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Variations on a Proof of a Borderline Bourgain-Brezis Sobolev Embedding Theorem*

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(*To Haïm Brezis in admiration and friendship*)

Abstract This paper offers a variant of a proof of a borderline Bourgain-Brezis Sobolev embedding theorem on $\mathbb{R}^n$. The authors use this idea to extend the result to real hyperbolic spaces $\mathbb{H}^n$.

Keywords Bourgain-Brezis inequalities, Divergence-free vector fields, Sobolev inequalities, Real hyperbolic space
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1 Introduction

The Sobolev embedding theorem states that if $\dot{W}^{1,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space, obtained by completing the set of compactly supported smooth functions $C_c^\infty(\mathbb{R}^n)$ under the norm $\|\nabla u\|_{L^p(\mathbb{R}^n)}$, then $\dot{W}^{1,p}(\mathbb{R}^n)$ embeds into $L^{p^*}(\mathbb{R}^n)$, whenever $n \geq 2$, $1 \leq p < n$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. This fails when $p = n$, i.e., $\dot{W}^{1,n}(\mathbb{R}^n)$ does not embed into $L^\infty(\mathbb{R}^n)$. One of the well-known remedies of this failure is to say that $\dot{W}^{1,n}(\mathbb{R}^n)$ embeds into $\mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation. In [2, 4], Bourgain and Brezis established another remedy of the failure of this Sobolev embedding for $\dot{W}^{1,n}(\mathbb{R}^n)$. They proved, among other things, that if X is a differential ℓ-form on $\mathbb{R}^n$ with $\dot{W}^{1,n}(\mathbb{R}^n)$ coefficients, where $1 \leq \ell \leq n-1$, then there exists a differential ℓ-form Y, whose components are all in $\dot{W}^{1,n} \cap L^\infty(\mathbb{R}^n)$, such that

$$\mathrm{d}Y = \mathrm{d}X$$

with

$$\|Y\|_{\dot{W}^{1,n} \cap L^\infty} \leq C\|dX\|_{L^n}.$$

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(Such a theorem would have been trivial by Hodge decomposition, if $\dot{W}^{1,n}(\mathbb{R}^n)$ were to embed into $L^\infty(\mathbb{R}^n)$.) The existing proofs of the above theorem are all long and complicated. On the contrary, a weaker version of this theorem, where one replaces the space $\dot{W}^{1,n} \cap L^\infty$ by L^∞, can be obtained from the following theorem of Van Schaftingen [9], when $\ell \le n-2$.

Theorem 1.1 (see [9]) *Suppose that f is a smooth vector field on $\mathbb{R}^n$, with*

$$\operatorname{div} f = 0.$$

Then for any compactly supported smooth vector field ϕ on $\mathbb{R}^n$, we have

$$\Big| \int_{\mathbb{R}^n} \langle f, \phi \rangle \Big| \le C \|f\|_{L^1} \|\nabla \phi\|_{L^n}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the pointwise Euclidean inner product of two vector fields in $\mathbb{R}^n$.

See e.g. [4, 6]. We refer the interested reader to the survey in [10], for a more detailed account of this circle of ideas.

The original direct proof of Theorem 1.1 in [9] proceeds by decomposing

$$\int_{\mathbb{R}^n} \langle f, \phi \rangle = \sum_{i=1}^m \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{i-1} \times \{s\} \times \mathbb{R}^{n-i}} f_i \phi_i \Big) \, \mathrm{d}s,$$

and by estimating first directly the innermost $(n-1)$-dimensional integral. This gives the impression that the strategy is quite rigid. The first goal of this note is to prove Theorem 1.1 by averaging a suitable estimate over all unit spheres in $\mathbb{R}^n$.

In a second part of this paper, we adapt this idea of averaging over families of sets to prove an analogue of Theorem 1.1, in the setting where $\mathbb{R}^n$ is replaced by the real hyperbolic space $\mathbb{H}^n$.

