Porous Medium Flow with Both a Fractional Potential Pressure and Fractional Time Derivative*

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(*Dedicated to Haim Brezis on the occasion of his* 70*th birthday*)

Abstract The authors study a porous medium equation with a right-hand side. The operator has nonlocal diffusion effects given by an inverse fractional Laplacian operator. The derivative in time is also fractional and is of Caputo-type, which takes into account "memory". The precise model is

$$\mathcal{D}_t^\alpha u - \operatorname{div}(u(-\Delta)^{-\sigma}u) = f, \quad 0 < \sigma < \frac{1}{2}.$$

This paper poses the problem over $\{t \in \mathbb{R}^+, x \in \mathbb{R}^n\}$ with nonnegative initial data $u(0,x) \geq 0$ as well as the right-hand side $f \geq 0$. The existence for weak solutions when $f, u(0,x)$ have exponential decay at infinity is proved. The main result is Hölder continuity for such weak solutions.

Keywords Caputo derivative, Marchaud derivative, Porous medium equation, Hölder continuity, Nonlocal diffusion
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1 Introduction

In this paper we study both existence and regularity for solutions to a porous medium equation. The pressure is related to the density via a nonlocal operator. This diffusion takes into account long-range effects. The time derivative is nonlocal and fractional and therefore takes into account the past. In the typical derivation of the porous medium equation (see [16]) the equation one considers is

$$\partial_t u + \operatorname{div}(\mathbf{v}u) = 0$$

with $u(t,x) \geq 0$. By Darcy's law in a porous medium, $\mathbf{v} = -\nabla p$ arises as a potential where p is the pressure. According to a state law $p = f(u)$. In our case, we consider a potential which takes into account long-range interactions, namely $p = (-\Delta)^{-\sigma}u$. A porous medium equation with a pressure of this type

$$\partial_t u = \operatorname{div}(u(-\Delta)^{-\sigma}u) \tag{1.1}$$

has been recently studied. For $0 < \sigma < 1$ with $\sigma \neq \frac{1}{2}$, existence of solutions was shown in [6] while regularity and further existence properties were studied in [5]. Uniqueness for the range $\frac{1}{2} \leq \sigma < 1$ was shown in [19]. Another model of the porous medium equation

$$D_t^\alpha u - \operatorname{div}(\kappa(u)Du) = f$$

was introduced by Caputo in [7]. In the above equation, D_t^α is the Caputo derivative and the diffusion is local. Solvability for a more general equation was recently studied in [17]. The fractional derivative takes into account models in which there is "memory". The Caputo derivative has also been recently shown (see [8–9]) to be effective in modeling problems in plasma transport. See also [13, 18] for further models that utilize fractional equations in both space and time to account for long-range interactions as well as the past.

The specific equation we study is

$$\mathcal{D}_t^\alpha u(t,x) - \operatorname{div}(u\nabla(-\Delta)^{-\sigma}u) = f(t,x). \tag{1.2}$$

The operator $\mathcal{D}_t^\alpha$ is of Caputo-type and is defined by

$$\mathcal{D}_t^\alpha u := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} [u(t,x) - u(s,x)]K(t,s,x) \, \mathrm{d}s.$$

When $K(t,s,x) = (t-s)^{-1-\alpha}$, this is exactly the Caputo (or Marchaud) derivative (see Section 2), which we denote by D_t^α. We assume the following bounds on the kernel K:

$$\frac{1}{\Lambda(t-s)^{1+\alpha}} \leq \frac{\alpha}{\Gamma(1-\alpha)}K(t,s,x) \leq \frac{\Lambda}{(t-s)^{1+\alpha}}. \tag{1.3}$$

Our kernel in time then can be thought of as having "bounded, measureable coefficients". We also require the following relation on the kernel:

$$K(t,t-s) = K(t+s,t). \tag{1.4}$$

The relation (1.4) allows us to give a weak (in space and in time) formulation of (1.2). This weak formulation is given in Section 2.

In this paper, we also restrict ourselves to the range $0 < \sigma < \frac{1}{2}$. In [5], the use of a transport term was made to work in the range $\frac{1}{2} < \sigma < 1$. We have not yet found the correct manner in which to prove our results for $\frac{1}{2} < \sigma < 1$ when dealing with the nonlocal fractional time derivative $\mathcal{D}_t^\alpha$.

1.1 Accounting for the past

Nonlocal equations are effective in taking into account long-range interactions and taking into account the past. However, the nonlocal aspect of the equation provides both advantages and disadvantages in studying local aspects of the equation. One advantage is that there is a relation between two points built into the equation. Indeed, we utilize two nonlocal terms that are not present in the classical porous medium equation to prove Lemma 5.3. One disadvantage of nonlocal equations is that when rescaling of the form $v(t,x) = Au(Bt, Cx)$, the far away portions of u cannot be discarded, and hence v begins to build up a "tail". Consequently,

the usual test function $(u - k)_+$ or $F((u - k)_+)$ for some function F and a constant k is often insufficient. One must instead consider $F((u-\phi)_+)$, where ϕ is a constant close by but has some "tail" growth at infinity. This difficulty of course presents itself with the Caputo derivative. One issue becomes immediately apparent. If we choose $F((u - \phi)_+)$ as a test function, then

$$\int_a^T F((u - \phi)_+)D_t^\alpha u \ \mathrm{d}t = \int_a^T F((u - \phi)_+)D_t^\alpha((u - \phi)_+ - (u - \phi)_-) \ \mathrm{d}t$$
$$+ \int_a^T F((u - \phi)_+)D_t^\alpha \phi \ \mathrm{d}t.$$

The second term will no longer be identically zero if ϕ is not constant. When using energy methods, this second term can be treated as part of the right-hand side, and hence it becomes natural to consider an equation of the form (1.2) with the right-hand side. The main challenge with accomodating a nonzero right-hand side is that the natural test function $\ln u$ used in [5–6] is no longer available since the function u can evaluate zero. Indeed, if the initial data for a solution is compactly supported, then the solution is compactly supported for every time $t > 0$ (see Remark 4.2). We choose as our basic test function u^γ for $\gamma > 0$. For small σ, we will have to choose small γ. We then can accomodate the right-hand side as well as avoiding delicate integrability issues involved when using $\ln u$ as a test function. Using careful analysis, it is still most likely possible to utilize $\ln u$ as a test function for our equation (1.2) with zero right-hand side, but we find it more convenient to use u^γ and prove the stronger result that includes the right-hand side. Our method using u^γ should also work for the equation (1.1) to be able to prove existence and regularity with the right-hand side. One benefit of accomodating the right-hand side in L^∞ is that we obtain immediately regularity up to the initial time for smooth initial data (see Theorem 1.2).

1.2 Overview of the main results

We will prove our results for a class of weak solutions (2.4) later formulated in Section 2. Our first main result is existence. We use an approximating scheme as in [6] as well as discretizing in time as in [1]. We prove the following theorem.

Theorem 1.1 *Let* $0 \le u_0(x), f(t, x) \le Ae^{-|x|}$ *for some* $A \ge 0$. *Assume further that* $u_0 \in C^2$. *Then there exists a solution* u *to* (2.4) *in* $(0, \infty) \times \mathbb{R}^n$ *that has initial data* $u(0, x) = u_0(x)$.

Remark 1.1 Our constructions are made via recursion over a finite time interval $(0, T_1)$. Since our constructions are made via recursion, if $T_2 = mT_1$ for $m \in \mathbb{N}$, it is immediate that if u_i is the solution constructed on $(0, T_i)$, then $u_2 = u_1$ on $(0, T_1)$.

Remark 1.2 For technical reasons seen in the proof of Lemma 4.4, when $n = 1$, we make the further restriction $0 < \sigma < \frac{1}{4}$.

The main result of the paper is an interior Hölder regularity result. As expected, the Hölder norm will depend on the distance from the interior domain to the initial time t_0. However, by assuming the intial data u_0 is regular enough (for intance C^2) then we obtain regularity up to the initial time. This is a benefit of allowing the right-hand side. By extending the values of our solution $u(t, x) = u(0, x)$ for $t < 0$, we have (2.4) on $(-\infty, \infty) \times \mathbb{R}^n$ with the right-hand

side in L^∞. The right-hand side f for $t \leq 0$ will not necessarily satisfy $f \geq 0$. However, this nonnegativity assumption on f was only necessary to guarantee the existence of a solution $u \geq 0$. It is not a necessary assumption to prove regularity. From Remark 1.1, the solution constructed on $(-\infty, T)$ will agree with the original solution over the interval $(0, T)$.

Theorem 1.2 *Let u be a solution to* (2.4) *obtained via approximation from Theorem* 1.1 *on* $[0, T] \times \mathbb{R}^n$ *with* $0 \leq u_0(x), f(t, x) \leq A\mathrm{e}^{-|x|}$. *Assume also* $u_0 \in C^2$. *Then u is C^β continuous on* $[0, T] \times \mathbb{R}^n$ *(for some exponent β depending on $\alpha, \Lambda, n, \sigma$) with a constant that depends on the L^∞ norm of u and f, the C^2 norm of u_0, and on T.*

1.3 Future Directions

We prove existence and regularity for solutions obtained via limiting approximations. In this paper, we do not address the issue of uniqueness. As mentioned earlier, uniqueness for (1.1) for the range $\frac{1}{2} \leq \sigma < 1$ was shown in [19]. The issue of uniqueness for (1.2) is not trivial because of the nonlinear aspect of the equation as well as the lack of a comparison principle. The equation (2.4) which we consider also should present new difficulties because of the weak/very-weak formulation in time as well as the minimal "bounded, measurable" assumption (1.3) on the kernel $K(t, s, x)$. An interesting problem would be to then address the issue of uniqueness for solutions of (2.4).

Theorems 1.1–1.2 can most likely be further refined by making less assumptions on $u(0, x)$, assuming the right-hand side $f \in L^p$ as was done for a similar problem in [17], and proving the estimates uniform as $\sigma \to 0$ and recovering Hölder continuity for the local diffusion problem. Also, as mentioned earlier the theorems can be improved to include the range $\frac{1}{2} \leq \sigma < 1$.

Finally, just like in the local porous medium equation (see [16]) as well as for (1.1), the equation (2.4) has the property of finite propagation (see Remark 4.2). Therefore, it is of interest to study the free boundary $\partial\{u(t, x) > 0\}$.

1.4 Outline

The outline of this paper will be as follows. In Section 2, we state basic results for the Caputo derivative. We also give the weak formulation of the equation we study. In Section 3, we state some results for the discretized version of $\mathcal{D}_t^\alpha$ that we will use to prove the existence of solutions. In Section 4, we use the estimates and follow the approximation method from [6] combined with the method of discretization and the estimates presented in [1] to prove existence. In Section 5, we state the main lemmas that we will need to be able to prove Hölder regularity. In Section 6, we prove the most technically difficult Lemma 5.1 of the paper. This Lemma 5.1 most directly handles the degenerate nature of the problem. In Section 7, we prove an analogue of Lemma 5.1. In Section 8, we prove the final lemmas we need which give a one-sided decrease in oscillation from above. The one-sided decrease in oscillation combined with Lemma 5.1 is enough to prove the Hölder regularity and this is explained in Section 9.

1.5 Notations

We list here the notations that will be used consistently throughout the paper. The following letters are fixed throughout the paper and always refer to the following:

(1) α—the order of the Caputo derivative.

(2) σ—the order of inverse fractional Laplacian $(-\Delta)^{-\sigma}$. We use σ for the order because s will always be a variable for time.

(3) a—the initial time for which our equation is defined.

(4) D_t^α—the Caputo derivative as defined in Section 2.

(5) $\mathcal{D}_t^\alpha$—the Caputo-type fractional derivative with "bounded, measurable" coefficients with bounds (1.3) and relation (1.4).

(6) Λ—the constant appearing in (1.3).

(7) $\mathcal{D}_\epsilon^\alpha$—the discretized version of $\mathcal{D}_t^\alpha$ as defined in (3.1).

(8) ϵ refers to the time length of the discrete approximations as defined in Section 3.

(9) n will always refer to the space dimension.

(10) Γ_m—the parabolic cylinder $(-m, 0) \times B_m$.

(11) $W^{\beta, p}$—the fractional Sobolev space as defined in [10].

(12) $L^p(0, T; W^{\beta, p})$—the set of functions $u(t, x)$ for which $\int_0^T \|u\|_{W^{\beta, p}}^p \, dt < \infty$.

(13) $u_\pm$—the positive and negative parts respectively so that $u = u_+ - u_-$.

(14) $\widetilde{u}$—the extension $\widetilde{u}(t) = u(\epsilon j)$ for $\epsilon j - 1 < t < \epsilon j$.

2 Caputo Derivative

In this section, we state various properties of the Caputo derivative that will be useful. The proofs and methods of proof were shown in [1]. The Caputo derivative for $0 < \alpha < 1$ is defined by

$$_aD_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^\alpha} \, ds.$$

By using integration by parts, we have

$$\Gamma(1-\alpha) \, _aD_t^\alpha u(t) = \frac{u(t) - u(a)}{(t-a)^\alpha} + \alpha \int_a^t \frac{u(t) - u(s)}{(t-s)^{\alpha+1}} \, ds. \tag{2.1}$$

For the remainder of the paper, we will drop the subscript a when the initial point is understood. We now recall some properties of the Caputo derivative that were proven in [1].

For a function $g(t)$ defined on $[a, t]$, it is advantageous to define $g(t)$ for $t < a$. Then we have the formulation

$$_aD_t^\alpha g(t) = \, _{-\infty}D_t^\alpha g(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{g(t) - g(s)}{(t-s)^{1+\alpha}} \, ds,$$

utilized in [1]. (See also [14] under Marchaud derivative and [2–3] for properties of this one-sided nonlocal derivative.) This looks very similar to $(-\Delta)^\alpha$ except the integration only occurs for $s < t$. In this manner, the Caputo derivative retains directional derivative behavior while at the same time sharing certain properties with $(-\Delta)^\alpha$. This is perhaps best illustrated by the following integration by parts formula for the Caputo derivative.

