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On the Motion Law of Fronts for Scalar Reaction-Diffusion Equations with Equal Depth Multiple-Well Potentials

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(*Dedicated to Haïm Brezis on the occasion of his* 70*th birthday*)

Abstract Slow motion for scalar Allen-Cahn type equation is a well-known phenomenon, precise motion law for the dynamics of fronts having been established first using the so-called geometric approach inspired from central manifold theory (see the results of Carr and Pego in 1989). In this paper, the authors present an alternate approach to recover the motion law, and extend it to the case of multiple wells. This method is based on the localized energy identity, and is therefore, at least conceptually, simpler to implement. It also allows to handle collisions and rough initial data.

Keywords Reaction-diffusion systems, Parabolic equations, Singular limits
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1 Introduction

1.1 Motivation and setting

This paper is a follow-up of a previous work with Orlandi [3], where we derived an upper bound for the motion of front for gradient systems with potentials having several minimal wells of equal depth. Our approach there is based on the local energy inequality combined with some appropriate parabolic estimates. Our aim in this paper is to extend the analysis in order to derive the precise motion laws for fronts: The approach is however restricted at this stage to scalar equations. We will take advantage in particular of the fact that in the scalar case, stationary solutions can be completely integrated, allowing for refined energy estimates.

It is presumably needless to recall that the study of the motion of fronts for scalar reaction-diffusion equations has already a very long history. In particular, equations of Allen-Cahn type, that is, when the potential possesses only two distinct local minimizers which are non-degenerate, have been extensively studied. Under suitable preparedness assumptions on the initial datum, the precise motion law for the fronts has been derived in the seminal works of Carr and Pego [6] (see also [10]). Their approach relies on a careful study of the linearized problem around the stationary front, in particular from the spectral point of view. This type of approach is also sometimes termed the geometric approach (see, e.g., [8]), since it involves ideas related to central manifold theory. Alternate methods, usually termed energy methods relying

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on global energy estimates have later been worked out (see [5, 11, 12]). They are presumably more direct to capture the essence of the slow-motion or metastability of pattern phenomenon, but have been unable at this stage to yield the precise motion law. One of the aims of this paper is therefore to fill the gap between the two methods, and raise the energy methods to the same degree of accuracy as the geometric one.

The motivations of this paper are however manifold. First, as mentioned relying on the results in [3], we wish to recover the precise motion law of Carr and Pego, providing therefore an alternate approach which eludes the use of spectral theory and which also allows for a larger class of initial data. Second, whereas most of the existing literature is devoted to Allen-Cahn type potentials, our method can handle also potentials with several equal-depth wells. Notice that a major difference in the later case is that, whereas only attractive forces between the fronts are present in the case of two wells, repulsive forces may be present when there are more than two wells, inducing important differences in the limiting ordinary differential equations.

Besides this, we are able to handle collisions and splittings, and extend the analysis past these events: Similar issues were addressed and solved in the Allen-Cahn case[1] by Chen [8], relying crucially on a comparison principle worked out by Fife and McLeod [9]. In our opinion, such an argument cannot be extended for potentials with more than two wells, and when hence repulsive forces are present[2]. Finally, last but not least, we expect that the approach we develop here can be extended and be used as a model in order to derive the motion law in the case the potential wells are degenerate as well as the case of systems, with possibly additional assumptions on the stationary solutions.

To be more specific, we consider and analyze the behavior of solutions v of one-dimensional reaction-diffusion equations of the following form:

$$(\mathrm{PGL})_\varepsilon \qquad \partial_t v_\varepsilon - \partial_{xx} v_\varepsilon = -\frac{1}{\varepsilon^2} V'(v_\varepsilon),$$

where $0 < \varepsilon < 1$ denotes a (small) parameter, v denotes a scalar function of the space variable $x \in \mathbb{R}$ and the time variable $t \geq 0$, the function V, usually termed the potential, denotes a smooth scalar function on $\mathbb{R}$, and V' denotes its derivative. Notice that equation $(\mathrm{PGL})_\varepsilon$ actually corresponds to the L^2 gradient-flow of the energy functional $\mathcal{E}_\varepsilon$ which is defined for a function $u : \mathbb{R} \mapsto \mathbb{R}$ by the formula

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} e_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon \frac{|\dot{u}|^2}{2} + \frac{V(u)}{\varepsilon}. \tag{1.1}$$

Our assumptions on the potential V express the fact that it possesses several minimizers which are non-degenerate and are formulated as follows. We assume throughout that V is smooth and satisfies the three conditions:

$$(\mathrm{H}_1) \qquad \inf V = 0 \text{ and the set of minimizers } \Sigma \equiv \{y \in \mathbb{R}, V(y) = 0\}$$

is a finite set, with at least two distinct elements, that is

$$\Sigma = \{\sigma_1, \cdots, \sigma_q\}, \quad q \geq 2, \quad \sigma_i < \sigma_j, \quad \forall 1 \leq i < j \leq q. \tag{1.2}$$

[1] Actually, only collisions occur in the Allen-Cahn case, splittings do not.

[2] As a matter of fact, Proposition 3.1 in [8], which rephrases the Fife-McLeod result, simply does not hold when there are more than two wells.

(H_2) We have that $\lambda_i \equiv V''(\sigma_i) > 0$ is positive for each point σ_i of Σ.

(H_3) We have $V(u) \to +\infty$, as $|u| \to +\infty$.

A canonical example is given by the function

$$V_{\rm AC}(u) = \frac{(1-u^2)^2}{4}, \tag{1.3}$$

whose minimizers are $\sigma_1 = +1$ and $\sigma_2 = -1$, with $\lambda_1 = \lambda_2 = 2$ and which is a potential of Allen-Cahn type. Another example we have in mind and we wish to handle is given by

$$V_\infty(u) = (1 + \cos u), \tag{1.4}$$

for which $\Sigma = \{(2k+1)\pi, k \in \mathbb{Z}\}$ and $\lambda_i = 1$. Clearly, the potential given by (1.4) does not satisfy conditions (H_1) nor (H_3), since it has infinitely many minimizers and does not converge to $+\infty$ at infinity. However, the analysis can be carried over for this type of potentials, as Theorem 1.5 below will show.

As in [3], the assumption in this paper on the initial datum $v_\varepsilon^0(\cdot) = v_\varepsilon(\cdot, 0)$ is that its energy is finite. More precisely, given an arbitrary constant $M_0 > 0$, we assume throughout the paper that

$$(\mathrm{H}_0) \qquad \mathcal{E}_\varepsilon(v_\varepsilon^0) \leq M_0 < +\infty.$$

In particular, in view of the classical energy identity

$$\mathcal{E}_\varepsilon(v_\varepsilon(\cdot, T_2)) + \varepsilon \int_{T_1}^{T_2} \int_{\mathbb{R}} \left| \frac{\partial v_\varepsilon}{\partial t} \right|^2 (x,t) \mathrm{d}x\mathrm{d}t = \mathcal{E}_\varepsilon(v_\varepsilon(\cdot, T_1)), \quad \forall\, 0 \leq T_1 \leq T_2\,, \tag{1.5}$$

we have, $\forall t > 0$,

$$\mathcal{E}\left(v_\varepsilon(\cdot, t)\right) \leq M_0, \tag{1.6}$$

so that for every given $t \geq 0$, we have $V(v(x,t)) \to 0$ as $|x| \to \infty$. It is then quite straightforward to deduce from assumptions (H_0)–(H_2) as well as the energy identity (1.5), that $v(x,t) \to \sigma_\pm$ as $x \to \pm\infty$, where $\sigma_\pm \in \Sigma$ does not depend on t.

1.2 Regularized fronts and their evolution

The notion of regularized fronts is presumably central in this paper. It describes a situation where, at some given time $t_0 \geq 0$, the solution v_ε to $(\mathrm{PGL})_\varepsilon$ is close to a chain of stationary solutions which are well separated, and suitably glued together. This occurs, as we will see, when the solution has already undergone a parabolic regularization. The rate of accuracy of the regularization, is described by a parameter $\delta > 0$, homogeneous to a length, and which is also related to the distance between two fronts.

We recall that for $i \in \{1, \cdots, q-1\}$, there exists a unique (up to translations) solution ζ_i^+ to the stationary equation with $\varepsilon = 1$,

$$v_{xx} + V'(v) = 0 \quad \text{on } \mathbb{R} \tag{1.7}$$

with, as conditions at infinity, $v(-\infty) = \sigma_i$ and $v(+\infty) = \sigma_{i+1}$. We also set, for $i \in \{2, \cdots, q\}$, $\zeta_i^-(\cdot) \equiv \zeta_i(-\cdot)$, so that ζ_i^- is the unique, up to translations solution to (1.7) such that $v(+\infty) = \sigma_i$ and $v(-\infty) = \sigma_{i-1}$. A remarkable fact is that there are no other non-trivial solutions to equation (1.7) than the solutions $\zeta_i^\pm$: In particular there are no solutions connecting minimizers which are not neighbors[3]. Some relevant properties of these solutions ζ_i will be collected in Section 3. For $i = 1, \cdots, q-1$, let z_i be a point in the interval (σ_i, σ_{i+1}) where the potential V restricted to $[\sigma_i, \sigma_{i+1}]$ achieves its maximum, and set $\mathcal{Z} = \{z_1, \cdots, z_{q-1}\}$. Since we consider only the one-dimensional case, any solution ζ_i takes once and only once the value z_i.

Next let $t_0 \geq 0$, $\delta > \alpha_1 \varepsilon$, and $\mathrm{r} \geq \delta$ be given, where $\alpha_1 > 0$ denotes some constant which will be specified in Subsection 3.2.

Definition 1.1 *We say that v_ε satisfies the preparedness assumption $\mathcal{WP}_\varepsilon(\delta, t)$ if it satisfies the energy assumption* (H_0) *and if there exists a collection of points $\{a_k(t)\}_{k \in J(t)}$ in $\mathbb{R}$, with $J(t) = \{1, \cdots, \ell(t)\}$, such that the following conditions are fulfilled:*

(WP1) *For each $k \in J(t)$, there exist a number $i(k) \in \{1, \cdots, q\}$, such that*

$$v_\varepsilon(a_k(t), t) = z_{i(k)}. \tag{1.8}$$

(WP2) *For each $k \in J(t)$, there exists a symbol $\dagger_k \in \{+, -\}$, such that*

$$\Big\| v_\varepsilon(\cdot, t) - \zeta_{i(k)}^{\dagger_k}\Big(\frac{\cdot - a_k(t)}{\varepsilon}\Big)\Big\|_{C^1_\varepsilon(I_k)} \leq \exp\Big(-\rho_1 \frac{\delta}{\varepsilon}\Big), \tag{1.9}$$

where $I_k = ([a_k(t) - \delta, a_k(t) + \delta]$ for each $k \in J(t)$.

(WP3) *Set $\Omega(t) = \mathbb{R} \setminus \overset{\ell(t)}{\underset{k=1}{\cup}} I_k$. We have the energy estimate*

$$\int_{\Omega(t)} e_\varepsilon(v_\varepsilon(\cdot, t)) \mathrm{d}x \leq C M_0 \exp\Big(-\rho_1 \frac{\delta}{\varepsilon}\Big). \tag{1.10}$$

In the above definition $\rho_1 > 0$ denotes a constant which will be defined in Section 3 (see (3.24)). Notice that, if we consider more generally, for $t \geq 0$, the subset $\mathfrak{D}(t)$ of $\mathbb{R}$ is defined by

$$\mathfrak{D}(t) = \{x \in \mathbb{R}, \text{ such that } v_\varepsilon(x, t) \in \mathcal{Z}\}. \tag{1.11}$$

If $\mathcal{WP}_\varepsilon(\delta, t)$ holds, then we have for $\delta \geq \alpha_1 \varepsilon$,

$$\mathfrak{D}(t) = \{a_k(t)\}_{k \in J(t)}. \tag{1.12}$$

In particular, the points $a_k(t)$ are easily shown to be unique (see Section 4), and once their existence has been established, the main focus is then on their evolution in time. We introduce also the quantities

$$\mathfrak{d}_a^\pm(t) = \inf\{\sqrt{\lambda_{j^+(k)}}|a_k(t) - a_{k+1}(t)| \text{ for } k \in 1, \cdots, \ell - 1 \text{ such that } \dagger_k = \pm \dagger_{k+1}\}, \tag{1.13}$$

[3]The situation might be very different in the case of systems, where anyway the notion of neighbors is perhaps meaningless.

with the convention that the quantity is equal to $+\infty$ in case the defining set is empty, and where, for a given index $k \in J(t)$, we define the integers $j^{\pm}(k)$ as $j^{\pm}(k) = i(k) \pm 1$, if $\dagger_{i(k)} = +$, and $j^{\pm}(k) = i(k) \mp 1$, otherwise. We also set

$$\mathfrak{d}_a(t) = \inf\{\mathfrak{d}_a^+(t), \mathfrak{d}_a^-(t)\}.$$

Notice that if $\mathcal{WP}_\varepsilon(\delta, t)$ holds, then it is a simple exercise to show that, if α_1 is chosen sufficiently large, then we have

$$|a_k(t) - a_{k+1}(t)| \geq \delta, \tag{1.14}$$

so that $\delta \leq \sqrt{\lambda_{\min}}\mathfrak{d}_a(t)$, where $\lambda_{\min} = \inf \lambda_i$. Conversely, given the points $\{a_k\}$, the largest value of δ for which one may expect $\mathcal{WP}_\varepsilon(\delta, t)$ to hold is precisely of the same order as $\mathfrak{d}_a(t)$.

Our first result describes the situation, where the initial datum satisfies the assumption $\mathcal{WP}_\varepsilon(\delta, T)$. We will show that the motion law for the fronts is governed by a simple first order differential equation, which is of nearest neighbor interaction type. The strength of the interaction of the $(k+1)$-th fronts on the k-th fronts is governed by the quantity $\Gamma^+_{k,\varepsilon}(\{a_i(t)\})$ defined, for a collection of ordered points $\{a_1, \cdots, a_\ell\}$, with $a_1 < a_2 < \cdots < a_\ell$ and signs $\{\dagger_1, \cdots, \dagger_\ell\}$, by the formula

$$\Gamma^+_{k,\varepsilon}(\{a_i\}) = \dagger_k \, \dagger_{k+1} \, B^{\dagger_{i(k)}}_{i(k)} \, B^{-\dagger_{i(k+1)}}_{i(k+1)} \exp\Big(- \frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon} |a_k - a_{k+1}| \Big) \tag{1.15}$$

with the convention $\ell + 1 = +\infty$. The numbers $B_i^{\pm}$ entering in formula (1.15) depend only on the properties of the stationary front ζ_i and will be explicitly defined in Section 3 (see (3.9)). Let us however emphasize that $B_i^{\pm} > 0$. It follows in particular that $\Gamma^+_{k,\varepsilon}(\{a_i\}) > 0$ if the signs $\dagger_k$ and $\dagger_{k+1}$ are the same, and $\Gamma^+_{k,\varepsilon}(\{a_i\}) < 0$ if they are opposite. Notice also that the quantity $\Gamma^+_{k,\varepsilon}(\{a_i(t)\})$ decays exponentially as the distance between two neighboring fronts increases. We also set

$$\Gamma^-_{k,\varepsilon}(\{a_i(s)\}) = -\Gamma^+_{k-1,\varepsilon}(\{a_i(s)\}),$$

with the convention that

$$\Gamma^+_{0,\varepsilon}(\{a_i(s)\}) = \Gamma^-_{1,\varepsilon}(\{a_i(s)\}) = 0.$$

Our first main result shows that the evolution of regularized fronts is related to solutions of a differential equation of the type

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{b}_k(s) = -\mathfrak{S}^{-1}_{i(k)} \sum_{\dagger \in \{+,-\}} \Gamma^\dagger_{k,\varepsilon}(\{\mathrm{b}_i(s)\})[1 + \mathcal{C}^\dagger_k(s)], \tag{1.16}$$

where $\mathfrak{S}_i$ denotes a positive quantity[4] related to ζ_i and $\mathcal{C}^\dagger_k(s)$ stands for some error term which will be shown to be exponentially small.

Theorem 1.1 *Assume that the potential V satisfies assumptions* (H_1)–(H_3)*, let $\varepsilon > 0$, and let v_ε be a solution to* $(\mathrm{PGL})_\varepsilon$ *satisfying* (H_0)*. Let $T \geq 0$ be given. There exists constants*

[4] actually its energy

$\alpha_* > 0$, $c_* > 0$, $0 < \nu_* < 1$, $\rho_* > 0$, and $\mathcal{S}_* > 0$ depending only on V and M_0 and a time $\mathfrak{T} = \mathfrak{T}(T) > T$ satisfying

$$\mathfrak{T}(T) \geq \mathcal{T}_{\mathrm{ref}}(T, \mathfrak{d}_a(T)) \equiv \frac{\varepsilon^2}{2\mathcal{S}_*} \exp\Big(\frac{\mathfrak{d}_a(T)}{\varepsilon}\Big) + T, \tag{1.17}$$

such that, if $\delta \geq \alpha_* \varepsilon$ and property $\mathcal{WP}_\varepsilon(\delta, T)$ holds, then we may assert:

(i) *For any time $t \in [T, \mathfrak{T}]$ the points $\{a_k(t)\}_{k \in J(T)}$ satisfying (1.8) are unique and well-defined, whereas for any $t \in [T + c_*\varepsilon\delta, \mathfrak{T}]$, property $\mathcal{WP}_\varepsilon(\nu_*\delta, t)$ holds.*

(ii) *Property $\mathcal{WP}_\varepsilon(\nu_*\mathfrak{d}_a(T), t)$ holds for any t in $[T_{\mathrm{trans}}, \mathfrak{T}]$, where*

$$T_{\mathrm{trans}} = T + \varepsilon^2 \exp\Big(\frac{\mathfrak{d}_a(T)}{10\varepsilon}\Big) \leq T + \exp\Big(-\frac{\mathfrak{d}_a(T)}{2\varepsilon}\Big)(\mathcal{T}_{\mathrm{ref}} - T). \tag{1.18}$$

(iii) *We have $|\mathfrak{d}_a(T) - \mathfrak{d}_a(\mathfrak{T})| \geq \rho_* \mathfrak{d}_a(T)$.*

(iv) *For any time $t \in [T, \mathfrak{T}]$, there exists a collection of points $\{b_k(t)\}_{k \in J(T)}$ satisfying the differential equation (1.16) with*

$$|\mathcal{C}_k^\dagger(s)| \leq \exp\Big(-\rho_*\frac{\delta}{\varepsilon}\Big), \quad \forall s \in [T, \mathfrak{T}], \quad |\mathcal{C}_k^\dagger(s)| \leq \exp\Big(-\rho_*\frac{\mathfrak{d}_a(T)}{\varepsilon}\Big), \quad \forall s \in [T_{\mathrm{trans}}, \mathfrak{T}], \tag{1.19}$$

such that

$$\begin{cases} |\, a_k(s) - b_k(s)| \leq \varepsilon \exp\Big(-\rho_*\dfrac{\delta}{\varepsilon}\Big) \quad \text{for any } s \in [T, T_{\mathrm{trans}}], \\ |\, a_k(s) - b_k(s)| \leq \varepsilon \exp\Big(-\rho_*\dfrac{\mathfrak{d}_a(T)}{\varepsilon}\Big) \quad \text{for any } s \in [T_{\mathrm{trans}}, \mathfrak{T}]. \end{cases} \tag{1.20}$$

A few comments are in order. The first two statements describe how property $\mathcal{WP}_\varepsilon$ is propagated by the equation $(\mathrm{PGL})_\varepsilon$. Assertion (i) of Theorem 1.1 shows that property $\mathcal{WP}_\varepsilon$ remains true, except possibly on an initial boundary layer of order $\varepsilon\delta$, where the collection of points $\{a_k\}_{k\in J}$ is however still well-defined, and with some smaller length[5] $\delta' \equiv \nu_*\delta$. Assertion (ii) shows, that, after a transition period $[T, T_{\mathrm{trans}}]$, whose length is small compared to the length of $[T, \mathfrak{T}]$, the rate of the approximation has improved to $\delta' \equiv \nu_*\mathfrak{d}_a(T)$, which as mentioned, is the order of the best rate of approximation possible.

The approximation by the differential equation (1.16) is presented in assertion (iii). Turning first to the differential equation (1.16), we notice that two neighboring fronts with the same signs $\dagger$ repel, whereas they attract when these signs are opposite. In particular, we will show in Section 2 that, if there exists some $k \in \{1, \cdots, \ell\}$ such that $\dagger_k = -\dagger_{k+1}$, then collisions have to occur for the differential equation (1.16). Moreover, if the infimum in (1.13) is achieved at some fronts of opposite signs[6], then the maximal time of existence $T_{\max}$ of the differential equation (1.16) satisfies an estimate of the form

$$T_{\max} - T \propto \varepsilon^2 \exp\Big(\frac{\mathfrak{d}_a(T)}{\varepsilon}\Big), \tag{1.21}$$

[5] Recall that this parameter is supposed to describe the accuracy of the approximation by a chain of stationary solutions glued together.

[6] As a matter of fact, the purely attractive case, for which $\dagger_k = -\dagger_{k+1}$, for every k, and hence all forces are attractive, occurs for instance for the Allen-Cahn functional.

see inequality (2.5) for a precise statement. On the other hand, if all the signs $\dagger_k$ are identical, then the system is purely repulsive, and is then defined for all time, i.e., $T_{\max} = +\infty$. Moreover, in that case, the system has actually diffusive properties (see Proposition 2.4 below).

Comparing property (1.21) of the differential equation with (1.17) for the partial differential equation $(\mathrm{PGL})_\varepsilon$, we observe that the time $\mathfrak{T} - T$ is of the same order of magnitude as the one provided in (1.21), and therefore appropriate for comparing the two equations. Moreover, in view of assertion (ii) of Theorem 1.1, we see that a point at least as been moved by at least a distance of order of magnitude $\mathfrak{d}_a(T)$, which is indeed the appropriate length scale. On this length scale, it follows from assertion (iv) that the differential equation (7.8) describes, up to some lower order terms, the motion of the front points.

1.3 Collisions of fronts

Whereas collisions in the ordinary differential equation (1.16) represent genuine singularities for the solutions and lead to a maximal time of existence, it is not the case for the partial differential equation $(\mathrm{PGL})_\varepsilon$, which in view of its parabolic nature possesses regular solutions for all positive time. The notion of fronts is however only well-defined, in the sense of the previous subsection, if the fronts remain sufficiently well-separated, since their mutual distance should be at least of order $\alpha_*\varepsilon$. Our results below show that collisions in (1.16) induce an intermediate time layer for solutions to $(\mathrm{PGL})_\varepsilon$ or order ε^2, where annihilation of fronts takes places. This time layer is actually described by two collisions times: The first one, $\mathcal{T}^-_{\mathrm{col}}$ corresponds to a time where two fronts with opposite signs become $\alpha_*\varepsilon$ close, a distance at which the approximation by the differential equation (1.16) no longer remains valid. The existence and properties of the time $\mathcal{T}^-_{\mathrm{col}}$ are provided in the following result.

Theorem 1.2 *Let $\varepsilon > 0$, $T \geq 0$ and $\delta \geq \beta_*\varepsilon$ be given, where $\beta_* \geq 2\alpha_*$ is some constant depending only on V and M_0. Assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds and that the signs $\{\dagger_k\}_{k\in J}$ are not all identical. Then there exists some time $\mathcal{T}^-_{\mathrm{col}}$, such that the following hold:*

(i) *For any $t \in [T, \mathcal{T}^-_{\mathrm{col}}]$, property $\mathcal{WP}_\varepsilon(\alpha_*\varepsilon, t)$ holds and we have $\mathfrak{d}^+_a(t) \geq \mathfrak{d}^+_a(T) - c_*\varepsilon$.*

(ii) *We have $\mathfrak{d}^-_a(\mathcal{T}^-_{\mathrm{col}}) \leq 2\alpha_*\varepsilon$.*

(iii) *We have the upper bound $\mathcal{T}^-_{\mathrm{col}} - T \leq C^*\varepsilon^2 \exp\left(\frac{\mathfrak{d}^-_a(T)}{\varepsilon}\right)$, for some constant $C^* > 0$ depending only on V and M_0.*

(iv) *For any $t \in [T, \mathcal{T}^-_{\mathrm{col}}]$, if $\mathfrak{d}_a(t) \leq (1 - \frac{\rho_*}{4})\mathfrak{d}^-_a(T)$, then property $\mathcal{WP}_\varepsilon(\frac{\nu_*}{2}\mathfrak{d}_a(t), t)$ holds.*

Notice that the fact that two fronts with opposite signs become close at time $\mathcal{T}^-_{\mathrm{col}}$ is stated in part (ii). In contrast, fronts with the same signs remain well-separated, as shown by the first assertion.

In order to analyze the annihilation of fronts, we provide first some definitions. Assume therefore that at some time t conditions $\mathcal{WP}_\varepsilon(\delta, t)$ are satisfied, with $\delta \geq \alpha_1\varepsilon$. We say that a point $a_{k_0}(t)$ for $k_0 \in J(t)$ is free, if and only if

$$|a_{k_0\pm1}(t) - a_{k_0}(t)| \geq \kappa_{\mathrm{f}}\varepsilon, \tag{1.22}$$

where the constant $\kappa_{\mathrm{f}} > 0$ depends only on V and M_0 and will be defined in Section 9, and with the convention that $a_0(t) = -\infty$ and $a_{\ell+1}(t) = +\infty$. We set

$$\mathfrak{S}_{\mathrm{free}}(t) = \{k \in J(t), \text{ such that } a_k(t) \text{ is free}\}.$$

Likewise, we say that a point $a_{k_0}(t)$ for $k_0 \in J(t)$ is purely repulsive if and only if $\dagger_{k_0} = \dagger_{k_0+1} = \dagger_{k_0-1}$, with the convention that $\dagger_0 = \dagger$ and $\dagger_{\ell+1} = \dagger_\ell$. We set

$$\mathfrak{S}_{\mathrm{rep}}(t) = \{k \in J(t), \text{ such that } a_k(t) \text{ is purely repulsive}\},$$

and $\mathfrak{S}_{\mathrm{attr}}(t) = \mathfrak{S}(T) \setminus \mathfrak{S}_{\mathrm{rep}}(T)$. Notice that, if a point $a_{k_0}(t)$ is purely repulsive, then we have $|a_{k_0\pm1}(t) - a_{k_0}(t)| \geq \sqrt{\lambda_{\max}^{-1}}\mathfrak{d}_a^+(t)$, and hence is free if $\mathfrak{d}_a^+(t)$ is sufficiently large, that is,

$$\mathfrak{S}_{\mathrm{rep}}(t) \subset \mathfrak{S}_{\mathrm{free}}(t), \tag{1.23}$$

provided $\mathfrak{d}_a^+(t) \geq \sqrt{\lambda_{\max}}\kappa_{\mathrm{c}}\varepsilon$. In view of assertion (i) in Theorem 1.2, this last condition is met in particular for $t \in [T, \mathcal{T}_{\mathrm{col}}^-]$ provided we choose β_* sufficiently large, what we assume from now on. The next results provide the annihilation of at least two fronts with opposite signs, within an additional time of order ε^2.

Theorem 1.3 *Let $\varepsilon > 0$, $T \geq 0$ and $\delta \geq \gamma_*\varepsilon$ be given, where γ_* is some constant depending only on V and M_0. Assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds, and that the signs $\{\dagger_k\}_{k\in J}$ are not all identical. There exists a time $\mathcal{T}_{\mathrm{col}}^+$ such that condition $\mathcal{WP}_\varepsilon(\alpha_*\varepsilon, \mathcal{T}_{\mathrm{col}}^+)$ holds, and such that for some constant Υ depending only on V and M_0,*

$$0 < \mathcal{T}_{\mathrm{col}}^+ - \mathcal{T}_{\mathrm{col}}^- \leq \Upsilon\varepsilon^2. \tag{1.24}$$

Moreover, the following holds:

(i) *We have the inclusion $\mathfrak{S}(\mathcal{T}_{\mathrm{col}}^+) \subset \mathfrak{S}(\mathcal{T}_{\mathrm{col}}^-) + [-\kappa_{\mathrm{c}}\varepsilon, \kappa_{\mathrm{c}}\varepsilon]$, where κ_{c} is some constant depending only on V and M_0.*

(ii) *We have $\sharp(\mathfrak{S}_{\mathrm{free}}(\mathcal{T}_{\mathrm{col}}^+)) = \sharp(\mathfrak{S}_{\mathrm{free}}(\mathcal{T}_{\mathrm{col}}^-))$, $\sharp(\mathfrak{S}_{\mathrm{rep}}(\mathcal{T}_{\mathrm{col}}^+)) = \sharp(\mathfrak{S}_{\mathrm{rep}}(\mathcal{T}_{\mathrm{col}}^-))$,*

$$\mathfrak{S}_{\mathrm{free}}(\mathcal{T}_{\mathrm{col}}^+) \subset \mathfrak{S}_{\mathrm{free}}(\mathcal{T}_{\mathrm{col}}^-) + [-\kappa_{\mathrm{c}}\varepsilon, \kappa_{\mathrm{c}}\varepsilon] \quad \text{and} \quad \mathfrak{S}_{\mathrm{rep}}(\mathcal{T}_{\mathrm{col}}^+) \subset \mathfrak{S}_{\mathrm{rep}}(\mathcal{T}_{\mathrm{col}}^-) + [-\kappa_{\mathrm{c}}\varepsilon, \kappa_{\mathrm{c}}\varepsilon].$$

(iii) *We have for some $m \geq 1$,*

$$\sharp(\mathfrak{S}_{\mathrm{attr}}(\mathcal{T}_{\mathrm{col}}^+)) \leq \sharp(\mathfrak{S}_{\mathrm{attr}}(\mathcal{T}_{\mathrm{col}}^-)) - 2m.$$

We notice, combining assertion (ii) and assertion (iii) that the total number of front points has decreased by $2m$, that is,

$$\sharp(\mathfrak{S}(\mathcal{T}_{\mathrm{col}}^+)) \leq \sharp(\mathfrak{S}(\mathcal{T}_{\mathrm{col}}^-)) - 2m, \quad m \geq 1,$$

so that the results in Theorem 1.3 do indeed describe the annihilation of at least two fronts, annihilation which occurs on a time interval of order ε^2, in view of (1.24). Moreover, in view of assertion (ii), we have a one to one correspondence between free or repulsive points at time $\mathcal{T}_{\mathrm{col}}^-$ and $\mathcal{T}_{\mathrm{col}}^+$, each of these points being moved at most at a distance of order ε. The annihilation occurs among the attractive points which are not free, among which m pairs disappear in the process. This annihilation process can then only occur a finite number of times, after which the system becomes purely repulsive, all fronts repelling each other.

1.4 Relaxing the preparedness assumptions

We relax now the preparedness assumptions, and extend our analysis to the case of bounded energy initial data. For that purpose, we make use of the framework and concept developed in [3], and define as there for a scalar function u on $\mathbb{R}$, its front set as the set $\mathcal{D}(u)$ defined by

$$\mathcal{D}(u) \equiv \{x \in \mathbb{R},\ \mathrm{dist}(u(x), \Sigma) \geq \mu_0\}.$$

This notion which might be understood as a substitute to the notion of front points defined so far only when assumption $\mathcal{WP}_\varepsilon$ holds. The constant $\mu_0 > 0$ which appears in this definition is chosen so that, for $i = 1, \cdots, q$, we have $B(\sigma_i, \mu_0) \cap B(\sigma_j, \mu_0) = \emptyset$ for all $i \neq j$ in $\{1, \cdots, q\}$ and $\frac{1}{2}\lambda_i \leq V''(y) \leq 2\lambda_i$ for all $i \in \{1, \cdots, q\}$ and $y \in B(\sigma_i, \mu_0)$. A few elementary arguments yield (see [3, Corollary 1]) that, if the map u satisfies the energy bound $\mathcal{E}_\varepsilon(u) \leq M_0$, then there exists ℓ points $x_1, \cdots, x_\ell$ in $\mathcal{D}(u)$, such that

$$\mathcal{D}(u) \equiv \{x \in \mathbb{R},\ \mathrm{dist}(u(x), \Sigma) \geq \mu_0\} \subset \bigcup_{k=1}^{\ell} [x_i - \varepsilon, x_i + \varepsilon] \tag{1.25}$$

with the bound $\ell \leq \ell_0 = \frac{M_0}{\eta_0}$ on the number of points, where $\eta_0 > 0$ is some constant depending only on the potential V. In the context of equation $(\mathrm{PGL})_\varepsilon$, we set moreover $\mathcal{D}(t) = \mathcal{D}(v_\varepsilon(\cdot, t))$, so that

$$\mathcal{D}(t) \subset \bigcup_{k=1}^{\ell(t)} I_k(t), \tag{1.26}$$

where the intervals $I_k(t) \equiv [\mathfrak{a}_k^-(t), \mathfrak{a}_k^+(t)]$ are disjoint, with a length less than $\varepsilon\ell$ and $\sharp(J) \leq \ell$. It follows from our definitions of the front set, that in the intervals $[\mathfrak{a}_{k-1}^+(t), \mathfrak{a}_k^-(t)]$ the function $v_\varepsilon(\cdot, t)$ takes values near some of the minimizers, which we denote by $\sigma_{j^-(k)} = \sigma_{j^+(k-1)}$. The points $\mathfrak{a}_k^\pm$ play a role similar to the front points a_k in the definition $\mathcal{WP}_\varepsilon$, except that they are now only defined up to a scale of order ε, and that the function is not necessarily close to a stationary front in their neighborhood[7]. As a matter of fact, we notice that, if $\mathcal{WP}_\varepsilon(\delta, t)$ is satisfied, then, in view of (WP2), we have

$$\mathcal{D}(t) \subset \bigcup_{k \in J(t)} \{a_k(t)\} + [-\kappa_{\mathrm{w}}\varepsilon, \kappa_{\mathrm{w}}\varepsilon] \tag{1.27}$$

for some suitable constant κ_{w} depending only on V and M_0. However, the regularizing properties of equation $(\mathrm{PGL})_\varepsilon$ are at work, and drives the function towards a well-prepared case, as our next result shows.

Theorem 1.4 *Let $T \geq 0$ and $\alpha > \alpha_*$ be given and assume that (H_0) holds. Then there exists a time $t \in [T, T + \omega(\alpha)\varepsilon^2]$, such that $\mathcal{WP}_\varepsilon(\alpha\varepsilon, t)$ holds with $\omega(\alpha) = \mathrm{c_0}^2 M_0 \exp(2\rho_1\alpha)$. Moreover, we have*

$$\bigcup_{k \in J(t)} \{a_k(t)\} \subset \mathcal{D}^\varepsilon(T) + [-\kappa_*\varepsilon, \kappa_*\varepsilon].$$

[7] in contrast, the results described assuming $\mathcal{WP}_\varepsilon$ yield an accuracy of order $\varepsilon \log(-\rho_* \frac{\delta}{\varepsilon})$, hence extremely sharp when δ is of order 1.

A general principle might therefore be stated as follows: Up to an error term of order ε^2 in time and of order ε in space, the system behaves as if it were well-prepared according to assumption $\mathcal{WP}_\varepsilon$. More precisely, after an initial boundary layer in time of size at most $\omega(\alpha_*)\varepsilon^2$, during which the front set has been moved at distance of size at most $\kappa_*\varepsilon$, the preparedness assumption $\mathcal{WP}_\varepsilon(\alpha_*\varepsilon, t)$ is full-filled, so that we are in position to apply Theorems 1.1–1.3, which relate the dynamics to the ODE (1.16).