Theorem 1.2 *Suppose that f is a smooth vector field on $\mathbb{H}^n$, with*

$$\operatorname{div}_g f = 0,$$

where div_g is the divergence with respect to the metric g on $\mathbb{H}^n$. Then for any compactly supported smooth vector field ϕ on $\mathbb{H}^n$, we have

$$\Big| \int_{\mathbb{H}^n} \langle f, \phi \rangle_g \, \mathrm{d}V_g \Big| \le C \|f\|_{L^1(\mathbb{H}^n)} \|\nabla_g \phi\|_{L^n(\mathbb{H}^n)}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle_g$ and $\mathrm{d}V_g$ are the pointwise inner product and the volume measure with respect to g respectively, $\nabla_g \phi$ is the $(1,1)$ tensor given by the Levi-Civita connection of ϕ with respect to g, and

$$\|f\|_{L^1(\mathbb{H}^n)} = \int_{\mathbb{H}^n} |f|_g \, \mathrm{d}V_g, \quad \|\nabla_g \phi\|_{L^n(\mathbb{H}^n)} = \Big(\int_{\mathbb{H}^n} |\nabla_g \phi|_g^n \, \mathrm{d}V_g \Big)^{\frac{1}{n}}.$$

We note that the above theorem is formulated entirely geometrically on $\mathbb{H}^n$, without the need of specifying a choice of coordinate chart. As explained in Appendix A, Theorem 1.2 can be proved indirectly by patching together known estimates on $\mathbb{R}^n$ via a partition of unity, and by applying Hardy's inequality to get rid of lower order terms.

We shall prove Theorem 1.2 by averaging a suitable estimate over a family of hypersurfaces in $\mathbb{H}^n$, where the family of hypersurfaces is obtained from the orbit of a "vertical hyperplane" under all isometries in $\mathbb{H}^n$. The latter shares a similar flavour to the proof we will give below of Theorem 1.1. The innovation in the proof of the result for the hyperbolic space is in deducing Theorem 1.2 from Proposition 3.1, and in establishing Lemma 3.4 (see Section 3 for details).

2 Another Proof of Theorem 1.1

Theorem 1.1 will follow from the following proposition.

Proposition 2.1 *Let f, ϕ be as in Theorem 1.1. Write $\mathbb{B}^n$ for the unit ball $\{x \in \mathbb{R}^n \colon |x| < 1\}$ in $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ for the unit sphere (i.e., the boundary of $\mathbb{B}^n$). Also write $\mathrm{d}\sigma$ for the standard surface measure on $\mathbb{S}^{n-1}$, and ν for the outward unit normal to the sphere $\mathbb{S}^{n-1}$. Then*

$$\Big| \int_{\mathbb{S}^{n-1}} \langle f, \nu \rangle \langle \phi, \nu \rangle \, \mathrm{d}\sigma \Big| \leq C \|f\|_{L^1(\mathbb{R}^n \setminus \mathbb{B}^n)}^{\frac{1}{n}} \|f\|_{L^1(\mathbb{S}^{n-1})}^{1-\frac{1}{n}} \|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}. \tag{2.1}$$

Here $\|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})} = \|\phi\|_{L^n(\mathbb{S}^{n-1})} + \|\nabla_{\mathbb{S}^{n-1}} \phi\|_{L^n(\mathbb{S}^{n-1})}$, where $\nabla_{\mathbb{S}^{n-1}} \phi$ is the $(1,1)$ tensor on $\mathbb{S}^{n-1}$ given by the covariant derivative of the vector field ϕ.

The proof of Proposition 2.1 in turn depends on the following two lemmas. The first one is a simple lemma about integration by parts.

Lemma 2.1 *Let f, ν be as in Proposition 2.1. Then for any compactly supported smooth function ψ on $\mathbb{R}^n$, we have*

$$\int_{\mathbb{S}^{n-1}} \langle f, \nu \rangle \psi \ \mathrm{d}\sigma = - \int_{\mathbb{R}^n \setminus \mathbb{B}^n} \langle f, \nabla \psi \rangle \, \mathrm{d}x.$$

The second one is a decomposition lemma for functions on the sphere $\mathbb{S}^{n-1}$.

Lemma 2.2 *Let φ be a smooth function on $\mathbb{S}^{n-1}$. For any $\lambda > 0$, there exists a decomposition*

$$\varphi = \varphi_1 + \varphi_2 \quad \text{on } \mathbb{S}^{n-1},$$

and an extension $\widetilde{\varphi}_2$ of φ_2 to $\mathbb{R}^n \setminus \mathbb{B}^n$, such that $\widetilde{\varphi}_2$ is smooth and bounded on $\mathbb{R}^n \setminus \mathbb{B}^n$, with

$$\|\varphi_1\|_{L^\infty(\mathbb{S}^{n-1})} \leq C \lambda^{\frac{1}{n}} \|\nabla_{\mathbb{S}^{n-1}} \varphi\|_{L^n(\mathbb{S}^{n-1})},$$
$$\|\nabla \widetilde{\varphi}_2\|_{L^\infty(\mathbb{R}^n \setminus \mathbb{B}^n)} \leq C \lambda^{\frac{1}{n}-1} \|\nabla_{\mathbb{S}^{n-1}} \varphi\|_{L^n(\mathbb{S}^{n-1})}.$$