Proposition 2.1 *Let $g, h \in C^1(a, T)$. Then*

$$\int_a^T g D_t^\alpha h + h D_t^\alpha g = \int_a^T g(t) h(t) \left[\frac{1}{(T-t)^\alpha} + \frac{1}{(t-a)^\alpha} \right] \, dt$$

$$+ \alpha \int_a^T \int_a^t \frac{[g(t) - g(s)][h(t) - h(s)]}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t$$

$$- \int_a^T \frac{g(t)h(a) + h(t)g(a)}{(t-a)^\alpha} \, \mathrm{d}t. \tag{2.2}$$

Formula (2.1) is based on the following formal computation:

$$\int_a^T \int_a^t \frac{g(t) - g(s)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t = \int_a^T \int_a^t \frac{g(t)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t - \int_a^T \int_a^t \frac{g(s)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t$$

$$= \int_a^T \int_a^t \frac{g(t)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t - \int_a^T \int_s^T \frac{g(s)}{(t-s)^{1+\alpha}} \, \mathrm{d}t \, \mathrm{d}s$$

$$= \int_a^T \int_a^t \frac{g(t)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t - \int_a^T \int_t^T \frac{g(t)}{(s-t)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t$$

$$= \int_a^T g(t) \left(\int_0^{t-a} \frac{\mathrm{d}s}{s^{1+\alpha}} - \int_0^{T-t} \frac{\mathrm{d}s}{s^{1+\alpha}} \right) \, \mathrm{d}t$$

$$= \alpha^{-1} \int_a^T g(t) \left(\frac{1}{(T-t)^\alpha} - \frac{1}{(t-a)^\alpha} \right) \, \mathrm{d}t.$$

In the above computation to utilize the cancellation we only need a kernel $K(t, s)$ satsifying

$$K(t, t - s) = K(t + s, t).$$

To make the above computation rigorous, we will use the discretization in Section 3. An alternative, equivalent integration by parts formula is to extend $g(t) = g(a)$ for $t < a$. Then for any $h \in C^1$ with $h(t) = 0$ for any $t < b$ for some b, we have

$$\int_{-\infty}^T h(t) D_t^\alpha g(t) \, \mathrm{d}t = c_\alpha \int_{-\infty}^T \int_{-\infty}^t \frac{[g(t) - g(s)][h(t) - h(s)]}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t$$

$$+ c_\alpha \int_{-\infty}^T \int_{-\infty}^{2t-T} \frac{g(t)h(t)}{(t-s)^{1+\alpha}} \, \mathrm{d}s \, \mathrm{d}t$$

$$- \int_{-\infty}^T g(t) D_t^\alpha h(t) \, \mathrm{d}t \tag{2.3}$$

with $c_\alpha = \alpha \Gamma(1-\alpha)^{-1}$. Now (2.2) and (2.3) will both imply each other, so (2.3) is an alternative way of handling the initial condition $g(a)$. Furthermore, both (2.2) and (2.3) are weak formulations for the Caputo derivative that only require that $g \in H^{\frac{\alpha}{2}}((a, T))$. (2.3) will work for any kernel $K(t, s)$ satisfying the relation (1.4). In view of (2.3), we now give the exact formulation of our weak solutions. We assume the bounds (1.3) and the relation (1.4) on the kernel $K(t, s, x)$. For smooth initial data $u_0 \in C^2$, we assign $u(t, x) = u(a, x)$ for $t < a$. Then as stated earlier in the introduction, for $t \leq a$, a solution u will have the right-hand side

$$\mathrm{div}(u_0 (-\Delta)^{-\sigma} u_0) \in L^\infty.$$

We say that u is a weak solution, if for any $\phi \in C_0^\infty(-\infty, T) \times \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \int_{-\infty}^T \int_{-\infty}^t [u(t, x) - u(s, x)][\phi(t, x) - \phi(s, x)] K(t, s, x) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}x$$

$$+ \int_{\mathbb{R}^n} \int_{-\infty}^{T} \int_{-\infty}^{2t-T} u(t,x)\phi(t,x)K(t,s,x) \ ds \ dt \ dx$$

$$- \int_{\mathbb{R}^n} \int_{-\infty}^{T} u(t,x)\mathcal{D}_t^\alpha \phi(t,x) \ dt \ dx$$

$$+ \int_{-\infty}^{T} \int_{\mathbb{R}^n} \nabla\phi(t,x)u(t,x)\nabla(-\Delta)^{-\sigma}u \ dx \ dt$$

$$= \int_{-\infty}^{T} \int_{B_R} f(t,x)\phi(t,x) \ dx \ dt. \tag{2.4}$$

We will also utilize a fractional Sobolev norm that arises from the fractional derivative.

Lemma 2.1 *Let u be defined on $[a,T]$. We have for two constants c_1, c_2 depending on $\alpha, |T-a|$,*

$$\|u\|_{L^{\frac{2}{1-\alpha}}(a,T)} \leq c_1 \|u\|_{H^{\frac{\alpha}{2}}(a,T)}^2$$

$$\leq c_2 \Big(\alpha \int_a^T \int_a^t \frac{|u(t)-u(s)|^2}{|t-s|^{1+\alpha}} \ ds \ dt + \int_a^T \frac{u^2(t)}{(T-a)^\alpha} dt \Big).$$

The following estimate will be needed for our choice of cut-off functions.

Lemma 2.2 *Let*

$$h(t) := \max\{|t|^\nu - 1, 0\}$$

with $\nu < \alpha$. Then

$$\mathcal{D}_t^\alpha h \geq -c_{\nu,\alpha,\Lambda}$$

for $t \in \mathbb{R}$. Here, $c_{\nu,\alpha,\Lambda}$ is a constant depending only on α, ν, Λ.

Finally, we point out that if $g = g_+ - g_-$ the positive and negative parts respectively, then

$$\int_a^T g_\pm(t)\mathcal{D}_t^\alpha g_\mp(t)dt \geq 0. \tag{2.5}$$

3 Discretization in Time

To prove existence of solutions to (2.4), we will discretize in time. The discretization also allows us to make the computations involving the fractional derivative rigorous. This section contains properties of a discrete fractional derivative which we will utilize.

For future reference, we denote the discrete fractional derivative with kernel K as

$$_a\mathcal{D}_\epsilon^\alpha u(a+\epsilon j) := \epsilon \sum_{-\infty < i < j} [u(a+\epsilon j) - u(a+\epsilon i)]K(\epsilon j, \epsilon i). \tag{3.1}$$

We write $\mathcal{D}_\epsilon^\alpha$ when $a = 0$. The following is the discretized argument for the cancellation that appears in the formal computation of the version of (2.1) that we will need.

Lemma 3.1 *Assume $g(a) = 0$ and define $g(t) := 0$ for $t < a$. Assume relation (1.4) on K. Then*

$$\epsilon^2 \sum_{j \leq k} \sum_{i < j} [g(a+\epsilon j) - g(a+\epsilon i)]K(a+\epsilon j, a+\epsilon i)$$

$$= \epsilon^2 \sum_{j \le k} \sum_{i < 2j-k-1} g(a + \epsilon j) K(a + \epsilon j, a + \epsilon i).$$

Let $\widetilde{g}(t) := g(\epsilon j)$ for $\epsilon j - 1 < t \le \epsilon j$. If $g \ge 0$, then there exists c depending only on α, Λ, such that if $\epsilon < 1$, then

$$\epsilon^2 \sum_{j \le k} \sum_{i < j} [g(a + \epsilon j) - g(a + \epsilon i)] K(a + \epsilon j, a + \epsilon i) \ge c \int_{-\infty}^{T} \frac{\widetilde{g}(t)}{(T - t)^\alpha} \, dt. \qquad (3.2)$$

Proof For notational simplicity, we assume $a = 0$.

$$\epsilon^2 \sum_{j \le k} \sum_{i < j} [g(\epsilon j) - g(\epsilon i)] K(\epsilon j, \epsilon i)$$

$$= \epsilon^2 \sum_{j \le k} \sum_{i < j} g(\epsilon j) K(\epsilon j, \epsilon i) - \epsilon^2 \sum_{j \le k} \sum_{i < j} g(\epsilon i) K(\epsilon j, \epsilon i)$$

$$= \epsilon^2 \sum_{j \le k} \sum_{i < j} g(\epsilon j) K(\epsilon j, \epsilon i) - \epsilon^2 \sum_{i < k} \sum_{i < j \le k} g(\epsilon i) K(\epsilon j, \epsilon i)$$

$$= \epsilon^2 \sum_{j \le k} \sum_{i < j} g(\epsilon j) K(\epsilon j, \epsilon i) - \epsilon^2 \sum_{j < k} \sum_{j < i \le k} g(\epsilon j) K(\epsilon i, \epsilon j)$$

$$= \epsilon^2 \sum_{j \le k} \sum_{i < 2j-k-1} g(\epsilon j) K(\epsilon j, \epsilon i).$$

Inequality (3.2) follows from the estimates in [1].

Lemma 3.1 combined with the estimates in [1] can be used to show.

Lemma 3.2 *Let $u(0) = 0$ and assume $u \ge 0$. For fixed $0 < \epsilon < 1$, let $\widetilde{u}$ be the extension defined as in Lemma 3.1. Let K satisfy conditions (1.3)–(1.4). Then there exists two constants c_1, c_2 depending only on α, Λ, such that*

$$\epsilon^2 \sum_{j \le k} \sum_{i < j} u(\epsilon j) [u(\epsilon j) - u(\epsilon i)] K(\epsilon j, \epsilon i)$$

$$\ge c_1 \|\widetilde{u}\|_{H^{\frac{\alpha}{2}}}^2 \ge c_2 \Big(\int_{-\infty}^{T} \widetilde{u}^{\frac{2}{(1-\alpha)}} \, dt \Big)^{1-\alpha}.$$

This next lemma is analogous to Lemma 2.2 and was shown in [1].

Lemma 3.3 *Let h be as in Lemma 2.2. Then for $0 < \epsilon < 1$, there exists $c_{\nu, \alpha}$ depending on α and ν but independent of a, such that*

$$\mathcal{D}_\epsilon^\alpha h(t) \ge -c_{\nu, \alpha}$$

for $t \in \epsilon \mathbb{Z}$ and $a < t < 0$.

This last estimate we will use often the following lemma.

Lemma 3.4 *Let F be a convex function with $F'' \ge \gamma$, $F' \ge 0$, $F(0) = 0$. Assume $g \ge 0$, $g(a) = 0$. Then there exists c depending on α, Λ, such that*

$$\epsilon \sum_{j \le k} F'(g(\epsilon j)) \mathcal{D}_\epsilon^\alpha g(\epsilon j) \ge c\epsilon \sum_{j \le k} \frac{F(g(\epsilon j))}{(\epsilon(j-i))^\alpha} + c\frac{\gamma}{2} \epsilon^2 \sum_{j \le k} \sum_{e < j} \frac{[g(\epsilon j) - g(\epsilon i)]^2}{(\epsilon(j-i))^{1+\alpha}}.$$

Proof Assume for notational simplicity $a = 0$. Since F is convex,

$$F'(g(\epsilon j))[g(\epsilon j) - g(ei)] \geq F(g(\epsilon j)) - F(g(ei)) + \frac{\gamma}{2}[g(\epsilon j) - g(ei)]^2.$$

The result then follows from applying Lemma 3.1.

Finally, we point out that if g is a limit of $\widetilde{g}_\epsilon$ which are discretized problems with the assumptions in Lemma 3.4, it follows that

$$\int_a^T F'(g(t))\mathcal{D}_t^\alpha g(t) \, dt \geq c \int_a^T \frac{F(g(t))}{(T-a)^\alpha} \, dt + c\frac{\gamma}{2} \int_a^T \int_a^t \frac{[g(t) - g(s)]^2}{(t-s)^{1+\alpha}} \, ds \, dt. \qquad (3.3)$$

4 Existence

In this section, we prove the existence of weak solutions following the construction given in [6]. We will also discretize in time. We first consider a smooth approximation of the kernel $(-\Delta)^{-\sigma}$ as K_ζ. We start with the smooth classical solution to the elliptic problem

$$gu - \delta\mathrm{div}((u+d)\nabla u) - \mathrm{div}((u+d)\nabla K_\zeta u) = f \quad \text{on } B_R \qquad (4.1)$$

with $u \equiv 0$ on ∂B_R. $g, f \geq 0$ are smooth. The signs of f, g guarantee the solution is nonnegative. $\delta, d > 0$ are constants. The nonlocal part is computed in the expected way by extending $u = 0$ on B_R^c. To find such a solution, we first consider the linear problem

$$gu - \delta\mathrm{div}((v+d)\nabla u) - \mathrm{div}((v+d)\nabla K_\zeta u) = f \quad \text{on } B_R$$

for $v \in C_0^{0,\beta}(B_R)$ with $v \geq 0$. With fixed $d, \delta, R, \zeta > 0$, one can apply Schauder estimate theory to conclude

$$\|u\|_{C_0^{1,\beta}} \leq C\|v\|_{C_0^{0,\beta}}$$

with $u \geq 0$. The map $T : v \to u$ is then a compact map. The set $\{v\}$ with $v \geq 0$ and $v \in C_0^{0,\beta}$ is a closed convex set, and hence we can apply the fixed point theorem (see [12, Corollary 11.2]), to conclude there is a solution to (4.1). By bootstrapping, we conclude u is smooth.

Now we use the existence of solutions to (4.1) to obtain (via recursion) solutions to the discretized problem

$$\mathcal{D}_\epsilon^\alpha u - \delta\mathrm{div}((u+d)\nabla u) - \mathrm{div}((u+d)\nabla K_\zeta u) = f \quad \text{on } [0, T] \times B_R \qquad (4.2)$$

with $u(0, x) = u_0(x)$ an initially defined smooth function with compact support. $\epsilon = \frac{T}{k}$ for some $k \in \mathbb{N}$. We will eventually let $k \to \infty$, so that $\epsilon \to 0$.

For the next two lemmas, we will utilize the solution to

$$D_t^\alpha Y(t) = cY(t) + h(t), \qquad (4.3)$$

which as in [11] is given by

$$Y(t) = Y(0)E_\alpha(ct^\alpha) + \alpha \int_0^t (t-s)^{\alpha-1} E_\alpha'(c(t-s)^\alpha)h(t) \, dt,$$

where E_α is the Mittag-Leffler function of order α. We will utilize in the next lemmas two specific instances of (4.3). We define $Y_1(t)$ to be the solution to (4.3) with $Y(0) = \sup u(0, x)$, $c = 0$, $h = 2\Lambda f$. We define $Y_2(t)$ to be the solution to (4.3) with $c = C\Lambda^{-1}$, $Y_2(0) = 2$ and $h = 0$. The constant C will be chosen later.

Lemma 4.1 *Let u be a solution to (4.2). Let $Y_1(t)$ be defined as above. Then there exists ϵ_0 depending only on $T, \alpha, \|f\|_{L^\infty}$ such that if $\epsilon \leq \epsilon_0$, then*

$$u(\epsilon j) \leq Y_1(\epsilon j).$$

Proof Since $Y_1(t)$ is an increasing function

$$\mathcal{D}_\epsilon^\alpha Y_1(\epsilon j) \geq \Lambda^{-1} D_\epsilon^\alpha Y_1(\epsilon j).$$

Depending on T and $\|f\|_{L^\infty}$, there exists ϵ_0, such that if $\epsilon \leq \epsilon_0$, then

$$\mathcal{D}_\epsilon^\alpha Y_1(\epsilon j) \geq \Lambda^{-1} D_\epsilon^\alpha Y_1(\epsilon j) \geq \frac{2}{3\Lambda} D_t^\alpha Y_1(\epsilon j) = \frac{4}{3} f.$$

We use $(u(t, x) - Y_1(t))_+$ as a test function. Since $(u - Y)_+(0) = 0$, it follows from (2.5) and Lemma 3.4 that

$$\epsilon \sum_{j<k} (u - Y_1)_+ \mathcal{D}_\epsilon^\alpha [(u - Y_1)_+ - (u - Y_1)_-] \geq 0.$$

We define

$$\mathcal{B}_\zeta(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\zeta(x, y)[u(x) - u(y)][v(x) - v(y)] \, \mathrm{d}x \, \mathrm{d}y,$$

where $H_\zeta(x, y) = \Delta K_\zeta$. We have the identity (see [5])

$$\mathcal{B}_\zeta(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla u(x) \nabla K_\zeta v(y) \, \mathrm{d}x \, \mathrm{d}y.$$

Then for ϵ small enough and $j > 0$,

$$\epsilon \sum_{j \leq k} \int_{B_R} f(\epsilon j, x)(u(\epsilon j, x) - Y_1(\epsilon j))_+ \, \mathrm{d}x$$

$$= \int_{B_R} \epsilon \sum_{j \leq k} (u - Y_1)_+ \mathcal{D}_\epsilon^\alpha [(u - Y_1)_+ - (u - Y_1)_- + Y_1] \, \mathrm{d}x$$

$$+ \epsilon \sum_{j \leq k} \int_{B_R} (u + d) \nabla (u - Y_1)_+ \nabla u \, \mathrm{d}x$$

$$+ \epsilon \sum_{j \leq k} \int_{B_R} (u + d) \nabla (u - Y_1)_+ \nabla K_\zeta u \, \mathrm{d}x$$

$$\geq \int_{B_r} \epsilon \sum_{j \leq k} \frac{4}{3} f(u - Y_1)_+ \, \mathrm{d}x + \int_{\mathbb{R}^n} (u + d) \chi_{\{u > Y_1\}} \nabla u \nabla K_\zeta u \, \mathrm{d}x$$

$$= \int_{B_R} \epsilon \sum_{j \leq k} \frac{4}{3} f(u - Y_1)_+ \, \mathrm{d}x + \frac{\epsilon}{2} \sum_{j \leq k} \mathcal{B}_\zeta(\chi_{\{u > Y_1\}}(u + d)^2, u)$$

$$\geq \frac{4}{3} \epsilon \sum_{j \leq k} \int_{B_R} f(\epsilon j, x)(u(\epsilon j, x) - Y_1(\epsilon j))_+ \, \mathrm{d}x.$$

Thus $(u - Y_1)_+ \equiv 0$.

