpha^2\zeta_4\widetilde{b}_2(\alpha s, \alpha\theta, \alpha(t+h))\{\partial_s\phi_2 - \alpha\partial_\rho h\partial_t\phi_2\} \\ &+ \alpha^3\zeta_4(t+h)\widetilde{b}_3(\alpha s, \alpha\theta, \alpha(t+h))\partial_t\phi_2 + \zeta_4\widetilde{R}_{2,\alpha}, \end{aligned} \tag{5.14}$$

and from expression (4.7), we write

$$\begin{aligned} \widetilde{S}(u_1) = &-\alpha^2\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}w'(t) + \alpha^2\partial_i h\partial_j h w''(t) + \alpha^4\Delta_{\mathcal{M}}(|A_{\mathcal{M}}|^2)\psi_1(t) \\ &- \alpha^4\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\}|A_{\mathcal{M}}|^2\psi(t) + \alpha^4|A_{\mathcal{M}}|^4 t\partial_t\psi_1(t) \\ &- 2\alpha^4 a_{ij}^0(\alpha y)\partial_i h\partial_j(|A_{\mathcal{M}}|^2)\partial_t\psi_1(t) + \alpha^4 a_{ij}^0(\alpha y)\partial_i h\partial_j h|A_{\mathcal{M}}|^2\partial_{tt}\psi_1(t) \\ &- \alpha^3\zeta_4(t+h)a_{ij}^1(\alpha y, \alpha(t+h))(\partial_{ij}hw'(t) - \partial_i h\partial_j h w''(t)) \\ &- \alpha^3\zeta_4 b_i^1(\alpha y, \alpha(t+h))\partial_i h w'(t) - \alpha^4(t+h)\zeta_4 b_3^1(\alpha y, \alpha(t+h))w'(t)N(\phi_1) \\ &+ \alpha^5\zeta_4 R_{1,\alpha}(\alpha y, t, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h). \end{aligned} \tag{5.15}$$

Observe that $\widetilde{S}(u_1)$ and $\widetilde{S}(u_2)$ coincide with $S(u_1)$, $S(u_2)$ but the parts that are not defined for all $t \in \mathbb{R}$ are cut-off outside the support of ζ_4.

As for the boundary condition, we proceed in the same fashion by writing

$$\mathcal{B} = \zeta_4\Big[\sqrt{\widetilde{g}^{11}}\frac{\partial}{\partial n_\alpha} - \partial_s\Big].$$

It suffices to consider ϕ satisfying

$$\partial_{\tau_\alpha}\phi + \mathcal{B}(\phi) = \widetilde{B}(u_2),$$

where $\tau_\alpha = s$ is the normal inward direction to $\partial\mathcal{M}_\alpha$, and in expression (4.9), we cut-off the parts that are not defined for every t. We write also for further purposes

$$\begin{aligned}\widetilde{B}(u_2) &= -\alpha^3\partial_\rho(|A_{\mathcal{M}}|^2)\psi_1(t) + \alpha^3 I(\alpha\theta)|A_{\mathcal{M}}|^2\partial_t\psi_1(t)\\ &\quad + \alpha\{\partial_\rho h + I(\alpha\theta)h\}\partial_t\phi_2 - 2\alpha m_1(\alpha\theta)h\partial_s\phi_2 - \alpha^2(t+h)^2\zeta_4\widetilde{d}_1(\alpha\theta)\partial_s\phi_2\\ &\quad - 2\alpha^2 I(\alpha\theta)m_1(\alpha\theta)hw'(t) - \alpha^2 I(\alpha\theta)m_1(\alpha\theta)t^2\partial_t\phi_2 + \alpha^4\zeta_4\widetilde{B}_{0,\alpha}. \qquad (5.16)\end{aligned}$$

Observing that again, we have omitted the depedence on the end $M_{k,\alpha}$ for notational convenience.