Here $\|\nabla_{\mathbb{S}^{n-1}} \varphi\|$ is the norm of the gradient of the function φ on $\mathbb{S}^{n-1}$. We postpone the proofs of Lemmas 2.1–2.2 to the end of this section.

Now we are ready for the proof of Proposition 2.1.

Proof of Proposition 2.1 Let f, ϕ be as in the statement of Theorem 1.1. Apply Lemma 2.2 to $\varphi = \langle \phi, \nu \rangle$, where $\lambda > 0$ is to be chosen. Then since

$$\|\nabla_{\mathbb{S}^{n-1}} \langle \phi, \nu \rangle\|_{L^n(\mathbb{S}^{n-1})} \leq C \|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})},$$

there exists a decomposition

$$\langle \phi, \nu \rangle = \varphi_1 + \varphi_2 \quad \text{on } \mathbb{S}^{n-1},$$

and an extension $\widetilde{\varphi}_2$ of φ_2 to $\mathbb{R}^n \setminus \mathbb{B}^n$, such that $\widetilde{\varphi}_2 \in C^\infty \cap L^\infty(\mathbb{R}^n \setminus \mathbb{B}^n)$,

$$\|\varphi_1\|_{L^\infty(\mathbb{S}^{n-1})} \leq C \lambda^{\frac{1}{n}} \|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}$$

and

$$\|\nabla\widetilde{\varphi}_2\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \le C\lambda^{\frac{1}{n}-1}\|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}.$$

Now

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\langle f,\nu\rangle\langle\phi,\nu\rangle\,\mathrm{d}\sigma &= \int_{\mathbb{S}^{n-1}}\langle f,\nu\rangle\varphi_1\,\mathrm{d}\sigma + \int_{\mathbb{S}^{n-1}}\langle f,\nu\rangle\varphi_2\,\mathrm{d}\sigma\\ &= \mathrm{I}+\mathrm{I\!I}.\end{aligned}$$

In the first term, we estimate trivially

$$|\mathrm{I}| \le \|f\|_{L^1(\mathbb{S}^{n-1})}\|\varphi_1\|_{L^\infty(\mathbb{S}^{n-1})} \le C\lambda^{\frac{1}{n}}\|f\|_{L^1(\mathbb{S}^{n-1})}\|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}.$$

To estimate the second term, we let θ be a smooth cut-off function with compact support on $\mathbb{R}^n$, such that $\theta(x)=1$ whenever $|x|\le 1$. For $\varepsilon\in(0,1)$, let $\theta_\varepsilon(x)=\theta(\varepsilon x)$. Then $\theta_\varepsilon=1$ on $\mathbb{S}^{n-1}$, so we can rewrite $\mathrm{I\!I}$ as

$$\mathrm{I\!I} = \int_{\mathbb{S}^{n-1}}\langle f,\nu\rangle\varphi_2\theta_\varepsilon\,\mathrm{d}\sigma$$

for any $\varepsilon\in(0,1)$. We then integrate by parts using Lemma 2.1, with $\psi:=\widetilde{\varphi}_2\theta_\varepsilon$, and obtain

$$\mathrm{I\!I} = -\int_{\mathbb{R}^n\setminus\mathbb{B}^n}\langle f,\nabla\widetilde{\varphi}_2\rangle\theta_\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^n\setminus\mathbb{B}^n}\langle f,\nabla\theta_\varepsilon\rangle\widetilde{\varphi}_2\,\mathrm{d}x.$$