Lemma 4.2 *Let u be a solution to (4.2) in $[0, T] \times B_R$. Assume that*

$$0 \le u(0, x) \le Ae^{-|x|} \quad and \quad f \le Ae^{-|x|}. \tag{4.4}$$

If A is large, there exists constants

$$\mu_0, \delta_0, \zeta_0 \text{ depending only on } R, n, \sigma,$$

$$\epsilon_0 \text{ depending only on } T, \alpha, \Lambda,$$

$$C \text{ depending only on } n, \sigma, \|u(0, x)\|_{L^\infty}, \|f\|_{L^\infty},$$

such that if $\mu < \mu_0$, $\delta < \delta_0$, $\zeta < \zeta_0$, $\epsilon < \epsilon_0$, and $Y_2(t)$ is the solution defined earlier with constant C given above, then

$$0 \le u(\epsilon j, x) \le AY_2(\epsilon j)e^{-|x|}$$

for any $t = \epsilon j$.

Proof As before, there exists ϵ_0 depending only on T, α, such that for $\epsilon \le \epsilon_0$, we have

$$\mathcal{D}_\epsilon^\alpha Y_2(\epsilon j) \ge \frac{2}{3\Lambda} D_t^\alpha Y_2(\epsilon j).$$

Since u is smooth and hence continuous, $u \le LY_2(\epsilon j)e^{-|x|}$ for some $L > A$. We lower $L \ge A$ until it touches u for the first time. Since $u = 0$ on ∂B_R, this cannot happen on the boundary. Since u is smooth, this cannot happen at a point $(\epsilon j, 0)$. Also, $LY_2 \ge 2A \ge 2u(0, x)$, so this cannot occur at the initial time. We label a point of touching as (t_c, r_c). We compute the operator in nondivergence form and write $K_\zeta(u) = p$ and use the estimates in [6] to conclude for ϵ small enough that

$$
\begin{aligned}
\frac{2}{3}CLY_2(t_c)e^{-r_c} &= \frac{\Lambda}{3}LD_t^\alpha Y_2(t_c)e^{-r_c} \\
&\le \mathcal{D}_\epsilon^\alpha LY_2(t_c)e^{-r_c} \\
&\le \mathcal{D}_\epsilon^\alpha u(t_c) \\
&= \delta\operatorname{div}((u + d)\nabla u) + \operatorname{div}((u + d)\nabla K_\zeta u) + f(t_c, r_c) \\
&= \delta 2[LY_2(t_c)e^{-r_c}]^2 + \delta dLY_2(t_c)e^{-r_c} \\
&\quad - LY_2(t_c)e^{-r_c}\overline{\partial_r p} + (LY_2(t_c)e^{-r_c} + d)\overline{\Delta p} + f(t_c, r_c),
\end{aligned}
$$

where in the equation the bar above means evaluation at r_c. Then using again the estimates from [6], for small enough ζ, we have a universal constant M depending only on n, σ, such that

$$|\overline{\partial_r p}|, |\overline{\Delta p}| \le Y_1(T)M.$$

Now recalling also that $LY_2(t_c)e^{-r_c} \le Y_1(T)$,

$$
\begin{aligned}
\frac{2}{3}C &\le \delta(2 + d)Y_1(T) + MY_1(T) + \left(1 + \frac{d}{LY_2(t_c)}e^{r_c}\right)MY_1(T) + \frac{f(r_c)e^{r_c}}{LY_2(t_c)} \\
&\le \delta(2 + d)Y_1(T) + 2MY_1T\left(1 + \frac{d}{L}e^{r_c}\right) + \frac{A}{L}.
\end{aligned}
$$

Choosing δ, d small enough, the above inequality implies

$$C \leq 4MY_1(T) + 4.$$

If we choose now $C > 2MY_1(T) + 4$, we obtain a contradiction. We note that C will only depend on $n, \sigma, \|u(0,x)\|_{L^\infty}, \|f\|_{L^\infty}$.

We now give some Sobolev estimates. Because we have the right-hand side, we choose not to use $\ln(u)$ as the test function. For $0 < \gamma < 1$, we use $(u+d)^\gamma - d^\gamma$ as a test function. The function

$$F(t) = \frac{1}{\gamma+1}(t+d)^{\gamma+1} - d^\gamma t$$

will satisfy the conditions in Lemma 3.4. We now assume that u is a solution to (4.2) with assumptions as in Lemma 4.2, so that $|u| \leq Me^{-|x|}$ for some large M. As discussed in the introduction, we can extend $u(\epsilon j, x) = u(0, x)$ for $j < 0$, and u will be a solution to (4.2) on $(-\infty, T) \times \mathbb{R}^n$ with the right-hand side

$$\delta\mathrm{div}((u(0,x)+d)\nabla u(0,x)) + \mathrm{div}((u(0,x)+d)\nabla(-\Delta)^{-\sigma}u(0,x))$$

for $j \leq 0$. This right-hand side is not necessarily nonnegative. However, we only required the nonnegativity of the right-hand side to guarantee that our solution is nonnegative. In this case, we already know our solution is nonnegative. We fix a smooth cut-off $\phi(t)$ with $\phi(t) \geq M$ for $t \leq -2$ and $\phi(t) = 0$ for $t \geq -1$. We now take our test function as $\epsilon F'([u(t,x) - \phi(t)]_+)$. We define

$$u = (u-\phi)_+ - (u-\phi)_- + \phi =: u_\phi^+ - u_\phi^- + \phi.$$

We define $\widetilde{u} = u(t)$ for $\epsilon j - 1 < t \leq \epsilon j$. From Lemma 3.4 and the estimates in Section 3, there exist two constants c, C depending on α, T, Λ such that for $\epsilon < 1$,

$$\epsilon \sum_{j \leq k} F'(u_\phi^+(\epsilon j))\mathcal{D}_\epsilon^\alpha u(\epsilon j) \geq c \int_{-\infty}^T \int_{-\infty}^t \frac{[\widetilde{u}_\phi^+(t) - \widetilde{u}_\phi^+(s)]^2}{(t-s)^{1+\alpha}}\, \mathrm{d}s\, \mathrm{d}t + c \int_{-\infty}^T \frac{F(\widetilde{u}_\phi^+(t))}{(T-t)^\alpha}\, \mathrm{d}t$$

$$- C \int_{-\infty}^T F'(\widetilde{u}_\phi^+(t))D_t^\alpha \phi(t)\, \mathrm{d}t.$$

We now consider the nonlocal spatial term. We will also use the following property: For an increasing function V and a constant l,

$$\mathcal{B}_\zeta(V((u-l)_+), u) \geq \mathcal{B}_\zeta(V((u-l)_+), (u-l)_+) \geq 0.$$

We have for the nonlocal spatial terms,

$$\epsilon \sum_{j \leq k} \int_{B_R} \nabla F'(u_\phi^+(\epsilon j, x))(u+d)\nabla K_\zeta u\, \mathrm{d}x$$

$$= \epsilon \sum_{j \leq k} \int_{B_R} \nabla F'(u_\phi^+(\epsilon j, x))[(u_\phi^+) + d + \phi]\nabla K_\zeta u\, \mathrm{d}x$$

$$= \frac{\gamma}{\gamma+1} \int_{-2}^T \mathcal{B}_\zeta((\widetilde{u}_\phi^+ + d)^{\gamma+1}, u)\, \mathrm{d}t + \int_{-2}^T \phi(t)\mathcal{B}_\zeta((\widetilde{u}_\phi^+ + d)^\gamma, u)\, \mathrm{d}t$$

$$\geq \frac{\gamma}{\gamma+1} \int_{-2}^{T} \mathcal{B}_\zeta((\widetilde{u}_\phi^+ + d)^{\gamma+1}, u) \, dt$$

$$\geq \frac{\gamma}{\gamma+1} \int_{-2}^{T} \mathcal{B}_\zeta((\widetilde{u}_\phi^+ + d)^{\gamma+1}, u_\phi^+) \, dt.$$

From Proposition 10.1, if $u_\phi^+(x) - u_\phi^+(y) \geq 0$, then

$$(u_\phi^+ + d)^{\gamma+1}(x) - (u_\phi^+ + d)^{\gamma+1}(y) \geq (u_\phi^+(x) - u_\phi^+(y))^{\gamma+1}.$$

Then

$$\epsilon \sum_{j \leq k} \int_{B_R} \nabla F'(u_\phi^+(\epsilon j, x))(u+d)\nabla K_\zeta u \, dx \geq \frac{c\gamma}{\gamma+1} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\zeta(x,y)|\widetilde{u}(x) - \widetilde{u}(y)|^{2+\gamma} \, dx \, dy \, dt.$$

For the local spatial term, we have

$$\epsilon \sum_{j \leq k} \int_{B_R} \nabla F'(u_\phi^+(\epsilon j, x))(u+d)\nabla u \, dx \geq \gamma \int_0^T \int_{B_R} (\widetilde{u} + d)^\gamma |\nabla \widetilde{u}|^2 \, dx \, dt.$$

Now combining the previous estimates with the right-hand side term f, we have for a certain constant C depending on $n, \sigma, \alpha, \Lambda, \gamma, M, T$ which can change line by line,

$$\delta\gamma \int_0^T \int_{B_R} (\widetilde{u} + d)^\gamma |\nabla \widetilde{u}|^2 \, dx \, dt$$

$$+ \frac{c\gamma}{\gamma+1} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\zeta(x,y)|\widetilde{u}(x) - \widetilde{u}(y)|^{2+\gamma} \, dx \, dy \, dt$$

$$+ c \int_{B_R} \int_0^T \int_0^t \frac{[\widetilde{u}(t) - \widetilde{u}(s)]^2}{(t-s)^{1+\alpha}} \, ds \, dt \, dx + c \int_{B_R} \int_0^T \frac{F(\widetilde{u}(t))}{(T-t)^\alpha} \, dt \, dx$$

$$\leq C \int_{B_R} \int_{-2}^T F'(\widetilde{u}_\phi^+(t)) D_t^\alpha \phi(t) \, dt \, dx + \int_{-2}^T \int_{B_R} f(t,x) F'(\widetilde{u}_\phi^+) \, dx \, dt$$

$$\leq C \int_{-2}^T \int_{B_R} [(\widetilde{u}_\phi^+ + d)^\gamma - d^\gamma] \, dx \, dt$$

$$\leq C \int_{-2}^T \int_{B_R} [(Me^{-|x|} + d)^\gamma - d^\gamma] \, dx \, dt \leq C \int_{-2}^T \int_{B_R} M^\gamma e^{-\gamma|x|} \, dx \, dt \leq C.$$

The second to last inequality comes from Proposition 10.2. The value C is independent of $\zeta, d, R, \epsilon, \delta$ if

$$\zeta, \epsilon, \delta, d < 1 \quad \text{and} \quad R > 1.$$

Then as $\zeta, d \to 0$, we have uniform control and obtain the estimate

$$\delta\gamma \int_0^T \int_{B_R} (\widetilde{u} + d)^\gamma |\nabla \widetilde{u}|^2 \, dx \, dt + \int_0^T \|\widetilde{u}\|_{W^{\frac{2-2\sigma}{2+\gamma}, 2+\gamma}(B_R)}^{2+\gamma} \, dt + \int_{B_R} \|\widetilde{u}\|_{W^{\frac{\alpha}{2}, 2}(0,T)}^2 \, dx \leq C. \quad (4.5)$$

Notice that the constant C only depends on the exponential decay of f, u_0 and on σ, α, n, T, but not on R, δ. Letting $d, \zeta \to 0$, we obtain

$$\mathcal{D}_\epsilon^\alpha u - \delta \text{div}(u\nabla u) - \text{div}(u\nabla(-\Delta)^{-\sigma}u) = f \quad \text{on } [0,T] \times B_R. \quad (4.6)$$

We now give a compactness result.

Lemma 4.3 *Assume for any $v \in \mathcal{F}$,*

$$\int_0^T \|v(t,x)\|^{2+\gamma}_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}(B_R)} \, \mathrm{d}t + \int_{B_r} \|v(t,x)\|^2_{W^{\frac{\alpha}{2},2}(0,T)} \, \mathrm{d}x \le C. \tag{4.7}$$

Then $\mathcal{F}$ is totally bounded in $L^p([0,T] \times B_R)$ for $1 \le p \le 2$.

Proof We utilize the proof provided in [10] for compactness in fractional Sobolev spaces. We will show the result for $p = 2$, and it will follow for $p < 2$, since B_R is a bounded set. We divide T into k increments. (This k is unrelated to the number k for the ϵ approximations.) Let $l = \frac{T}{k}$. We define

$$v_l(x,t) := \frac{1}{l} \int_{jl}^{j(l+1)} v(x,s) \, \mathrm{d}s.$$

From [10],

$$\int_{B_R} \int_0^T [v_l(x,t) - v(x,t)]^2 \, \mathrm{d}t \, \mathrm{d}x \le c_\alpha l^\alpha \int_{B_R} \|v(x,\cdot)\|^2_{W^{\frac{\alpha}{2},2}(0,T)} \, \mathrm{d}x \le Cl^\alpha. \tag{4.8}$$

The above estimate is uniform for any v_j. We now utilize that $[0,T]$ is a finite measure space as well as Minkowski's inequality: The norm of the sum is less than or equal to the sum of the norm,

$$C \ge \sum_{j=0}^{k-1} \int_{lj}^{l(j+1)} \|v(\cdot,t)\|^{2+\gamma}_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}(B_R)} \, \mathrm{d}t$$

$$\ge \sum_{j=0}^{k-1} \frac{1}{l^{1+\gamma}} \left(\int_{lj}^{l(j+1)} \|v(\cdot,t)\|_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}(B_R)} \, \mathrm{d}t \right)^{2+\gamma}$$

$$\ge \sum_{j=0}^{k-1} \frac{1}{l^{1+\gamma}} \left(\left\| \int_{lj}^{l(j+1)} v(x,t) \, \mathrm{d}t \right\|_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}(B_R)} \right)^{2+\gamma}$$

$$= l \sum_{j=0}^{k-1} \left(\left\| \frac{1}{l} \int_{lj}^{l(j+1)} v(x,t) \, \mathrm{d}t \right\|_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}(B_R)} \right)^{2+\gamma}.$$

It then follows from the result in [10] that for every j and $\lambda > 0$, there exists finitely many $\{\beta_1, \cdots, \beta_{M_j}\}$, such that for any fixed j and $v \in \mathcal{F}$, there exists $\beta_i \in \{\beta_1, \cdots, \beta_{M_j}\}$, such that

$$\int_{B_R} \left| \beta_i - \frac{1}{l} \int_{lj}^{l(j+1)} v(x,t) \, \mathrm{d}t \right|^2 \, \mathrm{d}x \le \lambda.$$

Then combining the above estimate with (4.8), we obtain

$$\int_{B_R} \int_0^T |v - \beta_{i,j}|^2 \, \mathrm{d}t \, \mathrm{d}x \le \int_{B_R} \int_0^T |v - v_l|^2 \, \mathrm{d}t \, \mathrm{d}x + \int_{B_R} \int_0^T |v_l - \beta_{i,j}|^2 \, \mathrm{d}t \, \mathrm{d}x$$

$$= \int_{B_R} \int_0^T |v - v_l|^2 \, \mathrm{d}t \, \mathrm{d}x + l \sum_{j=0}^{k-1} \int_{B_R} |v_l - \beta_{i,j}|^2 \, \mathrm{d}x$$

$$\le Cl^\alpha + T\lambda.$$

Since l, λ can be chosen arbitrarily small, $\mathcal{F}$ is totally bounded.

The following result will guarantee that $\nabla(-\Delta)^{-\sigma} u \in L^p$ as $\delta \to 0, R \to \infty$.

Lemma 4.4 *Let u be a solution to* (4.6) *with the right-hand side f and u_0 both satisfying the exponential bound* (4.4). *Then*

$$\int_0^T \|(-\Delta)^{-\sigma} u(t,\cdot)\|^{2+\gamma}_{W^{\frac{2-2\sigma}{2+\gamma}+2\sigma,2+\gamma}} \, dt \leq C$$

with the constant C depending only on the exponential bounds in (4.4), n,γ,T.