Next, using Proposition 5.1, we solve equation (5.13) with $\psi = \Psi(\phi)$. Let us set

$$\mathbf{N}(\phi) := R(\phi) + (f'(u_2) - f'(w(t))\phi + \zeta_2(u_2 - \mathbb{H}(x))\Psi(\phi) + \zeta_2 N(\phi + \Psi(\phi)) \quad \text{in } \mathcal{M}_\alpha \times \mathbb{R}.$$

So, we only need to solve

$$\partial_{tt}\phi + \Delta_{\mathcal{M}_\alpha}\phi + f'(w(t))\phi = -\widetilde{S}(u_2) - \mathbf{N}(\phi) + c(y)w'(t) \quad \text{in } \mathcal{M}_\alpha \times \mathbb{R}, \qquad (5.17)$$

$$\partial_{\tau_\alpha}\phi + \mathcal{B}(\phi) = \widetilde{B}(u_2) \quad \text{on } \partial\mathcal{M}_\alpha \times \mathbb{R}, \qquad (5.18)$$

$$\int_{\mathbb{R}} \phi(\cdot,t)w'(t)\mathrm{d}t = 0 \quad \text{in } \mathcal{M}_\alpha. \qquad (5.19)$$

To solve problem (5.17)–(5.19), we solve a nonlinear problem in ϕ, that basically eliminates the parts of the error, that do not contribute to the projections.

The linear theory we develop to solve problem (5.17)–(5.19), considers right-hand sides and boundary data with a behavior similar to that of the error $\widetilde{S}(u_2)$ and $\widetilde{B}(u_2)$, that as we have seen, are basically of the form $\mathcal{O}(\alpha^3 \mathrm{e}^{-\sigma|t|})$.

Using the fact that $\mathbf{N}(\phi)$ is Lipschitz with small Lipschitz constant and contraction mapping principle in a ball of radius $\mathcal{O}(\alpha^3)$ in the norm $\|\cdot\|_{2,p,\sigma}$, we solve problem (5.17)–(5.19). This solution ϕ, defines a Lipschitz operator $\phi = \Phi(h)$. This information is collected in the following proposition.

Proposition 5.2 *Assume that* $3 < p \leq \infty$ *and* $\sigma > 0$ *is small. For every* $\alpha > 0$ *small, problem* (5.17)–(5.19) *has a unique solution* $\phi = \Phi(h)$*, satisfying*

$$\|\phi\|_{2,p,\sigma} \leq C\alpha^3$$

and

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\sigma} \leq C\alpha^2\|h_1 - h_2\|_*,$$

where the constant $C > 0$ *depends only on* p.

5.2 Adjusting h, to make the projection equal zero

We denote $c_0 = \|w'\|^2_{L^2(\mathbb{R})}$. To conclude the proof of Theorem 1.1, we adjust h so that

$$c(y) = \int_{\mathbb{R}} [\widetilde{S}(u_2) + \mathbf{N}(\phi)]w'(t)\mathrm{d}t = 0.$$

Integrating (5.17) against $w'(t)$ and using that the function β_α defined in Subsection 4.2 does not depend on the variable t, we compute

$$\int_{\mathbb{R}} \widetilde{S}(u_2)w'(t)\mathrm{d}t = \underbrace{(1-\beta_\alpha)\int_{\mathbb{R}} \widetilde{S}(u_1)w'(t)\mathrm{d}t}_{A} + \underbrace{\beta_\alpha \int_{\mathbb{R}} \widetilde{S}(u_2)w'(t)\mathrm{d}t}_{B} + \mathcal{O}_{L^\infty(M_\alpha)}(\alpha^4).$$

From (5.15), we compute

$$
\begin{aligned}
\int_{\mathbb{R}} \widetilde{S}(u_1)w'(t)\mathrm{d}t = & -\alpha^2 c_0\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\} \\
& - \alpha^3 \int_{\mathbb{R}} \zeta_4(t+h) a^1_{ij}(\alpha y, \alpha(\alpha(t+h)))\{\partial_{ij}h w'(t) - \partial_i h \partial_j h w''(t)\} w'(t)\mathrm{d}t \\
& - \alpha^3 \int_{\mathbb{R}} \zeta_4(t+h) b^1_1(\alpha y, \alpha(t+h))\partial_i h (w'(t))^2 \mathrm{d}t \\
& + \alpha^4 \int_{\mathbb{R}} (t+h)^3 \zeta_4 b^1_3(\alpha y, \alpha(t+h))(w'(t))^2 \mathrm{d}t \\
& + \alpha^4 |A_{\mathcal{M}}|^2 \int_{\mathbb{R}} t\partial_t \psi_1(t) w'(t)\mathrm{d}t + \alpha^5 P_1(\alpha y, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h).
\end{aligned}
$$