1.5 Relaxing the assumptions on V

The assumptions on the potential can be modified and in fact actually weakened to handle also other kind of potentials, for instance periodic potentials like (1.4). For that aim, we introduce an alternate set of assumptions on the potential V, which can be stated as follows:

$(\mathrm{H})_{1\mathrm{bis}}$ We have that $\inf V = 0$ and that the set of minimizers Σ is a discrete set which contains at least two elements.

We may hence write $\Sigma = \{\sigma_i\}_{i\in J}$, where $J \subset \mathbb{Z}$, with $\sigma_i < \sigma_j$, if $i < j$. If i_0 is a maximal element (resp. minimal) in J, then we set $\sigma_{i_0+1} = +\infty$ (resp. $\sigma_{i_0-1} = -\infty$).

$(\mathrm{H})_{2\mathrm{bis}}$ The potential V is of class C^3 with $\|V'\|_{C^2(\mathbb{R})} < \infty$. Moreover, we have

$$\lambda_{\min} \equiv \inf_{i\in J} V''(\sigma_i) > 0.$$

$(\mathrm{H})_{3\mathrm{bis}}$ There exists some number $\nu > 0$ such that, if $i \in J$ or $i+1 \in J$, then we have

$$\inf_{s\in[\sigma_i,\sigma_{i+1}]} V(s) \geq \nu \quad \text{for } i \in J.$$

Obviously, the potential given in (1.4) satisfies these assumptions, as well as actually any smooth periodic potential having non-degenerate minimizers. We have the following theorem.

Theorem 1.5 *The results in Theorems* 1.1–1.4 *hold true if we replace the assumptions* (H_1), (H_2) *and* (H_3) *on the potential* V *by assumptions* $(\mathrm{H}_{1\mathrm{bis}})$, $(\mathrm{H}_{2\mathrm{bis}})$ *and* $(\mathrm{H}_{3\mathrm{bis}})$, *respectively.*

The argument of the proof of Theorem 1.5 actually relies on an elementary observation.

Proposition 1.1 *Assume that the potential* V *satisfies assumptions* $(\mathrm{H}_{1\mathrm{bis}})$, $(\mathrm{H}_{2\mathrm{bis}})$ *and* $(\mathrm{H}_{3\mathrm{bis}})$, *and let* u *be such that* $\mathcal{E}_\varepsilon(u) \leq M_0$. *Then the limits* $u(\pm\infty) \equiv \lim_{x\to\pm\infty} u(x)$ *exist and we have, for some constant* A *depending only on* $\|V'\|_{C^2(\mathbb{R})} < \infty$, ν, *and* $\lambda_{\min}$,

$$u(x) \in [u(+\infty) - A, u(+\infty) + A], \quad \forall x \in \mathbb{R}. \tag{1.28}$$

Moreover, if v_ε *is a solution to* $(\mathrm{PGL})_\varepsilon$ *satisfying* (H_0), *then the limits* $u(\pm\infty, t) \equiv \lim_{x\to\pm\infty} u(x,t)$ *do not depend on the time* t *and hence*

$$u(x,t) \in [u_0(+\infty) - A, u_0(+\infty) + A], \quad \forall x \in \mathbb{R},\ t \in \mathbb{R}. \tag{1.29}$$

In order to prove Theorem 1.5, we then observe that relation (1.29) shows that the solution takes values only on a finite interval of $\mathbb{R}$: We therefore may modify the potential outside of this interval without changing the solution, so that assumptions (H_1)–(H_3) are fulfilled. We may then rely on our previous results.

1.6 Elements in the proofs

The proofs of our main results contain several distinct ingredients. The starting point is the study of solutions to the perturbed stationary equation, which writes for a scalar function u defined on $\mathbb{R}$ as

$$u_{xx} = \varepsilon^{-2} V'(u) + f \quad \text{on } \mathbb{R}. \tag{1.30}$$

Our main result concerning equation (1.30), which is completely elementary since it relies essentially on Gronwall's lemma, is given in Proposition 3.1. It states that, if the function verifies an energy bound of the form $\mathcal{E}_\varepsilon(u) \leq M_0$ and if $\varepsilon^{\frac{3}{2}} \|f\|_{L^2(\mathbb{R})}$ is sufficiently small, then the function u is close to a chain of stationary solutions, i.e., heteroclinic solutions, as described in property $\mathcal{WP}_\varepsilon(\delta, t)$, with a parameter δ proportional to $-\varepsilon \log(\varepsilon^{\frac{3}{2}} \|f\|_{L^2(\mathbb{R})})$. We use this result with $u \equiv v_\varepsilon(\cdot, t)$ and $f(\cdot) \equiv \partial_t v_\varepsilon$, so that smallness of the dissipation $\|\partial_t v_\varepsilon(\cdot, t)\|^2_{L^2(\mathbb{R})}$ at some time t yields property $\mathcal{WP}_\varepsilon(\delta, t)$, with a parameter δ large when dissipation becomes small. Combining this property with the energy identity (1.5), which allows to control dissipation, we show that the flow drives to well-preparedness. A similar result was already established in [3, Theorem 3]. However, here we take advantage of an important specificity of the scalar case, which is actually the only one which is used in this paper: Stationary solutions are perfectly known, and can even be integrated thanks to the method of separation of variables. In particular, assumption $\mathcal{WP}_\varepsilon$ implies a kind of quantization of the energy, which, in turn, allows to improve bounds on the dissipation.

The next step is to introduce more dynamics in our arguments. For that purpose, as in [3, Lemma 2], we use extensively the localized version of (1.5), a tool which turns out to be perfectly adapted to track the evolution of fronts, and which writes, for a smooth test function χ with compact support in $\mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \chi(x)\, e_\varepsilon(v_\varepsilon) \mathrm{d}x + \int_{\mathbb{R} \times \{t\}} \varepsilon \chi(x) |\partial_t v_\varepsilon|^2 \mathrm{d}x = \mathcal{F}_S(t, \chi, v_\varepsilon), \tag{1.31}$$

where the term $\mathcal{F}_S$ is given by

$$\mathcal{F}_S(t, \chi, v_\varepsilon) = \int_{\mathbb{R} \times \{t\}} \left(\left[\varepsilon \frac{\dot{v}_\varepsilon^2}{2} - \frac{V(v_\varepsilon)}{\varepsilon} \right] \ddot{\chi} \right) \mathrm{d}x \equiv \varepsilon^{-1} \int_{\mathbb{R} \times \{t\}} \xi(v(\cdot, t)) \ddot{\chi} \mathrm{d}x. \tag{1.32}$$

The first term on the right-hand side of identity (1.31) stands for local dissipation, whereas the second is a flux. The quantity ξ is defined for a scalar function u by

$$\xi(u) \equiv \varepsilon^2 \frac{\dot{u}^2}{2} - V(u), \tag{1.33}$$

sometimes referred to as the discrepancy term in the literature. It is constant for stationary solutions on some given interval I, i.e., for solutions to

$$-u_{xx} + \varepsilon^{-2} V'(u) = 0 \quad \text{on } I, \tag{1.34}$$

and it vanishes for finite energy solutions to (1.34) on $I = \mathbb{R}$. Using (1.31) for appropriate choices of test functions, combined with several parabolic estimates, we have shown in [3] the following theorem.

Theorem 1.6 *Let $T > 0$ be given, and assume that* (H_0) *holds. There exist constants* $\rho_0 > 0$ *and* $\alpha_0 > 0$*, depending only on the potential* V *and on* M_0*, such that if* $r \geq \alpha_0\varepsilon$*, then for every* $t \geq 0$*,*

$$\mathcal{D}(t + \Delta t) \subset \mathcal{D}(t) + [-r, r], \tag{1.35}$$

provided $0 \leq \Delta t \leq \rho_0^2 r^2 \exp\big(\rho_0 \dfrac{r}{\varepsilon}\big)$.

Actually, Theorem 1.6 is established in [3] for general systems, under assumptions on the potential V which are the higher dimensional analogs of (H_1)–(H_3). In particular, a rather remarkable fact is that the result does not involve any assumption of any kind on the stationary solutions[8]. A central idea in the proof is to derive appropriate upper bounds on the discrepancy in region which are far from the front set, as well as a suitable choice of test functions χ for (1.31): They are chosen to be affine near the front sets, so that the second derivative vanishes there, and the flux term needs only to be estimates off the front set.

Theorem 1.6 provides a first estimate of the velocity. This estimate combined with the results of Proposition 3.1, and the energy identity (1.5) is actually already sufficient to prove Theorem 1.4.

In order to establish Theorem 1.1 and derive actually an efficient motion law, we need to derive a far more precise estimate for the discrepancy. In order to sketch the argument, assume that $\mathcal{WP}_\varepsilon(\delta, t)$ holds for some $\delta > 0$, and let $a_k(t)$ and $a_{k+1}(t)$ be two front points, with $k \in J(t)$. In order to estimate the interaction between these two points, we evaluation $\xi(\cdot, t)$ near the middle point $a_{k+\frac{1}{2}}(t) = \frac{1}{2}(a_k(t) + a_{k+1}(t))$. To that aim, we use several observations as follows:

(1) The behavior of v_ε near the points a_k is described with high accuracy using the appropriate heteroclinic solutions near the points a_k and a_{k+1}, say on intervals of the form $[a_k(t), a_k(t) + \widetilde{\delta}]$ and $[a_{k+1}(t) - \widetilde{\delta}, a_{k+1}(t)]$, where $\widetilde{\delta}$ is of the same order as δ. We will term this region the inner region.

(2) The heteroclinic solutions are known.

(3) The evolution of the points a_k is known to be small thanks to Theorem 1.6.

(4) In the outer-region $[a_k(t) + \widetilde{\delta}, a_{k+1}(t) - \widetilde{\delta}]$, the solution is well approximated by the solution to the linearized equation near the minimizer $\sigma_{j(k)^+}$, which turns out to be

$$\partial_t \mathfrak{u}_\varepsilon - \partial_{xx}^2 \mathfrak{u}_\varepsilon + \varepsilon^{-2} \lambda_{j(k)^+} u_\varepsilon = 0.$$

The boundary conditions are deduced from the values of the heteroclinc solutions at $a_k(t) + \widetilde{\delta}$ and $a_{k+1}(t) - \widetilde{\delta}$.

(5) It relaxes very quickly to the solution to the corresponding stationary equation: This time relaxation is described by factors involving terms of the form $\exp\big(-\frac{\lambda_{j(k)^+} t}{\varepsilon^2}\big)$.

The expression of the discrepancy ξ for the stationary solution in the outer region then offers, after an appropriate small relaxation time, a good approximation of ξ near the point $a_{k+\frac{1}{2}}(t)$. We then use identity (1.31) with functions χ which are affine, except possibly near

[8] which is a far more difficult question for systems than in the scalar case

the points $a_{k+\frac{1}{2}}(t)$, so that the previous expansion can be used. We show that this yields a good approximation of the motion of the front points, leading to the proof of Theorem 1.1.

The proof of Theorem 1.2 is based on the approximation provided by Theorem 1.1 as well as some properties of the system of ordinary differential equations (1.16). The proof of Theorem 1.3 uses extensively, besides the results in Theorem 1.1–1.2, the quantization of the energy.

We describe now the outline of the paper. Since our arguments involve several ordinary differential equations, in particular equations (1.7), (1.16) and (1.30), and since the properties involved are all completely elementary, we wish to present them first. Therefore, we start in Section 2 with some result concerning equation (1.16): These results are only used in the proof of Theorems 1.2–1.3, the reader may therefore skip this part in a first reading of the paper. Section 3 presents some properties of the stationary equations (1.7) and (1.30), in particular properties of the heteroclinic orbits, which are obtained through the method of separation of variables, as well as the statement of proof of Proposition 3.1. In Section 4, we describe several properties related to the well-preparedness assumption $\mathcal{WP}_\varepsilon$, in particular the quantization of the energy, how it relates to dissipation, and its numerous implications for the dynamics. In Section 5, we set up a toolbox, which presents various parabolic linear estimates. These estimates are then extensively used in Section 6, where they provide estimates for $(\mathrm{PGL})_\varepsilon$ on parabolic cylinders, assuming that the map takes values to one of the minimizers σ_i. A major emphasis is put on the expansion of the quantity ξ, which is estimated sharply near the middle of the cylinder. Section 7 is devoted to the proof of Theorem 1.1, based on formula (1.31) as well as on the expansions of ξ provided in Section 6. Section 8 is devoted to the proof of Theorem 1.2 whereas Section 9 is devoted to the proof of Theorem 1.3. Finally in Section 10, we outline the proof of Theorem 1.5.

2 Some Remarks on the Differential Equation (1.16)

2.1 Statement of results

This section, which is independent of our previous analysis, focuses on general properties of the ordinary differential equations (1.16), with an emphasis on estimates for the possible collision time. Therefore, we assume that we are given an integer $\ell \in \mathbb{N}^*$, a mapping $\dagger$ from J to $\{+,-\}$, where $J = \{1, \cdots, \ell\}$, a solution $t \mapsto b(t) = b_1(t), \cdots, b_\ell(t)$ to the system (1.16), where the constants $\Gamma^{\pm}_{k,\varepsilon}$ are defined according to the definition (1.15), which requires that the value of one of the numbers $i(k)$, for instance $i(1)$ is also given. We consider the solution on its maximal interval of existence, that is, $[0, T_{\max}]$. We assume moreover throughout this section that the correction terms $\mathcal{C}^\dagger_k(s)$ satisfy the smallness assumption

$$|\mathcal{C}^\dagger_k(s)| \leq \frac{\mathfrak{q}_{\min}}{2\mathfrak{q}_{\max} 8^\ell}, \tag{2.1}$$

where $\mathfrak{q}_{\min} = \inf\{\mathfrak{q}_i\}$ and $\mathfrak{q}_{\max} = \sup\{\mathfrak{q}_i\}$. In order to describe the behavior of this system, in particular possible collisions, we are led to introduce the quantity

$$\mathfrak{d}_b(t) = \inf\{\sqrt{\lambda_{j^+(k)}}|b_k(t) - b_{k+1}(t)| \quad \text{for } k \in 1, \cdots, \ell(t_0) - 1\}. \tag{2.2}$$

It turns out that this quantity controls the motion of the points as our next result shows.

Proposition 2.1 *Let $b = (b_1, \cdots, b_\ell)$ be a solution to* (1.16) *on its maximal interval of existence* $[0, T_{\max}]$ *and assume that* (2.1) *is satisfied. Let $0 \le t_1 \le t_2 \le T_{\max}$ be given. For $k = 1, \cdots, \ell$, we have the bound*

$$|b_k(t_1) - b_k(t_2)| \le \mathcal{S}_0 |\mathfrak{d}_b(t_1) - \mathfrak{d}_b(t_2)| + \mathcal{S}_1 \varepsilon,$$

where we have set $\mathcal{S}_0 = 16^m (m+1)^{-2} \mathfrak{q}_{\max} \lambda_{\min}^{-1} \lambda_{\max}^{\frac{1}{2}} \mathcal{B}_{\max}^4 \mathcal{B}_{\min}^{-6}$ and $\mathcal{S}_1 = 8 \mathfrak{q}_{\min}^{-1} \mathcal{B}_{\max}^2 \log((m+1)\lambda_{\min}^{-\frac{1}{2}} \mathcal{B}_{\max}^2)$.

Notice that we have also the more straightforward inequality

$$|\mathfrak{d}_b(t_1) - \mathfrak{d}_b(t_2)| \le \sum_k |b_k(t_1) - b_k(t_2)|,$$

which is a simple consequence of the triangle inequality.

The proof of Proposition 2.1 will be given later. In view of the previous result, it is therefore of importance to derive bounds for $\mathfrak{d}_b$. In this direction, we first have the following result.

Proposition 2.2 *Let $b = (b_1, \cdots, b_\ell)$ be a solution to* (1.16) *on its maximal interval of existence* $[0, T_{\max}]$ *and assume that* (2.1) *is satisfied. Then, we have, for any $t \in [0, T_{\max}]$,*

$$\log\Big[1 - \varepsilon^{-2} \mathcal{S}_2 t \exp\Big(-\frac{\mathfrak{d}_b(0)}{\varepsilon}\Big)\Big] \le \frac{\mathfrak{d}_b(t) - \mathfrak{d}_b(0)}{\varepsilon} \le \log\Big[1 + \varepsilon^{-2} \mathcal{S}_2 t \exp\Big(-\frac{\mathfrak{d}_b(0)}{\varepsilon}\Big)\Big],$$

where we have set $\mathcal{S}_2 = 8\varepsilon^{-1} \sqrt{\lambda_{\max}} \mathfrak{q}_{\min}^{-1} \mathcal{B}_{\max}^2$.

Proof It follows from (1.16) and (2.1) that, for any $k = 1, \cdots, \ell$, we have

$$\varepsilon \Big| \frac{\mathrm{d}}{\mathrm{d}t} [b_k(t)] \Big| \le 4 \mathfrak{q}_{\min}^{-1} \mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_b(t)}{\varepsilon}\Big), \tag{2.3}$$

and hence $\varepsilon \big| \frac{\mathrm{d}}{\mathrm{d}t} \sqrt{\lambda_{j^+(k)}} [b_k(t) - b_{k+1}(t)] \big| \le \mathcal{S}_2 \exp\big(-\frac{\mathfrak{d}_b(t)}{\varepsilon}\big)$. Integrating, we obtain

$$\varepsilon |\mathfrak{d}_b(t) - \mathfrak{d}_b(0)| \le \mathcal{S}_2 \int_0^t \exp\Big(-\frac{\mathfrak{d}_b(s)}{\varepsilon}\Big) \mathrm{d}s,$$

and the assertion follows as a standard exercise.

In order to derive more refined estimates, we need to take into account the signs of the interactions. For that purpose, we introduce the quantities

$$\mathfrak{d}_b^{\pm}(t) = \inf\{\sqrt{\lambda_{j^+(k)}} |b_k(t) - b_{k+1}(t)| \quad \text{for } k \in 1, \cdots, \ell - 1 \text{ such that } \dagger_k = \pm \dagger_{k+1}\},$$

with the convention that the quantity is equal to $+\infty$ in case the defining set is empty, and we also set $\mathcal{B}_{\max} = \sup\{B_i^{\pm}\}$ and $\mathcal{B}_{\min} = \inf\{B_i^{\pm}\}$. The main results on this section can be summarized as follows.

Proposition 2.3 *Let $b = (b_1, \cdots, b_\ell)$ be a solution to* (1.16) *on its maximal interval of existence* $[0, T_{\max}]$ *and assume that* (2.1) *is satisfied. Then, we have, for any time $t \in [0, T_{\max}]$,*

$$\begin{cases} \dfrac{\mathfrak{d}_b^+(t) - \mathfrak{d}_b^+(0)}{\varepsilon} \ge \log\Big[\mathcal{S}_3 + \mathcal{S}_4 \varepsilon^{-2} t \exp\Big(-\dfrac{\mathfrak{d}_b^+(0)}{\varepsilon}\Big)\Big] - \log \mathcal{S}_5, \\ \dfrac{\mathfrak{d}_b^-(t) - \mathfrak{d}_b^-(0)}{\varepsilon} \le \log\Big[\mathcal{S}_3' - \mathcal{S}_4 \varepsilon^{-2} t \exp\Big(-\dfrac{\mathfrak{d}_b^-(0)}{\varepsilon}\Big)\Big] - \log \mathcal{S}_5', \end{cases} \tag{2.4}$$

where $\mathcal{S}_3 = (m+2)\lambda_{\min}^{\frac{1}{2}}\mathcal{B}_{\max}^{-2}$, $\mathcal{S}_4 = 16^{-m}(m+1)^2\mathfrak{q}_{\max}^{-1}\lambda_{\min}\frac{\mathcal{B}_{\min}^4}{\mathcal{B}_{\max}^4}$, $\mathcal{S}_5 = (\lambda_{\max}^{\frac{1}{2}}\mathcal{B}_{\min}^{-2})$, *and* $\mathcal{S}'_3 = \sqrt{\lambda_{\max}}\mathcal{B}_{\min}^{-2}$ *and* $\mathcal{S}'_5 = \big(\frac{\sqrt{\lambda_{\min}}}{(m+1)\mathcal{B}_{\max}^2}\big)$. *If all signs* $\{\dagger_k\}_{k\in J}$ *have the same value, then* $T_{\max} = +\infty$. *Otherwise, we have the estimate*

$$T_{\max} \le \varepsilon^2 \frac{\mathcal{S}'_3}{\mathcal{S}_4} \exp\Big(\frac{\mathfrak{d}_b^-(0)}{\varepsilon}\Big). \tag{2.5}$$

Given any two times $0 \le t_1 \le t_2 \le T_{\max}$, *we have the estimate*

$$\int_{t_1}^{t_2} \exp\Big(-\frac{\mathfrak{d}_b(s)}{\varepsilon}\Big)\mathrm{d}s \le 2\mathcal{S}_4^{-1}\lambda_{\max}^{\frac{1}{2}}\mathcal{B}_{\min}^{-2}\varepsilon\,[\mathfrak{d}_b(t_2) - \mathfrak{d}_b(t_1)] + \mathcal{S}'_1\varepsilon^2, \tag{2.6}$$

where $\mathcal{S}'_1$ *is defined in Lemma* 2.3 *below. Moreover the following inequality holds, in the sense of distributions*

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{d}_b^-(t) \le 4\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_b^-(t)}{\varepsilon}\Big) \quad \text{on } [0, T_{\max}]. \tag{2.7}$$

Notice that the behavior of $\mathfrak{d}_b^+$ and $\mathfrak{d}_b^-$ are very different, the first one measuring the repulsive forces present in the system, whereas the second measures the attractive ones.

Remark 2.1 We have stressed so far the behavior of the equation (1.16) for positive times. The properties of the system are actually similar when time flows backwards, i.e., considering negative times. It suffices to change the attractive forms into repulsive ones and vice-versa to deduce the corresponding results. Notice in particular that $\mathfrak{d}_b^\pm$ is changed into $\mathfrak{d}_b^\mp$.

Proof of Proposition 2.1 (Assuming Proposition 2.3) Integrating inequality (2.3), we obtain

$$|b_k(t_1) - b_k(t_2)| \le 4\varepsilon^{-1}\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \int_{t_1}^{t_2} \exp\Big(-\frac{\mathfrak{d}_b(s)}{\varepsilon}\Big)\mathrm{d}s,$$

and the conclusion follows invoking (2.6).

The proof of Proposition 2.3, relies on several observations which we present next, the completion of the proof of Proposition 2.3 being presented in a separate subsection.

Our starting point is that, since the system (1.16) involves both attractive and repulsive forces, it is convenient to divide the collection $\{b_1(t), b_2(t), \cdots, b_\ell(t)\}$ into repulsive and attractive chains. Consider more generally a positive integer $\ell \in \mathbb{N}^*$, set $J = \{1, \cdots, \ell\}$ and let $\dagger$ be a function from J to $\{+, -\}$. We say that a subset A of J is a chain if A consists of consecutive elements.

Definition 2.1 *Let* $A = \{k, k+1, k+2, \cdots, k+m, k+m+1\}$ *be an ordered subset of* $m+2$ *consecutive elements in* J, *with* $m \ge 0$.

(i) *The chain* A *is said to be a repulsive chain, if and only if given two elements* i_1 *and* i_2 *in* J, *we have if* $\dagger_{i_1} = \dagger_{i_2}$. *It is said to be a maximal repulsive chain, if there does exists a repulsive chain which contains* A *strictly.*

(ii) *The chain* A *is said to be an attractive chain, if and only if given two elements* i_1 *and* i_2 *in* J, *such that* $|i_1 - i_2| = 1$, *we have* $\dagger_{i_1} = -\dagger_{i_2}$. *It is said to be a maximal attractive chain, if there does exists an attractive chain which contains* A *strictly.*

Notice that, in view of our definition, repulsive or attractive chains contain at least two elements. For a given map $\dagger$, consider its maximal repulsive chains, ordered according to increasing numbers $A_1,\ A_2, \cdots, A_p$. Consider two consecutive chains $A_i = \{k_i, k_i+1, k_i+2, \cdots, k_i+m_i, k_i+m_i+1\}$ and $A_{i+1} = \{k_{i+1}, k_{i+1}+1, k_{i+1}+2, \cdots, k_{i+1}+m_{i+1}, k_{i+1}+m_{i+1}+1\}$. It follows from Definition 2.1 that $k_i+m_i+1 < k_{i+1}$. we leave to the reader to check that the chain

$$B_i = \{k_i+m_i+1, \cdots k_{i+1}\}$$

is a maximal attractive chain. In particular, we may decompose J, in increasing order, as

$$J = B_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots \cup B_{p-1} \cup A_p \cup B_p, \tag{2.8}$$

where the chains A_i are maximal repulsive chains, the sets B_i are maximal attractive chains for $i = 1, \cdots, p-1$, and the sets B_0 and B_p are possibly void or maximal attractive chains. Moreover, we have, for $i = 1, \cdots, p$,

$$A_i \cap B_i = \{k_i+m_i+1\}, \quad B_i \cap A_{i+1} = \{k_{i+1}\}.$$

2.2 Maximal repulsive chains

In this subsection, we restrict ourselves to the behavior of a maximal repulsive chain $A = \{j, j+1, \cdots, j+m\}$, $m \leq \ell - 2$ within the general system (1.16). Without loss of generally, we may assume that $\dagger_i = +$ for $i \in A$. Setting $\mathfrak{u}_k = b_{k+j}$, we are led to study the function $\mathfrak{U} = (\mathfrak{u}_0, \mathfrak{u}_1, \cdots, \mathfrak{u}_{m+1})$. It follows from the fact that b satisfies (1.16), $\mathfrak{U}$ is moved through a system of m ODE's, and two differential inequalities as follows:

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \mathfrak{u}_k(s) = -\mathfrak{q}_k^{-1} \sum_{\dagger \in \{+,-\}} \Gamma^{\dagger}_{k,\varepsilon}(\{\mathfrak{U}(s)\})[1 + \mathcal{C}^{\dagger}_k(s)] \tag{2.9}$$

and

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{u}_{m+1}(s) \geq \mathfrak{q}_{m+1}^{-1} \Gamma^{-}_{m+1,\varepsilon}(\{\mathfrak{U}(s)\})(1 + \mathcal{C}^{-}_m(s)) \geq 0, \\ \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{u}_0(s) \leq \mathfrak{q}_0^{-1} \Gamma^{+}_{0,\varepsilon}(\{\mathfrak{U}(s)\})(1 + \mathcal{C}^{+}_0(s)) \leq 0. \end{cases} \tag{2.10}$$

We assume that the solution is defined on $I = [0, T_{\max}]$, and that at initial time, we have

$$\mathfrak{u}_0(0) < \mathfrak{u}_1(0) < \cdots < \mathfrak{u}_m(0) < \mathfrak{u}_{m+1}(0). \tag{2.11}$$

The behavior of this system is related to the function F_ε defined on $\mathbb{R}^{m+2}$ by

$$F_\varepsilon(U) = \sum_{k=0}^{m} F_\varepsilon^{k+\frac{1}{2}}(U), \tag{2.12}$$

where, for $k = 1, \cdots, m-1$ and $u = (u_0, \cdots, u_{m+1})$, we set

$$F_\varepsilon^{k+\frac{1}{2}}(U) = \varepsilon \lambda_{j^+(k)}^{-\frac{1}{2}} \mathcal{B}^+_k \mathcal{B}^-_{k+1} \exp\Big(- \sqrt{\lambda_{j^+(k)}} \frac{|u_{k+1} - u_k|}{\varepsilon} \Big) \geq 0,$$

the numbers $\mathfrak{q}_k > 0$, $\mathcal{B}_k > 0$ and $\lambda_{j^+(k)} > 0$ being computed thanks to (1.15). In the case $q_0 < u_1 < \cdots < u_m < q_{m+1}$, the value of $F_\varepsilon(u)$ describes a nearest neighbor repulsive interactions between the points u_k. We have, for $k = 0, \ldots, m$,

$$\frac{\partial F}{\partial u_k}(U) = \Gamma^+_{k,\varepsilon}(U) - \Gamma^-_{k,\varepsilon}(U), \tag{2.13}$$

where $\Gamma^+_{m+1,\varepsilon}(U) = 0$, $\Gamma^-_{0,\varepsilon}(U) = 0$, for $k = 0, \cdots, m$, we have set

$$\Gamma^+_{k,\varepsilon}(U) = \frac{\partial F_\varepsilon^{k+\frac{1}{2}}}{\partial u_k}(U) = \mathcal{B}_k^+ \mathcal{B}_{k+1}^- \exp\Big(- \sqrt{\lambda_{j^+(k)}} \frac{|u_{k+1} - u_k|}{\varepsilon}\Big),$$

for $k = 1, \cdots, m+1$, we have set

$$\Gamma^-_{k,\varepsilon}(U) = -\frac{\partial F_\varepsilon^{k-\frac{1}{2}}}{\partial u_k}(U) = -B_k^- \mathcal{B}_{k-1}^+ \exp\Big(- \sqrt{\lambda_{j^-(k)}} \frac{|u_k - u_{k-1}|}{\varepsilon}\Big).$$

Notice in particular that $\Gamma^+_{k,\varepsilon} = \Gamma^-_{k+1,\varepsilon}$ for $k = 0, \cdots, m$. We consider

$$\rho_{\min}(U) = \inf\{\sqrt{\lambda^+_{j(k)}}|u_{k+1} - u_k|,\ k = 0, \cdots, m\} \quad \text{and} \quad \mathfrak{d}_{\mathfrak{u}}(t) = \rho_{\min}(\mathfrak{u}(t)).$$

We prove the following proposition in this subsection.

Proposition 2.4 *Assume that* (2.1) *is satisfied and that the function* $\mathfrak{U}$ *satisfies* (2.9)–(2.10) *on* $[0, T_{\max}]$ *with* (2.11). *Then, we have, for any* $t \in [0, T_{\max}]$,

$$\frac{\mathfrak{d}_{\mathfrak{u}}(t) - \mathfrak{d}_{\mathfrak{u}}(0)}{\varepsilon} \geq \log\Big[\mathcal{S}_3 + \mathcal{S}_4 \varepsilon^{-2} t \exp\Big(- \frac{\mathfrak{d}_{\mathfrak{u}}(0)}{\varepsilon}\Big)\Big] - \log \mathcal{S}_5. \tag{2.14}$$

The proof relies on several elementary observations, which we present first before completing the proof of Proposition 2.3. We start with some specific properties of the functional F, which are stated in the next lemma.

Lemma 2.1 *Let* $U = (u_0, \cdots, u_{m+1})$ *be such that* $u_0 < u_1 < \cdots < u_m < u_{m+1}$. *We have*

$$\lambda_{\max}^{-\frac{1}{2}} \mathcal{B}^2_{\min} \exp\Big(- \frac{\rho_{\min}(U)}{\varepsilon}\Big) \leq \frac{F(U)}{\varepsilon} \leq (m+1)\lambda_{\min}^{-\frac{1}{2}} \mathcal{B}^2_{\max} \exp\Big(- \frac{\rho_{\min}(U)}{\varepsilon}\Big), \tag{2.15}$$

$$|\nabla F(U)| \leq (m+2)\sqrt{\lambda_{\max}} \frac{\mathcal{B}^2_{\max}}{\mathcal{B}^2_{\min}} \varepsilon^{-1} F(U), \tag{2.16}$$

and for every $k = 0, \cdots, m+1$,

$$|\nabla F_\varepsilon(U)| \geq \frac{1}{4^m} |\Gamma^{\pm}_{k,\varepsilon} U)| \geq \frac{\mathcal{B}^2_{\min}}{4^m} \exp\Big(- \frac{\rho_{\min}(U)}{\varepsilon}\Big) \geq \frac{\mathcal{B}^2_{\min}\sqrt{\lambda_{\min}}}{(m+1)4^m \mathcal{B}^2_{\max}} \frac{F(U)}{\varepsilon}. \tag{2.17}$$

Proof Inequalities (2.15)–(2.16) are direct consequences of the definition (2.12) of F. In view of formula (2.13), if $k = 0$ or $k = m+1$, there is nothing to prove, provided that we choose $\mu_0 \leq 1$. Next, let $k = 1, \cdots, m$ and consider for instance $\Gamma^+_{k,\varepsilon}$. we distinguish the following two cases.

Case 1 $|\Gamma^-_{k,\varepsilon}| \leq \frac{1}{2}|\Gamma^+_{k,\varepsilon}|$. If this case occurs, then, we have, in view of (2.13),

$$|\Gamma^+_{k,\varepsilon}| \leq 2\Big|\frac{\partial F}{\partial u_k}(U)\Big| \leq 2|\nabla F(U)|,$$

and we are done with a choice of $\gamma_0 \le \frac{1}{2}$.

Case 2 $|\Gamma^+_{k-1,\varepsilon}| = |\Gamma^-_{k,\varepsilon}| \ge \frac{1}{2}|\Gamma^+_{k,\varepsilon}|$. In this case, we repeat the argument with k replaced by $k-1$. Then either

$$|\Gamma^+_{k-1,\varepsilon}| \le 2|\nabla F(U)|,$$

so that $|\Gamma^+_{k,\varepsilon}| \le 4|\nabla F(U)|$, and we are done, or $|\Gamma^+_{k-2,\varepsilon}| \ge \frac{1}{2}|\Gamma^+_{k-1,\varepsilon}|$, and we repeat the argument. Since we have to stop at $k=0$, this leads to the desired inequality.

The next result emphasizes the gradient flow structure of (1.16).

Lemma 2.2 *Let* $\mathfrak{U}$ *be a solution to* (2.9)–(2.10) *on* $[0,T]$, *such that* (2.11) *and* (2.1) *hold. Then, we have, for every* $t \in [0, T_{\max}]$,

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} F(\mathfrak{U}(t)) \le -\frac{\mathfrak{q}_{\max}^{-1}}{2} |\nabla F(\mathfrak{U}(t))|^2 \le -\mathcal{S}_4 \varepsilon^{-2} F(\mathfrak{U}(t))^2. \tag{2.18}$$

In particular, $F(\mathfrak{u}(t)) \le \varepsilon\big[\mathcal{S}_4\varepsilon^{-2}t + \frac{\varepsilon}{F(\mathfrak{U}(0))}\big]^{-1} \le F(\mathfrak{u}(0))$.

Proof Combining equations (2.9) and (2.10) with the chain rule, we are led to

$$\begin{aligned}
\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} F_\varepsilon(\mathfrak{U}(t)) &= \varepsilon \sum_{k=0}^{m+1} \frac{\partial F}{\partial u_k}(\mathfrak{U}(t)) \frac{\mathrm{d}\mathfrak{u}_k}{\mathrm{d}t}(t) \\
&\le -\mathfrak{q}_{\max}^{-1} |\nabla F_\varepsilon(\mathfrak{U}(t))|^2 + 2\mathfrak{q}_{\max}^{-1} |\nabla F(\mathfrak{U}(t))| \sup_{k,\dagger} |\Gamma_{k,\varepsilon}(\mathfrak{U}(t))| \sup_{k,\dagger} |\mathcal{C}_k^\dagger(t)| \\
&\le -\mathfrak{q}_{\max}^{-1} |\nabla F_\varepsilon(\mathfrak{U}(t))|^2 + \frac{\mathfrak{q}_{\max}^{-1}}{2} |\nabla F_\varepsilon(\mathfrak{U}(t))|^2 = -\frac{\mathfrak{q}_{\max}^{-1}}{2} |\nabla F(\mathfrak{U}(t))|^2,
\end{aligned}$$

where for the last inequality, we have invoked Lemma 2.1 and inequality (2.1). The second inequality in (2.18) is then a direct consequence of (2.17). Finally, the last inequality of the lemma follows by integration of the differential inequality (2.18).