Proof u is extended to be zero outside of B_R. We utilize inequality (4.5) and the lifting property for Besov spaces. We use the following results found in [15]. To begin, we have the following characterization of homogenous Besov spaces:

$$W^{\beta,p}(\mathbb{R}^n) = B^{\beta}_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{B}^{\beta}_{p,p}. \tag{4.9}$$

We also have the lifting property of the Riesz potential for the homogeneous Besov spaces

$$\|(-\Delta)^{-\sigma} u\|_{\dot{B}^{\beta+2\sigma}_{p,p}} \leq C\|u\|_{\dot{B}^{\beta}_{p,p}}.$$

To bound u in the nonhomogeneous Besov space, we recall

$$\|(-\Delta)^{-\sigma} u\|_{L^{\frac{nq}{n-2\sigma q}}(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}$$

for any $1 \leq q < \frac{n}{2\sigma}$. From the exponential bounds (4.4) and growth we have that u is uniformly in L^q for all $1 \leq q \leq \infty$. Letting

$$q = \frac{(2+\gamma)n}{n+2\sigma(2+\gamma)} > 1 \quad \text{for } \sigma < \frac{1}{2}, n \geq 2$$

(or let $\sigma < \frac{1}{4}$ for $n = 1$), we obtain by the finite length of T

$$\int_0^T \|(-\Delta)^{-\sigma} u\|^{2+\gamma}_{L^{2+\gamma}(\mathbb{R}^n)} \, dt \leq C.$$

Using again the characterization of homogeneous Besov spaces in (4.9), we obtain the result.

Corollary 4.1 *Let u_k be a sequence of solutions to* (4.6) *with $R \to \infty$ and $\delta \to 0$. For fixed $\rho > 0$, there exists a subsequence and limit with*

$$u_k \to u_0 \in L^p(B_\rho \times (0,T)) \quad \text{for } 1 \leq p \leq 2,$$
$$u_k \rightharpoonup u_0 \in L^{2+\gamma}(0,T;W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}).$$

Furthermore, for any compactly supported ϕ,

$$\epsilon \sum_{j\leq k} \int_{\mathbb{R}^n} [\phi(x,\epsilon j)\mathcal{D}^{\alpha}_{\epsilon} u_0(\epsilon j,x) + u_0 \nabla\phi\nabla(-\Delta)^{-\sigma} u_0] \, dx = \epsilon \sum_{j\leq k} \int_{\mathbb{R}^n} f\phi \, dx. \tag{4.10}$$

Proof The strong and weak convergence is an immediate result of the bound (4.5) and Lemma 4.3. For γ small enough depending on σ, then

$$\frac{2-2\sigma}{2+\gamma} + 2\sigma > 1.$$

Then from Lemma 4.4, we have that

$$\nabla(-\Delta)^{-\sigma} u_k \rightharpoonup \nabla(-\Delta)^{-\sigma} u_0 \in L^{2+\gamma}(0, T; W^{\frac{2-2\sigma}{2+\gamma}+2\sigma-1, 2+\gamma}),$$

and in particular

$$\nabla(-\Delta)^{-\sigma} u_k \rightharpoonup \nabla(-\Delta)^{-\sigma} u_0 \in L^{2+\gamma}((0, T) \times \mathbb{R}^n). \tag{4.11}$$

Then it is immediate from the weak and strong convergence that u_0 is a solution.

We now show the proof of Theorem 1.1.

Proof of Theorem 1.1 We first assume f, u_0 smooth and satisfying the exponential bounds (4.4). Consider solutions u_ϵ to (4.10) over a finite interval $(0, T)$. As before, as $\epsilon \to 0$, there exists a subsequence and a limit $u_\epsilon \to u_0$ with the weak convergence as in (4.11) and strong convergence over compact sets for $1 \le p \le 2$ just as in Lemma 4.3. Then for fixed $\phi \in C_0^\infty$, that u_0 is a solution follows from this convergence. The spatial piece and right-hand side is straightforward to show, and the nonlocal time piece is taken care of as in [1]. We now consider a sequence of solution $\{u_j\}$ with $\{f_j\} \in C^\infty$ with $f_j \to f$ in weak* L^∞. Then again there exists a limit solution u with the right-hand side f. From Remark 1.1 and Lemma 4.1, we can let $T \to \infty$.

Remark 4.1 Theorem 1.1 provides a solution u with initial data u_0 in the sense of (2.4). However, since the initial data is C^2, we have later from Theorem 1.2 that u is actually continuous up to the initial time $u(0, x)$.

Remark 4.2 In this section, we have shown how the estimates in [6] work for equations of the form (2.4). In the same way, one can show that the method of "true (exaggerated) super-solutions" as shown in [6] for $\sigma < \frac{1}{2}$ will also work to prove the property of finite propagation for solutions to (2.4). As the main result of this paper is Hölder regularity of solutions, we will not make this presentation here.

5 Continuity: Method and Lemmas

In this section, we outline the method used to prove Hölder regularity of solutions to (2.4). We follow the method used in [5], which is an adaption of the ideas originally used by De Giorgi. We prove a decrease in oscillation on smaller cylinders, and then utilize the scaling property that if u is a solution to (2.4), then $v(t, x) = A(Bt, Cx)$ is also a solution to (2.4) if $A = B^\alpha C^{2-2\sigma}$. Because of the degenerate nature of the problem, the decrease in oscillation will only occur from above. Since we do not have a decrease in oscillation from below, we will need a lemma that says in essence that if the solution u is above $\frac{1}{2}$ on most of the space time, then u is a distance from zero on a smaller cylinder. To prove the lemmas in this section, we will use energy methods, and thus we will want to use as a test function $F(u)$ for some F. If u is a solution to (2.4), then

$$u \in L^{2+\gamma}(0, T; W^{\frac{2-2\sigma}{2+\gamma}, 2+\gamma}),$$

and it is not clear that $\nabla F(u)$ will be a valid test function. We therefore prove the lemmas for the approximate problems

$$\mathcal{D}_t^\alpha u - \delta \text{div}(u\nabla u) - \text{div}(u\nabla(-\Delta)^{-\sigma}u) = f \quad \text{on } [0,T] \times B_R \tag{5.1}$$

for some large $R > 0$ and small $\delta > 0$ with $u \equiv 0$ on B_R^c. It is actually only necessary to prove the energy inequalities that we will utilize with constants uniform as $\delta \to 0$ and $R \to \infty$. We could also prove the lemmas for the approximate problems (4.6). However, for notational convenience and to make the proofs more transparent, we have chosen to let $\epsilon \to 0$. Because our solution is a limit of discretized solutions, we then are allowed to make the formal computations involved with $\mathcal{D}_t^\alpha u$ even though u may not be regular enough for $\mathcal{D}_t^\alpha u$ to be defined. One simply proves the energy inequalities (and hence the lemmas) for the discretized solutions as was done in [1].

Because of the one-sided nature of our problem, we prove the lemmas for solutions to the equation with the modified term $\text{div}(D(u)\nabla(-\Delta)^{-\sigma}u)$, where $D(u) = d_1 u + d_2$. We assume $0 \le d_1, d_2 \le 2$ and either $d_1 = 1$ or $d_2 \ge \frac{1}{2}$. As will be seen later, when $d_2 \ge \frac{1}{2}$, the proofs are simpler because the problem is no longer degenerate. We now define the exact class of solutions for which we prove the lemmas of this section. u is a solution if $u \equiv 0$ on $B_r^c \times (0,T)$ and for every $\phi \in C_0^\infty((-\infty,T) \times B_R)$, we have

$$\int_{B_R} \int_{-\infty}^T \int_{-\infty}^t [u(t,x) - u(s,x)][\phi(t,x) - \phi(s,x)]K(t,s,x) \, ds \, dt \, dx$$

$$+ \int_{B_R} \int_{-\infty}^T \int_{-\infty}^{2t-T} u(t,x)\phi(t,x)K(t,s,x) \, ds \, dt \, dx$$

$$- \int_{B_R} \int_{-\infty}^T u(t,x)\mathcal{D}_t^\alpha \phi(t,x) \, dt \, dx$$

$$+ \int_{-\infty}^T \int_{B_R} \nabla\phi(t,x)D(u)\nabla u(t,x) \, dx \, dt$$

$$+ \int_{-\infty}^T \int_{B_R} \nabla\phi(t,x)D(u)\nabla(-\Delta)^{-\sigma}u \, dx \, dt$$

$$= \int_{-\infty}^T \int_{B_R} f(t,x)\phi(t,x) \, dx \, dt. \tag{5.2}$$

By Lemmas 4.3–4.4, the lemmas stated in this section will be true when $R \to \infty$ and $\delta \to 0$.

Before stating the lemmas, we define the following function for small $0 < \tau < \frac{1}{4}$:

$$\overline{\Psi}(x,t) := 1 + (|x|^\tau - 2)_+ + (|t|^\tau - 2)_+.$$

We now state the lemmas we will need.

Lemma 5.1 *Let u be a solution to (5.2) with the right-hand side $|f| \le 1$ and $R > 4$ and assume*

$$1 - \overline{\Psi} \le u \le \overline{\Psi} \quad \text{for } \tau < \tau_0.$$

Given $\mu_0 \in \left(0, \frac{1}{2}\right)$ and $\tau_0 < \frac{1}{4}$, there exists $\kappa > 0$ depending on $\mu_0, \tau_0, \sigma, \alpha, n$, such that if

$$\left|\left\{u \ge \frac{1}{2}\right\} \cap \Gamma_4\right| \ge (1 - \kappa)|\Gamma_4|,$$

then $u \geq \mu_0$ on the smaller cylinder Γ_1.

We have a similar lemma from the above.

Lemma 5.2 *Under the same assumptions as Lemma 5.1, given $\mu_1 \in \left(0, \frac{1}{2}\right)$ and $\tau_0 < \frac{1}{4}$, there exists $\kappa > 0$ depending on $\mu_1, \tau_0, \sigma, \alpha, n$, such that if*

$$\left|\left\{u \geq \frac{1}{2}\right\} \cap \Gamma_2\right| \leq \kappa|\Gamma_2|,$$

then $u \leq 1 - \mu_1$ on the smaller cylinder Γ_1.

Lemma 5.2 in essence states that if the set of points for which $u \geq \frac{1}{2}$ is very small, then u pulls down from 1 in a smaller neighborhood. To prove the regularity, we will need a stronger result that in essence states that if the set of points for which $u \geq \frac{1}{2}$ is not all of Γ_4, then u still pulls down from 1 in a smaller neighborhood. This is given by the following.

Lemma 5.3 *Under the same assumptions as Lemma 5.1, assume further for fixed k_0,*

$$\left|\left\{u < \frac{1}{2}\right\} \cap \Gamma_4\right| \geq (1 - \kappa_0)|\Gamma_4|. \tag{5.3}$$

There exists $\lambda_0 > 0$ depending on κ_0, such that if $|f| \leq \lambda_0^3$, then $u \leq 1 - \mu_2$ on Γ_1 for some μ_2 depending on κ_0.

We will choose κ_0 to equal the κ in Lemma 5.1.

6 Pulling the Solution up from Zero

In this section, we provide the proof of Lemma 5.1. This lemma is the most technical to prove. We first prove the lemma in the most difficult case when $D(u) = u + d$ with $0 \leq d \leq 2$. Afterwards, we show how the proof is much simpler when $D(u) = d_1 u + d_2$ with $d_2 \geq \frac{1}{2}$ and $0 \leq d_1 \leq 2$.

We will need the following technical lemma. The proof is found in the appendix.

Lemma 6.1 *Let u, ϕ be two functions, such that $0 \leq u \leq \phi \leq 1$. Let $0 < \gamma < 1$ be a constant. If $|u(x) - u(y)| \geq 4|\phi(x) - \phi(y)|$, then*

$$\frac{2}{5}\left(\frac{4}{5}\right)^\gamma |u_\phi^-(x) - u_\phi^-(y)|^{1+\gamma} \leq \left|\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)}\right| \leq \frac{14}{3}|u_\phi^-(x) - u_\phi^-(y)|. \tag{6.1}$$

Also, if

$$0 \leq \frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)},$$

then $0 \leq u_\phi^-(x) - u_\phi^-(y)$.

If instead we assume $|u(x) - u(y)| \leq 4|\phi(x) - \phi(y)|$, then

$$\left|\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)}\right| \leq 14|\phi(x) - \phi(y)|. \tag{6.2}$$

Remark 6.1 *When $0 \leq u \leq \phi \leq 3$, Lemma 6.1 will hold with new constants by applying the lemma to $\frac{u}{3}, \frac{\phi}{3}$.*

We will use a sequence of cut-off functions $\{\phi_k\}$ which will be chosen to be smooth cut-off functions in space, and smooth increasing cut-off functions in time. We recall that for small $0 < \tau < \frac{1}{2}$,

$$\overline{\Psi}(x,t) := 1 + (|x|^\tau - 2)_+ + (|t|^\tau - 2)_+.$$

We now recall the construction of a sequence of smooth radial cut-offs θ_k from [5] that satisfy

(1) $\theta_k(x) \le \theta_{k-1}(x) \le \cdots \le \theta_0(x)$,

(2) $\frac{|\nabla \theta_k|}{\theta_k} \le C^k \theta_k^{-\frac{1}{m}}$ with $m \ge 2$,

(3) $\theta_{k-1} - \theta_k \ge (1 - \mu_0)2^{-k}$ in the support of θ_k,

(4) $\theta_k \to \mu_0 \chi_{B_2}$ as $k \to \infty$,

(5) the support of θ_k is contained in the set where θ_{k-1} achieves its maximum.

We also have $\theta_0 \equiv 1$ on B_3 and the support of θ_0 is contained in B_4. As a cut-off in time, we consider a sequence $\{\xi_k\}$ satisfying

(1) $\xi_k(t) \le \xi_{k-1}(t)$,

(2) $\xi_k'(t) \le C^k$,

(3) $\xi_k \to \chi_{\{t > -2\}}$ as $k \to \infty$,

(4) $\xi_k = \max \xi_k = 1$ on the interval $[-2 - 2^{-k}, 0]$,

(5) the support of ξ_k is contained in the set where ξ_{k-1} achieves its maximum.

We now define

$$\phi_k(x,t) := 1 - \overline{\Psi}(x,t) + \frac{1}{2}\xi_k(t)\theta_k(x).$$

We use the convention for negative part that $u = u_+ - u_-$. We also write $u_{\phi_k}^- := (u - \phi_k)_-$. We now consider the convex function

$$F(x) := \frac{1}{\gamma + 1}(1 - x)^{\gamma+1} + x - \frac{1}{\gamma + 1}. \tag{6.3}$$

Because of the degenerate nature of our equation, we will want to utilize the test function

$$-F'\left(\frac{u_{\phi_k}^-}{\phi_k + d}\right) = \left(1 - \frac{(u - \phi_k)_-}{\phi_k + d}\right)^\gamma - 1 = -\left[\left(\frac{u + d}{\phi_k + d}\right)^\gamma - 1\right]_-. \tag{6.4}$$

The basic idea of the proof of Lemma 5.1 is as follows. We first seek to obtain an energy inequality of the form

$$\int_{\mathbb{R}^n} \|(u - \phi_k)_-\|_{H^{\frac{\alpha}{2}}}^2 + \int_{-\infty}^0 \|(u - \phi_k)_-\|_{W^{\frac{2-2\sigma}{2+\gamma},2+\gamma}}^{2+\gamma} \le C^k \iint \chi_{\{u < \phi_k\}}.$$

The first step of the proof consists of obtaining the first term on the left-hand side (or the energy) in time. The second step of the proof consists of obtaining the second term on the left-hand side (or the energy) in space. The third step of the proof consists of bounding all remaining terms by the right-hand side and thus obtaining the inequality. In the fourth step, we utilize the fractional Sobolev embedding theorem to obtain the inequality

$$\left(\iint_{U_k} (u - \phi_k)_-^p\right)^{\frac{b}{p}} \le C^k \iint_{U_{k-1}} \chi_{\{u < \phi_k\}},$$

where U_k is the set on which ϕ_k obtains its maximum. We then utilize Tchebychev's inequality to obtain

$$\iint_{U_k} (u - \phi_k)^p_- \leq C^k \left(\iint_{U_{k-1}} (u - \phi_{k-1})^p_- \right)^{\frac{p}{b}}$$

with $p > b$. This nonlinear recursion relation allows us to conclude the proof.