On the other hand, from (5.14), the reduced error near the boundary reads as

$$
\begin{aligned}
\int_{\mathbb{R}} \widetilde{S}(u_2) = & \int_{\mathbb{R}} \widetilde{S}(u_1) - \alpha^2 |A_{\mathcal{M}}|^2 \int_{\mathbb{R}} t\partial_t \phi_2 w'(t)\mathrm{d}t - 2\alpha \frac{I(\alpha\theta)}{l_1(\alpha\theta) l_2(\alpha\theta)} h \int_{\mathbb{R}} \partial_{st}\phi_2 w'(t)\mathrm{d}t \\
& + 2\alpha^2 \frac{I(\alpha\theta)}{l_1(\alpha\theta) l_2(\alpha\theta)} \partial_\rho h \int_{\mathbb{R}} (t+h)\partial_{tt}\phi_2 w'(t)\mathrm{d}t \\
& - \alpha l_1^{-2}(\alpha\theta)\partial_{\upsilon\upsilon} h \int_{\mathbb{R}} \partial_{\theta t}\phi_2 w'(t)\mathrm{d}t - \alpha l_2^{-2}(\alpha\theta)\partial_{\rho\rho} h \int_{\mathbb{R}} \partial_{st}\phi_2 w'(t)\mathrm{d}t \\
& + \alpha^2 \int_{\mathbb{R}} \zeta_4 \widetilde{a}_1(\alpha s, \alpha\theta, t)\{\partial_{\theta t}\phi_2 - \alpha\partial_\upsilon h \partial_{tt}\phi_2\} w'(t)\mathrm{d}t \\
& + \alpha^2 \int_{\mathbb{R}} \zeta_4 \widetilde{a}_2(\alpha s, \alpha\theta, t)\{\partial_{st}\phi_2 - \alpha\partial_\rho h \partial_{tt}\phi_2\} w'(t)\mathrm{d}t \\
& + \alpha^2 \int_{\mathbb{R}} \zeta_4 \widetilde{b}_1(\alpha s, \alpha\theta, t)\{\partial_{\theta}\phi_2 - \alpha\partial_\upsilon h \partial_{t}\phi_2\} w'(t)\mathrm{d}t \\
& + \alpha^2 \int_{\mathbb{R}} \zeta_4 \widetilde{b}_2(\alpha s, \alpha\theta, t)\{\partial_{s}\phi_2 - \alpha\partial_\rho h \partial_{t}\phi_2\} w'(t)\mathrm{d}t \\
& + \alpha^3 \int_{\mathbb{R}} \zeta_4(t+h) \widetilde{b}^1_3(\alpha s, \alpha\theta, \alpha(t+h))\partial_t\phi_2 w'(t)\mathrm{d}t \\
& - 2\alpha^2 \frac{I(\alpha\theta)}{l_1(\alpha\theta) l_2(\alpha\theta)} (t+h) \int_{\mathbb{R}} \{\partial_{st}\phi_3 - \alpha\partial_{\rho\rho} h \partial_{tt}\phi_3\} w'(t)\mathrm{d}t \\
& + \int_{\mathbb{R}} [f'(u_1) - f'(w(t))](\phi_2 + \phi_3) w'(t)\mathrm{d}t + \alpha^3 \widetilde{R}_{0,\alpha}(\alpha s, \alpha\theta) \\
& + \alpha^4 \int_{\mathbb{R}} \widetilde{R}_\alpha(\alpha s, \alpha\theta, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h) w'(t)\mathrm{d}t.
\end{aligned}
$$