Proof of Proposition 2.4 Combining the last inequality of Lemma 2.2 with inequality (2.15), inequality (2.14) follows.

We complete this section with the following lemma.

Lemma 2.3 *Let* $\mathfrak{U}$ *be a solution to* (2.9)–(2.10) *on* $[0,T]$, *such that* (2.11) *and* (2.1) *hold. Given any time* $0 \le t_1 \le t_2$, *we have, with* $\mathcal{S}_1' = 2\log((m+1)\lambda_{\min}^{-\frac{1}{2}}\mathcal{B}_{\max}^2)$,

$$\int_{t_1}^{t_2} \exp\Big(-\frac{\mathfrak{d}_{\mathfrak{u}}(s)}{\varepsilon}\Big)\mathrm{d}s \le \mathcal{S}_4^{-1}\lambda_{\max}^{\frac{1}{2}}\mathcal{B}_{\min}^{-2}\varepsilon\,[\mathfrak{d}_{\mathfrak{u}}(t_2) - \mathfrak{d}_{\mathfrak{u}}(t_1)] + \mathcal{S}_1'\varepsilon^2.$$

Proof We have by the chain rule and inequality (2.15),

$$-\varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}t}[\log(F(\mathfrak{U}(t)))] = -\frac{\varepsilon^2}{F(\mathfrak{U}(t))}\frac{\mathrm{d}}{\mathrm{d}t}F(\mathfrak{U}(t)) \ge \mathcal{S}_4 \frac{F(\mathfrak{U}(t))}{\varepsilon} \ge \mathcal{S}_4 \lambda_{\max}^{-\frac{1}{2}}\mathcal{B}_{\min}^2 \exp\Big(-\frac{\mathfrak{d}_{\mathfrak{u}}(t)}{\varepsilon}\Big).$$

The conclusion follows by integration.

2.3 Maximal attractive chains

In this section, we provide a few properties of a maximal attractive chains $B = \{j, j+1, \cdots, j+m\}$, with $m \le \ell - 2$ within the general system (1.16): In particular, we show that it generates collisions in finite time, with an upper bound on the collision time. We may assume without loss of generally that $\dagger_j = +$, so that $\dagger_{j+k} = \mathrm{sign}(-1)^k$. Defining $\mathfrak{U}$ as above, the function $\mathfrak{U}$ still satisfies (2.9), but the inequalities (2.10) are now replaced by

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{u}_{m+1}(s) \leq \mathfrak{q}_{m+1} \Gamma^-_{m+1,\varepsilon}(\{\mathfrak{U}(s)\})(1 + \mathcal{C}^-_m(s)) \leq 0, \\ \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{u}_0(s) \geq \mathfrak{q}_0 \Gamma^+_{0,\varepsilon}(\{\mathfrak{U}(s)\})(1 + \mathcal{C}^+_0(s)) \geq 0. \end{cases} \tag{2.19}$$

The behavior of the chain B is now still related to the functional $F_\varepsilon(U)$, where F_ε is defined in (2.12), with $\mathcal{B}^\pm_k = B^{\pm\mathrm{sign}(-1)^k}_{i(j)}$ and hence takes only two values, and $\lambda_{j^+(k)} = \lambda_{j^+}$. However, the differential inequality (2.18) is now turned into

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} F(\mathfrak{U}(t)) \geq \frac{\mathfrak{q}^{-1}_{\max}}{2} |\nabla F(\mathfrak{U}(t))|^2 \geq \mathcal{S}_4 F(\mathfrak{U}(t))^2, \tag{2.20}$$

which, by integration yields $\frac{F(\mathfrak{U}(t))}{\varepsilon} \geq \left[\frac{\varepsilon}{F(\mathfrak{U}(0))} - \mathcal{S}_4 \varepsilon^{-2} t\right]^{-1} \geq \left[\mathcal{S}'_3 \exp\left(\frac{\mathfrak{d}_{\mathfrak{u}}(0)}{\varepsilon}\right) - \mathcal{S}_4 \varepsilon^{-2} t\right]^{-1}$.

Proposition 2.5 *Assume that* (2.1) *is satisfied and that the function* $\mathfrak{U}$ *satisfies the system* (2.9) *and* (2.19) *on* $[0, T_{\max}]$ *together with* (2.11). *Then, we have, for any* $t \in [0, T_{\max}]$,

$$\frac{\mathfrak{d}_{\mathfrak{u}}(t) - \mathfrak{d}_{\mathfrak{u}}(0)}{\varepsilon} \leq \log\left[\mathcal{S}'_3 - \mathcal{S}_4 \varepsilon^{-2} t \exp\left(-\frac{\mathfrak{d}_{\mathfrak{u}}(0)}{\varepsilon}\right)\right] - \log \mathcal{S}'_5.$$

The argument is similar to the proof of Proposition 2.4, we therefore omit it.

Lemma 2.4 *Assume that* (2.1) *is satisfied and that the function* $\mathfrak{U}$ *satisfies the system* (2.9) *and* (2.19) *on* $[0, T_{\max}]$ *together with* (2.11). *Then, we have the estimate*

$$\int_{t_1}^{t_2} \exp\left(-\frac{\mathfrak{d}_{\mathfrak{u}}(s)}{\varepsilon}\right) \mathrm{d}s \leq \mathcal{S}_1^{-1} \lambda_{\max}^{\frac{1}{2}} \mathcal{B}_{\min}^{-2} \varepsilon [\mathfrak{d}_{\mathfrak{u}}(t_2) - \mathfrak{d}_{\mathfrak{u}}(t_1)] + \mathcal{S}_2 \varepsilon^2.$$

2.4 Proof of Proposition 2.3 completed

Inequalities (2.4) and (2.6) of Proposition 2.3 follow immediately from Proposition 2.4 and Proposition 2.5 applied to each separate maximal chain provided by the decomposition (2.8): We leave the details of the proof to the reader. Inequality (2.5) is then a direct consequence of (2.4). For inequality (2.7), we consider again each maximal attractive chain and notice that, if b_k is an element of such a chain which is not at the end points, then we have

$$\varepsilon \left| \frac{\mathrm{d}}{\mathrm{d}t} [b_k(t)] \right| \leq 4 \mathfrak{q}^{-1}_{\min} \mathcal{B}^2_{\max} \exp\left(-\frac{\mathfrak{d}^-_b(t)}{\varepsilon}\right),$$

and a similar estimate holds for the points which are at the end of the chain. A few elementary arguments then lead to the conclusion.

3 Remarks on Stationary Solutions

In this section, we collect a few elementary results about stationary solutions to $(\mathrm{PGL})_\varepsilon$.

3.1 Stationary solutions in $\mathbb{R}$ with vanishing discrepancy

Stationary solutions on $\mathbb{R}$ may be described by using the method of separation of variable, a tool which cannot be extended to systems. As matter of fact, this simple fact turns out to be crucial, and explains for a large part why the analysis of this paper remains restricted to the scalar case.

Consider more generally an interval I of $\mathbb{R}$ and a solution u to (1.34). Multiplying equation (1.34) by u, we are led to the fact that, for any solution u of (1.34), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\xi(u) = 0, \tag{3.1}$$

so that ξ is a constant function on I. We restrict ourselves in this section to solutions with vanishing discrepancy, that is which verify

$$\xi = 0, \text{ that is, } \frac{\dot{u}^2}{2} = \varepsilon^{-2}V(u). \tag{3.2}$$

Differentiating (3.2), we verify that any smooth solution to (3.2) is actually a solution to (1.34). We finally solve equation (3.2) by separation of variables. Consider the function ζ_i defined on the interval (σ_i, σ_{i+1}) by

$$\gamma_i(u) = \int_{z_i}^{u} \frac{\mathrm{d}u}{\sqrt{2V(u)}}, \tag{3.3}$$

where z_i is defined in the introduction. The map γ_i is one-to-one from (σ_i, σ_{i+1}) to $\mathbb{R}$, so that we may define its inverse map

$$\zeta_i^+(x) = \gamma_i^{-1}(x) \tag{3.4}$$

from $\mathbb{R}$ to (σ_i, σ_{i+1}) as well as the map $\zeta_i^-(x) = \zeta_i^{-1}(-x)$. We verify that $\zeta_i^+\left(\frac{\cdot}{\varepsilon}\right)$ as well as $\zeta_i^-\left(\frac{\cdot}{\varepsilon}\right)$ solve (3.2) and hence (1.34). The next result, those proof is left to the reader, shows that we have actually obtained all solutions.

Lemma 3.1 *Let u be a solution to* (1.34) *on some interval I, such that* (3.2) *holds, and such that $u(x_0) \in (\sigma_i, \sigma_{i+1})$ for some $x_0 \in I$ and some $i \in 1, \cdots, q-1$. Then*

$$u(x) = \zeta_i^+\Big(\frac{x-a}{\varepsilon}\Big), \quad \forall x \in I \qquad \textit{or} \qquad u(x) = \zeta_i^-\Big(\frac{x-a}{\varepsilon}\Big), \quad \forall x \in I$$

for some $a \in \mathbb{R}$.

Next, we provide a few simple properties of the functions $\zeta_i^\pm$ which enter directly in our arguments. In view of the definition (3.4), we have

$$\zeta_i^\pm(0) = z_i, \quad {\zeta_i^+}'(0) = \sqrt{2V(z_i)} > 0, \tag{3.5}$$

whereas a change of variable shows that ζ_i has finite energy given by the formula

$$\mathfrak{S}_i \equiv \mathcal{E}(\zeta_i) = \int_{\sigma_i}^{\sigma_{i+1}} \sqrt{2V(u)}\mathrm{d}u. \tag{3.6}$$

It is also straightforward to establish that there exists some constant $\beta_1 > 0$, such that, if for some $s \in \mathbb{R}$, we have $|\zeta_i^{\pm}(s) - \sigma_i| \geq \frac{\mu_0}{2}$, then

$$|s| \leq \beta_1. \tag{3.7}$$

We introduce the constants

$$\begin{cases} A_i^- = -\displaystyle\int_{\sigma_i}^{z_i} \Big[\frac{1}{\sqrt{2V(u)}} - \frac{1}{\sqrt{\lambda_i}(u-\sigma_i)}\Big]\mathrm{d}u - \frac{1}{\sqrt{\lambda_i}}\log(z_i - \sigma_i), \\ A_i^+ = \displaystyle\int_{z_i}^{\sigma_{i+1}} \Big[\frac{1}{\sqrt{2V(u)}} - \frac{1}{\sqrt{\lambda_{i+1}}(\sigma_{i+1}-u)}\Big]\mathrm{d}u + \frac{1}{\sqrt{\lambda_{i+1}}}\log(\sigma_{i+1} - z_i), \end{cases}$$

so that we obtain the expansions, as $u \to \sigma_i^+$ and as $u \to \sigma_{i+1}^-$,

$$\begin{cases} \gamma_i(u) = A_i^- + \dfrac{1}{\sqrt{\lambda_i}}\log(u-\sigma_i) + \underset{u\to\sigma_i^+}{O}(u-\sigma_i), \\ \gamma_i(u) = A_i^+ - \dfrac{1}{\sqrt{\lambda_{i+1}}}\log(\sigma_{i+1}-u) + \underset{u\to\sigma_{i+1}^-}{O}(\sigma_{i+1}-u). \end{cases} \tag{3.8}$$

It follows that as $x \to -\infty$ and as $x \to +\infty$,

$$\begin{cases} \zeta_i^+(x) = \sigma_i + B_i^- \exp(\sqrt{\lambda_i}\, x) + \underset{x\to-\infty}{O}(\exp(2\sqrt{\lambda_i}\, x)), \\ \zeta_i^+(x) = \sigma_{i+1} - B_i^+ \exp(-\sqrt{\lambda_{i+1}}\, x) + \underset{x\to+\infty}{O}(\exp -(2\sqrt{\lambda_{i+1}}\, x)), \end{cases} \tag{3.9}$$

where $B_i^- = \exp(-A_i^-)$ and $B_i^+ = \exp(-A_i^+)$. Similar asymptotics hold for derivatives. For $0 < \varepsilon < 1$ given, and $i = 1, \cdots, q-1$, consider the scaled function $\zeta_{i,\varepsilon}^{\pm} = \zeta_i^{\pm}\left(\frac{\cdot}{\varepsilon}\right)$. Straighforward computations show that

$$\begin{cases} e_\varepsilon(\zeta_{i,\varepsilon})(x) = \dfrac{\lambda_i}{\varepsilon} B_i^- \exp\Big(\dfrac{2\sqrt{\lambda_i}x}{\varepsilon}\Big) + \underset{u\to-\infty}{O}\Big(\exp\Big(\dfrac{\sqrt{3\lambda_i}x}{\varepsilon}\Big)\Big), \\ e_\varepsilon(\zeta_{i,\varepsilon^+})(x) = \dfrac{\lambda_{i+1}}{\varepsilon} B_i^+ \exp\Big(\dfrac{2\sqrt{\lambda_{i+1}}x}{\varepsilon}\Big) + \underset{u\to+\infty}{O}\Big(\exp\Big(\dfrac{\sqrt{3\lambda_{i+1}}x}{\varepsilon}\Big)\Big), \end{cases} \tag{3.10}$$

so there is some constant $C > 0$ which does not depend on r and ε, such that

$$\mathfrak{S}_i \geq \int_{-r}^{r} e_\varepsilon(\zeta_{i,\varepsilon}^+)\mathrm{d}x \geq \mathfrak{S}_i - C\Big[\exp\Big(-\frac{2\sqrt{\lambda_i}\,r}{\varepsilon}\Big) + \exp\Big(-\frac{2\sqrt{\lambda_{i+1}}\,r}{\varepsilon}\Big)\Big]. \tag{3.11}$$

3.2 Study of the perturbed stationary equation

This section is devoted to some properties of solutions (1.30), that is to the perturbed differential equation $u_{xx} = \varepsilon^{-2}V'(u) + f$ on $\mathbb{R}$, where the function f belongs to $L^2(\mathbb{R})$. The main result of this section will be to show that, if u has bounded energy and if f is small, then u is close to several translations of the functions $\zeta_{i,\varepsilon}$ suitably glued together. More precisely, we assume throughout this section that

$$\mathcal{E}_\varepsilon(u) \leq M_0 \tag{3.12}$$

and consider the number

$$d_f = \frac{\varepsilon}{\rho_1}\log\Big(\frac{1}{\mathrm{c}_0\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R})}}\Big), \tag{3.13}$$

where ρ_1 and c_0 are constants depending possibly on M_0 and which will be determined later (see (3.24) for ρ_1 and the proof of Lemma 3.4 for c_0). Hence, we have

$$\|f\|_{L^2(\mathbb{R})} = \frac{\varepsilon^{-\frac{3}{2}}}{\mathrm{c}_0}\exp\Big(-\frac{\rho_1 d_f}{\varepsilon}\Big). \tag{3.14}$$

We assume throughout this subsection that

$$d_f \geq \alpha_1\varepsilon > 0, \tag{3.15}$$

where $\alpha_1 > 0$ is some constant depending only on V, which will be fixed in the proof of Lemma 3.4 below. This assumption implies in particular

$$\mathrm{c}_0\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R})} \leq 1. \tag{3.16}$$

If I is some interval of $\mathbb{R}$ and g is a C^1 function defined on $\mathbb{R}$, it is convenient to introduce the notation

$$\|g\|_{C^1_\varepsilon(I)} = \sup_{x\in I}|g(x)| + \varepsilon\sup_{x\in I}|g'(x)|. \tag{3.17}$$

The main result of this section can be stated as follows.

Proposition 3.1 *Let u be a solution to* (1.30) *satisfying assumptions* (3.12) *and* (3.15). *Then, there exists a collection of points $\{a_k\}_{k\in J}$ in $\mathbb{R}$, such that the following conditions are fulfilled:*

(1) $\sharp(J) \leq \frac{2M_0}{\mathfrak{S}_0}$, *where* $\mathfrak{S}_0 = \inf\{\mathfrak{S}_i, i = \{1\cdots,q\}$.

(2) *For each $k \in J$, there exists a number $i(k) \in \{1,\cdots,q\}$, such that*

$$u(a_k) = z_{i(k)}. \tag{3.18}$$

(3) *The points are well-separated, that is, we have, for $k \neq k'$,*

$$\mathrm{dist}(a_k, a_{k'}) > \frac{d_f}{2}. \tag{3.19}$$

(4) *For each $k \in J$, there exists a symbol $\dagger_i \in \{+,-\}$, such that we have the estimate, for* $I_k = \big[a_k - \frac{d_f}{2}, a_k + \frac{d_f}{2}\big]$,

$$\Big\|u - \zeta^{\dagger_i}_{i(k),\varepsilon}\Big(\frac{\cdot - a_k}{\varepsilon}\Big)\Big\|_{C^1_\varepsilon(I_k)} \leq \exp\Big(-\frac{\rho_1 d_f}{4\varepsilon}\Big). \tag{3.20}$$

(5) *Set* $\Omega_{\mathrm{r}}(t_0) = \mathbb{R}\setminus\bigcup_{k=1}^{J} I_k$. *We have the energy estimate*

$$\int_{\Omega_{\mathrm{r}}(t_0)} e_\varepsilon(v_\varepsilon(\cdot,t_0))\mathrm{d}x \leq \exp\Big(-\frac{\rho_1 d_f}{2\varepsilon}\Big). \tag{3.21}$$

The proof of Proposition 3.1 will be decomposed into several lemmas. Following the approach of [3], we recast equation (1.30) as a system of two differential equations of first order. For that purpose, we set $w = \varepsilon u_x$, so that (1.30) is equivalent to the system

$$u_x = \frac{1}{\varepsilon}w \text{ and } w_x = \frac{1}{\varepsilon}V'(u) + \varepsilon f, \tag{3.22}$$

which we may write in a more condensed form as

$$U_x = \frac{1}{\varepsilon} G(U) + \varepsilon F \quad \text{on } \mathbb{R}, \tag{3.23}$$

where, for x in $\mathbb{R}$, we have set $U(x) = (u(x), w(x))$ and $F(x) = (0, f(x))$, and where G denotes the vector field on $\mathbb{R}^2$ given by $G(u_1, u_2) = (u_2, V'(u_1))$. Notice that $|\nabla G(u_1, u_2)| \leq N(|u_1|)$, where $N \geq 1$ is some continuous non-decreasing scalar function. On the other hand, since u is assumed to satisfy the energy bound (3.12), we have

$$\|u\|_{L^\infty(\mathbb{R})} \leq C(M_0 + 1),$$

so that we are led to set

$$\rho_1 = N(C(M_0 + 1) + 1). \tag{3.24}$$

We next compare a given global bounded solution u of (1.30) to a possible local solution u^0 of the unperturbed equation

$$u_{xx} = \varepsilon^{-2} \nabla V(u) \tag{3.25}$$

with comparable initial condition at some point $x_0 \in \mathbb{R}$. We denote accordingly $U^0 = (u^0, \varepsilon^{-1} u^0_x)$ on its maximal interval of existence. As a consequence of Gronwall's identity, we have the following lemma.

Lemma 3.2 *Let $A = (-b, b)$ be an interval of $\mathbb{R}$, u be a solution to* (1.30) *on A and u^0 be a local solution to* (3.25). *Assume that for some number a satisfying $b \geq a > 0$, we have the inequality*

$$|U(0) - U^0(0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2\rho_1}} \|f\|_{L^2([-b,b])} \leq \exp\Big(-\frac{\rho_1 a}{\varepsilon}\Big). \tag{3.26}$$

Then u^0 is well-defined on $[-a, +a]$, and we have

$$\|U - U^0\|_{L^\infty([-a,+a])} \leq \Big(|U(0) - U^0(0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2\rho_1}} \|f\|_{L^2}\Big) \exp\Big(\frac{\rho_1 a}{\varepsilon}\Big). \tag{3.27}$$

Proof Let I be the largest interval containing 0, such that

$$\|u^0\|_{L^\infty(I)} \leq \|u\|_\infty + 1. \tag{3.28}$$

On I, since $(U - U^0)_x = G(U) - G(U^0) + \varepsilon F$, we obtain the inequality

$$|(U - U^0)_x| \leq \frac{\rho_1}{\varepsilon} |U - U^0| + \varepsilon |F|.$$

It follows from Gronwall's inequality, that, for $x \in I$,

$$|(U - U^0)(x)| \leq \exp\Big(\frac{\rho_1 |x|}{\varepsilon}\Big) |(U - U^0)(0)| + \Big| \int_0^x \varepsilon |F(x-y)| \exp\Big(\frac{\rho_1 |x|}{\varepsilon}\Big) \mathrm{d}y \Big|,$$

so that by the Cauchy-Schwarz inequality, we are led to the bound, for $x \in I$,

$$|(U - U^0)(x)| \leq \Big(|U(0) - U^0(0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2\rho_1}} \|f\|_{L^2}\Big) \exp\Big(\frac{\rho_1 |x|}{\varepsilon}\Big). \tag{3.29}$$

Hence, if (3.26) is verified, then $[-a, a] \subset I$ and (3.27) follows.

We will combine the previous lemma with the following lemma.

Lemma 3.3 *Let u be a solution to* (1.30) *on $\mathbb{R}$, such that $\mathcal{E}_\varepsilon(u) \leq M_0 < +\infty$. Then*

$$\|\xi_\varepsilon(u)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2M_0}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R})}.$$

Proof This is a direct consequence of the equality $\frac{\mathrm{d}}{\mathrm{d}x}\xi_\varepsilon(u) = \varepsilon^2 f \frac{\mathrm{d}}{\mathrm{d}x}u$, $\frac{\mathrm{d}}{\mathrm{d}x}u$, Cauchy-Schwarz inequality, and the fact that it is zero at infinity since u has finite energy.

Lemma 3.4 *Let u be a solution to* (1.30) *on $\mathbb{R}$ satisfying assumptions* (3.12) *and* (3.15) *and let $x_0 \in \mathcal{D}(u)$. There exists some point $y_0 \in \mathbb{R}$, some $i \in \{1, \cdots, q\}$ and some symbol $\dagger \in \{+, -\}$, such that for every $0 < a < d_f$, we have*

$$\|u - \zeta^\dagger_{i,\varepsilon}(\cdot - y_0)\|_{C^1_\varepsilon([x_0-a,x_0+a])} \leq \exp\Big(-\frac{\rho_1}{\varepsilon}(d_f - a)\Big). \tag{3.30}$$

Moreover, there exists some constant $\alpha_1 > 0$ depending only on the potential V, such that, if $d_f \geq \gamma_1\varepsilon$, then $y_0 \in [x_0 - \frac{d_f}{32}, x_0 + \frac{d_f}{32}]$ and

$$\Big|\int_{y_0-\frac{d_f}{2}}^{y_0+\frac{d_f}{2}} e_\varepsilon(u) - e_\varepsilon(\zeta^\dagger_{i,\varepsilon}(\cdot - y_0))\mathrm{d}x\Big| \leq C\frac{d_f}{\varepsilon}\exp\Big(-\frac{\rho_1 d_f}{4\varepsilon}\Big), \tag{3.31}$$

where the constant $C > 0$ depends only on the potential V.

Remark 3.1 (1) Since $\zeta^\dagger_i(0) = z_i$, it is straightforward to deduce from (3.30) applied with $a = \frac{7d_f}{8}$ and the properties of the functions $\zeta^\dagger_i$ (see (3.9)) that, if the constant α_1 is choosing sufficiently large, then there exists some point $\widetilde{y}_0$, such that $|y_0 - \widetilde{y}_0| \leq \frac{d_f}{32}$ and $u(\widetilde{y}_0) = z_i$.

(2) We also notice that if the constant α_1 is chosen sufficiently large,

$$\mathcal{D}(u) \cap \Big[\widetilde{y}_0 + \frac{d_f}{32}, \widetilde{y}_0 + \frac{3d_f}{4}\Big] = \emptyset \quad \text{and} \quad \mathcal{D}(u) \cap \Big[\widetilde{y}_0 - \frac{3d_f}{4}, \widetilde{y}_0 - \frac{d_f}{32}\Big] = \emptyset. \tag{3.32}$$

(3) Set

$$\rho_2 = \inf\Big\{\frac{\rho_1}{4}, \sqrt{\lambda_i}, i = 1, \cdots, q\Big\}. \tag{3.33}$$

Then, it follows combining (3.11) and (3.31) that

$$\Big|\int_{y_0-\frac{d_f}{2}}^{y_0+\frac{d_f}{2}} e_\varepsilon(u)\mathrm{d}x - \mathfrak{S}_i\Big| \leq C\frac{d_f}{\varepsilon}\exp\Big(-\frac{\rho_2 d_f}{\varepsilon}\Big), \tag{3.34}$$

since the function $s \mapsto s\exp(-s)$ is decreasing for large values of $s > 0$ choosing the constant α_1 sufficiently large, and if we assume $d_f \geq \alpha_1\varepsilon$, we are led to

$$\int_{y_0-\frac{d_f}{2}}^{y_0+\frac{d_f}{2}} e_\varepsilon(u)\mathrm{d}x \geq \frac{\mathfrak{S}_0}{2}, \tag{3.35}$$

where $\mathfrak{S}_0 = \inf\{\mathfrak{S}_i, i = 1, \cdots, q-1\} > 0$.

Proof Going back to Lemma 3.2, we consider as solution u^0 to the unperturbed equation (3.25), the solution obtained choosing as initial conditions $u^0(x_0) = u(x_0)$ and the derivative $u^0_x(x_0)$ in such a way that $\xi_\varepsilon(u^0)(x_0) = 0$. Obviously, it suffices therefore to choose

$$(u^0_x(x_0))^2 = \frac{V(u_\varepsilon(x_0))}{\varepsilon^2}.$$

We impose moreover the sign of $u^0_x(x_0)$ to be the same as the sign of $u_x(x_0)$, which does not vanish, so that u_0 is uniquely defined. Since by construction $\xi_\varepsilon(u^0) = 0$, it follows from Lemma 3.1 that

$$u^0(\cdot) = \zeta^\dagger_{i,\varepsilon}(\cdot - y_0)$$

for some point $y_0 \in \mathbb{R}$, some $i \in \{1, \cdots, q\}$ and some $\dagger \in \{+,-\}$. Notice that, since $x_0 \in \mathcal{D}(u)$, there exists a constant $c_0 > 0$ (depending only on the choice of μ_0 and on the numbers λ_i, hence on the properties of the potential V), such that

$$\frac{\varepsilon|u_x(x_0)|^2}{2} = \frac{V(u(x_0))}{\varepsilon} + \varepsilon^{-1}\xi_\varepsilon(u)(x_0) \geq \frac{c_0}{\varepsilon} + \varepsilon^{-1}\xi_\varepsilon(u)(x_0),$$

so that by Lemma 3.3,

$$\frac{\varepsilon|u_x(x_0)|^2}{2} \geq \frac{c_0}{\varepsilon} - \sqrt{2M_0\varepsilon}\|f\|_{L^2(\mathbb{R})}.$$

Since by assumption (3.15), we have $\mathrm{c}_0\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R})} < 1$ and deduce

$$\frac{\varepsilon|u_x(x_0)|^2}{2} \geq \frac{\mathrm{c}_0}{\varepsilon} - \frac{\sqrt{2M_0}\mathrm{c}_0{}^{-1}}{\varepsilon}.$$

We next impose as first condition on c_0 that $\mathrm{c}_0{}^2 \geq 2\sqrt{2}M_0$, so that we obtain, since $0 < \varepsilon \leq 1$,

$$\varepsilon|u_x(x_0)| \geq \sqrt{\mathrm{c}_0}. \tag{3.36}$$

Combining the identity $\varepsilon^2(|u_x(x_0)|^2 - |u^0_x(x_0)|^2) = 2\xi_\varepsilon(u)(x_0)$, the bound (3.36) and Lemma 3.3, that are led to

$$|\varepsilon(u_x - u^0_x)(x_0)| \leq \frac{2\sqrt{2M_0}}{\sqrt{\mathrm{c}_0}}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R})}.$$

Since $u(x_0) = u^0(x_0)$, we deduce

$$\big|U(x_0) - U^0(x_0)\big| \leq \frac{2\sqrt{2M_0}}{\sqrt{\mathrm{c}_0}}\varepsilon^{\frac{3}{2}}\||f\|_{L^2(\mathbb{R})}. \tag{3.37}$$

In view of Lemma 3.2, we estimate

$$|U(0) - U^0(0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2\rho_1}}\|f\|_{L^2(\mathbb{R})} \leq \Big(\frac{2\sqrt{2M_0}}{\sqrt{\mathrm{c}_0}} + \frac{1}{2}\Big)\varepsilon^{\frac{3}{2}}\||f\|_{L^2(\mathbb{R})}.$$

Imposing the additional condition $\mathrm{c}_0 \geq 2\sqrt{2}\frac{M_0}{\sqrt{\mathrm{c}_0}} + \frac{1}{2}$, we completely determine c_0. In view of the definition definition of d_f, we are hence led to

$$|U(0) - U^0(0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2\rho_1}}\|f\|_{L^2(\mathbb{R})} \leq \exp\Big(-\frac{\rho_1 d_f}{\varepsilon}\Big).$$

The inequality (3.30) then follows from Lemma 3.2. For the second assertion, we specify inequality (3.30) for the point $x = x_0$ with $a = \frac{d_f}{2}$, so that

$$|\zeta^\dagger_{i,\varepsilon}(x_0 - y_0) - u(x_0)| \leq \exp\Big(-\frac{\rho_1 d_f}{2\varepsilon}\Big) \leq \exp\Big(-\frac{\rho_1 \gamma_1}{2}\Big) \leq \frac{\mu_0}{2},$$

provided, for the last inequality that α_1 is chosen sufficiently large. Since $x_0 \in \mathcal{D}(u)$, we have either $|u(x_0) - \sigma_i| \geq \mu_0$ or $|u(x_0) - \sigma_{i+1}| \geq \mu_0$, and hence

$$|\zeta^\dagger_{i,\varepsilon}(x_0 - y_0) - \sigma_i| \geq \mu_0 \quad \text{or} \quad |\zeta^\dagger_{i,\varepsilon}(x_0 - y_0) - \sigma_{i+1}| \geq \mu_0.$$

Invoking (3.7), we are led to

$$|x_0 - y_0| \leq \beta_1 \varepsilon.$$

Choosing α_1 possibly even larger so that $\alpha_1 \geq 16\beta_1$, we are led to $y_0 \in [x_0 - \frac{d_f}{16}, x_0 + \frac{d_f}{16}]$, that is the second assertion follows. We finally turn to the proof of (3.31). For that purpose, we choose $a = \frac{3d_f}{4}$, so that $[y_0 - \frac{d_f}{2}, y_0 + \frac{d_f}{2}] \subset [x_0 - a, x_0 + a]$, and hence, by inequality (3.30), we may decompose u as $u = \zeta^\dagger_{i,\varepsilon}(\cdot - y_0) + w$, where

$$\|w\|_{C^1_\varepsilon([y_0 - \frac{d_f}{2}, y_0 + \frac{d_f}{2}])} \leq \exp\Big(-\frac{\rho_1 d_f}{4\varepsilon}\Big). \tag{3.38}$$

Expending accordingly the energy, we derive estimate (3.31).

Remark 3.2 The conclusion (3.30) of Lemma 3.4 remains essentially unchanged, if instead of a solution u defined on the whole real line $\mathbb{R}$, we consider a solution on a bounded interval A. In that case, however, we have to replace in our computation the quantity $\|f\|_{L^2(\mathbb{R})}$ by the quantity $\|f\|_{L^2(A)} + \frac{\xi_\varepsilon(u(z))}{\sqrt{2}M_0}$, where z is some arbitrary point, which lead, in the conclusion, to changing the constant d_f by the constant

$$\widetilde{d} \equiv -\frac{\varepsilon}{\rho_1}\log\Big(c_0\varepsilon^{\frac{3}{2}}[\|f\|_{L^2(A)} + \frac{\xi_\varepsilon(u(z))}{\sqrt{2}M_0}]\Big), \tag{3.39}$$

Conditions (3.15)–(3.16) also have been changed accordingly.

Proof of Proposition 3.1 (Completed) We distinguish two cases.

Case 1 $\mathcal{D}(u) = \emptyset$. In this case, we take $J = \emptyset$, and the only thing to be established in this case is estimate (3.21). Since $\mathcal{D}(u) = \emptyset$, there exists some $i \in \{1, \cdots, q\}$, such that $|u - \sigma_i| \leq \mu_0$. Multiplying equation (1.30) by $u - \sigma_i$, we are led to

$$\begin{aligned}\int_{\mathbb{R}} u_x^2 + \varepsilon^{-2}\lambda_0(u - \sigma_i)^2 \mathrm{d}x &\leq \int_{\mathbb{R}} u_x^2 + \varepsilon^{-2}V'(u)(u - \sigma_i)\mathrm{d}x \\ &= \int_{\mathbb{R}} f.(u - \sigma_i)\mathrm{d}x \\ &\leq \|f\|_{L^2(\mathbb{R})}\|u - \sigma_i\|_{L^2(\mathbb{R})}\end{aligned} \tag{3.40}$$

and hence, by Cauchy-Schwarz, $-\!\!\int_{\mathbb{R}} \varepsilon u_x^2 + \varepsilon^{-1}\lambda_0(u - \sigma_i)^2 \mathrm{d}x \leq \lambda_0^{-1}\varepsilon^3\|f\|^2_{L^2(\mathbb{R})}$. In view of (3.14) and the fact that $0 < \varepsilon < 1$, this yields to (3.21).