(The cut-off function θ_ε is inserted so that ψ has compact support.) We now let $\varepsilon\to 0^+$. The second term then tends to 0, since it is just

$$-\varepsilon\int_{\mathbb{R}^n\setminus\mathbb{B}^n}\langle f(x),(\nabla\theta)(\varepsilon x)\rangle\widetilde{\varphi}_2(x)\,\mathrm{d}x,$$

where $f\in L^1$, $\nabla\theta(\varepsilon\cdot)\in L^\infty$ and $\widetilde{\varphi}_2\in L^\infty$ on $\mathbb{R}^n\setminus\mathbb{B}^n$. On the other hand, the first term tends to

$$-\int_{\mathbb{R}^n\setminus\mathbb{B}^n}\langle f,\nabla\widetilde{\varphi}_2\rangle\,\mathrm{d}x$$

by dominated convergence theorem, since $f\in L^1$ and $\nabla\widetilde{\varphi}_2\in L^\infty$ on $\mathbb{R}^n\setminus\mathbb{B}^n$. As a result,

$$\mathrm{I\!I} = -\int_{\mathbb{R}^n\setminus\mathbb{B}^n}\langle f,\nabla\widetilde{\varphi}_2\rangle\,\mathrm{d}x,$$

from which we see that

$$|\mathrm{I\!I}| \le \|f\|_{L^1(\mathbb{R}^n\setminus\mathbb{B}^n)}\|\nabla\widetilde{\varphi}_2\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \le C\lambda^{\frac{1}{n}-1}\|f\|_{L^1(\mathbb{R}^n\setminus\mathbb{B}^n)}\|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}.$$

Together, by choosing $\lambda=\frac{\|f\|_{L^1(\mathbb{R}^n\setminus\mathbb{B}^n)}}{\|f\|_{L^1(\mathbb{S}^{n-1})}}$, we get

$$\Big|\int_{\mathbb{S}^{n-1}}\langle f,\nu\rangle\langle\phi,\nu\rangle\,\mathrm{d}\sigma\Big| \le C\|f\|_{L^1(\mathbb{R}^n\setminus\mathbb{B}^n)}^{\frac{1}{n}}\|f\|_{L^1(\mathbb{S}^{n-1})}^{1-\frac{1}{n}}\|\phi\|_{W^{1,n}(\mathbb{S}^{n-1})}$$

as desired.

We will now deduce Theorem 1.1 from Proposition 2.1. The idea is to average (2.1) over all unit spheres in $\mathbb{R}^n$.

Proof of Theorem 1.1 First, for each fixed $x \in \mathbb{R}^n$, we have

$$\langle f(x), \phi(x)\rangle = c\int_{\mathbb{S}^{n-1}} \langle f(x), \omega\rangle\langle \phi(x), \omega\rangle \,\mathrm{d}\sigma(\omega), \tag{2.2}$$

where we are identifying $\omega \in \mathbb{S}^{n-1}$ with the corresponding unit tangent vector to $\mathbb{R}^n$ based at the point x. Hence to estimate $\int_{\mathbb{R}^n}\langle f(x), \phi(x)\rangle \,\mathrm{d}x$, it suffices to estimate

$$\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} \langle f(x), \omega\rangle\langle \phi(x), \omega\rangle \,\mathrm{d}\sigma(\omega)\,\mathrm{d}x,$$

which is the same as

$$\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} \langle f(z+\omega), \omega\rangle\langle \phi(z+\omega), \omega\rangle \,\mathrm{d}\sigma(\omega)\,\mathrm{d}z$$

by a change of variable $(x,\omega) \mapsto (z+\omega, \omega)$. Now when $z = 0$, the inner integral can be estimated by Proposition 2.1; for a general $z \neq 0$, one can still estimate the inner integral by Proposition 2.1, since the proposition is invariant under translations. Thus the above double integral is bounded, in absolute value, by

$$C\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}\int_{\mathbb{R}^n}\|f(z+\cdot)\|_{L^1(\mathbb{S}^{n-1})}^{1-\frac{1}{n}}\|\phi(z+\cdot)\|_{W^{1,n}(\mathbb{S}^{n-1})}\,\mathrm{d}z.$$

Applying Hölder's inequality to the last integral in z, one bounds this by

$$C\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}\Big(\int_{\mathbb{R}^n}\|f(z+\cdot)\|_{L^1(\mathbb{S}^{n-1})}\,\mathrm{d}z\Big)^{1-\frac{1}{n}}\Big(\int_{\mathbb{R}^n}\|\phi(z+\cdot)\|_{W^{1,n}(\mathbb{S}^{n-1})}^n\,\mathrm{d}z\Big)^{\frac{1}{n}}.$$