Proof of Lemma 5.1 First Step: Obtaining the energy piece in time We note that for $0 \leq x \leq 1$, $F(x)$ is convex, $F'(x) \geq 0$, and $F''(x) \geq \gamma$. From the convexity and second derivative estimate, we also conclude for $0 \leq x, y \leq 1$

$$F'(x)(x - y) \geq F(x) - F(y) + \left(\frac{\gamma}{2}\right)(x - y)^2, \tag{6.5}$$

$$F(x) \approx x^2. \tag{6.6}$$

We now consider $-F'\left(\frac{u^-_{\phi_k}}{\phi_k + d}\right)\mathcal{D}^\alpha_t u$, and rewrite $u = u^+_{\phi_k} - u^-_{\phi_k} + \phi_k$. To obtain an energy in time, we first consider

$$\int_{-\infty}^0 F'\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right)\mathcal{D}^\alpha_t u^-_{\phi_k}(t) \, dt$$

$$= \int_{-\infty}^0 \int_{-\infty}^0 F'\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right)\left[u^-_{\phi_k}(t) - u^-_{\phi_k}(s)\right]K(t, s) \, ds \, dt$$

$$= \int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)F'\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right)\frac{u^-_{\phi_k}(t) - u^-_{\phi_k}(s)}{\phi_k(t) + d}K(t, s) \, ds \, dt$$

$$\geq \int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)F'\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right)\left[\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d} - \frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right]K(t, s) \, ds \, dt$$

$$\geq \int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)\left[F\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right) - F\left(\frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right)\right]K(t, s) \, ds \, dt$$

$$+ \int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)\frac{\gamma}{2}\left[\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d} - \frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right]^2 K(t, s) \, ds \, dt$$

$$\geq \int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)\left[F\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right) - F\left(\frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right)\right]K(t, s) \, ds \, dt$$

$$+ \int_{-2-2^{-k}}^0 \int_{-2-2^{-k}}^0 \frac{\gamma}{2}[u^-_{\phi_k}(t) - u^-_{\phi_k}(s)]^2 K(t, s) \, ds \, dt$$

$$= (1) + (2).$$

In the first inequality, we used that ϕ_k is increasing in t and positive for $t \geq -4$ as well as $u^-_{\phi_k}(s) = 0$ for $s \leq -4$, and in the second inequality we used (6.5). Term (2) is half of what we will need for the Sobolev embedding (see Lemma 2.1). To gain the other half, we consider term (1). For c, C^k depending on Λ, α and the Lipschitz constant of ϕ_k, we have

$$\int_{-\infty}^0 \int_{-\infty}^0 (\phi_k(t) + d)\left[F\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right) - F\left(\frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right)\right]K(t, s) \, ds \, dt$$

$$= \int_{-\infty}^0 \int_{-\infty}^0 \left[(\phi_k(t) + d)F\left(\frac{u^-_{\phi_k}(t)}{\phi_k(t) + d}\right) - (\phi_k(s) + d)F\left(\frac{u^-_{\phi_k}(s)}{\phi_k(s) + d}\right)\right]K(t, s) \, ds \, dt$$

$$+ \int_{-\infty}^0 \int_{-\infty}^0 [\phi_k(s) - \phi_k(t)] F\Big(\frac{u_{\phi_k}^-(s)}{\phi_k(t) + d}\Big) K(t, s) \; ds \; dt$$

$$\geq c \int_{-\infty}^0 (\phi_k(t) + d) F\Big(\frac{u_{\phi_k}^-(t)}{\phi_k(t) + d}\Big) \frac{1}{(0 - t)^\alpha} \; dt - C^k \int_{-\infty}^0 \chi_{\{u(t) < \phi_k(t)\}} \; dt$$

$$\geq c \int_{-2-2^{-k}}^0 \frac{(u_{\phi_k}^-)^2(t)}{(\phi_k(t) + d)(0 - t)^\alpha} \; dt - C^k \int_{-\infty}^T \chi_{\{u(t) < \phi_k(t)\}} \; dt.$$

The last inequality coming from (6.6).

Now

$$- \int_{-\infty}^0 F'\Big(\frac{u_{\phi_k}^-(t)}{\phi_k(t) + d}\Big) \mathcal{D}_t^\alpha u_{\phi_k}^+(t) \; dt \geq 0,$$

and in this proof, we ignore this term which will be on the left hand side. Now for the term involving ϕ_k, we have

$$- \int_{-\infty}^0 F'\Big(\frac{u_{\phi_k}^-(t)}{\phi_k(t) + d}\Big) \mathcal{D}_t^\alpha \phi_k(t) \; dt \geq -C^k \int_{-\infty}^0 \chi_{\{u < \phi_k\}} \; dt.$$

Then utilizing the embedding theorem for fractional Sobolev spaces (see [10]) combined with the above inequalities, we obtain

$$\int_{-\infty}^0 F'\Big(\frac{u_{\phi_k}^-}{\phi_k + d}\Big) \mathcal{D}_t^\alpha u(t) \; dt \geq c \int_{-2-2^{-k}}^0 \frac{(u_{\phi_k}^-)^2(t)}{(0 - t)^\alpha} \; dt$$

$$+ c \int_{-2-2^{-k}}^0 \int_{-2-2^{-k}}^t \frac{[u_{\phi_k}^-(t) - u_{\phi_k}^-(s)]^2}{(t - s)^{1+\alpha}} \; ds \; dt$$

$$- C^k \int_{-\infty}^0 \chi_{\{u < \phi_k\}} \; dt$$

$$\geq c \Big(\int_{-2-2^{-k}}^0 (u_{\phi_k}^-)^{\frac{2}{1-\alpha}} \; dt \Big)^{1-\alpha} - C^k \int_{-\infty}^0 \chi_{\{u < \phi_k\}} \; dt. \qquad (6.7)$$

After integrating in the spatial variable, we have

$$c \int_{\mathbb{R}^n} \Big(\int_{-2-2^{-k}}^0 (u - \phi_k)_-^{\frac{2}{1-\alpha}} \; dt \Big)^{1-\alpha} \; dx$$

$$- \int_{-\infty}^0 \int_{\mathbb{R}^n} (u + d) \nabla F'\Big(\frac{u_{\phi_k}^-}{\phi_k + d}\Big) \nabla(-\Delta)^{-\sigma} u \; dx \; dt$$

$$- \int_{-\infty}^0 \int_{\mathbb{R}^n} (u + d) \nabla F'\Big(\frac{u_{\phi_k}^-}{\phi_k + d}\Big) \nabla u \; dx \; dt$$

$$\leq - \int_{-\infty}^0 \int_{\mathbb{R}^n} f F'\Big(\frac{u_{\phi_k}^-}{\phi_k + d}\Big) \; dx \; dt + C^k \int_{\mathbb{R}^n} \int_{-\infty}^0 \chi_{\{u < \phi_k\}} \; dt \; dx$$

$$\leq C^k \int_{-\infty}^0 \int_{\mathbb{R}^n} \chi_{\{u < \phi_k\}} \; dt \; dx.$$

Second Step: Obtaining the energy piece in space We now turn our attention to the elliptic portion of the problem. We recall from [5] the identity

$$\mathcal{B}(v, w) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[v(x) - v(y)][w(x) - w(y)]}{|x - y|^{n+2-2\sigma}} \; dx \; dy = c_{n,\sigma} \int_{\mathbb{R}^n} \nabla v \nabla(-\Delta)^{-\sigma} u \; dx.$$

We multiply by our test function (6.4) and integrate by parts. On the left-hand side of the equation, we have

$$\chi_{\{u<\phi_k\}}\nabla\Big[\Big(\frac{u+d}{\phi_k+d}\Big)^{\gamma}-1\Big](u+d)\nabla(-\Delta)^{-\sigma}u$$

$$=\chi_{\{u<\phi_k\}}\gamma\Big(\frac{u+d}{\phi_k+d}\Big)^{\gamma-1}\nabla\Big(\frac{u+d}{\phi_k+d}\Big)(u+d)\nabla(-\Delta)^{-\sigma}u$$

$$=\chi_{\{u<\phi_k\}}\frac{\gamma}{\gamma+1}\nabla\Big[\Big(\frac{u+d}{\phi_k+d}\Big)^{\gamma+1}-1\Big](\phi_k+d)\nabla(-\Delta)^{-\sigma}u$$

$$=\chi_{\{u<\phi_k\}}\frac{\gamma}{\gamma+1}\nabla\Big[\frac{(u+d)^{\gamma+1}}{(\phi_k+d)^{\gamma}}-(\phi_k+d)\Big]\nabla(-\Delta)^{-\sigma}u$$

$$\quad-\frac{\gamma}{\gamma+1}\chi_{\{u<\phi_k\}}\Big[\Big(\frac{u+d}{\phi_k+d}\Big)^{\gamma+1}-1\Big]\nabla\phi_k\nabla(-\Delta)^{-\sigma}u$$

$$:=(1)+(2).$$

We now focus on (1) which will give us the energy term we need. For the term $(-\Delta)^{-\sigma}u$, we rewrite $u=(u-\phi_k)_++(u-\phi_k)_-+\phi_k:=u_{\phi_k}^+-u_{\phi_k}^-+\phi_k$. Then we rewrite $(1)=(1a)+(1b)+(1c)$. We focus on the term (1b). We rewrite

$$(1b)=(1bi)+(1bii)$$

$$:=-\chi_{\{u<\phi_k\}}\frac{\gamma}{\gamma+1}\nabla\Big[\frac{(u+d)^{\gamma+1}}{(\phi_k+d)^{\gamma}}\Big]\nabla(-\Delta)^{-\sigma}u_{\phi_k}^-$$

$$\quad+\chi_{\{u<\phi_k\}}\frac{\gamma}{\gamma+1}\nabla\phi_k\nabla(-\Delta)^{-\sigma}u_{\phi_k}^-.$$

The term (1bi) will give us the energy term in space that we will need,

$$(1bi)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\gamma}{\gamma+1}\nabla\Big[\frac{(u+d)^{\gamma+1}}{(\phi_k+d)^{\gamma}}(x)\Big]\frac{1}{|x-y|^{n-2\sigma}}\nabla u_{\phi_k}^-(y)\ dx\ dy$$

$$=c_{n,\sigma}\frac{\gamma}{\gamma+1}\mathcal{B}(\chi_{\{u<\phi_k\}}\frac{(u+d)^{\gamma+1}}{(\phi_k+d)^{\gamma}},-u_{\phi_k}^-).$$

We define the set

$$A_k:=\{|u(x)-u(y)|\geq 4|\phi_k(x)-\phi_k(y)|\}.$$

It is clear that A_k contains the set $V_k\times V_k$, where we define V_k as the set on which θ_k achieves its maximum. From Lemma 6.1 and Remark 6.1, we have

$$\iint\limits_{A_k}\Big[\frac{(u+d)^{\gamma+1}(y)}{(\phi_k+d)^{\gamma}(y)}-\frac{(u+d)^{\gamma+1}(x)}{(\phi_k+d)^{\gamma}(x)}\Big]\frac{[u_{\phi_k}^-(x)-u_{\phi_k}^-(y)]}{|x-y|^{n+2-2\sigma}}\ dx\ dy$$

$$\geq c\iint\limits_{A_k}\frac{|u_{\phi_k}^-(x)-u_{\phi_k}^-(y)|^{2+\gamma}}{|x-y|^{n+2-2\sigma}}\ dx\ dy.$$

We now label U_k as the set where ϕ_k achieves its maximum. Notice that $U_k=[-2-2^{-K},0]\times V_k$. To utilize the fractional Sobolev embedding on $V_k\times V_k$, we also will need an L^p norm of $u_{\phi_k}^-$ on V_k. We utilize half of the integral of $u_{\phi_k}^-$ that we gained from the following fractional time term:

$$\int_{\mathbb{R}^n}\int_{-2-2^{-k}}^{0}\frac{(u_{\phi_k}^-)^2(t)}{(0-t)^{\alpha}}\ dt\ dx$$

$$\geq \frac{1}{2} \iint\limits_{U_k} \frac{(u_{\phi_k}^-)^2(t)}{(0-t)^\alpha} \; \mathrm{d}x \; \mathrm{d}t + \frac{1}{2} \iint\limits_{U_k} \frac{(u_{\phi_k}^-)^{2+\gamma}(t)}{(0-t)^\alpha} \; \mathrm{d}x \; \mathrm{d}t.$$

The inequality comes from the fact that $0 \leq u_{\phi_k}^- \leq 1$. Now from the fractional Sobolev embedding (see [10]),

$$\int_{-2-2^{-k}}^0 \iint\limits_{V_k \times V_k} \frac{|u_{\phi_k}^-(x) - u_{\phi_k}^-(y)|^{2+\gamma}}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}t + \frac{1}{2} \iint\limits_{U_k} (u - \phi_k)_-^{2+\gamma}(t) \; \mathrm{d}x \; \mathrm{d}t$$

$$\geq c_{n,\sigma,\gamma} \int_{-2-2^{-k}}^0 \left(\int_{V_k} (u_{\phi_k}^-)^{\frac{n(2+\gamma)}{n-2+2\sigma}} \; \mathrm{d}x \right)^{\frac{n-2+2\sigma}{n}} \; \mathrm{d}t. \tag{6.8}$$

This is the helpful spatial term on the left-hand side that we will return to later.

Third Step: Bounding the remaining terms We will now show that everything left in our equation can be bounded by

$$C^k \int_{-\infty}^0 \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x \; \mathrm{d}t. \tag{6.9}$$

We will denote

$$X_k(x,y) := \chi_{\{u(x)<\phi_k(x)\}} + \chi_{\{u(y)<\phi_k(y)\}}.$$

For the remainder of term (1bi), we have

$$\left| \iint\limits_{A_k^c} \left[\frac{(u+d)^{\gamma+1}(y)}{(\phi_k+d)^\gamma(y)} - \frac{(u+d)^{\gamma+1}(x)}{(\phi_k+d)^\gamma(x)} \right] \frac{[u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y \right|$$

$$\leq C \iint\limits_{A_k^c} X_k(x,y) \frac{|\phi_k(x) - \phi_k(y)|^2}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y$$

$$\leq C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x.$$

The last inequality is due to the Lipschitz constant of ϕ_k when x, y are close, and the tail growth of ϕ_k when x, y are far apart.

We now control the term (1bii). Again, we split the region of integration over A_k and A_k^c. Using Hölder's inequality (provided $2\sigma > \frac{\gamma}{1+\gamma}$ and therefore we must choose γ small when σ is small) as well as the Lipschitz and sup bounds on ϕ_k, we have

$$c_{n,\sigma} \iint\limits_{A_k} \frac{[\phi_k(x) - \phi_k(y)][u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y$$

$$\leq \eta \iint\limits_{A_k} \frac{[u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]^{2+\gamma}}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y$$

$$+ C \iint\limits_{A_k} \frac{[\phi_k(x) - \phi_k(y)]^{\frac{2+\gamma}{1+\gamma}}}{|x-y|^{n+2-2\sigma}} X_k(x,y) \; \mathrm{d}x \; \mathrm{d}y$$

$$\leq \eta \iint\limits_{A_k} \frac{[u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]^{2+\gamma}}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y + C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x.$$

The first term is absorbed into the left-hand side and the second term is controlled exactly as before.

We now consider the integration over A_k^c.

$$\iint\limits_{A_k^c} \frac{[\phi_k(x) - \phi_k(y)][u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq \iint\limits_{A_k^c} \frac{[\phi_k(x) - \phi_k(y)]^2}{|x-y|^{n+2-2\sigma}} X_k(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq C^k \int_{\mathbb{R}^n} \chi_{\{u < \phi_k\}} \, \mathrm{d}x.$$

We now turn our attention to the term (1c). By Lemma 6.1, we have

$$\left| \mathcal{B}\left(\chi_{\{u < \phi_k\}} \left(\frac{(u+d)^{\gamma+1}}{(\phi_k+d)^\gamma - \phi_k} \right), \phi_k \right) \right|$$

$$\leq \left| \mathcal{B}\left(\chi_{\{u < \phi_k\}} \frac{(u+d)^{\gamma+1}}{(\phi_k+d)^\gamma}, \phi_k \right) \right| + \left| \mathcal{B}(\chi_{\{u < \phi_k\}} \phi_k), \phi_k) \right|.$$

Both of the above terms are handled exactly as before by using Lemma 6.1 and splitting the region of integration over A_k and A_k^c.