Case 2 $\mathcal{D}(u) \neq \emptyset$. we construct the points a_i by an inductive argument, which stops in a finite numbers of steps. Let first x_0 be an arbitrary point in $\mathcal{D}(u)$. Applying Lemma 3.4 to x_0 as well as Remark 3.1, we deduce the existence of a point $\widetilde{y}_0$, such that $u(\widetilde{y}_0) = z_i$, $|y_0 - x_0| \leq \frac{d_f}{16}$,

$$\mathcal{D}(u) \cap \Big[\widetilde{y}_0 + \frac{d_f}{32}, \widetilde{y}_0 + \frac{3d_f}{4}\Big] = \emptyset \quad \text{and} \quad \mathcal{D}(u) \cap \Big[\widetilde{y}_0 - \frac{3d_f}{4}, \widetilde{y}_0 - \frac{d_f}{32}\Big] = \emptyset. \tag{3.41}$$

Applying (3.30) with $a = \frac{3d_f}{4}$, we are led to

$$\|u - \zeta_{i,\varepsilon}^{\dagger}(\cdot - \widetilde{y}_0)\|_{C_\varepsilon^1([\widetilde{y}_0 - \frac{d_f}{2}, \widetilde{y}_0 + \frac{d_f}{2}])} \leq \exp\Big(-\frac{\rho_1}{4\varepsilon} d_f\Big). \tag{3.42}$$

We set $a_1 = \widetilde{y}_0$, so that (3.18) as well as (3.20) are satisfied for $k = 1$. Moreover, it follows from (3.35) that

$$\int_{\mathbb{R}\setminus[a_1 - \frac{d_f}{2}, a_1 + \frac{d_f}{2}]} e_\varepsilon(u)\mathrm{d}x \leq M_0 - \frac{\mathfrak{S}_0}{2}. \tag{3.43}$$

We iterative the process considering next $\Omega_1 = \mathcal{D}(u) \setminus [a_1 - \frac{d_f}{2}, a_1 + \frac{d_f}{2}]$. If this set is empty, then we take $J = \{1\}$, $i(1) = i$ and $\dagger_1 = \dagger$, and we stop. Otherwise, we choose some $x_1 \in \Omega_1$, and argue as we did above, now with x_1 instead of x_0: This yields a point $\widetilde{y}_1$, some number $i(2) \in \{1, \cdots, q\}$, some sign $\dagger_2 \in \{+, -\}$, such that $|\widetilde{y}_2 - x_1| \leq \frac{d_f}{16}$,

$$\|u - \zeta_{i(2),\varepsilon}^{\dagger_2}(\cdot - \widetilde{y}_1)\|_{C_\varepsilon^1([\widetilde{y}_1 - \frac{d_f}{2}, \widetilde{y}_1 + \frac{d_f}{2}])} \leq \exp\Big(-\frac{\rho_1}{4\varepsilon} d_f\Big), \tag{3.44}$$

$$\mathcal{D}(u) \cap \Big[\widetilde{y}_0 + \frac{d_f}{32}, \widetilde{y}_1 + \frac{3d_f}{4}\Big] = \emptyset, \quad \mathcal{D}(u) \cap \Big[\widetilde{y}_1 - \frac{3d_f}{4}, \widetilde{y}_1 - \frac{d_f}{32}\Big] = \emptyset. \tag{3.45}$$

Setting $a_2 = \widetilde{y}_1$, and invoking (3.35) again, we are led to

$$\int_{\mathbb{R}\setminus \bigcup\limits_{k=1}^{2}[a_k - \frac{d_f}{2}, a_k + \frac{d_f}{2}]} e_\varepsilon(u)\mathrm{d}x \leq M_0 - \mathfrak{S}_0. \tag{3.46}$$

Notice also that, by construction, $|a_1 - a_2| \geq \frac{d_f}{2}$.

We construct the set $\{a_k\}_{k\in J}$ repeating the previous construction inductively. Since in each iteration the energy in estimates (3.46) decreases by at least an amount of $\frac{\mathfrak{S}_0}{2}$, we stop in at most $\frac{2M_0}{\mathfrak{S}_0}$ number of steps, and all assertions, except (3.21) have been verified. In order to establish (3.21), we argue as in case one. We have, integrating by parts in (1.30) for $k = 1, \cdots, q-1$

$$\begin{aligned}
&\int_{a_i + \frac{d_f}{2}}^{a_{i+1} - \frac{d_f}{2}} (u_x^2 + \varepsilon^{-2} V'(u)(u - \sigma_i))\mathrm{d}x \\
&= \int_{a_i + \frac{d_f}{2}}^{a_{i+1} - \frac{d_f}{2}} (f(u - \sigma_i) + u'(a_{i+1})u(a_{i+1}) - u'(a_i)u(a_i))\mathrm{d}x \\
&\leq \|f\|_{L^2(\mathbb{R})} \|u - \sigma_i\|_{L^2([a_i + \frac{d_f}{2}, a_{i+1} - \frac{d_f}{2})} + \varepsilon^{-1}\exp\Big(-\frac{\rho_1 d_f}{2\varepsilon}\Big),
\end{aligned} \tag{3.47}$$

so that

$$\varepsilon \int_{a_i + \frac{d_f}{2}}^{a_{i+1} - \frac{d_f}{2}} (u_x^2 + \varepsilon^{-2}\lambda_0(u - \sigma_i))\mathrm{d}x \leq C\exp\Big(-\frac{3\rho_1 d_f}{4\varepsilon}\Big).$$

By summation, we obtain (3.21), and the proof of Proposition3.1 is complete.

As a by-product of the Proposition 3.1, we may also derive a global estimate.

Lemma 3.5 *Let u be a solution to* (1.30) *satisfying assumptions* (3.12) *and* (3.15). *then, we have*

$$\Big|\mathcal{E}_\varepsilon(u) - \sum_{k\in J}\mathfrak{S}_{i(k)}\Big| \leq CM_0 \frac{d_f}{\varepsilon}\exp\Big(-\frac{\rho_2 d_f}{\varepsilon}\Big).$$

Proof It suffices to combine (3.34) and (3.21).

4 Regularized Fronts

4.1 First properties

In this section, we provide some properties of the solution v_ε to $(\mathrm{PGL})_\varepsilon$ on time slices on which it has already undergone a parabolic regularization, that is when fronts become close to the stationary ones, and are well separated. Such a situation is described in Definition 1.1.

Lemma 4.1 *If $\mathcal{WP}_\varepsilon(\delta, t_0)$ holds, then $\mathcal{WP}_\varepsilon(\delta', t_0)$ holds for any $\alpha_1\varepsilon \le \delta' \le \delta$.*

We leave the proof to the reader. Next, we consider for $k \in J(t_0)$ the function $w^\pm_{k,\varepsilon} = v_\varepsilon - \sigma_{j^\pm(k)}$.

Lemma 4.2 *Let $t_0 \ge 0$, $\delta > \alpha_1\varepsilon$ be given, and assume that v_ε satisfies condition $\mathcal{WP}_\varepsilon(\delta, t_0)$. Then, we have, for any $0 < \delta' \le \delta$,*

$$\Big|(w^\pm_\varepsilon(a_k \pm \delta', t_0) - \mathcal{B}^\pm_k \exp\Big(-\frac{\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\delta'\Big)\Big| \le K\Big[\exp\Big(-\frac{2\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\delta'\Big) + \exp\Big(-\rho_1\frac{\delta}{\varepsilon}\Big)\Big],$$

where we have set $\mathcal{B}^+_k = -\dagger_k B^{\dagger_k}_{i(k)}$ and *$\mathcal{B}^-_k = \dagger_k B^{-\dagger_k}_{i(k)}$, with the numbers $B^\pm_i > 0$ having been introduced in* (3.9).

Proof We have, in view of the definition of condition $\mathcal{WP}_\varepsilon(\delta, t_0)$,

$$\Big|v_\varepsilon(a_k \pm \delta', t_0) - \zeta^{\dagger_i}_{i(k)}\Big(\pm\frac{\delta'}{\varepsilon}\Big)\Big| \le \exp\Big(-\rho_1\frac{\delta}{\varepsilon}\Big), \tag{4.1}$$

whereas by (3.9), we have

$$\Big|\zeta^{\dagger_i}_{i(k)}\Big(\pm\frac{\delta'}{\varepsilon}\Big) - \Big(\sigma_{j^\pm(k)} - B^{\dagger_k}_{i(k)}\exp\Big(-\frac{\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\,\delta'\Big)\Big)\Big| \le K\exp\Big(-\frac{2\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\delta'\Big). \tag{4.2}$$

In several place, we will assume additionally that $\delta' \le \frac{\rho_1}{2\sqrt{\lambda_{\max}}}\delta$, where $\lambda_{\max} = \sup\{\lambda_i\}$. Then we obtain under the assumption of Lemma 4.2

$$\Big|(w^\pm_\varepsilon(a_k \pm \delta', t_0) - \mathcal{B}^\pm_k \exp\Big(-\frac{\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\delta'\Big)\Big| \le K\Big[\exp\Big(-\frac{2\sqrt{\lambda_{j^\pm(k)}}}{\varepsilon}\delta'\Big)\Big]. \tag{4.3}$$

For the outer region, we have the following lemma.

Lemma 4.3 *Let $t_0 \ge 0$, $\delta \ge \alpha_1\varepsilon$ be given, and assume that v_ε satisfies condition $\mathcal{WP}_\varepsilon(\delta, t_0)$. Then, we have, for $x \in [a_k + \delta, a_{k+1} - \delta]$ (resp. $x \in [a_{k-1} + \delta, a_k - \delta]$),*

$$|w^+_{k,\varepsilon}(x, t_0)| \le C\exp\Big(-\rho_1\frac{\delta}{2\varepsilon}\Big) \quad \Big(\text{resp. } |w^\pm_{k,\varepsilon}(x, t)| \le C\exp\Big(-\rho_1\frac{\delta}{2\varepsilon}\Big)\Big).$$

Proof We write $|\frac{\mathrm{d}}{\mathrm{d}x}(w^+_{k,\varepsilon})^2| = 2|\varepsilon^{-\frac12} w^+_{k,\varepsilon}\varepsilon^{\frac12}\frac{\mathrm{d}}{\mathrm{d}x}(w^+_{k,\varepsilon})| \le 2e_\varepsilon(w^+_{k,\varepsilon})$, so that

$$|w^+_{k,\varepsilon}(x, t_0) - w^+_{k,\varepsilon}(a_k(t_0) + \delta)| \le 2\int_{a_k(t_0)+\delta}^x e_\varepsilon(w^+_{k,\varepsilon})(x, s)\mathrm{d}s.$$

The conclusion then follows from Lemma (4.1) and assumption WP3.

We complete the subsection with energy estimates.

Lemma 4.4 *Let $t_0 \geq 0$, $\delta \geq \alpha_1\varepsilon$ be given, and assume that v_ε satisfies condition $\mathcal{WP}_\varepsilon(\delta, t_0)$. Then, we have*

$$|\mathcal{E}_\varepsilon(v_\varepsilon(\cdot, t_0)) - \mathfrak{E}(t_0)| \leq M_0 \frac{\delta}{\varepsilon} \exp\Big(- \frac{\rho_2 \delta}{\varepsilon}\Big), \tag{4.4}$$

where ρ_2 is defined in (3.33) *and where the front energy $\mathfrak{E}(t_0)$ is defined by $\mathfrak{E}(t_0) = \sum_{k \in J(t_0)} \mathfrak{S}_{i(k)}$.*

The proof is similar to the proof of Lemma 3.5, and we omit it.

Notice that the front energy $\mathfrak{E}(t_0)$ may take only a finite number of values, and is hence quantized. We emphasize also that, at this stage, the front energy $\mathfrak{E}(t_0)$ is only defined assuming that condition $\mathcal{WP}_\varepsilon(\delta, t_0)$ holds. However, we leave to the reader to check that the value of $\mathfrak{E}(t_0)$ does not depend on δ, provided of course that $\delta \geq \alpha_1\varepsilon$, so that it suffices ultimately, in order to define $\mathfrak{E}(t_0)$, to check that condition $\mathcal{WP}_\varepsilon(\alpha_1\varepsilon, t_0)$ is full-filled.

Choosing possibly a larger value for the constant α_1, an immediate consequence of Lemma 4.4 as well as the fact that $\mathfrak{E}(t_0)$ may take only a finite number of values is as follows.

Lemma 4.5 *Let $T_1 \geq T \geq 0$ be given, and assume that conditions $\mathcal{WP}_\varepsilon(\alpha_1\varepsilon, T)$ and $\mathcal{WP}_\varepsilon(\alpha_1\varepsilon, T_1)$ hold. Then, we have $\mathfrak{E}(T_1) \leq \mathfrak{E}(T)$. Moreover, there exists a positive constant $\mu_1 > 0$, such that, if $\mathfrak{E}(T_1) < \mathfrak{E}(T)$, then we have $\mathfrak{E}(T_1) \leq \mathfrak{E}(T) + \mu_1$.*

We next discuss the case of equality $\mathfrak{E}(T_1) = \mathfrak{E}(T)$, in particular with respect to the L^2 norm of the dissipation, which is central in several of our arguments. For that purpose consider two times T and T_1, such that $T_1 \geq T \geq 0$, and set

$$\mathrm{dissip}\,[T, T_1] = \varepsilon \int_{\mathbb{R}\times[T,T_1]} \Big|\frac{\partial v_\varepsilon}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t = \mathcal{E}_\varepsilon\left(v_\varepsilon(\cdot, T)\right) - \mathcal{E}_\varepsilon\left(v_\varepsilon(\cdot, T_1)\right). \tag{4.5}$$

As a direct consequence of Lemma 4.4 and the global energy identity (1.5), we have the following corollary.

Corollary 4.1 *Assume $\delta > \alpha_1\varepsilon$, and let $T_1 \geq T \geq 0$ be such that both $\mathcal{WP}_\varepsilon(\delta, T)$ and $\mathcal{WP}_\varepsilon(\delta, T_1)$ hold and that $\mathfrak{E}(T) = \mathfrak{E}(T_1)$. Then, we have*

$$\mathrm{dissip}\,[T, T_1] \leq 2M_0 \frac{\delta}{\varepsilon} \exp\Big(- \frac{\rho_2\delta}{\varepsilon}\Big).$$

4.2 Finding regularized fronts

Occurrences of well-prepared time slices may be found thanks to Proposition 3.1 and a rough mean-value argument. In many places, we rely on the following observation.

Lemma 4.6 *Let $T \geq 0$, $\Delta T > 0$ and $\delta \geq \alpha_1\varepsilon$ be given. If*

$$\Delta T \geq \mathrm{c_0}^2 \varepsilon^2\, \mathrm{dissip}\,[T, T + \Delta T] \exp\Big(2\rho_1 \frac{\delta}{\varepsilon}\Big), \tag{4.6}$$

then, there exists some time $t_0 \in [T, T + \Delta T]$, such that $\mathcal{WP}_\varepsilon(\delta, t_0)$ holds.

Proof By the mean-value argument, there exists some time $t_0 \in [T, T + \Delta T]$, such that

$$\|\partial_t(\cdot, t_0)\|^2_{L^2(\mathbb{R})} \leq \varepsilon^{-1} \frac{\mathrm{dissip}\,[T, T + \Delta T]}{\Delta T} \leq \frac{\varepsilon^{-3}}{\mathrm{c_0}^2} \exp\Big(- 2\rho_1 \frac{\delta}{\varepsilon}\Big).$$

Consider next the map $u = v_\varepsilon(\cdot, t_0)$, so that u is now a solution to (1.30), with source term $f = \partial_t(\cdot, t_0)$. Hence f satisfies (3.14) with $d_f = 2\delta$. The conclusion then follows from Proposition 3.1.

As a matter of fact, one may initiate the search of regularized fronts using (1.5), that is

$$\operatorname{dissip}[T, T_1] \leq M_0. \tag{4.7}$$

Therefore, it follows from Lemma 4.6 that it suffices to impose

$$\Delta T \geq \mathrm{c_0}^2 \varepsilon^2 M_0 \exp\left(2\rho_1 \frac{\delta}{\varepsilon}\right) \tag{4.8}$$

to deduce the existence of a time $t_0 \in [T, T + \Delta T]$, such that $\mathcal{WP}_\varepsilon(\delta, t_0)$ holds. In particular, taking δ of the form $\delta = \alpha\varepsilon$ with $\alpha \geq \alpha_1$, we see that given any $T \geq 0$, there exists a time $t \in [T, T + \omega(\alpha)\varepsilon^2]$, such that $\mathcal{WP}_\varepsilon(\alpha\varepsilon, t)$ holds with

$$\omega(\alpha) = \mathrm{c_0}^2 M_0 \exp\left(2\rho_1 \alpha\right). \tag{4.9}$$

Since we assume $\alpha \geq \alpha_1$ the front energy $\mathfrak{E}(t)$ is then well-defined. In other words, on each time interval of size $\omega(\alpha_1)\varepsilon^2$, there exists some time for which the front energy is well-defined. On the other hand, this energy is non-increasing takes only a finite number of values, so that we may expect to find large time intervals, where it remains constant. This is the situation we analyze in the next subsection.

4.3 Propagating regularized fronts

We assume throughout this subsection, that we are given $\delta > \alpha_1\varepsilon$ and two time $T_1 \geq T \geq 0$, such that

$$\begin{cases} \mathcal{WP}_\varepsilon(\delta, T) \text{ and } \mathcal{WP}_\varepsilon(\delta, T_1) \text{ hold,} \\ \mathfrak{E}(T) = \mathfrak{E}(T_1). \end{cases} \tag{4.10}$$

Under that assumption, our first result shows that v_ε remains regularized on almost the whole time interval $[T, T_1]$, with a smaller δ though.

Proposition 4.1 *Assume that assumption* (4.10) *holds with* $\delta \geq \alpha_2\varepsilon$, *where* $\alpha_2 \geq \alpha_1$ *is some constant depending only on the potential* V *and the constant* M_0. *There exists some constant* $0 < \nu_1 < 1$, *such that given any time* $t \in [T + \mathrm{c_1}\varepsilon\delta, T_1]$, *property* $\mathcal{WP}_\varepsilon(\nu_1\delta, t)$ *holds, where* $\mathrm{c_1} = 2M_0\mathrm{c}_0^2$.

The proof of Proposition 4.1 involves the next result, of possible independent interest.

Lemma 4.7 *Assume that assumption* (4.10) *holds with* $\delta \geq \alpha_1\varepsilon$. *We have the estimate, for* $t \in [T + \varepsilon^2, T_1]$,

$$|\partial_t v_\varepsilon(x, t)| \leq C M_0 \varepsilon^{-2} \sqrt{\frac{\delta}{\varepsilon}} \exp\Big(- \rho_2 \frac{\delta}{2\varepsilon}\Big).$$

Proof Differentiating equation (PGL$_\varepsilon$) with respect to time, we are led to

$$|\partial_t(\partial_t v_\varepsilon) - \partial_{xx}(\partial_t v_\varepsilon)| \leq \frac{C}{\varepsilon^2}|\partial_t v_\varepsilon|. \tag{4.11}$$

It follows from standard parabolic estimates, working on the cylinder $\Lambda_\varepsilon = [x-\varepsilon, x+\varepsilon]\times[t-\varepsilon^2, t]$ that

$$|\partial_t v_\varepsilon(x,t)| \leq C\varepsilon^{-\frac{3}{2}}\|\partial_t v_\varepsilon\|_{L^2(\Lambda_\varepsilon)} \leq CM_0\varepsilon^{-2}\sqrt{\frac{\delta}{\varepsilon}}\exp\Big(-\rho_2\frac{\delta}{2\varepsilon}\Big),$$

where the last inequality follows from Corollary 4.1, and which yields the conclusion.

Proof of Proposition 4.1 We divide the proof into several steps.

Step 1 Given any time $t \in [T + c_1\varepsilon\delta, T_1]$, we may find some time $\widetilde{t} \in [t - c_1\varepsilon\delta, t]$, such that $\mathcal{WP}_\varepsilon(\nu_2\delta, \widetilde{t})$ holds, where $0 < \nu_2 < 1$ is some positive constant.

Proof of Step 1 In view of Corollary 4.1, we have dissip $[t - c_1^2\varepsilon\delta, t] \leq 2M_0\frac{\delta}{\varepsilon}\exp\big(-\frac{\rho_2\delta}{\varepsilon}\big)$. The conclusion follows directly from Lemma 4.6, applied with $T = t - c_1\varepsilon\delta$ and $\Delta T = c_1\varepsilon\delta$, the definition (3.33) of ρ_2, and the choice $\nu_2 = \frac{\rho_2}{4\rho_1}$. Concerning the constant ν_1, our choice will be

$$\nu_1 = \frac{1}{2}\inf\Big\{\rho_1\nu_3, \frac{\rho_1\rho_2}{5}, \nu_2\Big\} \quad \text{with } \nu_3 = \inf\Big\{\frac{\rho_2}{4\rho_1}, \nu_2\Big\}. \tag{4.12}$$

Step 2 Set $U_k^0 = [a_k(t) + \nu_1\delta, a_{k+1}(t) - \nu_1\delta]$ and $U_k^1 = [a_k(t) + \nu_2\delta, a_{k+1}(t) - \nu_2\delta]$. Then, we have, for any $s \in [\widetilde{t}, t]$, $i = 0, 1$ and provided that α_2 is chosen sufficiently large,

$$\sum_k \int_{U_k^i} e_\varepsilon((v_\varepsilon(x,s))\mathrm{d}x \leq \exp\Big(-\nu_i\rho_1\frac{\delta}{\varepsilon}\Big).$$

Proof of Step 2 Since $\nu_1 \leq \nu_2$ and in view of property $\mathcal{WP}_\varepsilon(\nu_2\delta, \widetilde{t})$ inequality (1.10), we have for $i = 0, 1$,

$$\sum_k \int_{U_k^i} e_\varepsilon((v_\varepsilon(x,\widetilde{t}))\mathrm{d}x \leq CM_0\exp\Big(-\nu_i\rho_1\frac{\delta}{\varepsilon}\Big).$$

Consider the cylinder $\Lambda_k = [a_k(t) + \frac{\nu_1\delta}{2}, a_k(t) + \frac{\nu_1\delta}{2}] \times [\widetilde{t}, t]$, and set $\theta_{\mathrm{bd}} = \mathrm{Max}\{\theta_{\mathrm{bd}}^0, \theta_{\mathrm{bd}}^1\}$, where, for $i = 0, 1$, we have defined

$$\theta_{\mathrm{bd}}^i = \mathrm{Max}\Big\{\Big|v_\varepsilon\Big(a_{k+i}(t) + \frac{\nu_1\delta}{2}, \widetilde{t}\Big) - v_\varepsilon\Big(a_{k+i}(t) + \frac{\nu_1\delta}{2}, s\Big)\Big|,\, s \in [\widetilde{t}, t]\Big\}.$$

It follows from Lemma 4.7 that $|\theta_{bd}| \leq CM_0\big(\frac{\delta}{\varepsilon}\big)^{\frac{3}{2}}\exp\big(-\rho_2\frac{\delta}{2\varepsilon}\big)$. The conclusion that follows from the estimates provided in Proposition 6.1.

Step 3 There exists some point $z \in [a_k(t) + \delta, a_k(t) + 2\delta]$, such that

$$|\xi_\varepsilon(z,t)| \leq \frac{\varepsilon}{\delta}\exp\Big(-\nu_2\rho_1\frac{\delta}{\varepsilon}\Big).$$

Proof of Step 3 It is a direct consequence of the inequality $\varepsilon^{-1}|\xi_\varepsilon| \leq e_\varepsilon(v_\varepsilon)$, Step 2 for $i = 1$ and a mean-value argument.

Step 4 (Proof of Proposition 4.1 Completed) To prove that $\mathcal{WP}_\varepsilon(\nu_1\delta, t)$ holds, we have to establish that conditions $WP3(\nu_1\delta)$ and $WP4(\nu_1\delta)$ hold at time t. Condition $WP3(\nu_1\delta)$ is actually an immediate consequence of Step 1. For $WP2(\nu_1\delta)$, we apply Remark 3.2 to the map $v_\varepsilon(\cdot, t)$ on the interval $A = [a_k(t) - 2\delta, a_k(t) + 2\delta]$ with $f = \partial v_\varepsilon(., t)$, so that, in view of Lemma 4.7, we derive $\varepsilon^{\frac{3}{2}}\|f\|_{L^2(A)} \leq CM_0\delta\exp\big(-\rho_2\frac{\delta}{2\varepsilon}\big)$. Computing $\widetilde{d}$ according to (3.39) we find, thanks to the definition of ν_2,

$$\widetilde{d} \geq \frac{\rho_2}{5\varepsilon}\delta,$$

provided that α_2 is chosen sufficiently large. This yields

$$\|u-\zeta^{\dagger_k}_{i(k),\varepsilon}(\cdot-y_0)\|_{C^1_\varepsilon([x_0-a,x_0+a])}\leq \exp\Big(-\frac{\rho_1}{2\varepsilon}\widetilde{d}_f\Big)\leq \exp\Big(-\frac{\rho_1\rho_2}{10\varepsilon}\delta\Big),$$

which establishes $\mathcal{WP}_\varepsilon 3(\nu_1\delta)$ at time t for our choice (4.12) of the constant ν_1.

We complete this subsection deriving a simple consequence of Corollary 4.1, which will be used in several places. Consider an arbitrary point $a\in\mathbb{R}$, numbers $d>0$, $r>0$ and set

$$\theta^{\pm}_{(a,t)}(d,r)=\mathrm{Max}\{|v_\varepsilon(a\pm d,t)-v_\varepsilon(a\pm d,s)|,\ s\in[t-\varepsilon r,t+\varepsilon r]\cap[T,T_1]\}. \tag{4.13}$$

Lemma 4.8 *Assume that assumption* (4.10) *holds with* $\delta\geq\alpha_2\varepsilon$. *Let* $t\in[T,T_1]$, $a\in\mathbb{R}$, $d>0$ *and* $r>0$ *be given. There exists some* $\widetilde{d}\in[\frac{d}{2},d]$, *such that*

$$\theta^{\pm}_{(a,t)}(\widetilde{d},r)\leq 2\sqrt{M_0\frac{\delta r}{\varepsilon d}}\exp\Big(-\rho_2\frac{\delta}{2\varepsilon}\Big).$$

Proof Set $t_1=\inf\{t-\varepsilon r,T\}$. It follows from Corollary 4.1, the definition of dissip and a standard mean-value argument that, for some $\widetilde{d}\in[\frac{d}{2},d]$, we have the inequality

$$\int_{t_1}^{t+\varepsilon r}(|\partial_t v_\varepsilon(a+\widetilde{d},s)|^2+|\partial_t v_\varepsilon(a-\widetilde{d},s)|^2)\mathrm{d}s\leq\frac{4M_0\delta}{\varepsilon^2 d}\exp\Big(-\rho_2\frac{\delta}{\varepsilon}\Big). \tag{4.14}$$

The results follows by integration and invoking Cauchy-Schwarz inequality.

4.4 First properties of trajectories

In this subsection, we discuss a few elementary properties of the subset $\mathfrak{O}(t)$ of $\mathbb{R}$, defined in (1.11) in particular in connection with the property $\mathcal{WP}_\varepsilon(\delta,t)$. As a matter of fact, this is the true for all positive times, as a consequence of a general result of Angenent on parabolic scalar equations[9] (see [2]): Moreover, the number of elements in $\mathcal{WP}_\varepsilon(\delta,t)$ can only decrease. Going back to Proposition 4.1, we see that if (4.10) holds with $\delta\geq\alpha_2\varepsilon$, then the number of points in $\mathfrak{O}(t)$ is constant on the interval $[T,T_1]$, and therefore, we may write, for $t\in[T,T_1]$,

$$\mathfrak{O}(t)=\{a_k(t)\}_{k\in J(T)}. \tag{4.15}$$

Concerning the motion of the individual points $a_k(\cdot)$, we have the following proposition.

Proposition 4.2 *Let* $0\leq T\leq T_1$ *and* δ *be given, and assume that condition* (4.10) *holds with* $\delta\geq\alpha_3\varepsilon$, *where* $\alpha_3\geq\alpha_2$ *is some constant. Then given any times* t *and* $t'\in[T,T_1]$, *we have*

$$|a_k(t)-a_k(t')|\leq\Big(\frac{|t-t'|}{\varepsilon}+\mathrm{c}_1\delta\Big)\exp\Big(-\rho_3\frac{\delta}{\varepsilon}\Big). \tag{4.16}$$

Proof We divide the proof into steps.

Step 1 Given any times t and $t'\in[T,T_1]$, such that $|t-t'|\leq\mathrm{c}_1\varepsilon\delta$ and $\mathcal{WP}_\varepsilon(\nu_1\delta,t')$, we have

$$|a_k(t)-a_k(t')|\leq\frac{\varepsilon^2}{2\delta}\exp\Big(-2\rho_3\frac{\delta}{\varepsilon}\Big). \tag{4.17}$$

[9] This is the second place where we invoke the fact that the equation is scalar: However, this observation is not crucial in the proof.

Proof of Step 1 We apply Lemma 4.8 with $r = c_1\delta$, $a = a_k(t)$ and choose d of the form

$$d = \frac{\varepsilon^2}{\delta}\exp\Big(-\varrho\frac{\delta}{\varepsilon}\Big),$$

where the parameter ϱ is determined by $\varrho = \inf\big\{\frac{\rho_3}{4}, \frac{\nu_1\rho_1}{4}\big\}$. Since we impose that $4\varrho \leq \rho_3$, there exists, in view of Lemma 4.8, some $\widetilde{d} \in [\frac{d}{2}, d]$, such that

$$\begin{aligned}\theta^{\pm}_{(a,t)}(\widetilde{d}, c_0\delta) &\leq 2\sqrt{c_1 M_0\frac{\delta^3}{\varepsilon^3}}\exp\Big(-\frac{[\rho_3 - 2\varrho]\delta}{4\varepsilon}\Big) \leq 2\sqrt{c_1 M_0\frac{\delta^3}{\varepsilon^3}}\exp\Big(-\frac{\rho_3\delta}{8\varepsilon}\Big)\\ &\leq \exp\Big(-\frac{\rho_3\delta}{10\varepsilon}\Big).\end{aligned}\tag{4.18}$$

Since the assumption $0 < \varepsilon < 1$ holds, the last inequality holds if $\delta \geq \alpha_3\varepsilon$, provided that $\alpha_3 \geq \alpha_2 > 0$ is chosen sufficiently large. On the other hand, since $\mathcal{WP}_\varepsilon(\nu_1\delta, t')$ holds, we have for any $-\nu_1\delta \leq l \leq \nu_1\delta$,

$$\Big|v_\varepsilon(a_k(t') + l, t') - \zeta^{\dagger_i}_{i(k)}\Big(\frac{l}{\varepsilon}\Big)\Big| \leq \exp\Big(-\rho_1\frac{\nu_1\delta}{\varepsilon}\Big),\tag{4.19}$$

while we have that for any $l \in \mathbb{R}$ the estimate $\big|\zeta^{\dagger_i}_{i(k)}\big(\frac{l}{\varepsilon}\big) - z_i(k)\big| \geq K\inf\big\{\frac{l}{\varepsilon}, 1\big\}$, where K is some constant. We deduce that for any $-\nu_1\delta \leq l \leq \nu_1\delta$, we have

$$|v_\varepsilon(a_k(t') + l, t') - z_{i(k)}| \geq K\inf\Big\{\frac{l}{\varepsilon}, 1\Big\} - \exp\Big(-\rho_1\frac{\nu_1\delta}{\varepsilon}\Big).\tag{4.20}$$

Next assume by contradiction that

$$a_k(t') - a_k(t) \geq 4\frac{\varepsilon^2}{\delta}\exp\Big(-\varrho\frac{\delta}{2\varepsilon}\Big).\tag{4.21}$$

If (4.21) holds, then we have

$$a_k(t') - [a_k(t) + \widetilde{d}] \geq \frac{3\varepsilon^2}{\delta}\exp\Big(-\varrho\frac{\delta}{2\varepsilon}\Big).$$

Using (4.20) with $l = a_k(t') - [a_k(t) + d]$, we are led to

$$|v_\varepsilon(a_k(t) \pm \widetilde{d}, t') - z_{i(k)}| \geq K\frac{\varepsilon}{\delta}\exp\Big(-\varrho\frac{\delta}{2\varepsilon}\Big) - \exp\Big(-\nu_1\rho_1\frac{\delta}{\varepsilon}\Big) \geq \exp\Big(-\varrho\frac{\delta}{4\varepsilon}\Big),\tag{4.22}$$

where the last inequality holds since we impose $0 < \varrho \leq \frac{\nu_1\rho_1}{4}$, provided $\delta \geq \alpha_3\varepsilon$ and that α_3 is chosen sufficiently large. Combining this inequality with (4.18), we are led to

$$|v_\varepsilon(a_k(t) \pm \widetilde{d}, t) - z_{i(k)}| \geq K\frac{\varepsilon}{\delta}\exp\Big(-\frac{\varrho\delta}{4\varepsilon}\Big) - \exp\Big(-\frac{\rho_3\delta}{10\varepsilon}\Big) \geq \exp\Big(-\frac{\varrho\delta}{8\varepsilon}\Big),\tag{4.23}$$

provided $\delta \geq \alpha_3\varepsilon$ and that α_3 is chosen sufficiently large. Since $|\dot{v_\varepsilon}| \leq K\varepsilon^{-1}$, we deduce, setting $\mathrm{Min}_{\mathrm{bd}} = \inf\{|v_\varepsilon(a_k(t) + l, t) - z_{i(k)}|,\ l \in [-\widetilde{d}, \widetilde{d}]\}$,

$$\mathrm{Min}_{\mathrm{bd}} \geq \exp\Big(-\frac{\varrho\delta}{8\varepsilon}\Big) - \frac{Kd}{\varepsilon} \geq \exp\Big(-\frac{\varrho\delta}{8\varepsilon}\Big) - \frac{K\varepsilon}{\delta}\exp\Big(-\frac{\varrho\delta}{\varepsilon}\Big) \geq \exp\Big(-\frac{\varrho\delta}{2\varepsilon}\Big) > 0,\tag{4.24}$$

provided $\delta \geq \alpha_3\varepsilon$ and that α_3 is chosen sufficiently large. On the other hand, it follows from the definition of $a_k(t)$ that $|v_\varepsilon(a_k(t), t) - z_{i(k)}| = 0$, so that $\mathrm{Min}_{\mathrm{bd}} = 0$, a contradiction, and so that (4.21) does not hold, if $\delta \geq \alpha_3\varepsilon$. Similarily, one shows

$$a_k(t) - a_k(t') \leq 4\frac{\varepsilon^2}{\delta}\exp\Big(-\varrho\frac{\delta}{2\varepsilon}\Big),$$

which leads to the conclusion (4.17) with $\rho_3 = \frac{\varrho}{4}$ by choosing α_4 sufficiently large.

Step 2 Given any times t and $t' \in [T, T_1]$, such that $|t - t'| \leq c_1\varepsilon\delta$, we have

$$|a_k(t) - a_k(t')| \leq \frac{\varepsilon^2}{\delta}\exp\Big(-2\rho_3\frac{\delta}{\varepsilon}\Big). \tag{4.25}$$

Proof of Step 2 Without loss of generality, we may assume that $T \leq t \leq t' \leq T_1$. This is an immediate consequence of Step 1 and Proposition 4.1. Indeed, if $t' > T + c_1\varepsilon\delta$, then it satisfies, in view of Proposition 4.1, $\mathcal{WP}_\varepsilon(\nu_1\delta, t')$, and the conclusion then follows directly from Step 1. Otherwise, we have $t' - T \leq c_1\varepsilon\delta$. Since assumptions $\mathcal{WP}_\varepsilon(\delta, T)$ holds, we deduce from Step 1 that

$$|a_k(t) - a_k(T)| \leq \frac{\varepsilon^2}{2\delta}\exp\Big(-\rho_3\frac{\delta}{\varepsilon}\Big),$$

and the same inequality with t replaced by t'. Combining these two inequalities, the conclusion follows.

Step 3 (Proof of Proposition 4.2 Completed) We introduce the intermediate times $t_n = t + kc_1\varepsilon\delta$ for $n \in \{0, 1, \cdots, n_f\}$, where n_f is the largest integer less than $\frac{|t-t'|}{c_1\varepsilon\delta}$ with $t_{k_f+1} = t'$. In view of Step 2, we have for $n = 0, \cdots, n_f$,

$$|a_k(t_{n+1}) - a_k(t_n)| \leq \frac{\varepsilon^2}{\delta}\exp\Big(-2\rho_3\frac{\delta}{\varepsilon}\Big),$$

so that adding these inequalities, we are led to

$$|a_k(t) - a_k(t')| \leq (n_f + 1)\frac{\varepsilon^2}{\delta}\exp\Big(-\rho_4\frac{\delta}{\varepsilon}\Big) \leq \Big(\frac{|t-t'|}{\varepsilon} + \delta\Big)\frac{\varepsilon^2}{\delta^2}\exp\Big(-2\rho_3\frac{\delta}{\varepsilon}\Big),$$

and the conclusion follows for a suitable choice of the constant α_3.