From assumption (4.3), and the estimates in Section 7 for the nonlocal terms, we have

$$
\mathrm{Q}(\alpha y, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h) := \int_{\mathbb{R}} \mathbf{N}(\phi) w'(t)\mathrm{d}t, \quad \|\mathrm{Q}(\cdot, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h)\|_{L^{\mathcal{M}}} \le C\alpha^{4-\frac{2}{p}}.
$$

Therefore,

$$
\begin{aligned}
\alpha^{-2} \int_{\mathbb{R}} (\widetilde{S}(u_2) + \mathbf{N}(\phi)) w'(t)\mathrm{d}t = & -c_0\{\Delta_{\mathcal{M}}h + |A_{\mathcal{M}}|^2 h\} + \alpha P_0(y, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h) \\
& + \alpha^{2-\frac{2}{p}} P_1(y, h, \nabla_{\mathcal{M}}h, D^2_{\mathcal{M}}h), \qquad (5.20)
\end{aligned}
$$

where

$$|P_0| + |P_1| \leq C, \quad |DP_0| + |DP_1| \leq C.$$

As for the boundary condition, directly from (5.16) we ask h to satisfy

$$\partial_\rho h + I(\upsilon)h = \alpha c_1 I(\upsilon) m_1(\upsilon) + 2m_1(\upsilon) c_2\Big(\frac{\upsilon}{\alpha}\Big), \tag{5.21}$$

where we recall that

$$\|c_2\|_{L^\infty(\partial\mathcal{M}_\alpha)} \leq C\alpha$$

and the right-hand side in (5.21) does not depend on h.

We solve then

$$c_0\{\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h\} = \alpha P_0(y, h, \nabla_{\mathcal{M}} h, D^2_{\mathcal{M}} h) + \alpha^{2-\frac{2}{p}} P_1(y, h, \nabla_{\mathcal{M}} h, D^2_{\mathcal{M}} h) \tag{5.22}$$

with the boundary condition (5.21) by a direct application of the theory developed in Section 2 and a fixed point argument for h in a ball or order $\mathcal{O}(\alpha)$ in the topology induced by the norm $\|\cdot\|_*$ described in (4.3). This completes the proof of our theorem.

6 Projected Linear Problem

In this part, we provide the linear theory for the problem

$$\partial_{tt}\phi + \Delta_{\mathcal{M}_\alpha}\phi + f(w(t))\phi = g + c(y)w'(t) \quad \text{in } \mathcal{M}_\alpha \times \mathbb{R}, \tag{6.1}$$

$$\frac{\partial\phi}{\partial\tau_\alpha} = G \quad \text{on } \partial\mathcal{M}_\alpha \times \mathbb{R}, \tag{6.2}$$

which relies strongly on the fact that solutions to

$$\partial_{tt}\phi + \Delta_{\mathcal{M}_\alpha}\phi + f(w(t))\phi = 0 \quad \text{in } \mathcal{M}_\alpha \times \mathbb{R},$$

$$\frac{\partial\phi}{\partial\tau_\alpha} = 0 \quad \text{on } \partial\mathcal{M}_\alpha \times \mathbb{R}$$

are the scalar multiples of $w'(t)$. The proof follows the same lines of Lemma 5.1 in [10]. We simply remark that when decomposing the solution ϕ as

$$\phi = c(y)w'(t) + \phi^\perp$$

from maximum principle one obtains that $|\phi^\perp(y,t)| \leq C\mathrm{e}^{-\sigma|t|}$ for some $0 < \sigma < \sqrt{2}$. Defining

$$\psi(y) = \int_R |\phi(y,t)^\perp|^2 \mathrm{d}t,$$

it follows that for certain positive constant λ

$$-\Delta_{\mathcal{M}_\alpha}\psi + \lambda\psi \leq 0, \quad \frac{\partial\psi}{\partial\tau_\alpha} = 0,$$

where τ_α is the inward unit normal to $\partial\mathcal{M}_\alpha$ in $\mathcal{M}_\alpha$. Clearly it follows that $\psi = 0$ and

$$\Delta_{\mathcal{M}_\alpha} c = 0 \quad \text{in } \mathcal{M}_\alpha, \qquad \frac{\partial c}{\partial\tau_\alpha} = 0 \quad \text{on } \partial\mathcal{M}_\alpha.$$

Consequently, $c(y)$ is a constant function.