The term (1a) is

$$(1a) = \mathcal{B}(\chi_{\{u < \phi_k\}} \frac{u^{\gamma+1}}{\phi_k^\gamma} - (\phi_k + d), u_{\phi_k}^+)$$

$$= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\{u(x) < \phi_k(x)\}} \left[\phi_k(x) + d - \frac{(u+d)^{\gamma+1}(x)}{(\phi_k+d)^\gamma(x)} \right] \frac{u_{\phi_k}^+(y)}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y \geq 0. \quad (6.10)$$

The factor of 2 comes form the symmetry of the kernel. We will utilize this nonnegative term shortly.

We now consider the term (2) which we recall as

$$-\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\gamma}{\gamma+1} \chi_{\{u < \phi_k\}} \left(\left(\frac{u+d}{\phi_k+d} \right)^{\gamma+1} - 1 \right) \nabla\phi_k \nabla L(x-y)[u(y) - u(x)] \, \mathrm{d}x \, \mathrm{d}y.$$

In the above, $L = \nabla(-\Delta)^{-\sigma}$, and we have

$$\nabla L(x-y) \approx |x-y|^{-(n+1-2\sigma)}.$$

We again write $u = u_{\phi_k}^+ - u_{\phi_k}^- + \phi_k$. To control the term involving ϕ_k, we integrate over the two sets $\{|x-y| \leq 8\}$ and $\{|x-y| > 8\}$. We use that $|\phi_k(x) - \phi_k(y)| \leq C^k |x-y|$ when $|x-y| \leq 8$, and $|\phi_k(x) - \phi_k(y)| \leq |x-y|^\tau$ when $|x-y| > 8$, as well as the bound $|\nabla\phi_k| \leq C^k$, to obtain

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\gamma}{\gamma+1} \chi_{\{u < \phi_k\}} \left(\left(\frac{u+d}{\phi_k+d} \right)^{\gamma+1} - 1 \right) \nabla\phi_k \nabla L(x-y)[\phi_k(y) - \phi_k(x)] \, \mathrm{d}x \, \mathrm{d}y \right|$$

$$\leq C^k \iint\limits_{|x-y| \leq 8} \chi_{\{u < \phi_k\}} |x-y|^{-(n-2\sigma)} \, \mathrm{d}x \, \mathrm{d}y$$

$$+ C^k \iint\limits_{|x-y| > 8} \chi_{\{u < \phi_k\}} |x-y|^{-(n+1-2\sigma-\tau)} \, \mathrm{d}x \, \mathrm{d}y$$

$$\le C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x.$$

We now use the same set decomposition with $-u^-_{\phi_k}$, the inequality $|u^-_{\phi_k}| \le 1$ as well as Hölder's inequality

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\gamma}{\gamma+1} \chi_{\{u<\phi_k\}} \left(\left(\frac{u+d}{\phi_k+d} \right)^{\gamma+1} - 1 \right) \nabla \phi_k \nabla L(x-y) [u^-_{\phi_k}(y) - u^-_{\phi_k}(x)] \; \mathrm{d}x \; \mathrm{d}y \right|$$

$$\le C^k \iint\limits_{|x-y|\le 8} \chi_{\{u<\phi_k\}} |x-y|^{-(n-2\sigma+\frac{\gamma}{1+\gamma})} \; \mathrm{d}x \; \mathrm{d}y$$

$$+ \zeta \iint\limits_{|x-y|\le 8} \frac{[u^-_{\phi_k}(y) - u^-_{\phi_k}(x)]^{2+\gamma}}{|x-y|^{n+2-2\sigma}} \; \mathrm{d}x \; \mathrm{d}y$$

$$+ C^k \iint\limits_{|x-y|>8} \chi_{\{u<\phi_k\}} |x-y|^{-(n+1-2\sigma-\tau)} \; \mathrm{d}x \; \mathrm{d}y.$$

The third term is bounded by

$$C \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x$$

provided $\tau < 1 - 2\sigma$ as well as the first term provided again that $2\sigma > \frac{\gamma}{1+\gamma}$. The second term can be bounded as before by splitting the region of integration over A_k and A_k^c and absorbing the region over A_k into the left-hand side.

We now turn our attention to the last term involving $u^+_{\phi_k}$. We first remark that the integral becomes

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\{u(x)<\phi_k(x)\}} \left(\left(\frac{u+d}{\phi_k+d} \right)^{\gamma+1} - 1 \right) \nabla \phi_k \nabla L(x-y) u^+_{\phi_k}(y) \; \mathrm{d}x \; \mathrm{d}y.$$

We first consider the set $|x-y|>8$. Since $u^+_{\phi_k} \le \overline{\Psi}$,

$$\left| \iint\limits_{|x-y|>8} \chi_{\{u(x)<\phi_k(x)\}} \left(\left(\frac{u+d}{\phi_k+d} \right)^{\gamma+1} - 1 \right) \nabla \phi_k \nabla L(x-y) u^+_{\phi_k}(y) \; \mathrm{d}x \; \mathrm{d}y \right|$$

$$\le C^k \left| \iint\limits_{|x-y|>8} \chi_{\{u(x)<\phi_k(x)\}} |x-y|^{-(n+1-2\sigma+\tau)} \; \mathrm{d}x \; \mathrm{d}y \right|$$

$$\le C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \; \mathrm{d}x.$$

When $|x-y|<8$, we make the further decomposition

$$\frac{|\nabla \phi_k(x)|}{\phi_k(x)} |x-y| \le \eta$$

to absorb the integral by the nonnegative quantity (6.10). In the complement, when

$$\frac{|\nabla \phi_k(x)|}{\phi_k(x)} |x-y| > \eta,$$

we use $\phi_k^{-\frac{1}{m}} C^k \ge \frac{|\nabla \phi_k|}{\phi_k}$ and integrate in y

$$\left| \int_{B_8} \nabla L(x-y) u^+_{\phi_k}(y) \mathrm{d}y \right| \le \int_{\eta \phi_k^{\frac{1}{m}} C^{-k}}^{8} \frac{r^{n-1}}{r^{n+1-2\sigma}} \le \max\{C, (\eta C^k)^{2\sigma-1} \phi_k^{\frac{2\sigma-1}{m}}\}.$$

The remainder of the terms are bounded by $|\nabla \phi_k| \leq C^k \phi_k^{1-\frac{1}{m}}$. By multiplying by the term $\chi_{\{u<\phi_k\}}$ and integrating, we end up in the worst case with

$$C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \phi_k^{1-\frac{1}{m}+\frac{2\sigma-1}{m}} \, \mathrm{d}x \leq C^k \int_{\mathbb{R}^n} \chi_{\{u<\phi_k\}} \, \mathrm{d}x,$$

since $m \geq 2$. The last term to consider is the local spatial term. We use the Cauchy-Schwarz inequality

$$\delta \int_{B_R} \chi_{\{u<\phi_k\}} \nabla \Big[\Big(\frac{u+d}{\phi_k+d} \Big)^\gamma - 1 \Big] (u+d)\nabla u \, \mathrm{d}x$$

$$= \delta \int_{B_R} \gamma \Big(\frac{u+d}{\phi_k+d} \Big)^\gamma \frac{|\nabla u|^2}{\phi_k+d} \, \mathrm{d}x$$

$$- \delta \int_{B_R} \gamma \Big(\frac{u+d}{\phi_k+d} \Big)^{\gamma+1} \nabla u \nabla \phi_k \, \mathrm{d}x$$

$$\geq (1-\eta)\delta\gamma \int_{B_R} \Big(\frac{u+d}{\phi_k+d} \Big)^\gamma |\nabla u|^2 \, \mathrm{d}x$$

$$- C\delta \int_{B_R} \Big(\frac{u+d}{\phi_k+d} \Big)^{\gamma+2} |\nabla \phi_k|^2 \chi_{\{u<\phi_k\}} \, \mathrm{d}x$$

$$\geq (1-\eta)\delta\gamma \int_{B_R} \Big(\frac{u+d}{\phi_k+d} \Big)^\gamma |\nabla u|^2 \, \mathrm{d}x - C^k \delta \int_{B_R} \chi_{\{u<\phi_k\}} \, \mathrm{d}x.$$

Retaining the energy from (6.8) on the left-hand side and moving everything else to the right-hand side which is bounded by (6.9), our energy inequality becomes

$$c \int_{V_k} \Big(\int_{-2-2^{-k}}^0 (u-\phi_k)_-^{\frac{2}{1-\alpha}} \, \mathrm{d}t \Big)^{1-\alpha} \mathrm{d}x$$

$$+ c \int_{-2-2^{-k}}^0 \Big(\int_{V_k} (u_{\phi_k}^-)^{\frac{n(2+\gamma)}{n-2+2\sigma}} \, \mathrm{d}x \Big)^{\frac{n-2+2\sigma}{n}} \mathrm{d}t$$

$$\leq C^k \int_{\mathbb{R}^n} \int_{-\infty}^0 \chi_{\{u<\phi_k\}} \, \mathrm{d}t \, \mathrm{d}x \leq C^k \iint_{U_{k-1}} \chi_{\{u<\phi_k\}} \, \mathrm{d}x \, \mathrm{d}t. \qquad (6.11)$$

Fourth Step: The nonlinear recursion relation We now (as in [1]) use Hölder's inequality twice with the relations

$$\frac{\beta}{p_1} + \frac{1-\beta}{p_2} = \frac{1}{p} = \frac{\beta}{p_3} + \frac{1-\beta}{p_4}.$$

For a function v, we obtain

$$\iint v^p \leq \iint v^{p\beta} v^{p(1-\beta)}$$

$$\leq \int \Big(\int v^{p_1} \Big)^{\frac{p\beta}{p_1}} \Big(\int v^{p_2} \Big)^{\frac{p(1-\beta)}{p_2}}$$

$$\leq \Big(\int \Big(\int v^{p_1} \Big)^{\frac{p_3}{p_1}} \Big)^{\beta \frac{p}{p_3}} \Big(\int \Big(\int v^{p_2} \Big)^{\frac{p_4}{p_2}} \Big)^{(1-\beta)\frac{p}{p_4}}.$$

We now choose

$$p_1 = 2, \quad p_2 = \frac{n(2+\gamma)}{n-(2-2\sigma)}, \quad p_3 = \frac{2}{1-\alpha}, \quad p_4 = 2+\gamma,$$

so that if $r = 2 - 2\sigma$, then

$$p = 2\frac{r + \alpha n\frac{2+\gamma}{2}}{(1-\alpha)r + \alpha n} \quad \text{and} \quad \beta = \frac{r}{r + \alpha n\frac{2+\gamma}{2}}.$$

We now use Hölder's inequality one more time to obtain

$$\left(\iint v^p\right)^{\frac{b}{p}} \leq \left(\int\left(\int v^{p_1}\right)^{\frac{p_3}{p_1}}\right)^{\beta\frac{b}{p_3}}\left(\int\left(\int v^{p_2}\right)^{\frac{p_4}{p_2}}\right)^{(1-\beta)\frac{b}{p_4}}$$

$$\leq \frac{1}{\omega}\left(\int\left(\int v^{p_1}\right)^{\frac{p_3}{p_1}}\right)^{\frac{\beta b\omega}{p_3}} + \frac{\omega-1}{\omega}\left(\int\left(\int v^{p_2}\right)^{\frac{p_4}{p_2}}\right)^{\frac{(1-\beta)b\omega}{p_4(\omega-1)}}.$$

We choose

$$\omega = \frac{(2+\gamma\beta)}{(2+\gamma)\beta} \quad \text{and} \quad b = \frac{2(2+\gamma)}{2+\gamma\beta} < p,$$

so that

$$\left(\iint v^p\right)^{\frac{b}{p}} \leq \frac{1}{\omega}\left(\int\left(\int v^2\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} + \frac{\omega-1}{\omega}\left(\int\left(\int v^{\frac{n(2+\gamma)}{n-r}}\right)^{\frac{n-r}{n}}\right)$$

$$\leq \frac{1}{\omega}\int\left(\int v^{\frac{2}{1-\alpha}}\right)^{1-\alpha} + \frac{\omega-1}{\omega}\left(\int\left(\int v^{\frac{n(2+\gamma)}{n-r}}\right)^{\frac{n-r}{n}}\right), \qquad (6.12)$$

where we used Minkowski's inequality in the last inequality. Substituting $u_{\phi_k}^-$ for v in (6.12) and utilizing (6.8), we obtain

$$\left(\iint_{U_k}(u-\phi_k)_-^p\right)^{\frac{b}{p}} \leq C^k\iint_{U_{k-1}}\chi_{\{u<\phi_k\}}. \qquad (6.13)$$

We first recall that $\phi_{k-1} \geq \phi_k + (1-\mu_0)2^{-k}$. We now utilize Tchebychev's inequality

$$\iint_{U_k}\chi_{\{u<\phi_k\}} \leq \left(\frac{2^k}{1-\mu_0}\right)^p\iint_{U_{k-1}}(u-\phi_{k-1})_-^p.$$

Combining the above inequality with (6.13), we conclude

$$\iint_{U_k}(u-\phi_k)_-^p \leq C^k\left(\iint_{U_{k-1}}(u-\phi_{k-1})_-^p\right)^{\frac{p}{b}}.$$

If we define

$$M_k := \iint_{U_k}(u-\phi_k)_-^p,$$

then

$$M_k \leq C^k M_{k-1}^{\frac{p}{b}}.$$

Since $p > b$, if M_0 is sufficiently small (depending on C and $\frac{p}{b}$), we obtain that $M_k \to 0$, and hence $u \geq \mu_0$.

We now prove Lemma 5.1 in the case when $D(x) = (d_1 x + d_2)$ with $d_1 \leq 2$ and $d_2 \geq \frac{1}{2}$. This is actually much simpler, because we can utilize the test function $-(u-\phi_k)_-$ as when dealing with a linear equation.

Proof We choose as the test function $-(u - \phi_k)_-$. Notice that

$$\nabla u_{\phi_k}^-(d_1 u + d_2) = \nabla u_{\phi_k}^- d_1 u + \nabla u_{\phi_k}^- d_2.$$

The fact that $d_2 \geq \frac{1}{2}$ gives a nondegenerate linear term which we utilize. From the computations in [1], we then have

$$\int_{\mathbb{R}^n} \int_{-\infty}^0 u_{\phi_k}^- \mathcal{D}_t^\alpha u \, dx \, dt$$

$$\geq c \int_{\mathbb{R}^n} \Big(\int_{-\infty}^0 (u_{\phi_k}^-)^{\frac{2}{1-\alpha}} \, dt \Big)^{1-\alpha} \, dx$$

$$- C^k \int_{\mathbb{R}^n} \int_{-\infty}^0 \chi_{\{u < \phi_k\}} \, dt \, dx,$$

and even more importantly

$$\int_{-\infty}^0 \int_{\mathbb{R}^n} \nabla u_{\phi_k}^- \nabla(-\Delta)^{-\sigma} u \, dx \, dt$$

$$\geq c \int_{-\infty}^0 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]^2}{|x - y|^{n+2-2\sigma}} \, dx \, dy \, dt$$

$$- C^k \int_{-\infty}^0 \int_{\mathbb{R}^n} \chi_{\{u < \phi_k\}} \, dx \, dt. \tag{6.14}$$

Notice that in the above inequality, we have the power $|\cdot|^2$ rather than $|\cdot|^{2+\gamma}$.