If follows from Propositions 4.1–4.2 that if assumption (4.10) is satisfied for some $\delta \geq \alpha_4\varepsilon$, then the number of front points $a_k(t)$ does not change on the time interval $[T, T_1]$, the front points $a_k(t)$ are perfectly labelled, continuous in time. Likewise the numbers $i(k)$ and the signs $\dagger_i(k)$ do not depend on t. Moreover, an elementary, yet important observation is:

Lemma 4.9 *Let $0 \leq T \leq T_1$ and δ be given, and assume that condition* (4.10) *holds with $\delta \geq \alpha_3\varepsilon$. Then, given any $k_1 \neq k_2 \in J(T)$ and $t \in [T, T_1]$, we have*

$$|a_{k_1}(t) - a_{k_2}(t)| \geq \nu_1\delta.$$

Proof If $t \geq T + c_1\varepsilon\delta$, then the conclusion follows immediately from the fact that property $\mathcal{WP}_\varepsilon(\nu_1\delta, t)$ holds. If $T \leq t \leq T + c_1\varepsilon\delta$, then we have for $j = 1, 2$,

$$|a_{k_j}(t) - a_{k_j}(T)| \leq 2c_1\delta\exp\Big(-\rho_3\frac{\delta}{\varepsilon}\Big) \leq \frac{(1-\nu_1)\delta}{4},$$

provided that α_3 is chosen sufficiently large. On the other hand, it follows from property $\mathcal{WP}_\varepsilon(\delta, T)$ that $|a_{k_1}(T) - a_{k_2}(T)| \geq \delta$, and the conclusion follows combining the previous inequalities.

4.5 The stopping time $\mathcal{T}_{\rm sim}(\delta, T)$

Whereas the two previous subsections discussed some consequences of condition (4.10), we provide here a situation where such a condition is met. For that purpose, we will invoke for the first time so far the upper bound on the speed of the front set provided in Theorem 1.6. Given a time $T \geq 0$ and $\delta > 0$, we assume throughout this subsection that $\mathcal{WP}_\varepsilon(T, \delta)$ holds. We then set

$$\mathcal{T}_{\rm sim}(\delta, T) = \inf\Big\{s \geq T, \ \exists k \neq k' \in J(T), \text{ such that } |a_k(s) - a_{k'}(s)| \leq \frac{\delta}{2}\Big\}, \tag{4.26}$$

in the case that the set on the right-hand side is not void, and $\mathcal{T}_{\rm sim}(T, \delta) = +\infty$ otherwise. We have the following proposition.

Proposition 4.3 *Assume that $\mathcal{WP}_\varepsilon(T, \delta)$ holds with $\delta \geq \alpha_4\varepsilon$ for some constant $\alpha_4 \geq \alpha_3$. Then condition $\mathcal{C}(\nu_4\delta, T, \mathcal{T}_{\rm sim}(\delta, T))$ is met, where $0 < \nu_4 < 1$ is some constant. Moreover, we have*

$$\mathcal{T}_{\rm sim}(T, \delta) - T \geq 2c_1\varepsilon^2 \exp\Big(\rho_0 \frac{\delta}{8\varepsilon}\Big). \tag{4.27}$$

Proof We first establish inequality (4.27). In view of (1.27), we have $\mathcal{D}(T) \subset \overset{J(T)}{\underset{k=1}{\cup}} \{a_k(T)\} + [-\alpha_1\varepsilon, \alpha_1\varepsilon]$, so that, combining with Theorem 1.6, for $r \geq \alpha_0\varepsilon$, we are led to the inclusion

$$\mathcal{D}(T + \Delta T) \subset \bigcup_{k=1}^{J(T)} \{a_k(T)\} + [-\alpha_1\varepsilon - r, \alpha_1\varepsilon + r] \equiv\subset \bigcup_{k=1}^{J(T)} I_{k,r}, \tag{4.28}$$

provided

$$0 \leq \Delta T \leq (\Delta T)_0 = \rho_0^2 r^2 \exp\Big(\rho_0 \frac{r}{\varepsilon}\Big),$$

where the sets $I_{k,r}$ denote the intervals $I_{k,r} = [a_k(T) - \alpha_1\varepsilon - r, a_k(T) + \alpha_1\varepsilon + r]$. Choosing

$$r = \frac{\delta}{4} - \alpha_1\varepsilon \geq \frac{\delta}{8},$$

we deduce that

$$\text{dis}(I_k, I_{k'}) \geq \frac{\delta}{2} + 2\alpha_1\varepsilon \quad \text{and} \quad (\Delta T)_0 \geq \frac{\rho_0^2\alpha_2^2\varepsilon^2}{16} \exp\Big(\rho_0 \frac{\mathrm{r}}{\varepsilon}\Big) \geq 4c_0{}^2\varepsilon^2 M_0 \exp\Big(\rho_0 \frac{\delta}{8\varepsilon}\Big),$$

where we assume that the constant α_2 is chosen sufficiently large. This proves (4.27).

For the first statement, we notice that, thanks to (4.8) there exists some time $T_1 \in [\mathcal{T}_{\rm sim} - \frac{(\Delta T)_0}{2}, \mathcal{T}_{\rm sim}]$ such that $\mathcal{WP}_\varepsilon(\tilde{\nu}_4\delta, T_1)$ holds, where we set

$$\tilde{\nu}_4 = \inf\Big\{\frac{\rho_2}{8\rho_1}, \frac{1}{4}\Big\}.$$

Next consider the time $\mathcal{T}_{\rm sim}(T_1, \tilde{\nu}_4\delta)$. Using a similar argument, we may find some time $T_2 \in [\mathcal{T}_{\rm sim}(T, \delta), \mathcal{T}_{\rm sim}(T_1, \tilde{\nu}_4\delta)]$, such that $\mathcal{WP}_\varepsilon(\tilde{\nu}_4^2\delta, T_2)$ holds. It follows that $\mathcal{C}(\nu_4^2\delta, T, T_2)$ holds, hence $\mathcal{C}(\nu_4^2\delta, T, \mathcal{T}_{\rm sim}(\delta, T))$ holds. Choosing $\nu_4 = \tilde{\nu}_4^2$, the proof is complete.

Combining the previous result with Proposition 4.1, inequality (4.16) of Proposition 4.2 as well as the identity (4.15), we are immediately led to the following statement.

Proposition 4.4 *Assume that* $\mathcal{WP}_\varepsilon(T,\delta)$ *holds with* $\delta \geq \alpha_4\varepsilon$. *Then for any* $t \in [T, \mathcal{T}_{\rm sim}(T,\delta)]$, *the points* $\{a_k(t)\}_{k\in J(T)}$ *satisfying* (1.8) *are well-defined. Moreover, assumption* $\mathcal{WP}_\varepsilon(\nu_0\delta, t)$ *holds for any* $t \in [T+c_2\varepsilon\delta, \mathcal{T}_{\rm sim}(T,\delta)]$, *where* $\nu_0 = \nu_1\nu_4$ *and* $c_2 = c_1\nu_4$. *Moreover, we have*

$$|a_k(t) - a_k(T)| \leq \varepsilon \exp\Big(-\rho_3\frac{\delta}{2\varepsilon}\Big) \quad \text{for any } t \in [t, T+\varepsilon^2],\ k \in J(T), \tag{4.29}$$

provided that the constant α_4 *is chosen sufficiently large.*

Inequality (4.29) will be used to handle small initial time boundary layers of size ε^2, which are related to the parabolic estimates provided in the next section.

5 Linear Parabolic Estimates

In this section, we single out a few linear and mostly elementary parabolic estimates, which will be used directly in the study of the nonlinear equation $(\mathrm{PGL})_\varepsilon$. We consider in this section $0 < \varepsilon < 1$ a small parameter, the standard space-time cylinder[10]

$$\Lambda = \Big[-\frac{1}{2}, \frac{1}{2}\Big] \times [0,1], \tag{5.1}$$

a smooth function c defined on Λ, such that for all $(x,t) \in \Lambda$, we have

$$c(x,t) \geq \lambda, \tag{5.2}$$

where $\lambda > 0$ is a given positive number, and linear parabolic equation

$$\partial_t u_\varepsilon - \partial^2_{xx} u_\varepsilon + \varepsilon^{-2} c(x,t) u_\varepsilon = 0 \quad \text{on } \Lambda. \tag{5.3}$$

as well as its special case,

$$\partial_t \mathfrak{u}_\varepsilon - \partial^2_{xx} \mathfrak{u}_\varepsilon + \varepsilon^{-2} \lambda u_\varepsilon = 0 \quad \text{on } \Lambda. \tag{5.4}$$

Notice that it follows from the maximum principle and (5.2) that we have the inequality

$$|u_\varepsilon| \leq \mathfrak{u}_\varepsilon, \tag{5.5}$$

where $\mathfrak{u}_\varepsilon$ is the solution to (5.4) satisfying $\mathfrak{u}_\varepsilon = |u_\varepsilon|$ on Π, where

$$\Pi \equiv \Pi_0 \cup \Pi_- \cup \Pi_+ \tag{5.6}$$

with $\Pi_0 = [-\frac{1}{2}, \frac{1}{2}] \times \{t\}$, $\Pi_- = \{-\frac{1}{2}\} \times [0,1]$ and $\Pi_+ = \{\frac{1}{2}\} \times [0,1]$. In several places, we will be led to assuming that the function c satisfies the additional condition

$$|c(x,t)| \leq C \quad \text{for } (x,t) \in \Lambda. \tag{5.7}$$

The main estimate of this section will be given in Proposition 5.1. It involves also the difference

$$\varsigma(x,t) = c(x,t) - \lambda \geq 0. \tag{5.8}$$

We start with a few preliminary results.

[10] More general cylinders and solutions may be handled by using translations and scalings.

5.1 Basic estimates

Lemma 5.1 *Let u_ε be a solution to* (5.3), *such that* $u_\varepsilon = 0$ on $\Pi_- \cup \Pi_+$. *We have*

$$|u_\varepsilon(x,t)| \leq \exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big)\|u_\varepsilon(\cdot,0)\|_{L^\infty[-\frac{1}{2},\frac{1}{2}]} \tag{5.9}$$

for any $(x,t) \in \Lambda$. *Moreover, if c satisfies* (5.7) *then, we have for* $\varepsilon^2 \leq t \leq 1$ *and* $-\frac{1}{2} + \varepsilon \leq x \leq \frac{1}{2} - \varepsilon$,

$$|\partial_x u_\varepsilon(x,t)| \leq \frac{C}{\varepsilon}\exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big)\|u_\varepsilon(\cdot,0)\|_{L^\infty[-\frac{1}{2},\frac{1}{2}]}. \tag{5.10}$$

Proof For the first statement, we notice that the function h defined by

$$h(x,t) = \exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big)\|u_\varepsilon(\cdot,0)\|_{L^\infty[-\frac{1}{2},\frac{1}{2}]}$$

is a solution to (5.4), and hence, by the maximum principle, we have $\mathfrak{u}_\varepsilon \leq h(x,t)$. Invoking (5.5), the conclusion (5.9) hence follows.

For the second statement, that is estimate (5.10), we invoke the regularization properties of the heat equation together with a scaling argument. Let $(x_0,t_0) \in \Lambda$ be given, such that $\varepsilon^2 \leq t_0 \leq 1$ and $-\frac{1}{2} + \varepsilon \leq x_0 \leq \frac{1}{2} - \varepsilon$. We perform the change of variable $\mathrm{x} \to \frac{x-x_0}{\varepsilon}$ and $\mathrm{t} \to \frac{t-t_0}{\varepsilon} + 1$, so that (x_0,t_0) corresponds in the new variables (x,t) to the point $(0,1)$. We consider the scaled map

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = u_\varepsilon(x_0 + \varepsilon\mathrm{x}, t_0 + \varepsilon^2(\mathrm{t}-1)), \tag{5.11}$$

which satisfies the parabolic equation

$$\partial_{\mathrm{t}}\mathrm{u} - \partial^2_{\mathrm{xx}}\mathrm{u} + \mathrm{c}(\mathrm{x},\mathrm{t})\mathrm{u} = 0, \tag{5.12}$$

on the large cylinder $\widetilde{\Lambda}_\varepsilon = [-\frac{1}{2\varepsilon} - \frac{1}{\varepsilon x_0}, \frac{1}{2\varepsilon} - \frac{1}{\varepsilon x_0}] \times [-\frac{t_0}{\varepsilon^2} + 1, 1]$ and where the function c is defined as

$$\mathrm{c}(\mathrm{x},\mathrm{t}) = c_\varepsilon(x_0 + \varepsilon\mathrm{x}, t_0 + \varepsilon^2(\mathrm{t}-1)). \tag{5.13}$$

It follows from assumption (5.7) for any given $|\mathrm{c}(\mathrm{x},\mathrm{t})| \leq C$. To conclude, we evoke the following standard parabolic estimate.

Lemma 5.2 *Let u be a smooth real-valued function on Λ and assume*

$$|\partial_t u - \partial_{xx} u| \leq b \quad on\ \Lambda \quad \text{and} \quad |u| \leq d \quad on\ \Lambda. \tag{5.14}$$

Then, if $\Lambda_{\frac{1}{2}}$ denotes the cylinder $\big[-\frac{1}{2},\frac{1}{2}\big] \times \big[\frac{3}{4},1\big]$, there exists a universal constant $C > 0$, such that

$$\|\partial_x u\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(b+d).$$

For a proof, we refer to [4, Lemma A.7], where closely related estimates are established.

Proof of Lemma 5.1 (Completed) We apply (5.14) to the equation (5.12), restricted to $\Lambda \subset \widetilde{\Lambda}_\varepsilon$ with $d = \|c(x,t)u_\varepsilon\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])}$ and $c = \|u_\varepsilon\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])}$. We are hence led to the inequality, using (5.7),

$$|\partial_{\mathrm{x}}\mathrm{u}(0,1)| \leq C\|\mathrm{u}\|_{L^\infty(\Lambda)} = C\ \|u_\varepsilon\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])}.$$

Invoking (5.9), and going back to the original variables, the conclusion (5.10) follows.

Lemma 5.3 *Let u_ε be a solution to* (5.3) *such that* $u_\varepsilon = 0$ on Π_0. *There exists a constant $C > 0$ which does not depend on ε nor on λ, such that for all $(x,t) \in \Lambda$,*

$$|u_\varepsilon(x,t)| \leq C\|u_\varepsilon\|_{L^\infty(\Pi_-\cup\Pi_+)}\Big[\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big)\Big]. \tag{5.15}$$

Moreover, if the function c satisfies condition (5.7)*, then, we have*

$$|\partial_x u_\varepsilon(x,t)| \leq \frac{C}{\varepsilon}\|u_\varepsilon\|_{L^\infty(\Pi_-\cup\Pi_+)}\Big[\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big)\Big]. \tag{5.16}$$

Proof It follows from the maximum principle that for all $(x,t) \in \Lambda$,

$$|u(x,t)| \leq \|u_\varepsilon\|_{L^\infty(\Pi_-\cup\Pi_+)}\Psi_\varepsilon, \tag{5.17}$$

where Ψ_ε is the stationary solution to (5.4) given by $\Psi_\varepsilon(x,t) = \Psi_{0,\varepsilon}$, where $\Psi_{0,\varepsilon}$ is the solution to the stationary problem (5.29) with boundary conditions

$$\Psi_{0,\varepsilon}\Big(-\frac{1}{2}\Big) = 1 \quad \text{and} \quad \Psi_{0,\varepsilon}\Big(-\frac{1}{2}\Big) = 1, \tag{5.18}$$

so that

$$\Psi_{0,\varepsilon}(x) = \cosh\Big(\frac{\sqrt{\lambda}x}{\varepsilon}\Big)\Big[\cosh\Big(\frac{\sqrt{\lambda}}{2\varepsilon}\Big)\Big]^{-1}, \tag{5.19}$$

and in particular,

$$0 \leq \Psi_{0,\varepsilon}(x) \leq 2\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big). \tag{5.20}$$

Combining (5.20) and (5.17), we derive (5.15). The proof of (5.16) follows the same arguments as the proof of inequality (5.10) of Lemma 5.1. Therefore, we omit it.

5.2 Equations with source terms

Next, we let $f \in C^2(\Lambda)$, we consider the equation with source term

$$\partial_t u_\varepsilon - \partial^2_{xx}u_\varepsilon + \varepsilon^{-2}c(x,t)u_\varepsilon = f \quad \text{on } \Lambda \tag{5.21}$$

with boundary condition

$$u_\varepsilon = 0 \quad \text{on } \Pi. \tag{5.22}$$

Lemma 5.4 *Let u_ε be a solution to* (5.21)–(5.22). *We have the estimate*

$$|u_\varepsilon(x,t)| \leq \frac{\varepsilon}{2\sqrt{\lambda}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\Big(- \frac{\sqrt{\lambda}|x-y|}{\varepsilon}\Big) \sup_{0\leq s\leq t} |f(y,s)| \mathrm{d}y. \tag{5.23}$$

Moreover, if c satisfies (5.7) *then, we have*

$$\begin{aligned} |\partial_x u_\varepsilon(x,t)| \leq & \frac{C}{2\sqrt{\lambda}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\Big(- \frac{\sqrt{\lambda}|x-y|}{\varepsilon}\Big) \sup_{0\leq s\leq t} |f(y,s)| \mathrm{d}y. \\ & + C\varepsilon \|f\|_{L^\infty([x_0-\varepsilon, x_0+\varepsilon]\times[t_0-\varepsilon^2, t_0])}. \end{aligned} \tag{5.24}$$

Proof By the maximum principle, it suffices to consider the case $f \geq 0$, what we assume throughout the rest of the proof. Invoking the maximum principle once more in that case, we conclude that $0 \leq u_\varepsilon \leq \mathfrak{u}_\varepsilon$, where $\mathfrak{u}_\varepsilon$ is the solution to

$$\partial_t \mathfrak{u}_\varepsilon - \partial_{xx}^2 \mathfrak{u}_\varepsilon + \varepsilon^{-2}\lambda \mathfrak{u}_\varepsilon = f \quad \text{on } \Lambda \tag{5.25}$$

with boundary condition $\mathfrak{u}_\varepsilon = 0$ on Π. We are led therefore to establish the bound (5.23) for the solution $\mathfrak{u}_\varepsilon$ only. We extend the function f to the whole of $\mathbb{R}\times[0,1]$ setting $f(x,s)=0,$ if $x \notin [-\frac{1}{2}, \frac{1}{2}]$. Given $t>0$, we consider also the function $\widetilde{f}_t(x)$ of the scalar variable x defined by

$$\widetilde{f}^t(x) = \sup_{0\leq s\leq t} f(x,s), \tag{5.26}$$

and the solution $\widetilde{u}^t$ of the differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\widetilde{u}^t(x) + \frac{\lambda}{\varepsilon^2}\widetilde{u}^t = f^t \quad \text{on } \mathbb{R}, \tag{5.27}$$

given by convolution with the corresponding kernel, namely

$$\widetilde{u}^t(x) = \frac{\varepsilon}{2\sqrt{\lambda}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\Big(- \frac{\sqrt{\lambda}|x-y|}{\varepsilon}\Big) \widetilde{f}^t(y) \mathrm{d}y. \tag{5.28}$$

Next, we consider the function $\widetilde{u}$ defined on $\mathbb{R}^+\times[0,1]$ by $\widetilde{u}^t(x,s) = \widetilde{u}^t(x)$, so that we immediately derive that

$$\partial_t u^t - \partial_{xx}\widetilde{u}^t + \frac{\lambda}{\varepsilon}\widetilde{u}^t = \widetilde{f}^t \geq f \quad \text{on } [0,t]\times\Big[-\frac{1}{2}, \frac{1}{2}\Big].$$

Invoking once more the maximum principle, we deduce that $\widetilde{u}^t \geq \mathfrak{u}_\varepsilon$ on $[0,t]\times\big[-\frac{1}{2},\frac{1}{2}\big]$ and the conclusion (5.23) follows.

Next, we turn to the proof of (5.24). We argue as in the proof of (5.10), and consider the change of variables and the scaled map given by (5.11), which satisfies in our setting the parabolic equation $\partial_{\mathrm{t}}\mathrm{u} - \partial_{\mathrm{xx}}^2\mathrm{u} + \mathrm{c}(\mathrm{x},\mathrm{t})\mathrm{u} = \mathrm{f}$, where the function f is defined by $\mathrm{f}(\mathrm{x},\mathrm{t}) = \varepsilon^2 f_\varepsilon(x_0 + \varepsilon\mathrm{x}, t_0 + \varepsilon^2(\mathrm{t}-1))$. Invoking Lemma 5.2 once more, we deduce that

$$|\partial_{\mathrm{x}}\mathrm{u}(0,1)| \leq C\ \|u_\varepsilon\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])} + \varepsilon^2\|f\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])},$$

which yields the conclusion (5.24).

5.3 Stationary solution to the linearized problem

In this subsection, we consider the interval $I = [-\frac{1}{2}, \frac{1}{2}]$ and the solution U_ε to the stationary problem for a given parameter $\lambda > 0$

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} U_\varepsilon + \lambda \varepsilon^{-2} U_\varepsilon = 0 \quad \text{for } x \in [-\dfrac{1}{2}, \dfrac{1}{2}], \\ U_\varepsilon(-\frac{1}{2}) = \gamma_\varepsilon^- \quad \text{and} \quad U_\varepsilon(\frac{1}{2}) = \gamma_\varepsilon^+, \end{cases} \tag{5.29}$$

where γ_ε^- and γ_ε^+ are given. The solution to (5.29) is easily integrated as

$$U_\varepsilon(x) = \mathcal{A}_\varepsilon \sinh\Big(\frac{\sqrt{\lambda} x}{\varepsilon}\Big) + \mathcal{B}_\varepsilon \cosh\Big(\frac{\sqrt{\lambda} x}{\varepsilon}\Big), \tag{5.30}$$

where the constants $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are deduced from the values of γ_ε^- and γ_ε^+ by

$$\mathcal{A}_\varepsilon = (\gamma_\varepsilon^+ - \gamma_\varepsilon^-)\Big[2\sinh\Big(\frac{\sqrt{\lambda}}{2\varepsilon}\Big)\Big]^{-1} \quad \text{and} \quad \mathcal{B}_\varepsilon = (\gamma_\varepsilon^+ + \gamma_\varepsilon^-)\Big[2\cosh\Big(\frac{\sqrt{\lambda}}{2\varepsilon}\Big)\Big]^{-1}.$$

We introduce a quadratic form related to the discrepancy defined for a scalar function u by

$$Q_\lambda(u) = \frac{1}{2}[\varepsilon^2 u_x^2 - \lambda u^2]. \tag{5.31}$$

Lemma 5.5 *The function $Q(U_\varepsilon)$ is constant on $[-\frac{1}{2}, \frac{1}{2}]$ with value*

$$Q(U_\varepsilon) = \frac{\lambda}{2}(\mathcal{A}_\varepsilon^2 - \mathcal{B}_\varepsilon^2) = \frac{\lambda}{2\big[\sinh\big(\frac{\sqrt{\lambda}}{\varepsilon}\big)\big]^2}\Big[-2\gamma_\varepsilon^+\gamma_\varepsilon^- \cosh\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big) + ((\gamma_\varepsilon^+)^2 + (\gamma_\varepsilon^-)^2)\Big].$$

The proof is a straightforward computation, which is left to the reader. Similarly, we also have, concerning the energy, for any $x \in [-\frac{1}{2}, +\frac{1}{2}]$,

$$D_\lambda(U_\varepsilon)(x) \equiv \varepsilon^2 \Big(\frac{\mathrm{d}U_\varepsilon}{\mathrm{d}x}\Big)^2(x) + \lambda U_\varepsilon^2(x) \le C\lambda[(\gamma_\varepsilon^+)^2 + (\gamma_\varepsilon^-)^2] \exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(x - \frac{1}{2}\Big)\Big). \tag{5.32}$$

In view of our subsequence analysis of the nonlinear problem $(\mathrm{PGL})_\varepsilon$, we are led to introduce various additional assumptions on $\gamma_\varepsilon^\pm$ and the function c. First, we consider the case, where γ_ε^+ and γ_ε^- are of the same order of magnitude, that satisfies an inequality of the type

$$\Big|\frac{\gamma_\varepsilon^+}{\gamma_\varepsilon^-}\Big| + \Big|\frac{\gamma_\varepsilon^-}{\gamma_\varepsilon^+}\Big| \le K_0, \tag{5.33}$$

where K_0 is some given positive constant. In that case, we have the expansion

$$Q(U_\varepsilon) = -\gamma_\varepsilon^+\gamma_\varepsilon^- \exp\Big(-\frac{\sqrt{\lambda}}{\varepsilon}\Big)[1 + R_{0,\varepsilon}], \tag{5.34}$$

where the error term $R_{0,\varepsilon}$ satisfies, for every $0 < \varepsilon < 1$, the bound

$$|R_{0,\varepsilon}| \le C(K_0^2 + 1) \exp\Big(-\frac{\sqrt{\lambda}}{\varepsilon}\Big). \tag{5.35}$$

5.4 Comparison with the stationary solution to (5.4)

Next, we set

$$\gamma_\varepsilon^- = u_\varepsilon\Big(-\frac{1}{2},0\Big) \quad \text{and} \quad \gamma_\varepsilon^+ = u_\varepsilon\Big(+\frac{1}{2},0\Big),$$

and consider the solution U_ε to (5.29) with corresponding boundary conditions. Our next results describes the possible relaxation of a given solution u_ε to (5.3) to the stationary solution U_ε. In order to state our result, we introduce appropriate notions of oscillations for the various parts of the boundary Π, namely first

$$\theta_0 = \|U_\varepsilon - u_\varepsilon(\cdot,0)\|_{L^\infty[-\frac{1}{2},\frac{1}{2}]} \tag{5.36}$$

and

$$\theta_{bd} = \sup\big(\{|\gamma_\varepsilon^- - u_\varepsilon(x,t)|,\ (x,t)\in\Pi_-\}\cup\{|\gamma_\varepsilon^+ - u_\varepsilon(x,t)|,\ (x,t)\in\Pi_+\}\big)\,. \tag{5.37}$$

Proposition 5.1 *With the notation above, we have the estimate, for* $(x,t)\in\Lambda$,

$$\begin{aligned}|u_\varepsilon(x,t)-U_\varepsilon(x)| \le \theta_0\exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big) + C\theta_{bd}\Big[\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big)\Big]\\ +\Big(\frac{\varepsilon}{\sqrt{\lambda}}+\frac{\varepsilon^2}{4\lambda}\Big)\|\varsigma\|_{L^\infty(\Lambda)}(|\gamma_\varepsilon^+|+|\gamma_\varepsilon^-|)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big).\end{aligned} \tag{5.38}$$

Moreover, if c satisfies (5.7)*, then, we have, for* $t\ge\varepsilon^2$,

$$\begin{aligned}|\partial_x(u_\varepsilon(x,t)-U_\varepsilon(x))| \le \frac{C}{\varepsilon}\theta_0\exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big) + \frac{C}{\varepsilon}\theta_{bd}\Big[\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big)\Big]\\ +\Big(\frac{1}{\sqrt{\lambda}}+\frac{\varepsilon}{4\lambda}\Big)\|\varsigma\|_{L^\infty(\Lambda)}(|\gamma_\varepsilon^+|+|\gamma_\varepsilon^-|)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big).\end{aligned} \tag{5.39}$$

Proof We may decompose u_ε as

$$u_\varepsilon(x,t) = U_\varepsilon(x) + \widetilde{U}_\varepsilon(x,t) + u_{1,\varepsilon}(x,t) + u_{2,\varepsilon}(x,t), \tag{5.40}$$

where $\widetilde{U}_\varepsilon$ is the solution to (5.3) defined on Λ, such that

$$\widetilde{U}_\varepsilon(x,0) = u_\varepsilon(x,0) - U_\varepsilon(x) \ \text{ on } \Pi_0 \quad \text{and} \quad \widetilde{U}_\varepsilon = 0 \quad \text{on } \Pi_-\cup\Pi_+,$$

where the function $u_{1,\varepsilon}$ is the solution to (5.3) defined on Λ, such that

$$u_{1,\varepsilon} = 0 \text{ on } \Pi_0 \quad \text{and} \quad u_\varepsilon(x,t) = u_\varepsilon(x,t) - U_\varepsilon(x,t) \quad \text{for } (x,t)\in\Pi_-\cup\Pi_+,$$

and finally $u_{2,\varepsilon}$ is the solution to (5.21) with boundary condition $u_{2,\varepsilon}=0$ on Π and source term f given by

$$f(x,t) = \varsigma(x,t)U_\varepsilon(x). \tag{5.41}$$

The function $u_{0,\varepsilon}$ is estimated thanks to Lemma 5.1, which yields

$$|\widetilde{U}_\varepsilon(x,t)| \le \theta_0\exp\Big(-\frac{\lambda t}{\varepsilon^2}\Big), \tag{5.42}$$

whereas the function $u_{1,\varepsilon}$ is estimated thanks to Lemma 5.3, which yields

$$|u_{1,\varepsilon}(x,t)| \leq C\|u_\varepsilon - w_{0,\varepsilon}\|_{L^\infty(\Pi_-\cup\Pi_+)}\Big[\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big)\Big]. \tag{5.43}$$

In order to estimate the function $u_{2,\varepsilon}$, we will invoke Lemma 5.4, and for that purpose, we need first to bound the source term f given by (5.41). We have

$$\sup_{t\in[0,1]}|f(y,t)| \leq \|\varsigma(\cdot,\cdot)\|_{L^\infty(\Lambda)}(|\gamma_\varepsilon^+|+|\gamma_\varepsilon^-|)\chi_{0,\varepsilon}(y) \tag{5.44}$$

$$\leq 2\|\varsigma(\cdot,\cdot)\|_{L^\infty(\Lambda)}(|\gamma_\varepsilon^+|+|\gamma_\varepsilon^-|)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|y|-\frac{1}{2}\Big)\Big). \tag{5.45}$$

In view of Lemma 5.4, we are therefore led to estimate the integral

$$I(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\Big(-\frac{\sqrt{\lambda}|x-y|}{\varepsilon}\Big)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|y|-\frac{1}{2}\Big)\Big)\mathrm{d}y.$$

We leave it as an exercise to the reader to verify that

$$0 \leq I(x) \leq \Big(2+\frac{\varepsilon}{2\sqrt{\lambda}}\Big)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big), \tag{5.46}$$

so that

$$|u_{2,\varepsilon}(x,t)| \leq \frac{C\varepsilon}{\sqrt{\lambda}}\Big(2+\frac{\varepsilon}{2\sqrt{\lambda}}\Big)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big). \tag{5.47}$$

Combining (5.42)–(5.43) and (5.47), we derive (5.38). For (5.39), we use the corresponding estimates, observing that

$$\|f\|_{L^\infty([x_0-\varepsilon,x_0+\varepsilon]\times[t_0-\varepsilon^2,t_0])} \leq C\|\varsigma\|_{L^\infty(\Lambda)}(|\gamma_\varepsilon^+|+|\gamma_\varepsilon^-|)\exp\Big(\frac{\sqrt{\lambda}}{\varepsilon}\Big(|x|-\frac{1}{2}\Big)\Big).$$

5.5 Estimates for the quadratic part of the discrepancy

In this subsection, we wish to derive some estimates for $Q_\lambda(u_\varepsilon)$, there Q_λ is defined in (5.31). Furthermore, we restrict ourselves to the case that there exists some given constant $\varrho > 0$, such that

$$\theta_{\mathrm{bd}} + \|\varsigma\|_{L^\infty(\Lambda)} \leq \exp\Big(-\frac{\varrho}{\varepsilon}\Big), \tag{5.48}$$

and we finally also assume that

$$|\gamma_\varepsilon^+| + |\gamma_\varepsilon^-| + \theta_0 \leq K_0. \tag{5.49}$$

Lemma 5.6 *Assume that u_ε is a solution to* (5.3)*, and that assumptions* (5.33) *and* (5.48)–(5.49) *are full-filled. Then, we have, for $(x,t)\in\Lambda$,*

$$Q(u_\varepsilon)(x,t) = -\gamma_\varepsilon^+\gamma_\varepsilon^-\exp\Big(-\frac{\sqrt{\lambda}}{\varepsilon}\Big)[1+\mathcal{R}_{0,\varepsilon}(x,t)] + \mathcal{C}_{0,\varepsilon}(x,t), \tag{5.50}$$

where the error term satisfies the estimate, for every $0 < \varepsilon < 1$ *and* $t \geq 0$,

$$|\mathcal{R}_{0,\varepsilon}(x,t)| \leq C(K_0^2+1)\Big[\exp\Big(\frac{1}{\varepsilon}(2\sqrt{\lambda}|x|-2\varrho)\Big)+\exp\Big(-\frac{\sqrt{\lambda}}{\varepsilon}\Big)\Big] \tag{5.51}$$

and

$$|\mathcal{C}_{0,\varepsilon}(x,t)| \leq C\theta_0^2\Big[\exp\Big(-\frac{2\lambda t}{\varepsilon^2}\Big)\Big]. \tag{5.52}$$

For the proof of Lemma 5.6, we expand $u_\varepsilon = U_\varepsilon + r_\varepsilon$, where $r_\varepsilon = (u_\varepsilon - U_\varepsilon)$, so that by Cauchy-Schwarz inequality,

$$|Q(u_\varepsilon)(x,t) - Q(U_\varepsilon)| \leq D_\lambda(r_\varepsilon) + [D_\lambda(r_\varepsilon)D_\lambda(U_\varepsilon)]^{\frac{1}{2}} \leq \frac{3}{2}D_\lambda(r_\varepsilon) + \frac{1}{2}D_\lambda(U_\varepsilon).$$

We then estimate the right-hand side of this inequality thanks to the estimates for r_ε provided in Proposition 5.1, inequality (5.32) for $D_\lambda(U_\varepsilon)$ as well as the expansions (5.34) and (5.35). We omit the details.