Proceeding as in [9, Section 3], it suffices to solve the case $G = 0$ and $\int_{\mathbb{R}} g(\cdot, t) \cdot w'(t)\mathrm{d}t = 0$. To prove existence, we set

$$\langle \phi, \psi \rangle := \int_{\mathcal{M}_\alpha \times \mathbb{R}} \nabla\phi \cdot \nabla\psi + 2\phi \cdot \psi,$$

and we consider the space H of function $\phi \in H^1(\mathcal{M}_\alpha \times \mathbb{R})$, such that

$$\int_{\mathcal{M}_\alpha \times \mathbb{R}} \phi \cdot w' = 0.$$

Since $f'(w(t)) = -2 + \mathcal{O}(\mathrm{e}^{-\sqrt{2}|t|})$ as $|t| \to \infty$, the equation can be put into the setting

$$(I + K)\phi = g \quad \text{in } H,$$

where $K : H \to H$ is a compact operator. From Fredholm alternative, we obtain a solution ϕ, such that

$$\|\phi\|_{L^2(\mathcal{M}_\alpha \times \mathbb{R})} \le C\|g\|_{L^2(\mathcal{M}_\alpha \times \mathbb{R})}.$$

As for the a priori estimates, we can proceed using a blow up argument following the same lines as in the local elliptic regularity developed in [10]. In our case, we also need to consider two limiting blow up situations: The case of $\mathbb{R}^2 \times \mathbb{R}$ when taking limit well inside $\mathcal{M}_\alpha \times \mathbb{R}$ and the case of the half space $\mathbb{R}_+ \times \mathbb{R}^2$ when taking the limit in coordinates close to $\partial\mathcal{M}_\alpha \times \mathbb{R}$ inside one of the sets $\mathcal{M}_{k,\alpha} \times \mathbb{R}$. The former case is reduced to the case of $\mathbb{R}^2 \times \mathbb{R}$ as limiting situation by using an odd reflection respect to the boundary of $\mathbb{R}_+ \times \mathbb{R}^2$.

Thus we have proven the following proposition.

Proposition 6.1 *For every $p > 3$ and for every $\alpha > 0$ small enough and given functions g defined in $\mathcal{M}_\alpha \times \mathbb{R}$ and G defined in $\partial\mathcal{M}_\alpha \times \mathbb{R}$ such that*

$$\|g\|_{p,\sigma} + \|G\|_{p,\sigma} < \infty,$$

there exists a unique pair (ϕ, c) solving problem (6.1)–(6.2) satisfying the a priori estimate

$$\|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \le C(\|g\|_{p,\sigma} + \|G\|_{p,\sigma}),$$

where the constant C dependens only on p.

7 Gluing Reduction and Solution to the Projected Problem

In this section, we prove Lemma 5.1 and then we solve the nonlocal projected problem (5.17)–(5.19). The notations we use in this section have been set up in Sections 4–5.

7.1 Solving the gluing system

Given a fixed ϕ such that $\|\phi\|_{2,p,\sigma} \le 1$, we solve problem (5.8) with boundary condition (5.10). To begin with, we observe that there exist constants $a < b$, independent of α, such that

$$0 < a \le Q_\alpha(x) \le b \quad \text{for every } x \in \mathbb{R}^3,$$

where $Q_\alpha(x) = 2 - (1 - \zeta_2)[f'(U) + 2]$. Using this remark, we study the problem

$$\begin{aligned} &\Delta\psi - Q_\alpha(x)\psi = \widehat{g}(x) \quad x \in \Omega_\alpha, \\ &\frac{\partial\psi}{\partial n_\alpha} = \widehat{G}(x) \quad \text{on } \partial\Omega_\alpha, \end{aligned} \tag{7.1}$$

for given $\widehat{g}, \widehat{G}$. Concerning solvability of this linear problem, we have the following lemma.