We now show how to bound the terms invovling $d_1 u$,

$$-\nabla u_{\phi_k}^-[u_{\phi_k}^+ - u_{\phi_k}^- + \phi_k] = \nabla \Big[\frac{(u_{\phi_k}^-)^2}{2} - u_{\phi_k}^- \phi_k \Big] + u_{\phi_k}^- \nabla \phi_k = (1) + (2).$$

Then multiplying (1) by $\nabla(-\Delta)^{-\sigma} u$ and integrating over $\mathbb{R}^n$, we have

$$\mathcal{B}\Big(\frac{(u_{\phi_k}^-)^2}{2} - u_{\phi_k}^- \phi_k, u_{\phi_k}^+ - u_{\phi_k}^- + \phi_k \Big).$$

Now

$$\Big[\frac{(u_{\phi_k}^-)^2}{2} - u_{\phi_k}^- \phi_k \Big](x) - \Big[\frac{(u_{\phi_k}^-)^2}{2} - u_{\phi_k}^- \phi_k \Big](y)$$

$$= \frac{1}{2}[u_{\phi_k}^-(y) - u_{\phi_k}^-(x)][\phi_k(x) + \phi_k(y) - u_{\phi_k}^-(x) - u_{\phi_k}^-(y)]$$

$$- \frac{1}{2}[u_{\phi_k}^-(y) + u_{\phi_k}^-(x)][\phi_k(x) - \phi_k(y)]$$

$$= (1a) + (1b).$$

We write $u = u_{\phi_k}^+ - u_{\phi_k}^-$. We break up our set into the two regions

$$F_k := \{(x, y) : |u_{\phi_k}^-(x) - u_{\phi_k}^-(y)| \geq 2|\phi_k(x) - \phi_k(y)|\}.$$

We notice that on the set F_k, we have

$$\phi_k(x) + \phi_k(y) - u_{\phi_k}^-(x) - u_{\phi_k}^-(y) \geq \frac{|u_{\phi_k}^-(x) - u_{\phi_k}^-(y)|}{2}.$$

Then integrating over F_k, we have for the term (1a) with right term $-u^-_{\phi_k}$

$$-\iint_{F_k} \frac{\left(\left[\frac{(u^-_{\phi_k})^2}{2} - u^-_{\phi_k}\phi_k\right](x) - \left[\frac{(u^-_{\phi_k})^2}{2} - u^-_{\phi_k}\phi_k\right](y)\right)(u^-_{\phi_k}(x) - u^-_{\phi_k}(y))}{|x-y|^{n+2-2\sigma}} \, dx \, dy$$

$$\geq \iint_{F_k} \frac{|u^-_{\phi_k}(x) - u^-_{\phi_k}(y)|^3}{|x-y|^{n+2-2\sigma}} \, dx \, dy \geq 0.$$

This is the nonnegative energy piece which we actually do not need having obtained a better piece in (6.14). All of the remaining terms in (1) can be bounded by breaking up the region of integration over F_k, F_k^c. Over F_k, we use Hölder's inequality with $p = 2$ rather than with $p = 2 + \gamma$ and absorb the small pieces by the term in (6.14). We use the same methods as before to bound the integration over F_k^c. Bounding the term (2) is done as before with slightly easier computations.

The local spatial term is bounded in the usual manner.

7 Pulling the Solution Down

In this section, we prove Lemma 5.2. We will need the following estimate that is analogous to Lemma 6.1.

Lemma 7.1 *Let u, ϕ be two functions such that $\frac{1}{2} \leq \phi \leq u \leq 1$. Let $0 < \gamma < 1$ be a constant. If $|u(x) - u(y)| \geq 4|\phi(x) - \phi(y)|$, then*

$$c_1|u^+_\phi(x) - u^+_\phi(y)|^{1+\gamma} \leq \left|\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)}\right| \leq c_2|u^+_\phi(x) - u^+_\phi(y)|. \tag{7.1}$$

Also, if

$$0 \leq \frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)},$$

then $0 \leq (u - \phi)_-(x) - (u - \phi)_-(y)$.

If instead we assume $|u(x) - u(y)| \leq 4|\phi(x) - \phi(y)|$, then

$$\left|\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)}\right| \leq 14|\phi(x) - \phi(y)|. \tag{7.2}$$

The proof is similar to the proof of Lemma 6.1. In this case, since $u > \phi$ one uses the bound above on u and the fact that ϕ is bounded by below.

Proof of Lemma 5.2 The proof is nearly identical. We mention the differences. We only consider $D(u) = u$, since the modifications for handling $D(u) = d_1u + d_2$ have already been shown in the proof of Lemma 5.1. We consider a similar test function

$$F(x) = \frac{1}{\gamma+1}(1+x)^{\gamma+1} - x + \frac{1}{\gamma+1}.$$

We then utilize

$$F'\left(\frac{u^+_{\phi_k}}{\phi_k}\right) = \left(1 + \frac{(u - \phi_k)_+}{\phi_k}\right)^\gamma - 1. \tag{7.3}$$

This time we consider the same test functions $\theta_k(x)$ in the space variable, but this time we multiply only by a single cut-off in time $\xi_0(t)$. We define our ϕ_k as

$$\phi_k := \overline{\Psi}(x,t) - \xi_0(t)\theta_k\Big(\frac{x}{2}\Big).$$

To obtain the same estimate in time, we only need to recognize that ϕ_k is now decreasing in time and bounded by below by $\frac{1}{2}$,

$$
F'\Big(\frac{u^+_{\phi_k}(t)}{\phi_k(t)}\Big)[u^+_{\phi_k}(t) - u^+_{\phi_k}(s)]
$$
$$
= \phi_k(t)F'\Big(\frac{u^+_{\phi_k}(t)}{\phi_k(t)}\Big)\frac{u^+_{\phi_k}(t) - u^+_{\phi_k}(s)}{\phi_k(t)}
$$
$$
= \phi_k(t)F'\Big(\frac{u^+_{\phi_k}(t)}{\phi_k(t)}\Big)\Big[\frac{u^+_{\phi_k}(t)}{\phi_k(t)} - \frac{u^+_{\phi_k}(s)}{\phi_k(s)}\Big] - \phi_k(t)u^+_{\phi_k}(s)F'\Big(\frac{u^+_{\phi_k}(t)}{\phi_k(t)}\Big)\Big[\frac{1}{\phi_k(t)} - \frac{1}{\phi_k(s)}\Big]
$$
$$
\geq \frac{1}{2}\Big[F\Big(\frac{u^+_{\phi_k}(t)}{\phi_k(t)}\Big) - F\Big(\frac{u^+_{\phi_k}(s)}{\phi_k(s)}\Big)\Big] + \frac{\gamma}{2}[u^+_{\phi_k}(t) - u^+_{\phi_k}(s)]^2 - Cu^+_{\phi_k}(s)(t-s). \tag{7.4}
$$

The negative constant comes from the fact that ϕ_k^{-1} is Lipschitz. Then everything proceeds as before. Since our cut-off is bounded by below, our L^p norm in time occurs over all of $(-\infty, 0)$. We obtain as before

$$
c\int_{\mathbb{R}^n}\Big(\int_{-\infty}^0 (u - \phi_k)_+^{\frac{2}{1-\alpha}}\,dt\Big)^{1-\alpha}\,dx + \int_{-\infty}^0\int_{\mathbb{R}^n} u\nabla F'\Big(\frac{u^+_{\phi_k}}{\phi_k}\Big)\nabla(-\Delta)^{-\sigma}u\,dx\,dt
$$
$$
\leq \int_{\mathbb{R}^n}\int_{-\infty}^0 \chi_{\{u > \phi_k\}}\,dt\,dx.
$$

The spatial portion of the problem is handled exactly as before.

8 Decrease in Oscillation

We define

$$F(t,x) := \frac{1}{4}\sup\{-1, \inf\{0, |x|^2 - 9\}\} + \frac{1}{4}\sup\{-1, \inf\{0, |t|^2 - 9\}\}.$$

We point out that F is Lipschitz, compactly supported in $[-3,0] \times B_3$ and equal to $-\frac{1}{2}$ in $[-2,0] \times B_2$. We also define for $0 < \lambda < \frac{1}{4}$,

$$\psi_\lambda(t,x) := ((|x| - \lambda^{-\frac{1}{\nu}})^\nu - 1)_+ + ((|t| - \lambda^{-\frac{1}{\nu}})^\nu - 1)_+ \quad \text{for } |t|, |x| \geq \lambda^{-1,\nu}$$

and zero otherwise. The value of ν will be determined later. Finally, we define for $i \in \{0, 1, 2, 3, 4\}$

$$\phi_i = 1 + \psi_{\lambda^3} + \lambda^i F.$$

Then $\frac{1}{2} \leq \phi_0 \leq \cdots \leq \phi_4 \leq 1$ in Γ_4.

Lemma 8.1 *Let κ be the constant defined in Lemma 5.2. Let u be a solution to (5.2). There exists a small constant $\rho > 0$ depending only on n, σ, α and λ_0 depending only on n, σ, α, ρ, such that for any solution u defined in $(a, 0) \times \mathbb{R}^n$ with $a < -4$ and*

$$-\psi_{\lambda^3} \leq u(t,x) \leq 1 + \psi_{\lambda^3} \quad \text{in } (a,0) \times \mathbb{R}^n$$

with $\lambda \leq \lambda_0$, and $f \leq \lambda^3$, then if

$$|\{u < \phi_0\} \cap (B_1 \times (-4, -2))| \geq \rho,$$

then

$$|\{u > \phi_4\} \cap (\mathbb{R}^n \times (-2, 0))| \leq \kappa.$$

Proof We will show the computations for $D(u) = u$. The general situation is handled as before as in Lemma 5.1.

First Step: Revisiting the energy inequality We return again to the energy inequality. This time, however, we will make use of the nonnegative terms. We seek to obtain a bound on the right-hand side of the form $C\lambda^{\frac{2+\gamma}{1+\gamma}}$.

We now consider the test function as in (7.3), but with cut-off ϕ_1. If $u > \phi_i$, then $\frac{1}{2} \leq \phi_i \leq u \leq 1$, and so

$$F'\Big(\frac{u}{\phi_i}\Big) = \chi_{\{u>\phi_i\}} \frac{u^\gamma - \phi^\gamma}{\phi^\gamma} \leq 2\gamma u_{\phi_i}^+ \leq 2\gamma \lambda^{2i}.$$

To take care of the piece in time, we first note that ϕ_1 is Lipschitz in time for $t \in [0, 4]$ with Lipschitz constant 2λ. Then as before

$$\int F'\Big(\frac{u}{\phi_1}\Big) \mathcal{D}_t^\alpha u \, dt = \int F'\Big(\frac{u}{\phi_1}\Big) \mathcal{D}_t^\alpha(u_{\phi_1}^+ - u_{\phi_1}^- + \phi_1) = (\text{T1}) + (\text{T2}) + (\text{T3}).$$

For (T1), we return to the inequality (7.4) and utilize the Lipschitz nature of ϕ_1^{-1}, to obtain

$$\iint F'\Big(\frac{u}{\phi_1}\Big) \mathcal{D}_t^\alpha u_{\phi_1}^+ \, ds \, dt \geq c\Big(\int_{-\infty}^0 (u - \phi_1)_+^{\frac{2}{1-\alpha}} \, dt\Big)^{1-\alpha}$$
$$+ \int_{-\infty}^0 \int_{-\infty}^t \phi_1(t) u_{\phi_1}^+(s) F'\Big(\frac{u_{\phi_1}^+(t)}{\phi_1(t)}\Big)(\phi_1^{-1}(t) - \phi_1^{-1}(s)) \, ds \, dt$$
$$\geq c\Big(\int_{-\infty}^0 (u - \phi_1)_+^{\frac{2}{1-\alpha}} \, dt\Big)^{1-\alpha} - C\int_{-4}^0 \lambda^2 \, dt.$$

The nonnegative piece (T2) will be utilized in the second step of this proof. For (T3), we note that since ϕ_i is decreasing, we have

$$0 \leq -\mathcal{D}_t^\alpha \phi_i \leq -\Lambda^{-1} D_t^\alpha \phi_i = -D_t^\alpha \psi_{\lambda^3} - D_t^\alpha \lambda^i F.$$

Clearly, $-D_t^\alpha \lambda^i F \leq C\lambda^i$ for $t \leq 0$ from the Lipschitz nature of F. For $-4 \leq t \leq 0$, we have

$$-D_t^\alpha \psi_{\lambda^3} \leq \int_{-\infty}^{\lambda^{-\frac{1}{\nu}}} \frac{|s|}{|s|^{1+\alpha}} \, ds \leq C_\alpha \lambda^{\frac{\alpha-\nu}{\nu}}.$$

We therefore pick ν small enough that $\frac{\alpha-\nu}{\nu} > 2$,

$$\int_{-\infty}^0 F'\Big(\frac{u}{\phi_1}\Big) \mathcal{D}_t^\alpha \phi_1 \, dt \leq C\int_{-4}^0 \lambda u_{\phi_1}^+ \, dt \leq C\lambda^2.$$

Our energy inequality becomes

$$c\Big(\int_{-\infty}^0 (u - \phi_1)_+^{\frac{2}{1-\alpha}} \, dt\Big)^{1-\alpha} + c\int_{\mathbb{R}^n} \iint \frac{[u_{\phi_1}^+(t) - u_{\phi_1}^+(s)][u_{\phi_1}^-(t) - u_{\phi_1}^-(s)]}{(t-s)^{1+\alpha}} \, ds \, dt \, dx$$

$$+ \text{ "Spatial Terms" } \leq C\lambda^2 + \iint f u_{\phi_1}^+ \, \mathrm{d}s \, \mathrm{d}t.$$

Since $f \leq \lambda^3$, everything is bounded on the right-hand side by $C\lambda^2$.

We now turn our attention to the elliptic portion. We consider the terms (1a), (1bi), (1bii), (1c), (2) as the analogous terms for those defined in the proof of Lemma 5.1. As before we obtain a nonnegative energy from the term (1bi). Everything else we will absorb into this energy or bound by $C\lambda^{\frac{2+\gamma}{1+\gamma}}$. The term from (1bi) over A_1^c is bounded as follows:

$$c_{n,\gamma,\sigma} \iint_{A_1^c} \frac{[\phi_1(x) - \phi_1(y)][u_{\phi_1}^+(x) - u_{\phi_1}^+(y)]}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq C \iint_{A_1^c} \frac{[\phi_1(x) - \phi_1(y)]^2 X_1(x,y)}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y.$$

We have the following inequality from the computations given in [4]:

$$\iint \frac{[\phi_1(x) - \phi_1(y)]^{\frac{2+\gamma}{1+\gamma}} X_1(x,y)}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y \leq C\lambda^{\frac{2+\gamma}{1+\gamma}} \tag{8.1}$$

with $\frac{2-2\sigma-2\nu}{\nu} \geq 2$. In particular, (8.1) will hold for $\gamma = 0$. For the term (1bii), we break up the region of integration into A_1 and A_1^c. On A_1, we use Hölder's inequality as before

$$c_{n,\gamma,\sigma} \iint_{A_1} \frac{[\phi_1(x) - \phi_1(y)][u_{\phi_1}^+(x) - u_{\phi_1}^+(y)]}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq C \iint_{A_1} \frac{[\phi_1(x) - \phi_1(y)]^{\frac{2+\gamma}{1+\gamma}}}{|x-y|^{n+2-2\sigma}} X_1(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

$$+ \eta \iint_{A_1} \frac{[u_{\phi_1}^+(x) - u_{\phi_1}^+(y)]^2}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq C\lambda^{\frac{2+\gamma}{1+\gamma}} + \eta \iint_{A_1} \frac{[u_{\phi_1}^+(x) - u_{\phi_1}^+(y)]^2}{|x-y|^{n+2-2\sigma}} \, \mathrm{d}x \, \mathrm{d}y.$$

The last term is absorbed into the left-hand side. The other term is bounded again from (8.1). The term (1c) is bounded in exactly the same way. (1a) is nonnegative and will be utilized later. We now turn our attention to the term (2). We rewrite $u = u_{\phi_1}^+ - u_{\phi_1}^- + \phi_1$. The term involving $u_{\phi_1}^+$ with $|x-y| \leq \eta$ is absorbed by the nonnegative term (1a) on the left-hand side. We now utilize the inequalities:

(1) $u_{\phi_1}^+ \leq \lambda$,

(2) $|\nabla L(x-y)| \approx \frac{1}{|x-y|^{n+1-2\sigma}}$,

(3) $|\nabla \phi_1| \leq C$ for all x,

(4) $|\nabla \phi_1| \leq C\lambda$ in the support of $u_{\phi_1}^+$,

(5) $\chi_{\{u>\phi_1\}} \left[\left(\frac{u}{\phi_1} \right)^{1+\gamma} - 1 \right] \leq 4u_{\phi_1}^+ \leq 4\lambda$.