In the asymptotic limit $\varepsilon \to 0$, the bound (5.51) shows the term $\mathcal{R}_{0,\varepsilon}(x,t)$ is indeed an error term only in the case x is small. In particular, if $|x| \leq \frac{\varrho}{2\sqrt{\lambda}}$ and $0 < \varepsilon < 1$, then, we have

$$|\mathcal{R}_{0,\varepsilon}(x,t)| \leq C(K_0^2+1)\Big[\exp\Big(-\frac{\varrho}{\varepsilon}\Big)+\exp\Big(-\frac{\sqrt{\lambda}}{\varepsilon}\Big)\Big]. \tag{5.53}$$

6 Relaxation to the Stationary Equation off the Front Set

The purpose of this section is to obtain a precise expansion of the discrepancy function $\xi(\ ,t)$, when computed far from the front set. For that purpose, we are led to analyze in details a typical situation we present next. Let (x_0,t_0) be a given arbitrary point in $\mathbb{R}\times\mathbb{R}^+$. For $r>0$, we consider the space-time cylinder

$$\Lambda_r(x_0,t_0) = \{x_0,t_0)\} + \Lambda_r = \Big[x_0 - \frac{r}{2}, x_0 + \frac{r}{2}\Big] \times [t_0, r^2 + t_0],$$

where $\Lambda_r = \Lambda_r(0,0) = \big[-\frac{r}{2}, +\frac{r}{2}\big] \times [0,r^2]$, and a solution v_ε to $(\mathrm{PGL})_\varepsilon$. We assume throughout this section that the front set of v_ε does not intersect $\Lambda_r(x_0,t_0)$, that is, we assume that there exists some $i \in \{1,\cdots,q\}$, such that, $\forall(x,t) \in \Lambda_r(x_0,t_0)$,

$$|v_\varepsilon(x,t) - \sigma_i| \leq \eta_0.$$

Following the notation introduced in Section 5, we set $\gamma_\varepsilon^- = v_\varepsilon\big(x_0 - \frac{r}{2}, t_0\big) - \sigma_i$, $\gamma_\varepsilon^+ = v_\varepsilon\big(x_0 + \frac{r}{2}, t_0\big) - \sigma_i$, and define $U_{\varepsilon,r}$ as the solution to

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}x^2}U_{\varepsilon,r} + \lambda_i\varepsilon^{-2}U_{\varepsilon,r} = 0 \quad \text{for } x \in \Big[-\frac{r}{2}, \frac{r}{2}\Big], \\ &U_\varepsilon(-\frac{r}{2}) = \gamma_\varepsilon^- \quad \text{and} \quad U_\varepsilon(\frac{r}{2}) = \gamma_\varepsilon^+. \end{aligned} \tag{6.1}$$

Set $\Pi \equiv \Pi_0 \cup \Pi_- \cup \Pi_+$ with $\Pi_0 = [-\frac{r}{2}, \frac{r}{2}] \times \{t_0\}$, $\Pi_- = \{-\frac{1}{2}\} \times [t_0, t_0 + r^2]$, $\Pi_+ = \{\frac{r}{2}\} \times [t_0, t_0 + r^2]$,

$$\theta_{\mathrm{bd}}(r,x_0,t_0)(v_\varepsilon) = \sup\{\theta_{bd}^+, \theta_{bd}^-\} \quad \text{and} \quad \theta_{0,r} = \|U_{\varepsilon,r} - v_\varepsilon(x,0)\|_{L^\infty[-\frac{r}{2},\frac{r}{2}]}, \tag{6.2}$$

where $\theta_{\rm bd}^{\pm} = \sup\{|\gamma_\varepsilon^{\pm} - w_\varepsilon(x,t)|,\ (x,t) \in \Pi_{\pm}(r,x_0,t_0)\}$. We assume in this section that $\theta_{\rm bd}$ satisfies the following smallness assumption: For some fixed constant $\varrho > 0$,

$$\theta_{\rm bd}(r,x_0,t_0)(u_\varepsilon) \le \exp\Big(-\frac{\varrho r}{\varepsilon}\Big). \tag{6.3}$$

We assume similarly that

$$|\gamma_\varepsilon^-| + |\gamma_\varepsilon^+| \le \exp\Big(-\frac{\varrho r}{2\varepsilon}\Big). \tag{6.4}$$

Notice in particular that, if (6.3)–(6.4) are satisfied, then we have

$$|v_\varepsilon - \sigma_i| \le \exp\Big(-\frac{\varrho r}{2\varepsilon}\Big) \quad \text{on } \Lambda_r(x_0,t_0). \tag{6.5}$$

The main result of this section is as follows.

Proposition 6.1 *Let* (x_0,t_0) *be in* $\mathbb{R}\times\mathbb{R}^+$, *let* $r \ge \varepsilon$ *be given, and let* v_ε *be a solution to* $(\mathrm{PGL})_\varepsilon$, *such that* (6.3)–(6.4), (5.33) *and* (H_0) *hold. Then, we have for* $(x,t) \in \Lambda_r(x_0,t_0)$,

$$\begin{aligned} &|v_\varepsilon(x,t) - U_{\varepsilon,r}(x)| \\ &\le C\theta_{0,r}\exp\Big(-\frac{\lambda_i(t-t_0)}{\varepsilon^2}\Big) + C\Big[\exp\Big(\frac{\sqrt{\lambda_i}}{\varepsilon}\Big(|x-x_0| - \frac{\varrho r}{\sqrt{\lambda_i}} - \frac{r}{2}\Big)\Big)\Big] \end{aligned} \tag{6.6}$$

and, if $t \ge \varepsilon^2$,

$$\begin{aligned} &|\partial_x(u_\varepsilon(x,t) - U_\varepsilon(x))| \\ &\le \frac{C}{\varepsilon}\theta_{0,r}\exp\Big(-\frac{\lambda_i(t-t_0)}{\varepsilon^2}\Big) + \frac{C}{\varepsilon}\Big[\exp\Big(\frac{\sqrt{\lambda_i}}{\varepsilon}\Big(|x-x_0| - \frac{\varrho r}{\sqrt{\lambda_i}} - \frac{r}{2}\Big)\Big)\Big]. \end{aligned} \tag{6.7}$$

Moreover, we have for every $0 < \varepsilon < 1$,

$$\xi(v_\varepsilon)(x,t) = -\gamma_\varepsilon^+\gamma_\varepsilon^-\exp\Big(-\frac{\sqrt{\lambda_i}r}{\varepsilon}\Big)[1 + \mathcal{R}_{1,\varepsilon}(x,t)] + \mathcal{C}_{1,\varepsilon}(x,t), \tag{6.8}$$

where the error terms satisfies the estimate

$$|\mathcal{R}_{1,\varepsilon}(x,t)| \le C(K_0^2+1)\Big[|\gamma_\varepsilon^+\gamma_\varepsilon^-|^{-1}\exp\Big(\frac{2}{\varepsilon}(\sqrt{\lambda_i}|x-x_0| - \varrho r)\Big) + \exp\Big(-\frac{\sqrt{\lambda_i}r}{\varepsilon}\Big)\Big] \tag{6.9}$$

and

$$|\mathcal{C}_{1,\varepsilon}(x,t)| \le C\theta_{0,r}^2\exp\Big(-\frac{\lambda(t-t_0)}{\varepsilon^2}\Big). \tag{6.10}$$

Proof In view of the scale and translation invariance of the equation $(\mathrm{PGL})_\varepsilon$, we are led to introduce the new parameter $\epsilon = \frac{\varepsilon}{r}$, which satisfies assumption $0 < \epsilon < 1$, to perform the change of variables $x \to \mathrm{x} = \frac{x-x_0}{r}$ and $t \to \mathrm{t} = \frac{t-t_0}{r^2}$, and finally to set

$$\mathrm{v}_\epsilon(\mathrm{x},\mathrm{t}) = v_\varepsilon(r\mathrm{x} + x_0, r^2\mathrm{t} + t_0),$$

so that v_ϵ is now a solution to $(\mathrm{PGL})_\epsilon$, and the original domain of interest $\Lambda_r(x_0,t_0)$ is changed into the standard cylinder Λ. On the other hand, since σ_i is a minimizer for the potential V which is assumed to be smooth, we may expend the potential V as

$$V(u) = \frac{\lambda_i}{2}(u-\sigma_i)^2 + \Phi(u)(u-\sigma_i)^4, \tag{6.11}$$

and its derivative V' near σ_i as

$$V'(u) = \lambda_i(u - \sigma_i) + \varphi(u)(u - \sigma_i)^3, \tag{6.12}$$

where Φ and φ are some smooth functions. Setting $w_\epsilon = \mathrm{v}_\epsilon - \sigma_i$ on Λ we are led to rewrite the equation $(\mathrm{PGL})_\varepsilon$ as

$$\partial_{\mathrm{t}} w_\epsilon - \partial^2_{\mathrm{xx}} w_\epsilon + \epsilon^{-2} c_i(\mathrm{x}, \mathrm{t}) w_\epsilon = 0 \quad \text{on } \Lambda, \tag{6.13}$$

where the function c_i is defined on the cylinder Λ as $c(\mathrm{x}, \mathrm{t}) = \lambda_i - \varsigma_\epsilon(\mathrm{x}, \mathrm{t})$ with

$$\varsigma_\epsilon(\mathrm{x}, \mathrm{t}) = \varphi\left(v_\epsilon(\mathrm{x}, \mathrm{t})\right)[v_\epsilon(\mathrm{x}, \mathrm{t}) - \sigma_i]^2 \quad \text{for } (\mathrm{x}, \mathrm{t}) \in \Lambda.$$

It satisfies therefore, in view of assumption (6.4), the estimate

$$|\varsigma_\varepsilon(\mathrm{x}, \mathrm{t})|| \leq C \exp\left(-\frac{\varrho r}{\varepsilon}\right). \tag{6.14}$$

We are hence in position to apply Proposition 5.1 to the equation (6.13): Estimates (5.38)–(5.39) combined with the inequalities (6.3) and (6.14), then lead directly to (6.6). Turning to (6.9), we write

$$\xi(v_\varepsilon)(x, t) = Q_{\lambda_i}(w_\varepsilon(\mathrm{x}, \mathrm{t})) + \Phi(w_\varepsilon + \sigma_i) w_\varepsilon^4(\mathrm{x}, \mathrm{t}),$$

so that (6.9) is a direct consequence of Lemma 5.6 together with the smallness assumptions (6.3)–(6.4), which lead to (6.5) and allow to bound suitably the term $\Phi(w_\varepsilon + \sigma_i) w_\varepsilon^4(\mathrm{x}, \mathrm{t})$.

Finally, we end the section with a crude estimate, which will also be used in some places.

Lemma 6.1 *Let (x_0, t_0) be in $\mathbb{R} \times \mathbb{R}^+$, let $r \geq \varepsilon$ be given, and let v_ε be a solution to* $(\mathrm{PGL})_\varepsilon$*, such that* (6.3)–(6.4), (5.33) *and* (H_0) *hold. Then, we have for $(x, t) \in \Lambda_r(x_0, t_0)$,*

$$e_\varepsilon(v_\varepsilon(x, t)) \leq \frac{Cr^2}{\varepsilon} M_0 \exp\Big(-\frac{2\lambda_i(t - t_0)}{\varepsilon^2}\Big) + \frac{Cr^2}{\varepsilon}\Big[\exp\Big(\frac{2\sqrt{\lambda_i}}{\varepsilon}\Big(|x - x_0| - \frac{\varrho r}{\sqrt{\lambda_i}} - \frac{r}{2}\Big)\Big)\Big].$$

The proof is a direct consequence of Proposition 6.1 and inequality (5.32).

6.1 Expansions and bounds for ξ assuming $\mathcal{WP}_\varepsilon(\delta, T)$

Appropriate expansion for ξ are the central tool in order to derivate the motion law of the fronts. Throughout this section, given $\delta > 0$, we assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds for some time $T \geq 0$ and given some $\delta > 0$. For times $t \in [T, \mathcal{T}_{\mathrm{sim}}(\delta, T)]$, we consider the intervals $[a_k(t), a_{k+1}(t)]$. Our purpose is to provide some accurate estimates for the discrepancy ξ on these intervals. It turns out actually that our estimates are essentially relevant only for points near the center

$$a_{k+\frac{1}{2}} = \frac{a_k(t) + a_{k+1}(t)}{2}$$

of the interval, provided that the length $d_k(t) = |a_{k+1}(t) - a_k(t)|$ of the interval is not too large compared to δ. For that purpose, we will use in some places the condition

$$\frac{d_k(t)}{\varepsilon} \leq \frac{L(\delta)}{\varepsilon} \equiv \sqrt{\lambda_{\min}}\nu_5 \exp\left(\rho_4 \frac{\delta}{8\varepsilon}\right) \leq \sqrt{\lambda_{\min}}\nu_5 \exp\left(\rho_3 \frac{\delta}{8\varepsilon}\right), \tag{6.15}$$

with the last inequality being a consequence of the inequality $\rho_4 \leq \rho_3$, where we set

$$\nu_5 = \frac{2}{3} \inf \Big\{ \nu_1, \frac{\rho_3}{8\sqrt{\lambda_{\max}}} \Big\}. \tag{6.16}$$

Our next result is central in the derivation of the motion law.

Proposition 6.2 *Let $T \geq 0$ be given, and assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds for some $\delta \geq \alpha_5 \varepsilon$, for some constant $\alpha_5 \geq \alpha_4$. Given any time $T + \varepsilon^2 \leq t \leq \mathcal{T}_{\rm sim}(\delta, T)$ and $k \in J(T)$, such that $\mathcal{WS}_k(t)$ holds, we have, for $x \in [a_k(t), a_{k+1}(t)]$,*

$$\xi(v_\varepsilon)(x,t) = \Gamma^+_{k,\varepsilon}(\{a_i(t)\})[1 + \mathcal{R}_{2,\varepsilon}(x,t)] + C_{2,\varepsilon}(x,t), \tag{6.17}$$

where $\Gamma^+_{k,\varepsilon}(\{a_i(t)\}$ is defined in (1.15) and the error terms satisfy the estimate, for positive constants $K_1 > 0$ and ϱ_6,

$$|\mathcal{R}_{2,\varepsilon}(x,t)| \leq K_1 \Big[\exp\Big(\frac{2\sqrt{\lambda_{j^+(k)}}|x - a_{k+\frac{1}{2}}(t)| - \varrho_6 \delta}{\varepsilon} \Big) + \exp\Big(- \frac{\sqrt{\lambda_{j^+(k)}} d_k(t)}{\varepsilon} \Big) \Big] \tag{6.18}$$

and

$$|\mathcal{C}_{2,\varepsilon}(x,t)| \leq K_1 M_0 \exp\Big(- \frac{\lambda_{j^+(k)}(t-T)}{\varepsilon^2} - \rho_1 \frac{\delta}{\varepsilon} \Big). \tag{6.19}$$

Notice that the previous result yields a precise expansion of the discrepancy provided that the following conditions are met:

(i) The distance between the points $a_k(t)$ and $a_{k+1}(t)$ can be compared to the length δ, i.e., condition $\mathcal{WS}_k(t)$ is met.

(ii) The point x is close to the center $a_{k+\frac{1}{2}}(t)$.

(iii) The time t is not too close to the initial time T.

More precisely, a direct consequence of Proposition 6.2 is as follows. If $\mathcal{WS}_k(t)$ is satisfied and

$$|x - a_{k+\frac{1}{2}}(t)| \leq \frac{\rho_6 \delta}{2\sqrt{\lambda_{\max}}}, \quad t \geq T + \frac{d_k(t)}{2\sqrt{\lambda_{\min}}} \varepsilon,$$

then, we have

$$\xi(v_\varepsilon)(x,t) = \Gamma^+_{k,\varepsilon}(\{a_i(t)\})[1 + \mathcal{R}_{3,\varepsilon}(x,t)], \tag{6.20}$$

where

$$|\mathcal{R}_{3,\varepsilon}(x,t)| \leq \exp\Big(- \frac{\rho_6}{2} \frac{\delta}{\varepsilon} \Big). \tag{6.21}$$

Proof of Propostion 6.2 We first describe the general outline of the proof. We will work on a cylinder of the form

$$\Lambda_k^-(t) = [a_k(t), a_{k+1}(t)] \times [t - \varepsilon \mathfrak{r}, t],$$

where the parameter $\mathfrak{r} > 0$ homogeneous to a length is defined by

$$\mathfrak{r} = \inf \Big\{ \frac{d_k(t)}{\sqrt{\lambda_{j^+(k)}}}, \frac{t-T}{\varepsilon} \Big\} \leq \varepsilon \exp\Big(\rho_3 \frac{\delta}{8\varepsilon} \Big). \tag{6.22}$$

We then divide this cylinder $\Lambda_k^-(t)$ into a region close to the front sets near $a_k(t)$ and $a_{k+1}(t)$, termed here the "inner region", and the rest of the cylinder, termed the "outer" region. In the inner region, we will show that the solution remains close to an optimal profile, whereas in the outer region, we are in position to apply Proposition 6.1, with the main point in the proof being somehow to glue together the inner and the out region.

In order to define the outer region, we argue as in Lemma 4.8, so that we may find some number $\check{\delta} \in [\frac{\nu_5}{2}\delta, \nu_5\delta]$, where ν_5 is defined in (6.16), such that

$$\theta^+_{(a_k(t),t)}(\check{\delta}, \mathbf{r}) + \theta^-_{(a_{k+1}(t),t)}(\check{\delta}, \mathbf{r}) \leq 4\sqrt{\frac{\mathbf{r}}{\nu_5 \varepsilon}} \exp\Big(-\rho_3 \frac{\delta}{2\varepsilon}\Big) \leq \exp\Big(-\rho_3 \frac{\delta}{4\varepsilon}\Big). \tag{6.23}$$

We then define the outer region as

$$\Lambda_{\text{out}}(t) = [a_k(t) + \check{\delta}, a_{k+1}(t) - \check{\delta}] \times [t - \varepsilon\mathbf{r}, t].$$

Adapting the notation of Section 6 to the present framework, we are led to set $w_\varepsilon = v_\varepsilon - \sigma_{j^+(k)}$,

$$\begin{aligned}
&\Pi_- = \{a_k(t) + \check{\delta}\} \times [t - \varepsilon\mathbf{r}, t], \quad \Pi_+ = \{a_{k+1}(t) + \check{\delta}\} \times [t - \varepsilon\mathbf{r}, t],\\
&\gamma_\varepsilon^- = w_\varepsilon(a_k(t - \varepsilon\mathbf{r}) + \widetilde{\delta}, t - \varepsilon\mathbf{r}), \quad \gamma_\varepsilon^+ = w_\varepsilon(a_{k+1}(t - \varepsilon\mathbf{r})) - \widetilde{\delta}, t - \varepsilon\mathbf{r}),\\
&\theta_{\text{bd}} = \sup\{\theta^+_{bd}, \theta^-_{bd}\}, \quad \text{where } \theta^\pm_{\text{bd}} = \sup\{|\gamma^\pm_\varepsilon - w_\varepsilon(x,t)|,\ (x,t) \in \Pi_\pm\},
\end{aligned}$$

and θ_0 accordingly (see (6.2)). Finally, we notice that there exist some time $T \leq t' < t$, such that $|t - t'| \leq c_0\varepsilon\delta$ and $\mathcal{WP}_\varepsilon(\nu_1\delta, t')$ holds. Indeed, if $t - T \geq c_2\varepsilon\delta$, then the existence of t' follows directly from Corollary 4.4. On the other hand, if $t - T \leq c_2\varepsilon\delta$, we simply choose $t' = T$, so that the same conclusion holds. Notice also that we have the inequality

$$|a_k(t) - a_k(t')| + |a_{k+1}(t) - a_{k+1}(t')| \leq \frac{2\varepsilon^2}{\delta} \exp\Big(-\rho_4 \frac{\delta}{\varepsilon}\Big). \tag{6.24}$$

In order to apply Proposition 6.1 to $\Lambda_{\text{out}}(t)$, we need to deduce several estimates for γ_ε^-, γ_ε^+, $\theta_{\text{bd}} \cdots$, which are mainly derived from estimates on the inner region. We divide the remainder of the proof into several steps.

Step 1 Estimates for γ_ε^- and γ_ε^+.

Set $\mathcal{B}^+ = -\dagger_k B^{\dagger_k}_{i(k)}$ and $\mathcal{B}^- = \dagger_{k+1} B^{-\dagger_{k+1}}_{i(k+1)}$. If $\frac{\delta}{\varepsilon}$ is sufficiently large, we have

$$\Big|\gamma_\varepsilon^\pm - \mathcal{B}^\pm \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big)\Big| \leq \exp\Big(-\rho_4 \frac{\delta}{4\varepsilon}\Big) \tag{6.25}$$

and

$$\frac{1}{2}|\mathcal{B}^\pm| \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big) \leq |\gamma_\varepsilon^\pm| \leq 2|\mathcal{B}^\pm| \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big). \tag{6.26}$$

Proof It follows from condition $\mathcal{WP}_\varepsilon(\nu_1\delta, t')$ that for the time t' defined above and for any $0 \leq \delta' \leq \frac{3}{2}\check{\delta}$, we have

$$\Big|w_\varepsilon(a_k(t') \pm \delta', t') - \mathcal{B}^\pm \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\delta'\Big)\Big| \leq K \exp\Big(-\frac{2\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\delta'\Big). \tag{6.27}$$

Indeed, in view of the definition (6.16) of ν_5 and since $\rho_3 \leq \rho_1$, we have

$$\delta' \leq \frac{3}{2}\check{\delta} \leq \frac{3}{2}\nu_5\delta \leq \frac{\rho_3}{8\sqrt{\lambda_{\max}}}\delta \leq \frac{\rho_1}{2\sqrt{\lambda_{\max}}}\delta,$$

so that Lemma 4.2 and the previous remark apply. We turn first to the estimate for γ_ε^+. We choose $\delta' = \check{\delta} + a_k(t) - a_k(t')$, so that $a_k(t') + \delta' = a_k(t) + \check{\delta}$, whereas

$$\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\delta'\Big) = \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big)\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}(a_k(t) - a_k(t'))\Big).$$

Hence, inequality (6.24) yields

$$\Big|\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\delta'\Big) - \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big)\Big| \leq \frac{K\varepsilon}{\delta}\exp\Big(-\Big(\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta} + \rho_4\frac{\delta}{\varepsilon}\Big)\Big). \tag{6.28}$$

Combining (6.27) with (6.28), we deduce that provided $\delta \geq \varepsilon$,

$$\Big|w_\varepsilon(a_k(t) + \check{\delta}, t') - \mathcal{B}_k^+\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big)\Big| \leq K\Big[\exp\Big(-\Big(\frac{2\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta} + \rho_4\frac{\delta}{\varepsilon}\Big)\Big)\Big]. \tag{6.29}$$

On the other hand, it follows from the definition of $\theta^\pm_{(a,t)}(\check{\delta}, \mathfrak{r})$ that

$$\big|w_\varepsilon(a_k(t) + \check{\delta}, t) - w_\varepsilon(a_k(t) + \check{\delta}, t')\big| \leq \theta^+_{(a_k(t),t)}(\check{\delta}, \mathfrak{r}). \tag{6.30}$$

Combining (6.27), (6.29)–(6.30), (6.23) and the fact that $\rho_4 \leq \rho_3$, we derive (6.25) for γ_ε. We derive the corresponding estimate for γ_ε^- using the same argument. For the proof of (6.26), we observe that, as a consequence of our construction of $\check{\delta}$ and the definition (6.16) of ν_5, we have

$$\sqrt{\lambda_{j^+(k)}}\check{\delta} \leq \frac{\rho_4\delta}{8},$$

so that, if $\frac{\delta}{\varepsilon}$ is sufficiently large, then $\frac{\mathcal{B}^\pm}{2}\exp\big(-\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\big) \geq \exp\big(-\rho_4\frac{\delta}{4\varepsilon}\big)$, from which we deduce the conclusion.

Step 2 Estimates for θ_{bd} and θ_0.

We have

$$\theta_{\mathrm{bd}} \leq \exp\Big(-\rho_4\frac{\delta}{4\varepsilon}\Big) \leq \exp\Big(-2\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big), \quad \theta_0 \leq C\exp\Big(-\rho_1\frac{\delta}{2\varepsilon}\Big).$$

Proof of Step 2 The estimate for θ_{bd} is a direct consequence of (6.23) together with the inequality $\rho_4 \leq \rho_3$, whereas the estimate for θ_0 follows directly from Lemma 4.3.

Step 3 Proof of Proposition 6.2 completed.

We are now in position to apply Proposition 6.1 on the cylinder $\Lambda_{\mathrm{out}}(t)$ with

$$r = d_k(t) - 2\check{\delta}, \quad \varrho r = \frac{31}{16}\sqrt{\lambda_{j^+(k)}}\check{\delta},$$

so that assumptions (6.3)–(6.4) and (5.33) are satisfied, with a constant K_0 depending only on the numbers $\mathcal{B}^\pm$, and provided that the ratio $\frac{\delta}{\varepsilon}$ is sufficiently large. We have therefore, for every $0 < \varepsilon < 1$ and every $s \in [t - \varepsilon\mathfrak{r}, t]$,

$$\xi(v_\varepsilon)(x,s) = -\gamma_\varepsilon^+\gamma_\varepsilon^-\exp\Big(-\frac{\sqrt{\lambda^+_{j(k)}}(d_k(t) - 2\widetilde{\delta})}{\varepsilon}\Big)[1 + \widetilde{\mathcal{R}}_{2,\varepsilon}(x,s)] + \mathcal{C}_{2,\varepsilon}(x,s), \tag{6.31}$$

where the error term satisfies the estimates

$$
\begin{aligned}
&|\widetilde{\mathcal{R}}_{2,\varepsilon}(x,s)| \\
&\leq C(K_0^2+1)\Big[|\gamma_\varepsilon^+\gamma_\varepsilon^-|^{-1}\exp\Big(\frac{2}{\varepsilon}\sqrt{\lambda_j}|x-a_{k+\frac{1}{2}}(t)|-\varrho r\Big)+\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}d_k(t)}{\varepsilon}\Big)\Big]
\end{aligned}
\tag{6.32}
$$

and

$$
|\mathcal{C}_{2,\varepsilon}(x,s)| \leq C\exp\Big(-\frac{\lambda^+_{j(k)}((t-\varepsilon\mathfrak{r})-s)}{\varepsilon^2}-\rho_1\frac{\delta}{\varepsilon}\Big). \tag{6.33}
$$

We next use formula (6.31) at time $s=t$, and distinguish two cases. If $t-T>\varepsilon\frac{d_k(t)}{\sqrt{\lambda_{j^+(k)}}}$, then $\mathfrak{r}=\frac{d_k(t)}{\sqrt{\lambda_{j^+(k)}}}$, so that

$$
|\mathcal{C}_{2,\varepsilon}(x,t)| \leq C\exp\Big(-\frac{\sqrt{\lambda_j^+(k)}d_k(t)}{\varepsilon}\Big),
$$

and then, we absorb this term in the second term on the right-hand side of (6.32) at the cost of a larger constant C. Otherwise, that is, if $\mathfrak{r}=\frac{T-t}{\varepsilon}$, then $\mathcal{C}_{2,\varepsilon}(x,t)$ has exactly the form provided in (6.19), and we are done for this part of the error terms. It remain to check that the other error terms have the announced behavior. For that purpose, we first notice that it follows from Step 1 that

$$
\gamma_\varepsilon^+\gamma_\varepsilon^- = \mathcal{B}^+\mathcal{B}^-\exp\Big(-2\frac{\sqrt{\lambda_{j^+(k)}}}{\varepsilon}\check{\delta}\Big)[1+R_3], \tag{6.34}
$$

where $R_3 \leq K\exp\left(-\rho_4\frac{\delta}{4\varepsilon}\right)$, and that

$$
|\gamma_\varepsilon^+\gamma_\varepsilon^-|^{-1}\exp(-2\varrho r) \leq \exp\Big(-\frac{31\sqrt{\lambda_{j^+(k)}}\check{\delta}}{16\varepsilon}\Big) \leq \exp\Big(-\frac{15\sqrt{\lambda_{\min}}\nu_5}{16}\frac{\delta}{\varepsilon}\Big). \tag{6.35}
$$

Combining (6.31) and (6.34)–(6.35), we obtain the desired result by choosing

$$
\rho_6 = \inf\Big\{\rho_5, \frac{\rho_4}{4}, \frac{15\sqrt{\lambda_{\min}}\nu_5}{32}\Big\}.
$$

We also need to handle the case, where the assumption $\mathcal{WS}_k(t)$ is not met. In that direction, we are not able to provide an expansion, but only an upper bound, which turns out to be sufficient for our further analysis.

Proposition 6.3 *Let $T \geq 0$ be given, and assume that $\mathcal{WP}_\varepsilon(\delta,T)$ holds for some $\delta \geq \alpha_6\varepsilon$. Given any time $t \in [T, \mathcal{T}_{\rm sim}(\delta,T)]$, such that inequality $\mathcal{WS}_k(t)$ does not hold, we have for $k=1,\cdots,\ell(t)-1$ and $x \in \left[a_k(t)+\frac{L(\delta)}{2}, a_{k+\frac{1}{2}}(t)+\frac{d_k(t)}{2}\right]$,*

$$
|\xi(v_\varepsilon)(x,t)| \leq K\Big[\exp\Big(-\frac{\lambda_{j^+(k)}(t-T)}{\varepsilon^2}-\rho_1\frac{\delta}{\varepsilon}\Big)+\exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}L(\delta)}{4\varepsilon}\Big)\Big],
$$

with $L(\delta)=\sqrt{\lambda_{\min}}\nu_5\varepsilon\exp(-\rho_4\frac{\delta}{8\varepsilon})$.

The proof is essentially the same as the proof of Proposition 6.2, with the main point being to replace the definition of $\mathfrak{r}$ given in (6.22) by the new choice $\mathfrak{r}=\inf\left\{L(\delta), \frac{t-T}{\varepsilon}\right\}$. We leave the details to the reader. Finally, in some place, we will need another estimate somewhat in the same spirit as Proposition 6.3 provided by the following lemma.

Lemma 6.2 *Let $T \geq 0$ be given, and assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds for some $\delta \geq \alpha_6 \varepsilon$. Given any time $t \in [T, \mathcal{T}_{\rm sim}(\delta, T)]$, such that $t \geq \varepsilon^2$, given $k \in 1, \cdots, \ell(t) - 1$ and $x \in [a_k(t), a_{k+1}(t)]$, we have*

$$|\xi_\varepsilon(x,t)| \leq K\Big[\exp\Big(-\frac{\lambda_{j^+(k)}(t-T)}{\varepsilon^2} - \rho_1\frac{\delta}{\varepsilon}\Big) + \exp\Big(-\frac{\sqrt{\lambda_{j^+(k)}}\gamma(x,t)}{4\varepsilon}\Big)\Big],$$

where $\gamma(x,t) = \inf\{|x - a_k(t)|, |x - a_{k+1}(t)\}$.

The proof is a rather direct consequence of inequality (5.32), therefore we omit it.

7 Motion Law for Fronts

The purpose of this section is to provide the proofs of Theorems 1.1–1.2. To that aim, we will combine the result of the previous section with the motion law for the local energy (1.31) by making use of test function of a special type, which we describe in this section.

7.1 Approximating the points $a_k(t)$ using the energy density

Let $t \geq 0$ and let $k \in J(t)$ be given. Since the expansion for the discrepancy ξ is only accurate near the points $a_{k\pm\frac{1}{2}}$, we are led to introduce intervals $\mathcal{I}_k(t)$ of the following form:

$$\mathcal{I}_k(t) = [a_k^-(t,\delta), a_k^+(t,\delta)] \equiv \Big[\widetilde{a}_{k-\frac{1}{2}}(t) - \frac{\rho_6\delta}{4\sqrt{\lambda_{\max}}}, \widetilde{a}_{k+\frac{1}{2}}(t) + \frac{\rho_6\delta}{4\sqrt{\lambda_{\max}}}\Big], \tag{7.1}$$

where, for the construction of the points $\widetilde{a}_{k\pm\frac{1}{2}}(t)$, we distinguish several cases.

Case 1 None of the conditions $\mathcal{WS}_k(t)$ and $\mathcal{WS}_{k-1}(t)$ holds. In that case, we set

$$\widetilde{a}_{k\pm\frac{1}{2}}(t) = a_k(t) \pm \frac{\ell(\delta)}{2}.$$

Case 2 At least, one of the conditions $\mathcal{WS}_k(t)$ or $\mathcal{WS}_{k-1}(t)$ holds. Without loss of generally, we may assume that $\mathcal{WS}_k$ holds, with the other case being handled in a similar way. Then, we distinguish once more two subcases. If

$$d_k^+(t) \leq 4\sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}d_k^-(t) \quad \text{with } d_k^-(t) = d_{k-1}(t), \tag{7.2}$$

then we set

$$\widetilde{a}_{k+\frac{1}{2}}(t) = a_{k+\frac{1}{2}}(t), \quad \widetilde{a}_{k-\frac{1}{2}}(t) = a_k(t) - 2d_k^+(t).$$

Otherwise, we set $\widetilde{a}_{k\pm\frac{1}{2}}(t) = a_{k\pm\frac{1}{2}}(t)$.

Notice that, if condition (7.2) is met, then $\sqrt{\lambda_{j^+(k)}}d_k^+(t) \leq 4\sqrt{\lambda_{j^-(k)}}d_k^-(t)$. We then construct a test function $\chi \equiv \chi_{k,t}$ having the following properties:

$$\begin{cases} \chi \text{ has compact support in } \mathcal{I}_k(t), \\ \chi(x) = x - a_k(t) \quad \text{on the interval } [\widetilde{a}_{k-\frac{1}{2}}(t), \widetilde{a}_{k+\frac{1}{2}}(t)], \\ |\ddot{\chi}| \leq 48\lambda_{\max}\rho_6^{-2}\delta^{-1}. \end{cases} \tag{7.3}$$

One may check that the set of functions verifying these three conditions is not void. Notice that $\ddot{\chi} = 0$ on the interval $[a_{k-\frac{1}{2}}(t), a_{k+\frac{1}{2}}(t)]$, and hence $\ddot{\chi}$ has support on $\mathcal{V}_k(t) = \mathcal{I}_k(t) \setminus$

$(a_{k-\frac{1}{2}}(t), a_{k+\frac{1}{2}}(t))$. In view of Proposition 6.2, we introduce that the stopping time $\mathfrak{T}_k(\delta, t)$ is defined by

$$\mathfrak{T}_k(\delta,t) = \inf\Big\{\mathcal{T}_{\rm sim}(\delta,t) \geq s \geq t, \text{ such that } |a_j(s) - a_j(t)| \leq \frac{\rho_6\delta}{4\sqrt{\lambda_{\max}}} \\ \text{for } j = k-1, k, k+1\Big\}. \tag{7.4}$$

We consider the energy density $\mathfrak{b}_{k,t}(s) = \int_{\mathbb{R}} \chi_{(k,t)} e_\varepsilon(v_\varepsilon(\cdot, s))\mathrm{d}x$, as well as the related quantity

$$\mathrm{b}_{k,t}(s) = \mathfrak{q}_k^{-1}[\mathfrak{b}_{k,t}(s) + \mathrm{dissip}\,(t, s, \chi_{(k,t)}) - \beta_k^0] + a_k(t),$$

where $\mathfrak{q}_k = \mathfrak{S}_{i(k)}$,

$$\mathrm{dissip}\,(t,s,\chi) = \varepsilon \int_t^s \int_{\mathbb{R}} \chi \Big|\frac{\partial v_\varepsilon}{\partial t}\Big|^2(x,u)\mathrm{d}x\mathrm{d}u, \quad \beta_k^0 = \int_{\mathbb{R}} x e_\varepsilon\Big(\zeta_{i(k)}^{\dagger_k}\Big(\frac{x}{\varepsilon}\Big)\Big)\mathrm{d}x. \tag{7.5}$$

We claim that $\mathrm{b}_{k,t}(s)$ offers a good approximation of $a_k(s)$.