Lemma 7.1 *Assume $3 < p \leq \infty$ and $\alpha > 0$ is small. For any given $\widehat{g}, \widehat{G}$ with*

$$\|\widehat{g}\|_{L^p(\Omega_\alpha)} + \|\widehat{G}\|_{L^p(\partial\Omega_\alpha)} < \infty,$$

equation (7.1) has a unique solution $\psi = \psi(g)$, satisfying the a priori estimate

$$\|\psi\|_X \leq C(\|\widehat{g}\|_{L^p(\Omega_\alpha)} + \|\widehat{G}\|_{L^p(\partial\Omega_\alpha)}).$$

The proof of this lemma is standard, and we refer the reader to [9, Section 2] for details.

Now we prove Proposition 5.1. Denote by X, the space of functions $\psi \in W^{2,p}(\Omega_\alpha)$ such that $\|\psi\|_X < \infty$ and let us denote by $\Gamma(\widehat{g}, \widehat{G}) = \psi$ the solution to the equation (7.1), from the previous lemma. We see that the bilinear map Γ is continuous, i.e.,

$$\|\Gamma(\widehat{g}, \widehat{G})\|_X \leq C(\|\widehat{g}\|_{L^p(\Omega_\alpha)} + \|\widehat{G}\|_{L^p(\partial\Omega_\alpha)}).$$

Thus, (5.8) is restated as a fixed point problem

$$\psi = -\Gamma\Big((1-\zeta_2)S(U) + (1-\zeta_2)N[\zeta_2\phi + \psi], (u_2 - \mathbb{H}(x))\frac{\partial\beta_\eta}{\partial n_\alpha} + \phi\frac{\partial\zeta_2}{\partial n_\alpha}\Big). \tag{7.2}$$

Using the norms described in (5.3) and (5.4), let us take ϕ and h satisfying

$$\|\phi\|_{2,p,\sigma} \leq 1, \quad \|h\|_* \leq \mathcal{K}\alpha.$$

We next estimate the size of the right-hand side in (7.2). Recall that $S(U) = \zeta_2\widetilde{S}(u_2) + E$, so that

$$|(1-\zeta_2)\widetilde{S}(u_2)| \leq C\alpha^2 \mathrm{e}^{-\sigma|t|}(1-\zeta_2) \leq C\alpha^2\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}.$$

This means that

$$|(1-\zeta_2)\widetilde{S}(u_2)| \leq C\alpha^2\mathrm{e}^{-\sigma\frac{\eta}{\alpha}},$$

and so $\|(1-\zeta_2)\widetilde{S}(U)\|_{L^p(\Omega_\alpha)} \leq C\alpha^2\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}$.

As for the second term in the right-hand side of (7.2), the following holds true:

$$\begin{aligned}|2\nabla\zeta_2 \cdot \nabla\phi + \phi\Delta\zeta_2| &\leq C(1-\zeta_2)\mathrm{e}^{-\sigma|t|}\|\phi\|_{2,p\sigma}\\ &\leq C\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}\|\phi\|_{2,p,\sigma}.\end{aligned}$$

This implies that

$$\|2\nabla\zeta_2 \cdot \nabla\phi + \phi\Delta\zeta_2\|_\infty \leq C\mathrm{e}^{-c\frac{\eta}{\alpha}}.$$

Proceeding in the same fashion, we obtain the estimate for the boundary condition

$$\Big\|(u_2 - \mathbb{H}(x))\frac{\partial\beta_\eta}{\partial n_\alpha} + \phi\frac{\partial\zeta_2}{\partial n_\alpha}\Big\|_\infty \leq C\mathrm{e}^{-\frac{\sigma\eta}{\alpha}}.$$

Finally, we check the Lipschitz character on ψ of the term $(1-\zeta_2)N[\zeta_2\phi+\psi]$. Take $\psi_1, \psi_2 \in X$ and notice that

$$\begin{aligned}&|(1-\zeta_2)N[\zeta_2\phi + \psi_1] - (1-\zeta_2)N[\zeta_2\phi + \psi_2]|\\ &\leq (1-\zeta_2)|f(U + \zeta_2\phi + \psi_1)\\ &\quad - f(U + \zeta_2\phi + \psi_2) - f'(U)(\psi_1 - \psi_2)|\\ &\leq C\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}(1-\zeta_2)\sup_{t\in[0,1]}|\zeta_1\phi + t\psi_1 + (1-t)\psi_2||\psi_1 - \psi_2|\\ &\leq C\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}(\|\phi\|_{\infty,\sigma} + \|\psi_1\|_\infty + \|\psi_2\|_\infty)|\psi_1 - \psi_2|,\end{aligned}$$