Then

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\gamma}{\gamma+1} \chi_{\{u<\phi_1\}} \left(\left(\frac{u}{\phi_1} \right)^{\gamma+1} - 1 \right) \nabla \phi_1 \nabla L(x-y)[u(y) - u(x)] \, \mathrm{d}x \, \mathrm{d}y \right|$$

$$\leq C\lambda^2 \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\{u<\phi_1\}} \nabla L(x-y)[u(y) - u(x)] \, \mathrm{d}x \, \mathrm{d}y \right|.$$

These terms are all bounded as before. Notice that we have λ^2 on the outside. Then all nonlocal terms on the right-hand side are bounded by $C\lambda^{\frac{2+\gamma}{1+\gamma}}$. The local term $\operatorname{div}(D(u)\nabla u)$ is handled in the usual manner by use of the Cauchy-Schwarz inequality.

Second Step: Using the nonnegative pieces in space and time We now utilize the two nonnegative pieces. From Proposition 10.1, we have

$$\left[\left(\frac{u}{\phi_1}\right)^{\gamma+1} - \phi_1\right]_+ \geq 4(u_{\phi_1}^+)^{1+\gamma}.$$

Then we conclude that

$$\int_{\mathbb{R}^n} \int_{-4}^0 \int_{-4}^0 \frac{u_{\phi_1}^+(t)u_{\phi_1}^-(s)^{1+\alpha}}{t-s} \, ds \, dt \, dx$$
$$+ \int_{-4}^0 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_{\phi_1}^+)^{\gamma+1}(x)u_{\phi_1}^-(y)}{|x-y|^{n+2-2\sigma}} \, ds \, dt \, dx$$
$$\leq C\lambda^{\frac{2+\gamma}{1+\gamma}}. \tag{8.2}$$

Since we used Ψ_{λ^3}, replacing ϕ_1 with ϕ_3, we have the same inequality but with the bound $C\lambda^{3\frac{2+\gamma}{1+\gamma}}$.

We now show how the inequality (8.2) and its analogue for ϕ_3 are enough to prove the remainder of the lemma as in [1]. We note that for the proof as written in [1] to work, we need $3\frac{2+\gamma}{1+\gamma} > 5$ which is achieved for γ small enough. We first utilize

$$\int_{-4}^0 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_{\phi_1}^+)^{\gamma+1}(x)u_{\phi_1}^-(y)}{|x-y|^{n+2-2\sigma}} \, dx \, dy \, dt \leq C\lambda^{\frac{2+\gamma}{1+\gamma}}.$$

From our hypothesis

$$|\{w < \phi_0\} \cap ((-4,-2) \times B_1)| \geq \rho.$$

Then the set of times $\Sigma \in (-4,-2)$ for which $|\{u(t,\cdot) < \phi_0\} \cap B_1| \geq \frac{\rho}{4}$ has at least measure $\frac{\rho}{2|B_1|}$. And so

$$C\lambda^{\frac{2+\gamma}{1+\gamma}} \geq c\rho \int_\Sigma \int_{\mathbb{R}^n} (u_{\phi_1}^+)^{1+\gamma} \, dx \, dt.$$

Now $(\{u-\phi_2 > 0\}\cap(\Sigma\times B_2)) \subset (\{u-\phi_1 > \frac{\lambda}{2}\}\cap(\Sigma\times B_2))$, and so from Chebychev's inequality,

$$|\{u - \phi_2 > 0\} \cap (\Sigma \times B_2)| \leq \frac{C}{\rho}\lambda^{\frac{2+\gamma}{1+\gamma}-(1+\gamma)}.$$

The exponent on λ is positive for γ small enough. We write this as

$$|\{u \leq \phi_2\} \cap (\Sigma \times B_2)| \geq |\Sigma \times B_2| - \frac{C}{\rho}\lambda^{\frac{2+\gamma}{1+\gamma}-(1+\gamma)} \geq \frac{\rho}{2} - \frac{C}{\rho}\lambda^{\frac{2+\gamma}{1+\gamma}-(1+\gamma)}.$$

This will be positive for λ small enough depending on $n, \sigma, \alpha, \gamma, \rho$.

The proof then proceeds just as in [1], where we then utilize $3\frac{2+\gamma}{1+\gamma} > 5$ as well as the analogue of (8.2) for ϕ_3.

This next lemma will imply Lemma 5.3. For this next lemma, we define

$$\psi_{\tau,\lambda} = \left(\left(\frac{|x|-1}{\lambda^{\frac{4}{\sigma}}}\right)^\tau - 1\right)_+ + \left(\left(\frac{|t|-1}{\lambda^{\frac{4}{\alpha}}}\right)^\tau - 1\right)_+.$$

Lemma 8.2 *Given $\rho > 0$, there exist $\tau > 0$ and μ_2, such that for any solution to (5.2) in $\mathbb{R}^n \times (a, 0)$ with $a < -4$ and $|f| \leq \lambda^3$ satisfying*

$$-\psi_{\tau,\lambda} \leq u \leq 1 + \psi_{\tau,\lambda},$$

where $\lambda \leq \lambda_0$ with λ_0 as in Lemma 8.1, if

$$|\{u < \phi_0\} \cap (B_1 \times (-4, -2))| > \rho,$$

then

$$\sup_{B_1 \times (-1, 0)} u \leq 1 - \mu_2.$$

Proof We consider the rescaled function $w(t, x) = \frac{u - (1 - \lambda^4)}{\lambda^4}$. We fix τ small enough, such that

$$\frac{(|x|^\tau - 1)_+}{\lambda^4} \leq (|x|^{\frac{\sigma}{4}} - 1)_+ \quad \text{and} \quad \frac{(|t|^\tau - 1)_+}{\lambda^4} \leq (|t|^{\frac{\sigma}{4}} - 1)_+.$$

Then w satisfies equation (5.2) with $D_2(x) = D_1(\lambda^4 x + (1 - \lambda^4))$, where D_1 is the coefficient for the equation u satisfies. From our hypothesis and Lemma 8.1,

$$\left|\left\{w > \frac{1}{2}\right\} \cap (B_2 \times [-2, 0])\right| = |\{u > \phi_4\} \cap (B_2 \times [-2, 0])| \leq \rho.$$

Then from Lemma 5.2, we conclude that $w \leq 1 - \mu_1$ on $(-1, 0) \times B_1$, and so $u \leq 1 - \lambda^4 \mu_1 = 1 - \mu_2$.

9 Proof of Regularity

With Lemmas 5.1–5.3 we are ready to finish the proof of Theorem 1.2. We first mention that solutions of (2.4) satisfy the following scaling property: If u is a solution on $(a, 0) \times \mathbb{R}^n$, then $v(t, x) = Au(Bt, Cx)$ is a solution on $(a/B, 0) \times \mathbb{R}^n$ if $A = B^\alpha C^{2\sigma - 2}$. The method of proof is given in [5] which we now briefly outline. We take any point $p = (x, t) \in \mathbb{R}^n \times (a, T)$ and prove that u is Hölder continuous around p. The Hölder continuity exponent will depend only on α, σ, n. The constant will depend on the L^∞ norm of u, f and on the C^2 norm of $u(a, x)$. By translation, we assume that $p = (0, 0)$. By scaling, we assume that $0 \leq u(t, x) \leq \overline{\Psi}(t, x)$ and $|f| \leq \lambda_0^3$ for λ_0 as defined in Lemma 8.1.

We now take a positive constant $M < \frac{1}{4}$, such that for $0 < K \leq M$,

$$\frac{1}{1 - \frac{\mu_2}{2}} \psi_{\tau,\lambda^3}(Kt, Mx).$$

M will depend only on λ, μ_2 and $\tau > 0$. During the iteration, we have the following alternative.

Alternative 1 Suppose that we can apply Lemma 5.3 repeatedly. We then consider the rescaled functions

$$u_{j+1}(t, x) = \frac{1}{1 - \frac{\mu_2}{4}} u_j(M_1 t, Mx), \quad M_1 = \left(\frac{M^{2 - 2\sigma}}{1 - \frac{\mu_2}{4}}\right)^{\frac{1}{\alpha}}.$$

Notice that $M_1 < M$. All the u_j satisfy the same equation. If we can apply Lemma 5.3 at every step, then $u_j \leq 1 - \mu_2$ on the cylinder Γ_1. This implies Hölder regularity around p and also implies $u(p) = 0$.

Alternative 2 If at some point the assumption (5.3) fails, then we are in the situation of Lemma 5.1 and

$$0 < \mu_0 \leq u_j(t, x) \leq 1.$$

Scaling the above situation our equations will have $D(u) = d_1 u + d_2$ with $d_2 > 0$. We may then repeat the procedure since Lemmas 5.1 and 5.3 apply also in this situation.

10 Appendix

Proof of Lemma 6.1 Since throughout this paper, we only require γ small when σ is small, we will prove the lemma for $\gamma = \frac{1}{k}$ for $k \in \mathbb{N}$. Now we assume without loss of generality that

$$\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)} \geq 0.$$

We first assume that $|u(x) - u(y)| \leq 4|\phi(x) - \phi(y)|$. We need to bound

$$\frac{u^{\gamma+1}(y)}{\phi^\gamma(y)} - \frac{u^{\gamma+1}(x)}{\phi^\gamma(x)}. \tag{10.1}$$

We first notice that the term above in (10.1) will be larger, if we assume that $u(y) \geq u(x)$ and $\phi(x) \geq \phi(y)$ without changing $|u(x) - u(y)|$ and $|\phi(x) - \phi(y)|$. Furthermore, the term in (10.1) will still be greater if $u(y) = \phi(y)$ and not changing $|u(y) - u(x)|$. We are then looking for the bound

$$u(y) - \frac{u^{1+\gamma}(x)}{\phi^\gamma(x)} \leq c_2|\phi(x) - \phi(y)| = c_2(\phi(x) - u(y)).$$

Thus, for a constant l, we need the bound

$$l - \frac{(l-v)^{1+\gamma}}{(l+\mu)^\gamma} \leq C\mu. \tag{10.2}$$

Recalling that we are assuming $4|\phi(x) - \phi(y)| \geq |u(x) - u(y)|$ (or $v \leq 4\mu$) the above term is maximized when v is largest or when $v = 4\mu$. Now

$$l - \frac{(l - 4\mu)^{1+\gamma}}{(l+\mu)^\gamma}$$
$$= \frac{l[(l+\mu)^\gamma - (l-4\mu)^\gamma]}{(l+\mu)^\gamma} + 4\frac{\mu(l-4\mu)^\gamma}{(l+\mu)^{-\gamma}}$$
$$:= L_1 + L_2.$$

It is clear that

$$L_2 \leq 4\mu.$$

To control L_1 we first consider when $l - 4\mu \leq \frac{l}{2}$. Then $l \leq 8\mu$ and it is clearly true that

$$L_1 \leq 8\mu.$$

Now when $l - 4\mu \geq \frac{l}{2}$, from the concavity of x^γ, we have

$$L_1 \leq \frac{l}{l - 4\mu} \frac{(l-4\mu)^\gamma}{(l+\mu)^\gamma} 5\mu \leq 10\mu.$$

Then

$$l - \frac{(l-\upsilon)^{1+\gamma}}{(l+\mu)^{\gamma}} \le 14\mu, \tag{10.3}$$

and (6.2) is proven with constant $c_2 = 14$.

We now assume $4|\phi(x) - \phi(y)| \le |u(x) - u(y)|$, and the left-hand side of (10.2) is maximized again when $4\mu = \upsilon$, and so we have

$$l - \frac{(l-\upsilon)^{1+\gamma}}{(l+\frac{\upsilon}{4})^{\gamma}} \le \frac{14}{4}\upsilon,$$

which is just (10.3) rewritten with the substitution $\mu = \frac{\upsilon}{4}$. Then

$$\frac{u^{\gamma+1}(y)}{\phi^{\gamma}(y)} - \frac{u^{\gamma+1}(x)}{\phi^{\gamma}(x)} \le \frac{14}{4}|u(x) - u(y)| \le \frac{14}{3}|u_{\phi}^{-}(x) - u_{\phi}^{-}(y)|, \tag{10.4}$$

and the right-hand side of (6.1) is shown.

Now

$$\frac{u^{\gamma+1}(y)}{\phi^{\gamma}(y)} - \frac{u^{\gamma+1}(x)}{\phi^{\gamma}(x)} = \frac{u^{\gamma+1}(y) - u^{\gamma+1}(x)}{\phi^{\gamma}(y)} - u^{\gamma+1}(x)\frac{\phi^{\gamma}(x) - \phi^{\gamma}(y)}{\phi^{\gamma}(x)\phi^{\gamma}(y)} := M_1 + M_2.$$

We suppose $\gamma = \frac{1}{k}$. By factoring, we have

$$|M_1| = \frac{|u(y) - u(x)|}{\phi^{\gamma}(y)} \frac{\displaystyle\sum_{j=0}^{k} u^{\frac{k-j}{k}}(y)u^{\frac{j}{k}}(x)}{\displaystyle\sum_{j=0}^{k-1} u^{\frac{k-1-j}{k}}(y)u^{\frac{j}{k}}(x)} \ge \frac{u(x)}{\phi^{\gamma}(y)} \frac{4|\phi(x) - \phi(y)|}{\displaystyle\sum_{j=0}^{k-1} u^{\frac{k-1-j}{k}}(y)u^{\frac{j}{k}}(x)},$$

$$|M_2| = \frac{u^{1+\frac{1}{k}}(x)}{\phi^{\frac{1}{k}}(x)\phi^{\frac{1}{k}}(y)} \frac{|\phi(x) - \phi(y)|}{\displaystyle\sum_{j=0}^{k-1} \phi^{\frac{k-1-j}{k}}(y)\phi^{\frac{j}{k}}(x)} \le \frac{u(x)}{\phi^{\gamma}(y)} \frac{|\phi(x) - \phi(y)|}{\displaystyle\sum_{j=0}^{k-1} u^{\frac{k-1-j}{k}}(y)u^{\frac{j}{k}}(x)}.$$

Thus $M_2 \le \frac{M_1}{4}$. Thus, M_1 is the dominant term. We then have from the convexity of $x^{\gamma+1}$,

$$M_1 + M_2 \ge \frac{M_1}{2} \ge \frac{1}{2}\frac{u^{\gamma+1}(y) - u^{\gamma+1}(x)}{\phi^{\gamma}(y)}$$

$$\ge \frac{1}{2}\frac{(u(y) - u(x))^{1+\gamma}}{\phi^{\gamma}(y)}$$

$$\ge \left(\frac{4}{5}\right)^{1+\gamma}\frac{1}{2}\frac{[u_{\phi}^{-}(y) - u_{\phi}^{-}(x)]^{1+\gamma}}{\phi^{\gamma}(y)}$$

$$\ge \frac{2}{5}\left(\frac{4}{5}\right)^{\gamma}\frac{[u_{\phi}^{-}(y) - u_{\phi}^{-}(x)]^{1+\gamma}}{\phi^{\gamma}(y)}.$$

Proposition 10.1 *Let F be a function satisfying $F'' \ge 0$ for $x \ge 0$. Assume also $F(0) = 0$. If $y \ge x \ge 0$, then*

$$F(y) - F(x) \ge F(y - x).$$

Proof For fixed $h > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{F(x+h) - F(x)}{h} = \frac{F'(x+h) - F'(x)}{h} \geq 0.$$

Then for $x, h \geq 0$,

$$\frac{F(x+h) - F(x)}{h} \geq \frac{F(0+h) - F(0)}{h} = \frac{F(h)}{h}.$$

Let $h = y - x$, and multiply both sides of the equation by $y - x$.

Proposition 10.2 *Let* $0 < \gamma < 1$ *and* $x, d \geq 0$. *Then*

$$(x + d)^\gamma - d^\gamma \leq 2^\gamma x^\gamma.$$

Proof First assume $x \leq d$. From the concavity of x^γ, we have

$$(x + d)^\gamma - d^\gamma \leq \gamma d^{\gamma-1} x = \gamma d^{\gamma-1} x^{1-\gamma} x^\gamma \leq \gamma x^\gamma.$$

If on the other hand $x > d$, then

$$(x + d)^\gamma - d^\gamma \leq (x + d)^\gamma \leq (2x)^\gamma.$$

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