Lemma 7.1 *Let $T \geq 0$ be given, and assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds for some $\delta \geq \alpha_6\varepsilon$. We have, for any $k \in J(T)$ and $s \in [T + \mathrm{c}_2\varepsilon\delta, \mathfrak{T}_k(\delta, T)]$,*

$$|a_k(s) - \mathrm{b}_{k,t}(s)| \leq K\mathfrak{D}_k(t)\exp\Big(-\rho_3\frac{\delta}{2\varepsilon}\Big) \leq K\varepsilon\exp\Big(-\rho_3\frac{\delta}{4\varepsilon}\Big),$$

where we have set $\mathfrak{D}_k(T) = \inf\{L(\delta), d_k^+(T), d_k^-(T)\}$ and for the last inequality at the cost of a possible larger choice of the constant α_6.

Proof Since in view of Proposition 4.4, property $\mathcal{WP}_\varepsilon(\nu_0\delta, s)$ holds for $s \in [T + \mathrm{c}_2\varepsilon\delta\mathfrak{T}_k(\delta, T)]$, we deduce from (1.9)–(1.10) that

$$\Big|\mathfrak{b}_{k,t}(s) - \int_{\mathbb{R}} (x - a_k(t)) e_\varepsilon\Big(\zeta_{i(k)}^{\dagger_k}\Big(\frac{\cdot - a_k(s)}{\varepsilon}\Big)\Big)\mathrm{d}x\Big| \leq KM_0\mathfrak{D}_k(t)\exp\Big(-\rho_1\nu_0\frac{\delta}{\varepsilon}\Big),$$

where $K > 0$ is some constant. On the other hand, we have, going back to (7.5),

$$\int_{\mathbb{R}} (x - a_k(t)) e_\varepsilon\Big(\zeta_{i(k)}^{\dagger_k}\Big(\frac{\cdot - a_k(s)}{\varepsilon}\Big)\Big)\mathrm{d}x = (a_k(s) - a_k(t))\mathfrak{S}_{i(k)} + \beta_k^0.$$

Finally, by Corollary 4.1, we also have $|\mathrm{dissip}\,(t, s, \chi)| \leq C\mathfrak{D}_k(t)\frac{\delta}{\varepsilon}\exp\big(-\rho_3\frac{\delta}{\varepsilon}\big)$, so that the conclusion follows combining the previous inequalities, and imposing additionally that $\rho_3 \leq \rho_1\nu_0$.

7.2 A first motion law for the points $a_k(s)$

Our previous estimates lead to us directly to the following result, which is the building block in the proof of Theorem 1.1.

Proposition 7.1 *Let $T \geq 0$ be given, and assume that $\mathcal{WP}_\varepsilon(\delta, T)$ holds for some $\delta \geq \alpha_6\varepsilon$. Let $t \in [T, \mathcal{T}_{\rm sim}(\delta, T)]$ and consider $k \in J(T)$. If $t \geq \varepsilon^2$, it holds for $s \in [t, \mathcal{T}_{\rm sim}(\delta, T)]$, that*

$$\Big|\varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{b}_{k,t}(s)\Big| \leq K_1\Big[\exp\Big(-\frac{\lambda_{\min}(t-T)}{\varepsilon^2} - \rho_1\frac{\delta}{\varepsilon}\Big) + \exp\Big(-\frac{\sqrt{\lambda_{\min}}\mathfrak{D}_k(t)}{4\varepsilon}\Big)\Big], \tag{7.6}$$

where $\mathfrak{D}_k(t) \equiv \inf\{d_k^+(t), d_k^-(t), L(\delta)\}$, *with* $L(\delta)$ *defined in Proposition* 6.3. *Moreover, if one of the conditions* $\mathcal{WS}_k(t)$ *or* $\mathcal{WS}_{k-1}(t)$ *holds, and*

$$t \geq T_k^+ = T_k^+(T) \equiv T + \frac{\inf\{d_k^+(t), d_k^-(t)\}}{2\sqrt{\lambda_{\min}}}\varepsilon, \tag{7.7}$$

then, we have for $s \in [t, \mathfrak{T}_k(t,\delta)]$, *where* $\mathfrak{T}_k$ *is defined in* (7.4), *the differential equation*

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{b}_{k,t}(s) = -\mathfrak{q}_k^{-1} \sum_{\dagger \in \{+,-\}} \Gamma_{k,\varepsilon}^{\dagger}(\{b_{i,t}(s)\})[1 + \mathcal{C}_k^{\dagger}(s)], \tag{7.8}$$

where the error term satisfies the estimate $|\mathcal{C}_k^{\dagger}(s)| \leq K \exp\big(-\frac{\rho_6}{2}\frac{\delta}{\varepsilon}\big)$.

Proof Inequality (7.6) is a rather direct consequence of Proposition 6.3, the definition of our test function, and the motion law for the local energy (1.31), which yields

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{b}_{k,t}(s) = \int_{\mathcal{V}_k^-(t) \cup \mathcal{V}_k^+ t)} \mathcal{F}_S(s, \chi, v_\varepsilon)\mathrm{d}x = \varepsilon^{-1} \int_{\mathbb{R}\times\{t\}} \xi(v(\cdot,t))\ddot{\chi}\mathrm{d}x. \tag{7.9}$$

where $\mathcal{V}_k^{\pm}(t) = \big[a_{k\pm\frac{1}{2}}(t), a_{k\pm\frac{1}{2}}(t) + \frac{\rho_6\delta}{4\sqrt{\lambda_{\max}}}\big] \subset \big[a_{k\pm\frac{1}{2}}(s) - \frac{\rho_6\delta}{2\sqrt{\lambda_{\max}}}, a_{k\pm\frac{1}{2}}(s) + \frac{\rho_6\delta}{2\sqrt{\lambda_{\max}}}\big]$, with the inclusion being a consequence of the definition (7.4) of the stopping time $\mathfrak{T}_k$.

Turning to (7.8) we may assume that, for instance $\mathcal{WS}_k(t)(\delta)$ holds, that is, we are either in Case 1 or Case 2 of the definition (7.1) of $\mathcal{I}_k$. We have for $s \in [t, \mathfrak{T}_k(\rho,t)]$,

$$\int_{a_k(t)}^{+\infty} \mathcal{F}_S(s,\chi,v_\varepsilon)\mathrm{d}x = \Gamma_{k,\varepsilon}^+(\{a_i(s)\}) \int_{\mathcal{V}_k^+(t)} \ddot{\chi}[1 + \mathcal{R}_{3,\varepsilon}(x,s)]\mathrm{d}x. \tag{7.10}$$

Next we remark that, in view of the properties of the χ, we have

$$\int_{\mathcal{V}_k^+(t)} \ddot{\chi}[1 + \mathcal{R}_{3,\varepsilon}(x,s)]\mathrm{d}x = -\dot{\chi}(a_{k+\frac{1}{2}}(t)) + \int_{\mathcal{V}_k^+(t)} \ddot{\chi}(x)\mathcal{R}_{3,\varepsilon}(x,s)\mathrm{d}x.$$

Moreover, in view of the construction of χ, we have $\dot{\chi}(a_{k+\frac{1}{2}}(t)) = 1$. Since we assume that $\mathcal{WS}_k(t)$ holds, we may invoke on the interval $\mathcal{V}_k^+(t)$, the bound (6.21), so that finally our computation yields

$$\int_{a_k(t)}^{+\infty} \mathcal{F}_S(s,\chi,v_\varepsilon)\mathrm{d}x = \Gamma_{k,\varepsilon}^+(\{a_i(s)\})[1 + \mathcal{C}_{(0,k)}k^+(s)]$$

with the estimate $|\mathcal{C}_{(0,k)}^+(s)| \leq K \exp\big(-\frac{\rho_6}{2}\frac{\delta}{\varepsilon}\big)$. In view of Lemma 7.1, we may write

$$\Gamma_{k,\varepsilon}^+(\{a_i(s)\})[1 + \mathcal{C}_{(0,k)}k^+(s)] = \Gamma_{k,\varepsilon}^+(\{\mathrm{b_{i,t}(s)}\})[1 + \mathcal{C}_{\mathrm{0,k}}^+(\mathrm{s})]$$

with the estimate $|\mathcal{C}_k^+(s)| \leq K \exp\big(-\frac{\rho_6}{2}\frac{\delta}{\varepsilon}\big)$. Similar estimates hold for the integral $\int_{-\infty}^{a_k(t)} \mathcal{F}_S\mathrm{d}x$, if $\mathcal{WS}_{k-1}$ holds, which lead to equation (7.8) in that case. Otherwise, we are in Case 2 of the definition (7.1) of $\mathcal{I}_k$, and then $\int_{-\infty}^{a_k(t)} \mathcal{F}_S\mathrm{d}x$ turns out to be lower order compared to $\int_{a_k(t)}^{+\infty} \mathcal{F}_S(s,\chi,v_\varepsilon)\mathrm{d}x$, so that (7.8) holds likewise.

We complete this section with a lower bound for $\mathfrak{T}(t,\delta)$.

Lemma 7.2 *Under the assumptions of Proposition* 7.1*, we have the lower bound*

$$\mathfrak{T}(t,\delta) \equiv \inf_{k\in J(t)} \mathfrak{T}_k(t,\delta) \geq \mathcal{T}_{\mathrm{ref}}(t,\mathfrak{d}_a(t)) \equiv \frac{\varepsilon^2}{2\mathcal{S}_2}\exp\Big(\frac{\mathfrak{d}_a(t)}{\varepsilon}\Big)+t,$$

where the constant $\mathcal{S}_2$ *is defined in Proposition* 2.2*.*

Proof We first observe, combining the definition (7.4) of $\mathfrak{T}(t,\delta)$ and the result of Lemma 7.1, that for some $k_0 \in J(t)$,

$$|b_{k_0}(\mathfrak{T}(t,\delta)) - b_{k_0}(t)| \geq \frac{\rho_6\delta}{6\sqrt{\lambda_{\max}}}.$$

We then invoke several properties of the differential equation (1.16) presented in Section 2[11]. First, in view of Proposition 2.1, we have, for any $k \in J(t)$,

$$|b_{k_0}(\mathfrak{T}(t,\delta)) - b_{k_0}(t)| \leq \mathcal{S}_0|\mathfrak{d}_b(\mathfrak{T}(t,\delta)) - \mathfrak{d}_b(t)| + \mathcal{S}_1\varepsilon,$$

so that, if α_6 is chosen sufficiently large, we obtain, combining the two previous inequalities,

$$|\mathfrak{d}_b(\mathfrak{T}(t,\delta)) - \mathfrak{d}_b(t)| \geq \frac{\rho_6\delta}{8\mathcal{S}_0\sqrt{\lambda_{\max}}}.$$

We finally invoke Proposition 2.2 to deduce that

$$-\log\Big[1-\varepsilon^{-2}\mathcal{S}_2[\mathfrak{T}(t,\delta)-t]\exp\Big(-\frac{\mathfrak{d}_a(t)}{\varepsilon}\Big)\Big] \geq \frac{\rho_6\delta}{8\mathcal{S}_0\varepsilon\sqrt{\lambda_{\max}}} \geq \frac{\rho_6\alpha_6}{8\mathcal{S}_0\sqrt{\lambda_{\max}}},$$

and the conclusion follows, choosing possibly α_6 sufficiently large.

We deduce from the previous result.

Corollary 7.1 *Let* $T \geq 0$ *be given, and assume that* $\mathcal{WP}_\varepsilon(\delta,T)$ *holds for some* $\delta \geq \alpha_7\varepsilon$*, where* $\alpha_7 \geq \alpha_6$ *is some constant. Then, we have for any times* $\inf\{\varepsilon^2,t\} \leq s_1 \leq s_2 \leq \mathfrak{T}(t,\delta)$*,*

$$|\mathrm{b}_{k,t}(s_1)-\mathrm{b}_{k,t}(s_2)| \leq K\Big[\varepsilon\exp\Big(-\rho_1\frac{\delta}{\varepsilon}\Big)+\frac{|s_2-s_1|}{\varepsilon}\exp\Big(-\frac{\sqrt{\lambda_{\min}}\mathfrak{D}_k(t)}{4\varepsilon}\Big)\Big]. \tag{7.11}$$

If moreover we have $\inf\{\varepsilon^2,t\} \leq s_1 \leq s_2 \leq t+\varepsilon^2\exp\big(\frac{\mathfrak{d}_a(T)}{9\varepsilon}\big)$*, then we have that for a constant* $0<\rho_7<\rho_1$*, the following holds:*

$$\begin{cases} |\mathrm{b}_{k,t}(s_1)-\mathrm{b}_{k,t}(s_2)| \leq \varepsilon\exp\Big(-\rho_7\dfrac{\delta}{\varepsilon}\Big), \\ |a_k(s_1)-a_k(s_2)| \leq \varepsilon\exp\Big(-\rho_7\dfrac{\delta}{\varepsilon}\Big). \end{cases} \tag{7.12}$$

If $\mathcal{D}_k(t) = L(\delta)$*, then* (7.11) *holds for all* $\inf\{\varepsilon^2,t\} \leq s_1 \leq s_2 \leq \mathfrak{T}(t,\delta)$*.*

Proof Inequality (7.11) is derived integrating inequality (7.6) in time. The first inequality in (7.12) is then derived immediately, noticing that $\mathfrak{d}_a(T) \leq 2\mathfrak{d}_a(t) \leq 2\sqrt{\lambda_{\min}}\mathfrak{D}_k(t)$. Finally, the second inequality of (7.12) follows combining estimate (4.29) of Proposition 4.4, Lemma 7.1 and (7.12) with possibly a judicious tuning of the constants α_7 and ρ_7. The last statement is proven similarly.

[11] which are actually independent of the present discussion.

7.3 Proof of Theorem 1.1

Step 1 Defining the time $\mathfrak{T}(T)$, proofs of (1.17) and assertions (i)–(ii).

We first consider the stopping time $\mathfrak{T}(T,\delta)$ defined in Lemma 7.2. We then choose the constant $\mathcal{S}_*$, such that $\mathcal{S}_* = \mathcal{S}_2$, so that we are immediately led to the inequality

$$\mathfrak{T}(T,\delta) \geq \mathcal{T}_{\mathrm{ref}}(T, \mathfrak{d}_a(T)), \tag{7.13}$$

where $\mathcal{T}_{\mathrm{ref}}$ is defined in (1.17). On the other hand, in view of definitions (4.26) and (7.4), we also have the inequality $\mathfrak{T}(T,\delta) \leq \mathcal{T}_{\mathrm{sim}}(T,\delta)$. Imposing furthermore that the constant α_* in the statement of Theorem 1.1 satisfies the condition $\alpha_* \geq \alpha_4$, we are in position to apply Proposition 4.4, which yields that $\mathcal{WP}_\varepsilon(\nu_0\delta, t)$ holds for any $t \in [T + \mathrm{c}_2\varepsilon\delta, \mathfrak{T}(T,\delta)]$ and that the points $\{a_k(t)\}_{k\in J(T)}$ are well-defined for $t \in [T, T + \mathrm{c}_2\varepsilon\delta]$, where the constants c_2 and ν_0 are defined in Proposition 4.4.

We next introduce a new length scale $\widetilde{\delta}$ defined by

$$\widetilde{\delta} = \delta \ \text{ if } \frac{\mathfrak{d}_a(T)}{22\rho_1} \leq 2\delta \quad \text{and} \quad \widetilde{\delta} = \frac{\mathfrak{d}_a(T)}{22\rho_1} \ \text{ otherwise.} \tag{7.14}$$

In order to define $\mathfrak{T}$, we distinguish two cases.

Case 1 $\widetilde{\delta} = \delta$.

In this case, we set

$$\mathfrak{T}(T) = \mathfrak{T}(T,\delta).$$

With this choice, (1.17) follows from (7.13), whereas as already mentioned, Proposition 4.4 shows that $\mathcal{WP}_\varepsilon(\nu_0\delta, t)$ holds for any $t \in [T + \mathrm{c}_2\varepsilon\delta, \mathfrak{T}]$. Since in the case considered here, we have $\mathfrak{d}_a(T) \leq 44\rho_1\delta$, it follows that property $\mathcal{WP}_\varepsilon(\nu_*\mathfrak{d}_a(T), t)$ holds for any $t \in [T + \mathrm{c}_2\varepsilon\delta, \mathfrak{T}]$, provided that the constant ν_* satisfies the conditions $\nu_* \leq \frac{\nu_0}{44\rho_1}$ and $\alpha_* \geq 44\rho_1\alpha_4$. We impose also $\mathrm{c}_* = \mathrm{c}_2$, and we verify that (1.17) as well as assertions (i)–(ii) have been established in the case considered here.

Case 2 $\widetilde{\delta} = \frac{\mathfrak{d}_a(T)}{22\rho_1} \geq 2\delta$.

We introduce first ΔT and $\widetilde{T}_{\mathrm{trans}}$ defined by

$$\Delta T \equiv \mathrm{c}_0{}^2\varepsilon^2 M_0 \exp\Big(\frac{\mathfrak{d}_a(T)}{11\varepsilon}\Big), \quad \widetilde{T}_{\mathrm{trans}} = T + \frac{\varepsilon^2}{2}\exp\Big(\frac{\mathfrak{d}_a(T)}{10\varepsilon}\Big) < T_{\mathrm{trans}}.$$

It follows from Lemma 4.6 and (4.8) that there exists some time $T_{\mathrm{reg}} \in [T, T + \Delta T]$, such that $\mathcal{WP}_\varepsilon(T_{\mathrm{reg}}, \widetilde{\delta})$ holds. We define $\mathfrak{T}$ as

$$\mathfrak{T}(T) = \mathfrak{T}(T_{\mathrm{reg}}, \widetilde{\delta}) = \mathfrak{T}\Big(T_{\mathrm{reg}}, \frac{\mathfrak{d}_a(T)}{22\rho_1}\Big) \geq \mathfrak{T}(T,\delta). \tag{7.15}$$

First notice that $\Delta T + \mathrm{c}_2\varepsilon\widetilde{\delta} \leq T - \widetilde{T}_{\mathrm{trans}} = \frac{\varepsilon^2}{2}\exp\big(\frac{\mathfrak{d}_a(T)}{10\varepsilon}\big)$, provided $\delta \geq \alpha_8\varepsilon$, where α_8 is some positive constant. Hence, imposing the additional condition $\alpha_* \geq \alpha_7$, we are led to $T + \Delta T + \mathrm{c}_2\varepsilon\widetilde{\delta} \in [T, \widetilde{T}_{\mathrm{trans}}]$, and hence $T_{\mathrm{reg}} \in [T, \widetilde{T}_{\mathrm{trans}}]$. We claim that

$$\mathfrak{T}(T) \geq \mathfrak{T}(T,\delta). \tag{7.16}$$

Indeed, in the case considered here, we have $\widetilde{\delta} \geq 2\delta$, so that $\mathfrak{T} = \mathfrak{T}(T_{\rm reg}, \widetilde{\delta}) \geq \mathfrak{T}(T_{\rm reg}, 2\delta)$. On the other hand, by Corollary 7.1 and (7.12), we have, for any $k \in J(T)$,

$$|a_k(t) - a_k(\widetilde{T}_{\rm trans})| \leq \varepsilon \exp\Big(-\rho_7 \frac{\delta}{\varepsilon}\Big) \leq \varepsilon \exp(-\alpha_8).$$

Hence, in view of the definition of $\mathfrak{T}$, we deduce $\mathfrak{T}(T_{\rm reg}, 2\delta) \geq \mathfrak{T}(T, \delta)$, provided that α_8 is chosen sufficiently large, and the claim (7.16) follows. Then this establishes inequality (1.17) in Case 2.

In order to establish assertion (ii), we invoke again Proposition 4.4. It yields that for any $t \in [T_{\rm reg} + c_2\varepsilon\widetilde{\delta}, \mathfrak{T}]$, and hence any $t \in [\widetilde{T}_{\rm trans}, \mathfrak{T}]$, property $\mathcal{WP}_\varepsilon(t, \nu_0\widetilde{\delta})$, i.e., property $\mathcal{WP}_\varepsilon(t, \frac{\nu_0}{22\rho_1}\boldsymbol{\delta}_a(T))$ holds. Hence, property $\mathcal{WP}_\varepsilon(t, \nu_*\boldsymbol{\delta}_a(T))$ holds for any $t \in [T_{\rm trans}, \mathfrak{T}]$, provided that ν_* is chosen so that $\nu_* \leq \frac{\nu_0}{22\rho_1}$. This establishes assertion (ii) whereas assertion (i) follows from the fact that $T_{\rm trans} \leq \mathfrak{T}(T, \delta)$. In view of the previous discussion, we are now in position to fix the value of the constant ν_* as

$$\nu_* = \frac{\nu_0}{44\rho_1}. \tag{7.17}$$

Notice that a number of our other constants have been determined so far, namely, besides ν_* also $\mathcal{S}_*$ and c_*. We however still have left open the choice for α_* and ρ_*.

Step 2 Defining the points $b_k(t)$, proof of assertion (iv).

In order to define the points $\{b_k(t)\}_{k\in J(T)}$, we distinguish two cases.

Case 1 $\mathcal{D}_k(T) < L(\delta)$.

In this case, one, at least of the conditions $\mathcal{WS}_k(t)$ or $\mathcal{WS}_{k-1}(t)$ holds. We then set

$$b_k(t) = b_{k,\widetilde{T}_{\rm trans}}(t) \quad \text{for } t \in [T_{\rm trans}, \mathfrak{T}], \tag{7.18}$$

and define the family $\{b_k\}_{k\in J(T)}$ on $[T, T_{\rm trans}]$ as the unique solution to the system of ordinary differential equations (1.16), with initial datum at time $T_{\rm trans}$ given by

$$b_k(T_{\rm trans}) = b_{k,\widetilde{T}_{\rm trans}}(T_{\rm trans}) \tag{7.19}$$

with the coefficients $\mathcal{C}_k^\pm$ taken as $\mathcal{C}_k^\pm(t) = \mathcal{C}_k \pm (T_{\rm trans})$ for any $t \in [T, T_{\rm trans}]$. Since the two definitions and the desired estimates are somewhat different, we handle the intervals $[T, T_{\rm trans}]$ and $[T_{\rm trans}, \mathfrak{T}]$ separately.

For the interval $[T_{\rm trans}, \mathfrak{T}]$, in view of the choice (7.18) of the function b_k, the statement of assertion (iv) on the time interval $[T_{\rm trans}, \mathfrak{T}]$ is essentially a consequence of Proposition 7.1 and Lemma 7.1. Indeed, condition (7.7) is clearly satisfied for $t \geq T_{\rm trans}$, since, in view of our constructions, we have for any $k \in J(T)$, $T_{\rm trans} \geq T_k^+(\widetilde{T}_{\rm trans})$, provided $\delta \geq \alpha_*\varepsilon$, and that the constant α_* is chosen sufficiently large. On the other hand, we know, thanks to assertion (ii) that $\mathcal{WP}_\varepsilon(2\nu_*\boldsymbol{\delta}_a(T), \widetilde{T}_{\rm trans})$ is satisfied. It follows that the functions b_k are solutions to a system of the form (1.16) on the interval $[T_{\rm trans}, \mathfrak{T}]$, and that estimate (1.19) holds for the whole interval $[t, \mathfrak{T}]$, provided that we choose

$$\rho_* \leq \frac{\rho_6\nu_*}{2}. \tag{7.20}$$

Turning to (1.20) for the interval $[T_{\rm trans}, \mathfrak{T}]$, we invoke Lemma 7.1 at time $\widetilde{T}_{\rm trans}$, since, as mentioned $\mathcal{WP}_\varepsilon(\nu_* \mathfrak{d}_a(T), \widetilde{T}_{\rm trans})$ holds, where ν_* is fixed in (7.17). It yields for $s \in [\widetilde{T}_{\rm trans} + {\rm c}_2\varepsilon\mathfrak{d}_a(T), \mathfrak{T}]$ that

$$|a_k(s) - {\rm b}_{k,\widetilde{T}_{\rm trans}}(t)| \leq K\varepsilon \exp\Big(- \rho_3\nu_* \frac{\mathfrak{d}_a(T)}{\varepsilon}\Big) \leq \varepsilon \exp\Big(- \rho_* \frac{\mathfrak{d}_a(T)}{\varepsilon}\Big), \tag{7.21}$$

where the last inequality holds, if we impose additionally

$$\rho_* \leq \nu_3 \nu_*, \tag{7.22}$$

and provided that α_* is chosen sufficiently large. Since $\widetilde{T}_{\rm trans} + {\rm c}_2\varepsilon\mathfrak{d}_a(T) \leq T_{\rm trans}$, provided $\delta \geq \alpha_8\varepsilon$, inequality (7.21) holds in particular for $s \in [T_{\rm trans}, \mathfrak{T}]$. Hence inequality (7.21) combined with the choice (7.18) of the functions ${\rm b}_k$ leads directly to (1.20) for the interval $[T_{\rm trans}, \mathfrak{T}]$, and establishes assertion (iv) on the interval $[T_{\rm trans}, \mathfrak{T}]$.

Turning to the interval $[T, T_{\rm trans}]$, we notice that it follows directly from our definition (7.19) that the function ${\rm b}_k$ is a solution to a differential equation of the desired form with the desired estimate (1.19) for the coefficients. It remains to establish (1.20) for the interval $[T, T_{\rm trans}]$. For that purpose, we relay on several distinct observations. First, it follows from Corollary 7.1, inequality (7.12) that

$$|a_k(t) - a_k(T_{\rm trans})| \leq \varepsilon \exp\Big(- \rho_7 \frac{\delta}{\varepsilon}\Big) \quad \text{for } t \in [T, T_{\rm trans}], \tag{7.23}$$

so that $\mathfrak{d}_b(T_{\rm trans}) = \mathfrak{d}_a(T_{\rm trans}) \leq \frac{9\mathfrak{d}_a(T)}{10}$, provided that α_* is choose sufficiently large.

Next, in view of the equation (1.16) for $\{{\rm b}_k\}_{k\in J(T)}$, we may apply Lemma 2.2 and Remark 2.1 to assert that

$$|\mathfrak{d}_b(T_{\rm trans}) - \mathfrak{d}_b(t)| \leq K\varepsilon \exp\Big(- \frac{\mathfrak{d}_a(T_{\rm trans})}{\varepsilon} + \frac{\mathfrak{d}_a(T)}{10\varepsilon}\Big) \leq \varepsilon \exp\Big(- \frac{\mathfrak{d}_a(T)}{5\varepsilon}\Big) \quad \text{for } t \in [T, T_{\rm trans}],$$

so that

$$\mathfrak{d}_b(t) \geq \frac{4\mathfrak{d}_a(T)}{5\varepsilon} \quad \text{for } t \in [T, T_{\rm trans}],$$

provided that α_* is chosen sufficiently large. Hence we deduce integrating inequality (2.3) between T and $T_{\rm trans}$, we are led to

$$|{\rm b}_k(s) - {\rm b}_k(T_{\rm trans})| \leq \varepsilon \exp\Big(- \frac{3\mathfrak{d}_a(T)}{5\varepsilon}\Big), \quad t \in [T, T_{\rm trans}]. \tag{7.24}$$

Combining (7.23)–(7.24) and (7.21) for $t = T_{\rm trans}$, we derive (1.20) on the interval $[T, T_{\rm trans}]$, if we impose, besides (7.20) and (7.22) the conditions $\rho_* \leq \rho_7$ and $\rho_* \leq \frac{3}{5}$.

Case 2 $\mathcal{D}_k(T) \geq L(\delta)$

In this case, define the family $\{{\rm b}_k\}_{k\in J(T)}$ on $[T, \mathfrak{T}]$ as the unique solution to the system of ordinary differential equations (1.16), with initial datum at time $T_{\rm trans}$ given

$${\rm b}_k(T_{\rm trans}) = a_k(T_{\rm trans}) \tag{7.25}$$

and with the coefficients $\mathcal{C}_k^\pm$ taken as $\mathcal{C}_k^\pm(t) = \mathcal{C}_k^\pm(T_{\rm trans})$ for any $t \in [T, \mathfrak{T}]$.

One verifies with the same argument as above that with this choice of function ${\rm b}_k$ inequalities (1.19) are automatically satisfied. For inequality (1.20), we apply the last statement in Corollary

7.1 and inequality (4.29) in Proposition 4.4: Since properties $\mathcal{WP}_\varepsilon(\delta, T)$ and $\mathcal{WP}_\varepsilon(\nu_* \mathfrak{d}_a(T))$ hold,

$$\begin{cases} |a_k(t) - a_k(T)| \le \varepsilon \exp\Big(- \rho_7 \dfrac{\delta}{\varepsilon}\Big) \quad \text{for } t \in [t, T_{\text{trans}}], \\ |a_k(t) - a_k(T_{\text{trans}})| \le \varepsilon \exp\Big(- \rho_7 \nu_* \dfrac{\mathfrak{d}_a(T)}{\varepsilon}\Big) \quad \text{for } t \in [T_{\text{trans}}, \mathfrak{T}]. \end{cases} \tag{7.26}$$

Using the same arguments as for (7.24) but now on the whole interval $[t, \mathfrak{T}]$, we obtain

$$|\mathrm{b}_k(s) - \mathrm{b}_k(T_{\text{trans}})| \le \varepsilon \exp\Big(- \frac{\mathfrak{d}_a(T)}{\varepsilon}\Big), \quad t \in [T, \mathfrak{T}]. \tag{7.27}$$

Combining (7.26), (7.27) and the definition (7.25), we derive the desired conclusion (1.20) in the case considered.

Step 3 Proof of assertion (iii).

The stopping time $\mathfrak{T}$ is defined in Step 1, where we distinguish two cases. We provide here a proof in the second case, the proof in the first case being readily the same (and even possibly a little simpler). In view of (7.23), we first observe that, we have, for any $k \in J(T)$, and if $\delta \ge \alpha_*$,

$$|a_k(T) - a_k(T_{\text{reg}})| \le 2\varepsilon \exp\Big(- \rho_7 \frac{\delta}{\varepsilon}\Big) \le \frac{2}{\alpha_*} \exp(-\rho_7 \alpha_*). \tag{7.28}$$

On the other hand, in view of the definition (7.15) of $\mathfrak{T}$, there exists some $k_0 \in J(T)$, such that

$$|a_{k_0}(\mathfrak{T}) - a_{k_0}(T_{\text{reg}})| \ge \frac{\rho_6}{88\rho_1\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T). \tag{7.29}$$

Combining (7.28) with (7.29), we are hence led to

$$|a_{k_0}(\mathfrak{T}) - a_{k_0}(T)| \ge \frac{\rho_6}{89\rho_1\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T),$$

provide that the constant α_* is chosen sufficiently large. Combining this inequality with (1.20), we deduce that

$$|\mathrm{b}_{k_0}(\mathfrak{T}) - \mathrm{b}_{k_0}(T)| \ge \frac{\rho_6}{90\rho_1\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T),$$

provided once more that the constant α_* is chosen sufficiently large. Since $\{\mathrm{b}_k\}_{k \in J(T)}$ is a solution to the differential equation (1.16), we may invoke Proposition 2.1 to assert that

$$|\mathfrak{d}_b(\mathfrak{T}) - \mathfrak{d}_b(T)| \ge \frac{\rho_6}{90\rho_1\mathcal{S}_0\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T) - \frac{\mathcal{S}_1}{\mathcal{S}_0}\varepsilon \ge \frac{\rho_6}{91\rho_1\mathcal{S}_0\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T), \tag{7.30}$$

provided again that the constant α_* is chosen sufficiently large. Invoking once more (1.20), we finally deduce

$$|\mathfrak{d}_a(\mathfrak{T}) - \mathfrak{d}_a(T)| \ge \frac{\rho_6}{92\rho_1\mathcal{S}_0\sqrt{\lambda_{\max}}} \mathfrak{d}_a(T),$$

which yields the desired result, provided that ρ_* is chosen sufficiently small.

8 Collisions

The proof of Theorem 1.2 relies the results in Theorem 1.1, combined with various properties of solutions to the differential equation (1.16). We present next the other main observations which lead to the proof of Theorem 1.2 as separate subsections.

8.1 Comparing with the differential equation (1.16)

We first notice that, under the assumptions of Theorem 1.1, a rather direct consequence of inequality (1.20) is that, for any $T \leq t \leq \mathfrak{T}(T)$ (resp. $T_{\rm trans} \leq T \leq \mathfrak{T}(T)$),

$$|\mathfrak{d}_a^{\pm}(t) - \mathfrak{d}_b^{\pm}(t)| \leq \varepsilon \exp\Big(- \rho_* \frac{\delta}{\varepsilon}\Big) \quad \Big[\text{resp. } |\mathfrak{d}_a^{\pm}(t) - \mathfrak{d}_b^{\pm}(t)| \leq \varepsilon \exp\Big(- \rho_* \frac{\mathfrak{d}_a(T)}{\varepsilon}\Big)\Big], \tag{8.1}$$

where the subscript b refers to the solution $\{\mathrm{b}_k\}_{k\in J(T)}$ to (1.16) described in Theorem 1.1. Hence, we may use the properties of the differential equation (1.16) presented in Section 2 to derive related results for the partial differential equation. For instance, applying Proposition 2.3 to $\{\mathrm{b}_k\}_{k\in J(T)}$, we are led, for $t \in [T, \mathfrak{T})]$ and $\delta \geq \alpha_* \varepsilon$, to the estimates

$$\begin{cases} \dfrac{\mathfrak{d}_a^+(t) - \mathfrak{d}_a^+(T)}{\varepsilon} \geq \log\Big[1 + \Lambda_0 \varepsilon^{-2}(t-T)\exp\Big(-\dfrac{\mathfrak{d}_a^+(T)}{\varepsilon}\Big)\Big] - \Lambda_1, \\ \dfrac{\mathfrak{d}_a^-(t) - \mathfrak{d}_a^-(T)}{\varepsilon} \leq \log\Big[1 - \Lambda_0 \varepsilon^{-2}(t-T)\exp\Big(-\dfrac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big)\Big] + \Lambda_1, \end{cases} \tag{8.2}$$

where $\Lambda_0 > 0$ and $\Lambda_1 > 0$ are two constant depending only on the constants in Proposition 2.3. As an immediate consequence of (8.2), we obtain for $t \in [T, \mathfrak{T})]$,

$$\mathfrak{d}_a^+(t) \geq \mathfrak{d}_a^+(T) - \gamma\varepsilon, \quad \mathfrak{d}_a^-(t) \leq \mathfrak{d}_a^-(T) + \gamma\varepsilon, \tag{8.3}$$

where $\gamma > 0$ is some constant. We next present a few observations and constructions which enter into the proof.