from where it follows that

$$\|(1-\zeta_2)N[\zeta_2\phi+\psi_1]-(1-\zeta_2)N[\zeta_2\phi+\psi_1]\|_\infty \le C\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}\|\psi_1-\psi_2\|_\infty,$$

and in particular we see that $\|(1-\zeta_2)N(\zeta_2\phi)\|_\infty \le C\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}$.

Consider $\widetilde{\Gamma}: X \to X$, $\widetilde{\Gamma}=\widetilde{\Gamma}(\psi)$ the operator given by the right-hand side of (7.2). From the previous remarks, we have that $\widetilde{\Gamma}$ is a contraction provided that α is small enough, and so we have found $\psi=\widetilde{\Gamma}(\psi)$ the solution to (5.8).

We can check directly that $\Psi(\phi)=\psi$ is Lipschitz in ϕ, i.e.,

$$\begin{aligned}\|\Psi(\phi_1)-\Psi(\phi_2)\|_X &\le C\|(1-\zeta_2)[N(\zeta_1\phi_1+\Psi(\phi_1))-N(\zeta_1\phi_2+\Psi(\phi_2))]\|_{\infty,\mu}\\ &\quad + C^{-\sigma\frac{\sigma\eta}{\alpha}}\|\phi_1-\phi_2\|_{2,p,\sigma}\\ &\le C\mathrm{e}^{-c\frac{\eta}{\alpha}}(\|\Psi(\phi_1)-\Psi(\phi_2)\|_X+\|\phi_1-\phi_2\|_{2,p,\sigma})\end{aligned}$$

Hence, for α small, we conclude

$$\|\Psi(\phi_1)-\Psi(\phi_2)\|_X \le C\mathrm{e}^{-c\frac{\eta}{\alpha}}\|\phi_1-\phi_2\|_{2,p,\sigma}.$$

7.2 Solving the projected problem

Now we solve problem (5.17)–(5.19) using the linear theory developed in Section 6, together with a fixed point argument. From the discussion in Subsection 7.1, we have a nonlocal operator $\psi=\Psi(\phi)$.

Recall that

$$\mathbf{N}(\phi) := R(\phi) + (f'(u_2)-f'(w(t)))\phi + \zeta_2(u_2-\mathbb{H}(x))\Psi(\phi) + \zeta_2 N(\phi+\Psi(\phi)) \quad \text{in } M_\alpha\times\mathbb{R}.$$

Let us denote

$$\begin{aligned}N_1(\phi) &:= R(\phi)+[f'(u_2)-f'(w(t))]\phi,\\ N_2(\phi) &:= \zeta_2(u_2-\mathbb{H}(x))\Psi(\phi),\\ N_3(\phi) &:= \zeta_2 N(\phi+\Psi(\phi)).\end{aligned}$$

We need to investigate the Lipschitz character of N_i, $i=1,2,3$. We see that

$$\begin{aligned}|N_3(\phi_1)-N_3(\phi_2)| &= \zeta_2|N(\phi_1+\Psi(\phi_1))-N(\phi_2+\Psi(\phi_2))|\\ &\le C\zeta_2\sup_{\tau\in[0,1]}|\tau(\phi_1+\Psi(\phi_1))+(1-\tau)(\phi_2+\Psi(\phi_2))|\\ &\quad\cdot|\phi_1-\phi_2+\Psi(\phi_1)-\Psi(\phi_2)|\\ &\le C[|\Psi(\phi_2)|+|\phi_1-\phi_2|+|\Psi(\phi_1)-\Psi(\phi_2)|+|\phi_2|]\\ &\quad\cdot[|\phi_1-\phi_2|+|\Psi(\phi_1)-\Psi(\phi_2)|].\end{aligned}$$