8.2 The stopping time $\mathfrak{T}_1$

We introduce a new stopping time $\mathfrak{T}_1$ defined by

$$\mathfrak{T}_1(T) = \inf\Big\{ s \in [T, \mathfrak{T}(T)], \text{ s.t. } |\mathfrak{d}_a(s) - \mathfrak{d}_a(T)| \geq \frac{\rho_*}{4}\mathfrak{d}_a(T)\Big\}, \tag{8.4}$$

so that $\mathfrak{T}_1(T) < \mathfrak{T}(T)$. Since the comparison between the partial differential equation and the ordinary differential equations holds only on a bounded interval of time, we are led to introduce an iterative construction in order to track more accurately the solution. More precisely, we construct inductively and whenever this as a meaning for $j \in N$,

$$\mathfrak{T}_{j+1}(T) = \mathfrak{T}_1(\mathfrak{T}_j(T)) = \mathfrak{T}_j(\mathfrak{T}_1(T)) \tag{8.5}$$

with the convention $\mathfrak{T}_0(T) = T$. Upper bounds for $\mathfrak{T}_1$, expressed in terms of $\mathfrak{d}_a(T)$ will be needed for our proofs. A first one is derived from (8.2), which yields

$$\mathfrak{T}_1(T) < \mathfrak{T}(T) \leq \mathfrak{T}^-_{\rm ref}(T) \equiv \Lambda_0^{-1}\varepsilon^2 \exp\Big(-\frac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big) + T. \tag{8.6}$$

Lemma 8.1 *Assume that property $\mathcal{WP}_\varepsilon(\delta, T)$ holds, and that $\delta \geq \beta_2\varepsilon$, where $\beta_2 \geq \beta_1$ is some constant. Assume that $\mathfrak{d}_a^+(T) \leq \mathfrak{d}_a^-(T)$. Then we have*

$$\mathfrak{T}_1(T) \leq \mathfrak{T}^+_{\rm ref}(T) \equiv \Lambda_2 \exp(-\Lambda_1))\Lambda_0^{-1}\varepsilon^2 \exp\Big(-\frac{\mathfrak{d}_a^+(T)}{\varepsilon}\Big(1 + \frac{\rho_*}{4}\Big)\Big) + T, \tag{8.7}$$

where $\Lambda_2 = \exp(-\Lambda_1)\Lambda_0^{-1}$. If furthermore,

$$\mathfrak{d}_a^+(T)\Big(1 + \frac{3\rho_*}{8}\Big) \leq \mathfrak{d}_a^-(T), \tag{8.8}$$

then, we have the identities $\mathfrak{d}_a(\mathfrak{T}_1(T)) = \mathfrak{d}_a^+(\mathfrak{T}_1(T)) = \mathfrak{d}_a^+(T)\big(1 + \frac{\rho_*}{4}\big)$.

Proof By the assumption, the inequality $\mathfrak{d}_a^+(T) \leq \mathfrak{d}_a^-(T)$ holds, so that we have, on one hand, $\mathfrak{d}_a(T) = \mathfrak{d}_a^+(T)$, whereas on the other hand, we have by the definition of $\mathfrak{T}_1$, the inequality $\mathfrak{d}_a(s) \leq \mathfrak{d}_a^+(T)(1 + \frac{\rho_*}{4})$ for $s \in [T, \mathfrak{T}_1]$. Going back to (8.2), we check that, if $\mathfrak{T}_{\rm ref}^+(T) \leq \mathfrak{T}(T)$, then $\mathfrak{d}_a^+(\mathfrak{T}_{\rm ref}^+(T)) > \mathfrak{d}_a^+(T)(1+\frac{\rho_*}{4})$, and the first assertion follows. We leave the second assertion to the reader.

Our previous discussion leads us to distinguish whether condition (8.8) is met or not.

8.3 The iterative construction (8.5) when condition (8.8) holds

When condition (8.8) holds, we introduce the integer n_f defined as

$$n_f = \inf\Big\{j \in \mathbb{N}, \mathfrak{d}_a^-(T_j) > \Big(1 + \frac{3\rho_*}{8}\Big)\mathfrak{d}_a^+(T_j)\Big\},$$

where the time T_j is defined by $T_j = \mathfrak{T}_j(T)$ for $j = 0, \cdots, n_f$. We denote by $b^j = \{b_k^j\}_{k\in J(T)}$ the solution to the ordinary differential equation (1.16) defined on $I_j = [T_j, T_{j+1}]$ obtained invoking Theorem 1.1. We describe first some elementary properties.

Lemma 8.2 *Assume that* (8.8) *and* $\mathcal{WP}_\varepsilon(\delta, T)$ *hold with* $\delta \geq \beta_2\varepsilon$. *For* $k = 0, \cdots, n_f$, *we have*

$$\mathfrak{d}_a(T_j) = \mathfrak{d}_a^+(T_j) = \mathfrak{d}_a^+(T)\Big(1 + \frac{\rho_*}{4}\Big)^j, \quad |\mathfrak{d}_a^+(T_{n_f}) - \mathfrak{d}_a^-(T_{n_f})| \leq \frac{\rho_*}{2}\mathfrak{d}_a(T_{n_f}) \tag{8.9}$$

and

$$0 < T_{j+1} - T_j \leq \Lambda_2\varepsilon^2 \exp\Big(\frac{\mathfrak{d}_a^+(T_{j+1})}{\varepsilon}\Big). \tag{8.10}$$

For any $j = 1, \cdots, n_f$ *and every* $s \in [T_j, T_{j+1}]$, *property* $\mathcal{WP}_\varepsilon(\nu_*\mathfrak{d}_a^+(T_{j-1}), s)$ *holds, and we have*

$$|a_k(t) - b_k^j(t)| \leq \varepsilon \exp\Big(-\rho_*\Big(1 + \frac{\rho_*}{4}\Big)^{j-1}\frac{\mathfrak{d}_a^+(T)}{\varepsilon}\Big). \tag{8.11}$$

Proof The identities (8.9) and (8.10) follow directly from Lemma 8.1, whereas the last assertions are a direct consequence of Theorem 1.1 assertion (ii).

Remark 8.1 As rather direct consequence of inequality (8.10) we deduce that for some constant $\Lambda_3 > 0$, and provided that β_2 is chosen sufficiently large,

$$0 < T_j - T \leq \Lambda_3\varepsilon^2 \exp\Big(\frac{\mathfrak{d}_a^+(T_j)}{\varepsilon}\Big). \tag{8.12}$$

The main result in this subsection are summarized in the following proposition.

Proposition 8.1 *Assume that* (8.8) *and* $\mathcal{WP}_\varepsilon(\delta, T)$ *hold with* $\delta \geq \beta_2\varepsilon$. *Then, we have the inequality* $\mathfrak{d}_a^-(s) \leq \mathfrak{d}_a^-(T) + \varepsilon$ *for every* $s \in [T, T_{n_f}]$ *and*

$$T_{n_f} \leq \Lambda_4\varepsilon^2 \exp\Big(\Big(1 - \frac{\rho_*}{8}\Big)\frac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big) + T. \tag{8.13}$$

Proof we introduce the stopping time s_1 defined by

$$s_1 = \inf\{s \in [t, T_{n_f}],\ \text{s.t. } \mathfrak{d}_a^-(s_1) = \mathfrak{d}_a^-(T) + \varepsilon\},$$

if the set on the right-hand side is not empty, and $s_1 = T_{n_f}$ otherwise. Let $j_1 \in \mathbb{N}$ be such that $s_1 \in (T_{j_1}, T_{j_1+1}]$. Since $\mathfrak{d}_a^-(s) \leq \mathfrak{d}_a^-(T) + \varepsilon$ on $[T, s_1]$, we deduce, combining with inequality (8.3) applied for the time T_{j_1}, that $\mathfrak{d}_a^-(s) \leq \mathfrak{d}_a^-(T) + (\gamma+1)\varepsilon$ on $[T, T_{j_1+1}]$. It then follows from the definition of n_f that

$$\mathfrak{d}_a^-(T) + (\gamma+1)\varepsilon \geq \mathfrak{d}_a^-(T_{j_1+1}) \geq \Big(1 + \frac{\rho_*}{2}\Big)\mathfrak{d}_a^+(T_{j_1+1}).$$

Hence, if we choose the constant β_2 sufficiently large, then we have $\big(1 + \frac{\rho_*}{4}\big)\mathfrak{d}_a^+(T_{j_1+1}) \leq \mathfrak{d}_a^-(T)$. It follows, invoking inequality (8.12) that

$$(T_{j_1+1} - T)\exp\Big(-\frac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big) \leq \varepsilon^2\Lambda_3 \exp\Big(-\rho_*\frac{\mathfrak{d}_a^-(T)}{8\varepsilon}\Big). \tag{8.14}$$

In view of inequality (2.7) of Proposition 2.3, we have

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{d}_{b^j}^-(s) \leq 4\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_{b^j}^-(s)}{\varepsilon}\Big) \leq 8\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_a^-(s)}{\varepsilon}\Big),$$

where the last inequality is a consequence of (8.11), provided that we choose the constant β_2 sufficiently large. We then deduce that for $s \in [s_0, s_1]$, we have

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{d}_{b^j}^-(s) \leq 8\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big).$$

Integrating this inequality and using again (8.11), we are led to

$$\begin{aligned}\mathfrak{d}_a^-(s_1) - \mathfrak{d}_a^-(T) &\leq 8\varepsilon^{-1}(s_1 - T)\mathfrak{q}_{\min}^{-1}\mathcal{B}_{\max}^2 \exp\Big(-\frac{\mathfrak{d}_a^-(T)}{\varepsilon}\Big) \\ &\quad + \varepsilon\sum_{k=1}^{n_f}\exp\Big(-\rho_*\Big(1+\frac{\rho_*}{4}\Big)^{j-1}\frac{\mathfrak{d}_a^+(T)}{\varepsilon}\Big) \\ &\leq \varepsilon\Lambda_4\Big[\exp\Big(-\rho_*\frac{\mathfrak{d}_a^-(T)}{8\varepsilon}\Big) + \exp\Big(-\rho_*\frac{\mathfrak{d}_a^+(T)}{8\varepsilon}\Big)\Big],\end{aligned} \tag{8.15}$$

where $\Lambda_4 > 0$ is some constant. If β_2 is chosen sufficiently large, then we deduce from (8.14) that $\mathfrak{d}_a^-(s_1) - \mathfrak{d}_a^-(T) \leq \frac{\varepsilon}{4}$. Hence, $s_1 = T_{n_f}$ which establishes the first assertion in Proposition 8.1. Inequality (8.13) follows going back to (8.14).

8.4 The iterative construction (8.5) when condition (8.8) fails

We begin this subsection with a preliminary results, which is somewhat a counterpart to Lemma 8.1.

Lemma 8.3 *Assume that property $\mathcal{WP}_\varepsilon(\delta, T)$ holds with $\delta \geq \beta_2\varepsilon$, and assume that* (8.8) *does not hold. Then we have $\mathfrak{d}_a(\mathfrak{T}_1(T)) = \mathfrak{d}_a^-(\mathfrak{T}_1(T)) = \big(1 - \frac{\rho_*}{4}\big)\mathfrak{d}_a^-(T)$.*

Proof In view of our assumption and since $0 < \rho_* \leq 1$, we have $\mathfrak{d}_a^-(T) \geq \mathfrak{d}_a(T) \geq \frac{8}{11}\mathfrak{d}_a^-(T)$. It follows on the other hand from (8.3) that $\mathfrak{d}_a(\mathfrak{T}_1(T)) \leq \mathfrak{d}_a^-(T) + \gamma\varepsilon$. In view of the definition of $\mathfrak{T}_1$, we have $|\delta_a(\mathfrak{T}_1(T)) - \delta_a(T)| = \frac{\rho_*}{4}\delta_a(T)$, so that either $\delta_a(\mathfrak{T}_1(T)) = (1 + \frac{\rho_*}{4})\delta_a(T)$ or $\delta_a(\mathfrak{T}_1(T)) = (1 - \frac{\rho_*}{4})\delta_a(T)$. The first equality is excluded: Indeed, if it were true, then we would have, combining with our previous inequality,

$$\Big(1 + \frac{\rho_*}{4}\Big)\delta_a(T) \leq \delta_a^-(T) + \gamma\varepsilon,$$

a contradiction with the fact that (8.8) does not hold, if we choose the constant β_2 sufficiently large. Hence we conclude that $\delta_a(\mathfrak{T}_1(T)) = \big(1 - \frac{\rho_*}{4}\big)\delta_a(T)$, which, combined once more with (8.3), leads to the conclusion.

When condition (8.8) fails, a new stopping time is introduced, related to the integer m_f defined by

$$m_f = \inf\{j \in N, \text{s.t. } \delta_a^-(T_j) \leq 2\alpha_*\varepsilon\}. \tag{8.16}$$

Proposition 8.2 *Assume that property $\mathcal{WP}_\varepsilon(\delta, T)$ holds with $\delta \geq \beta_2\varepsilon$, and assume that* (8.8) *does not hold. We have for $j = 1, \cdots, n_f$,*

$$\delta_a(T_j) = \delta_a^-(T_j) = \delta_a^-(T)\Big(1 - \frac{\rho_*}{4}\Big)^j. \tag{8.17}$$

For every $s \in [T_j, T_{j+1}]$, property $\mathcal{WP}_\varepsilon(\nu_*\delta_a^-(T_{j-1}), s)$ holds and $\delta_a^+(s) \geq \delta_a^+(T) - \gamma\varepsilon$. Moreover,

$$T_{m_f} - T \leq \Lambda_3\varepsilon^2 \exp\Big(\frac{\delta_a^-(T)}{\varepsilon}\Big), \quad \delta_a^-(T_{m_f}) \leq 2\alpha_*\varepsilon. \tag{8.18}$$

Proof The proof is very similar to the proof of Proposition 8.1 and relies both on Theorem 1.1 and Lemma 8.3. We therefore omit the details.

8.5 Proof of Theorem 1.2 completed

We choose throughout $\beta_* = \beta_2$. We distinguish two cases.

Case A Inequality (8.8) does not hold.

In this case, we are in position to apply the results of Proposition 8.2 in Subsection 8.4. As a matter of fact, we choose

$$\mathcal{T}_{\rm col}^- = T_{m_f},$$

where the value of m_f is provided by (8.16). For this choice, imposing $C^* \geq \lambda_3$ and $c_* \geq \lambda$, all the statements provided in Theorem 1.2 are provided by the results of Proposition 8.2.

Case B Inequality (8.8) holds.

Here, we are in position to apply first the results of Subsection 8.3. We introduce the time $\mathcal{T}_0 = T_{n_f}$, where T_{n_f} is provided by Proposition 8.1. Hence, we obtain the bounds $\delta_a^-(\mathcal{T}_0) \leq \delta_a^-(T) + \varepsilon$, $\delta_a^-(\mathcal{T}_0) > \big(1 + \frac{3\rho_*}{8}\big)\delta_a^+(\mathcal{T}_0)$ and

$$\mathcal{T}_0 - T \leq \Lambda_4\varepsilon^2 \exp\Big(\Big(1 - \frac{\rho_*}{8}\Big)\frac{\delta_a^-(T)}{\varepsilon}\Big) + T. \tag{8.19}$$

It follows that inequality (8.8) holds for the time $\mathcal{T}_0$, and we may therefore then argue as in the first step, setting

$$\mathcal{T}_{\rm col}^- = T_{m_f}(\mathcal{T}_0),$$

It follows from inequality (8.18) of Proposition 8.2 that

$$\mathcal{T}_{\rm col}^- - \mathcal{T}_0 \le \Lambda_3 \varepsilon^2 \exp\left(\frac{\delta_a^-(\mathcal{T}_0)}{\varepsilon}\right) \le e\Lambda_3 \varepsilon^2 \exp\left(\frac{\delta_a^-(T)}{\varepsilon}\right), \tag{8.20}$$

where the last inequality follows from the bound $\delta_a^-(\mathcal{T}_0) \le \delta_a^-(T) + \varepsilon$. Moreover, we have $\delta_a^-(T_{m_f}) \le 2\alpha_* \varepsilon$. The conclusion then follows combining (8.19)–(8.20) and an appropriate choice of the constants.

9 Annihilations

The purpose of this section is to provide the proof of Theorem 1.3.

9.1 Dissipation of energy

Proposition 9.1 *Let $0 \le T_1 < T_2$ be given and assume that $\mathcal{WP}_\varepsilon(\delta, T_i)$ holds for $i = 1, 2$ with $\delta \ge \beta_3 \varepsilon$, where $\beta_3 = \sup\{\beta_2, 4\nu_1^{-1}\alpha_*\}$, and ν_1 is the constant introduced in Lemma 4.9. Assume moreover that the signs $\{\dagger_k\}_{k\in J(T_1)}$ are not all identical. Then, it holds*

$$\mathfrak{E}(T_1) \ge \mathfrak{E}(T_2) + \mu_1, \tag{9.1}$$

provided $T_2 - T_1 \ge C_* \varepsilon^2 \exp\left(\frac{\delta_a^-(T_1)}{\varepsilon}\right)$, where the constant μ_1 is introduced in Lemma 4.5.

Proof Assume by contradiction that inequality (9.1) does not hold. In view of Lemma 4.5, we then necessarily have $\mathfrak{E}(T_1) = \mathfrak{E}(T_2)$, and we are in position to apply Lemma 4.9, which yields in particular,

$$\delta_a^-(t) \ge \nu_1 \delta \ge 4\alpha_* \varepsilon \quad \text{for any } t \in [T_1, T_2]. \tag{9.2}$$

In view of our assumption on $T_2 - T_1$, we deduce that $\mathcal{T}_{\rm col}^-$ defined in Theorem 1.2 belongs to $[T_1, T_2]$. From the very definition of $\mathcal{T}_{\rm col}^-$, we obtain $\delta_a^-(\mathcal{T}_{\rm col}^-) \le 2\alpha_* \varepsilon$, which contradicts (9.2) and hence completes the proof.

We notice that, under the assumptions of Theorem 1.2 that $\mathfrak{E}(s) = \mathfrak{E}(T)$ for any $s \in [T, \mathcal{T}_{\rm col}^-]$. We next define the time $\mathcal{T}_{\rm col}^+ > \mathcal{T}_{\rm col}^-$ for which a dissipation has undergone.

Proposition 9.2 *Let $T \ge 0$ and $\delta > 0$ be given, suppose that the assumptions of Theorem 1.2 are full-filled and that moreover $\delta \ge \beta_4 \varepsilon$, where $\beta_4 \ge \beta_3$ is some constant depending only on V and M_0. Then there exists some time $\mathcal{T}_{\rm col}^+ > \mathcal{T}_{\rm col}^-$, such that $\mathcal{WP}_\varepsilon(\alpha_*\varepsilon, \mathcal{T}_{\rm col}^+)$ holds, $\mathfrak{E}(\mathcal{T}_{\rm col}^+) \le \mathfrak{E}(T) + \mu_1$, and moreover (1.24) holds, for some constant Υ depending only on V and M_0.*

Proof We impose first that $\beta_4 \ge 2\nu_*^{-1}\beta_3$. Turning to Theorem 1.2, by continuity of the function $\delta^-(\cdot)$, we may assert that there exists some time $T_1 \in [T, \mathcal{T}_{\rm col}^-]$, such that

$$\delta^-(T_1) = 2\nu_*^{-1}\beta_3 \le \beta_4,$$

and, in view of assertion (iii), that property $\mathcal{WP}_\varepsilon(\beta_3\varepsilon, T_1)$ holds. On the other hand, assuming that the constant β_4 is given, it follows from Lemma 4.6 and relation (4.9) that there exists some time $T_1 \in [\mathcal{T}_{\rm col}^- + \omega(\beta_4)\varepsilon^2, \mathcal{T}_{\rm col}^- + 2\omega(\beta_4)\varepsilon^2]$, such that condition $\mathcal{WP}_\varepsilon(\beta_4\varepsilon, T_2)$ holds. Moreover, by definition, we have

$$\omega(\beta_4)\varepsilon^2 \leq T_2 - \mathcal{T}_{\rm col}^- \leq 2\omega(\beta_4)\varepsilon^2. \tag{9.3}$$

We choose next the constant β_4 sufficiently large, such that it satisfies the additional condition

$$\omega(\beta_4) \geq C_* \exp(\nu_*^{-1}\beta_3),$$

so that $T_2 - T_1 \geq T_2 - \mathcal{T}_{\rm col}^- \geq \omega(\beta_4)\varepsilon^2 \geq C_*\varepsilon^2 \exp \frac{\delta_a^-(T_1)}{\varepsilon}$. It follows hence from Proposition 9.1 that $\mathfrak{E}(T_1) \geq \mathfrak{E}(T_2) + \mu_1$. Setting $\mathcal{T}_{\rm col}^+ = T_2$ the conclusion follows with $\Upsilon = 2\omega(\beta_4)$.

9.2 The fate of fronts between $\mathcal{T}_{\rm col}^-$ and $\mathcal{T}_{\rm col}^+$

Proposition 9.3 *Let $T \geq 0$ and $\delta > 0$ be given, suppose that the assumptions of Theorem* 1.2 *are full-filled and that moreover $\delta \geq \beta_4\varepsilon$. There exists a constant $\kappa_c > 0$, such that*

$$\bigcup_{k\in J(\mathcal{T}_{\rm col}^+)} \{a_k(\mathcal{T}_{\rm col}^+)\} \subset \bigcup_{k\in J(\mathcal{T}_{\rm col}^+)} \{a_k(\mathcal{T}_{\rm col}^-)\} + [-\kappa_c\varepsilon, \kappa_c\varepsilon]. \tag{9.4}$$

If for some $k_0 \in J(\mathcal{T}_{\rm col}^+)$, $a_{k_0}(\mathcal{T}_{\rm col}^-)$ is well separated from the other fronts in the sense of (1.22), *then we have*

$$|a_{m_0}(\mathcal{T}_{\rm col}^+) - a_{k_{\rm f}}(\mathcal{T}_{\rm col}^-)| \leq \frac{\kappa_{\rm f}}{10}\varepsilon, \quad |a_{m'}(\mathcal{T}_{\rm col}^+) - a_{m_0}(\mathcal{T}_{\rm col}^+)| \geq \frac{9\kappa_{\rm f}}{10}\varepsilon \quad \text{for } m' \neq m_0.$$

Proof Recall that in view of Theorem 1.3, we have $\mathcal{T}_{\rm col}^+ - \mathcal{T}_{\rm col}^- \leq \Upsilon\varepsilon^2$, so that, in view of Theorem 1.6, we have, for some constant $\kappa_1 > 0$, such that, for any $s \in [\mathcal{T}_{\rm col}^-, \mathcal{T}_{\rm col}^+]$,

$$\mathcal{D}(s) \subset \mathcal{D}(\mathcal{T}_{\rm col}^-) + [-\kappa_1\varepsilon, \kappa_1\varepsilon]. \tag{9.5}$$

On the other hand, by (1.27), we have $\mathcal{D}(\mathcal{T}_{\rm col}^\pm) \subset \underset{k\in J(\mathcal{T}_{\rm col}^\pm)}{\cup} \{a_k(\mathcal{T}_{\rm col}^\pm)\} + [\kappa_{\rm w}\varepsilon, \kappa_{\rm w}\varepsilon]$, which yields (9.4), choosing $\kappa_c \geq 2\kappa_{\rm w} + 2\kappa_1$, the inclusion (9.4).

Assume next that (1.22) holds for some $k_0 \in J(\mathcal{T}_{\rm col}^+)$, with a constant $\kappa_{\rm f} > 0$ yet to be determined. We set

$$J_{k_0}(\mathcal{T}_{\rm col}^+) = \left\{ m \in J(\mathcal{T}_{\rm col}^+),\ a_m(\mathcal{T}_{\rm col}^+) \in \left[a_{k_0}(\mathcal{T}_{\rm col}^-) - \frac{\kappa_{\rm f}}{2}\varepsilon,\ a_{k_0}(\mathcal{T}_{\rm col}^-) + \frac{\kappa_{\rm f}}{2}\varepsilon \right] \right\}$$

and $\mathfrak{O}_{k_0}(\mathcal{T}_{\rm col}^+) = \{a_m(\mathcal{T}_{\rm col}^+),\ m \in J_{k_0}(\mathcal{T}_{\rm col}^+)\}$. If we impose the condition $\kappa_{\rm f} > 10\kappa_c$, then the second assertion of Proposition 9.3 essentially reduces to prove that these sets are singletons. Imposing also that $\kappa_{\rm f} > 4\kappa_1$, we see in view of (9.5) that $|v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^+) - \sigma_{j(k_0)^-}| \leq \mu_0$ on the interval $[a_{k_0-1}(\mathcal{T}_{\rm col}^-) + \kappa_1\varepsilon, a_{k_0}(\mathcal{T}_{\rm col}^-) - \kappa_1\varepsilon]$, which is not empty in view of our constraint on $\kappa_{\rm f}$, and similarily that $|v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^+) - \sigma_{j(k_0)^+}| \leq \mu_0$ on the interval $[a_{k_0}(\mathcal{T}_{\rm col}^-) + \kappa_1\varepsilon, a_{k_0+1}(\mathcal{T}_{\rm col}^-) - \kappa_1\varepsilon]$. Hence, $v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^+)$ needs to connect between the points $a_{k_0}(\mathcal{T}_{\rm col}^-) - \kappa_1\varepsilon$ and $a_{k_0}(\mathcal{T}_{\rm col}^-) + \kappa_1\varepsilon$ the values $\sigma_{j(k)^-}$ to $\sigma_{j(k)^+}$, and hence we deduce that

$$\mathfrak{O}_{k_0}(\mathcal{T}_{\rm col}^+) = \mathfrak{O}(\mathcal{T}_{\rm col}^+) \cap [a_{k_0}(\mathcal{T}_{\rm col}^-) - \kappa_1\varepsilon, a_{k_0}(\mathcal{T}_{\rm col}^-) + \kappa_1\varepsilon] \neq \emptyset.$$

To complete the proof, it remains finally to show that $\mathfrak{O}_{k_0}(\mathcal{T}_{\rm col}^+)$ reduces to a single point. For that purpose, we invoke once more the localized energy inequality (1.31), with a test function χ such that $0 \leq \chi \leq 1$ and

$$\begin{cases} \chi = 1 \quad \text{on } [a_{k_0}(\mathcal{T}_{\rm col}^-) - \frac{\kappa_{\rm f}}{4}\varepsilon,\ a_{k_0}(\mathcal{T}_{\rm col}^-) + \frac{\kappa_{\rm f}}{4}\varepsilon], \\ \chi = 0 \quad \text{on } \mathbb{R} \setminus [a_{k_0}(\mathcal{T}_{\rm col}^-) - \frac{\kappa_{\rm f}}{2}\varepsilon,\ a_{k_0}(\mathcal{T}_{\rm col}^-) + \frac{\kappa_{\rm f}}{2}\varepsilon], \\ |\ddot{\chi}| \leq 16\kappa_{\rm f}^{-2}\varepsilon^{-2}. \end{cases}$$

Writing (1.31) for this choice of test function, we are led to

$$\Big| \int \chi e_\varepsilon(v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^-)) - \int \chi e_\varepsilon(v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^+)) \Big| \leq 16 M_0 \kappa_{\rm f}^{-2}. \tag{9.6}$$

On the other hand, in view of the energy quantization property expressed in Lemma 4.4 which can easily be localized, we deduce that

$$\Big| \sum_{m \in J_{k_0}(\mathcal{T}_{\rm col}^+)} \mathfrak{S}_{i(m)} - \mathfrak{S}_{i(k_0)} \Big| \leq 16 M_0 \kappa_{\rm f}^{-2} + M_0 \frac{\delta}{\varepsilon} \exp\Big(-\frac{\rho_2 \delta}{\varepsilon}\Big).$$

Since $v_\varepsilon(\cdot, \mathcal{T}_{\rm col}^+)$ needs to connect between the points $a_{k_0}(\mathcal{T}_{\rm col}^-) - \kappa_1 \varepsilon$ and $a_{k_0}(\mathcal{T}_{\rm col}^-) + \kappa_{mf}\varepsilon$ the values $\sigma_{j(k)^-}$ to $\sigma_{j(k)^+}$, there exits some $m_0 \in J_{k_0}(\mathcal{T}_{\rm col}^+)$, such that $\mathfrak{S}_{i(m)} = \mathfrak{S}_{i(k_0)}$, and the previous inequality becomes

$$\sum_{m \in J_{k_0}(\mathcal{T}_{\rm col}^+) \setminus \{m_0\}} \mathfrak{S}_{i(m)} \leq 16 M_0 \kappa_{\rm f}^{-2} + M_0 \frac{\delta}{\varepsilon} \exp\Big(-\frac{\rho_2 \delta}{\varepsilon}\Big). \tag{9.7}$$

The right-hand side of this inequality can be made arbitrarily small, choosing $\kappa_{\rm f}$ and possibly also β_4 sufficiently large, whereas the right-hand side is either zero or bounded below by a positive constant. Hence for a suitable choice of the constants, we are led to $J_{k_0}(\mathcal{T}_{\rm col}^+) \setminus \{m_0\} = \emptyset$, yielding hence the desired conclusion.

9.3 Proof of Theorem 1.3 completed

First notice that the time $\mathcal{T}_{\rm col}^+$ has been defined in Proposition 9.2, and the estimate (1.24) as well as the condition $\mathcal{WP}_\varepsilon(\alpha_* \varepsilon, \mathcal{T}_{\rm col}^+)$ have been established there. It remains hence to prove assertions (i)–(iii). Assertion (i) rephrases the inclusion (9.4) of Proposition 9.3. Assertion (ii) follows from the second assertion in Proposition 9.3, as well, for the part concerning repulsive points, inclusion (1.23) and the discussion thereafter.

The proof of Assertion (iii) requires some additional discussion. Arguing as for (9.7), we deduce that

$$\sum_{a_k(\mathcal{T}_{\rm col}^+) \in \mathfrak{O}_{\rm rep}(\mathcal{T}_{\rm col}^+)} \mathfrak{S}_{i(k)} = \sum_{a_k(\mathcal{T}_{\rm col}^-) \in \mathfrak{O}_{\rm rep}(\mathcal{T}_{\rm col}^-)} \mathfrak{S}_{i(k)},$$

so that, in view of the inequality $\mathfrak{E}(\mathcal{T}_{\rm col}^+) \leq \mathfrak{E}(T) + \mu_1$, we are led to the inequality

$$\sum_{a_k(\mathcal{T}_{\rm col}^+) \in \mathfrak{O}_{\rm attr}(\mathcal{T}_{\rm col}^+)} \mathfrak{S}_{i(k)} \leq \sum_{a_k(\mathcal{T}_{\rm col}^-) \in \mathfrak{O}_{\rm attr}(\mathcal{T}_{\rm col}^-)} \mathfrak{S}_{i(k)} + \mu_1. \tag{9.8}$$

We next invoke several observations. The first one is that the set $\mathfrak{O}_{\mathrm{attr}}(\mathcal{T}^-_{\mathrm{col}})$ can be decomposed as $\mathfrak{O}_{\mathrm{attr}}(\mathcal{T}^-_{\mathrm{col}}) = \overset{p}{\underset{m=1}{\cup}} \mathfrak{A}^-_m$ where all of the sets $\mathfrak{A}^-_m$ are maximal attractive chains. We notice that a maximal attractive chain involves only one heteroclinic orbit denoted here ξ_m, so that

$$\sum_{a_k(\mathcal{T}^-_{\mathrm{col}}) \in \mathfrak{A}^-_m} \mathfrak{S}_{i(k)} = \sharp(\mathfrak{A}^-_m)\mathfrak{S}_m.$$

In view of the continuity of the front sets properties described in Subsection 9.2, each maximal attractive chain $\mathfrak{A}^-_m$ gives rise to a corresponding maximal attractive chain $\mathfrak{A}^+_m$ for $\mathfrak{O}_{\mathrm{attr}}(\mathcal{T}^+_{\mathrm{col}})$, so that $\mathfrak{O}_{\mathrm{attr}}(\mathcal{T}^+_{\mathrm{col}}) = \overset{p}{\underset{m=1}{\cup}} \mathfrak{A}^+_m$ the number of elements in $\mathfrak{A}^+_m$ is odd if the number of elements in $\mathfrak{A}^-_m$ is odd, even but possibly zero if the number of elements in $\mathfrak{A}^-_m$ is even. Moreover, invoking the localized energy identity (1.31) with appropriate test functions as for the proof of (7.16), we obtain

$$\sharp(\mathfrak{A}^+_m) \leq \sharp(\mathfrak{A}^-_m).$$

On the other hand, inequality (9.8) is turned into

$$\sum_{m=1}^{p} \sharp(\mathfrak{A}^+_m)\mathfrak{S}_m \leq \sum_{m=1}^{p} \sharp(\mathfrak{A}^-_m)\mathfrak{S}_m + \mu_1.$$

Hence, combining the two last inequalities, we deduce that there exists some m_0, such that $\sharp(\mathfrak{A}^+_{m_0}) < \sharp(\mathfrak{A}^-_{m_0})$. The conclusion follows, taking into account that the numbers involved are positive integers with the same parity.

10 Relaxing the Assumptions on the Potential

In this section, we outline the main points of the arguments of the proofs of Proposition 1.1 and Theorem 1.5. An important step is to check that the result stated in [3, Lemma 1] remains valid under assumptions on the potential we consider here. More precisely, we have the following lemma.

Lemma 10.1 *Assume that the potential V satisfies assumptions* (H_1)–(H_3). *Let u be such that $\mathcal{E}_\varepsilon(u) < +\infty$. There exist constants $\eta_0 > 0$ and $N > 0$ depending only on $\|V'\|_{C^2(\mathbb{R})}$, ν and $\lambda_{\min}$, such that, if, for $a \in \mathbb{R}$, we have*

$$\int_{[a,a+1]} e_\varepsilon(u(x))\mathrm{d}x \leq \eta_0,$$

then there exists some $\sigma_i \in \Sigma$, such that

$$|u(x) - \sigma_i| \leq N(\|V\|_{C^2(\mathbb{R})}, \nu, \lambda_{\min})\Big(\int_{[a,a+1]} e_\varepsilon(u(x))\mathrm{d}x\Big)^{\frac{1}{2}}, \quad \forall x \in [a, a+1]. \tag{10.1}$$

The proof is parallel to the proof of Lemma 1 in [3] and is left to the reader. We notice also that (1.25) remains also valid with $\ell \leq \ell_0$, where the constant ℓ_0 depends only on $\|V'\|_{C^2(\mathbb{R})}$, ν and $\lambda_{\min}$.

10.1 Sketch of the proof of Proposition 1.1

Since $\int_{[a,a+1]} e_\varepsilon(u(x))\mathrm{d}x \to 0$ as $a \to \pm\infty$, so that, in view of (10.1), we deduce that there exists some $\sigma_+ \in \Sigma$ (resp. $\sigma_- \in \Sigma$), such that $|u(x) - \sigma_+| \to 0$ as $x \to +\infty$ (resp. as $x \to -\infty$). In view of (1.25), we have

$$u(\mathbb{R}) \subset \bigcup_{k=1}^{\ell} u([x_i - \varepsilon, x_i + \varepsilon]) + [-\mu_0, \mu_0].$$

On the other hand, by embedding, we have

$$|u([x_i - \varepsilon, x_i + \varepsilon])| \leq \sqrt{2\varepsilon}\|\dot{u}\|_{L^2(\mathbb{R})} \leq 2M_0,$$

so that $|u(\mathbb{R})| \leq \ell_0 M_0$, and (1.28) follows. Finally, we leave the last assertion to the reader.

10.2 Sketch of the proof of Theorem 1.5

In view of Proposition 1.1 and relation (1.29), we know that v_ε takes values in some interval of the form $[\sigma_+ - A, \sigma_+ + A]$, with A depending only on $\|V'\|_{C^2(\mathbb{R})}$, ν and $\lambda_{\min}$. We may then construct a potential $\widetilde{V}$, such that $\widetilde{V} = V$ on the set $[\sigma_+ - 2A, \sigma_+ + 2A]$ which full-fills conditions (H_1)–(H_3). It follows that the function v_ε is also a solution to $(PGL)_\varepsilon$ for the potential $\widetilde{V}$, and we are hence in position to apply the results in Theorems 1.1–1.4, which leads to the desired conclusion. One may also check that the extension $\widetilde{V}$ might be constructed in such a way that all constants involved in the theorems depend only on $\|V'\|_{C^2(\mathbb{R})}$, ν and $\lambda_{\min}$.

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