Using the norm described in (5.2), we find that

$$\|N_3(\phi_1)-N_3(\phi_2)\|_{p,\sigma} \le C[\mathrm{e}^{-\sigma\frac{\eta}{\alpha}}+\|\phi_1\|_{p,\sigma}+\|\phi_2\|_{p,\sigma}]\cdot\|\phi_1-\phi_2\|_{p,\sigma}.$$

As for the term $N_1(\phi)$, we just have to pay attention to the term $R(\phi)$. Notice that $R(\phi)$ is linear on ϕ and

$$R(\phi) = -\alpha^2\{\Delta_{\mathcal{M}}h+|A_{\mathcal{M}}|^2(t+h)\}\partial_t\phi - 2\alpha\nabla_{\mathcal{M}}h\partial_t\nabla_{\mathcal{M}_\alpha}\phi + \alpha^2|\nabla_{\mathcal{M}}h|^2\partial_{tt}\phi + D_{\alpha,h}(\phi).$$

Hence, from the assumptions made on h, we have

$$\|N_1(\phi_1) - N_1(\phi_2)\|_{p,\sigma} \leq C\alpha\|\phi_1 - \phi_2\|_{2,p,\sigma}.$$

Observe also that under the assumption made on h we have

$$\|\widetilde{S}(u_2) + \alpha^2\{\Delta_{\mathcal{M}} h + |A_{\mathcal{M}}|^2 h\} w'(t)\|_{p,\sigma} \leq C\alpha^3.$$

Hence, for $\|\phi\|_{2,p,\sigma} \leq A\alpha^2$, we have that $\|N(\phi)\|_{p,\sigma} \leq C\alpha^4$.

As for the boundary condition, we check directly from expressions (3.18), (5.16) and (5.18) that on every end $M_{k,\alpha}$ the following estimates hold:

$$\|\widetilde{B}(u_2)\|_{\infty,\sigma} \leq C\alpha^3, \quad \|\mathcal{B}(\phi)\|_{\infty,\sigma} \leq C\alpha(\|\nabla\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma})$$

with $\mathcal{B}(\phi)$ linear in ϕ.

Setting $T(g, G) = \phi$ the bilinear operator given from Proposition 6.1, we recast problem (5.17)–(5.19) as the fixed point problem

$$\phi = T(-\widetilde{S}(u_2) - \mathbf{N}(\phi), \widetilde{B}(u_2) - \mathcal{B}(\phi)) =: \mathcal{T}(\phi)$$

in the ball

$$B_\alpha^X := \{\phi \in X / \|\phi\|_{2,p,\sigma} \leq A\alpha^3\},$$

where X is the space of function $\phi \in W^{2,p}_{loc}(\mathcal{M}_\alpha \times \mathbb{R})$ with the norm $\|\phi\|_{2,p,\sigma}$. Observe that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_X \leq C\|\mathbf{N}(\phi_1) - \mathbf{N}(\phi_2)\|_{p,\sigma} + C\alpha\|\phi_1 - \phi_2\|_{2,p,\sigma} \leq C\alpha\|\phi_1 - \phi_2\|_X, \quad \phi \in B_\alpha^X.$$

On the other hand, because C and A do not depend on $\alpha > 0$, we take A large enough, so that

$$\|\mathcal{T}(\phi)\|_X \leq C(\|\widetilde{S}(u_2)\|_{p,\sigma} + \|\mathbf{N}(\phi)\|_{p,\sigma} + \|\widetilde{B}(u_2)\|_{\infty,\sigma} + \|\mathcal{B}(\phi)\|_{\infty,\sigma}) \leq A\alpha^3, \quad \phi \in B_\alpha^X.$$

Hence, the mapping $\mathcal{T}$ is a contraction from the ball B_α^X onto itself. From the contraction mapping principle, we get a unique solution ϕ as required. We denote the solution to (5.17)–(5.19) for h fixed.

As for the Lipschitz character of $\Phi(h)$, it comes from a lengthy by direct computation. We left to the reader to check on the details of the proof of the following estimate:

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\sigma} \leq C\alpha^2\|h_1 - h_2\|_*,$$

and this completes the proof of Proposition 5.2.

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