Since

$$\int_{\mathbb{R}^n}\|f(z+\cdot)\|_{L^1(\mathbb{S}^{n-1})}\,\mathrm{d}z = c\|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\int_{\mathbb{R}^n}\|\phi(z+\cdot)\|_{W^{1,n}(\mathbb{S}^{n-1})}^n\,\mathrm{d}z \le c(\|\phi\|_{L^n(\mathbb{R}^n)}^n + \|\nabla\phi\|_{L^n(\mathbb{R}^n)}^n),$$

we proved that under the assumption of Theorem 1.1, we have

$$\Big|\int_{\mathbb{R}^n}\langle f, \phi\rangle\Big| \le C\|f\|_{L^1(\mathbb{R}^n)}(\|\nabla\phi\|_{L^n(\mathbb{R}^n)} + \|\phi\|_{L^n(\mathbb{R}^n)}). \tag{2.3}$$

This is almost the desired conclusion, except that we have an additional zeroth order term on ϕ on the right-hand side of the estimate. But that can be scaled away by homogeneity. In fact, if f and ϕ satisfies the assumption of Theorem 1.1, then so does the dilations

$$f_\varepsilon(x) := \varepsilon^{-n}f(\varepsilon^{-1}x), \quad \phi_\varepsilon(x) := \phi(\varepsilon^{-1}x), \quad \varepsilon > 0.$$

Applying (2.3) to f_ε and ϕ_ε instead, we get

$$\Big|\int_{\mathbb{R}^n}\langle f_\varepsilon, \phi_\varepsilon\rangle\Big| \le C\|f_\varepsilon\|_{L^1(\mathbb{R}^n)}(\|\nabla\phi_\varepsilon\|_{L^n(\mathbb{R}^n)} + \|\phi_\varepsilon\|_{L^n(\mathbb{R}^n)}),$$

i.e.,

$$\Big|\int_{\mathbb{R}^n}\langle f, \phi\rangle\Big| \le C\|f\|_{L^1(\mathbb{R}^n)}(\|\nabla\phi\|_{L^n(\mathbb{R}^n)} + \varepsilon\|\phi\|_{L^n(\mathbb{R}^n)}).$$

So, letting $\varepsilon \to 0^+$, we get the desired conclusion of Theorem 1.1.

Proof of Lemma 2.1 Note that $\langle f, \nu\rangle \psi = \langle \psi f, \nu\rangle$, and ν is the inward unit normal to $\partial(\mathbb{R}^n \setminus \mathbb{B}^n)$. So by the divergence theorem on $\mathbb{R}^n$, we have

$$\int_{\mathbb{S}^{n-1}} \langle f, \nu\rangle \psi \,\mathrm{d}\sigma = -\int_{\mathbb{R}^n \setminus \mathbb{B}^n} \operatorname{div}(\psi f)\,\mathrm{d}x.$$

But since $\operatorname{div} f = 0$, we have

$$\operatorname{div}(\psi f) = \langle f, \nabla\psi\rangle + \psi \operatorname{div} f = \langle f, \nabla\psi\rangle,$$

and the desired equality follows.

Proof of Lemma 2.2 Suppose that φ and λ are as in Lemma 2.2. We will construct first a decomposition $\varphi = \varphi_1 + \varphi_2$ on $\mathbb{S}^{n-1}$, so that both φ_1 and φ_2 are smooth on $\mathbb{S}^{n-1}$, and

$$\begin{aligned}
&\|\varphi_1\|_{L^\infty(\mathbb{S}^{n-1})} \leq C\lambda^{\frac{1}{n}} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})},\\
&\|\nabla_{\mathbb{S}^{n-1}}\varphi_2\|_{L^\infty(\mathbb{S}^{n-1})} \leq C\lambda^{\frac{1}{n}-1} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})}.
\end{aligned}$$

(Here $\nabla_{\mathbb{S}^{n-1}}\varphi_2$ is the gradient on $\mathbb{S}^{n-1}$.) Once this is established, the lemma will follow, by extending φ_2 so that it is homogeneous of degree 0; in other words, we will then define

$$\widetilde{\varphi}_2(x) := \varphi_2\Big(\frac{x}{|x|}\Big), \quad |x| \geq 1.$$

It is then straight forward to verify that

$$\|\nabla\widetilde{\varphi}_2\|_{L^\infty(\mathbb{R}^n \setminus \mathbb{B}^n)} \leq C\lambda^{\frac{1}{n}-1} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})},$$

since the radial derivative of $\widetilde{\varphi}_2$ is zero.

To construct the desired decomposition on $\mathbb{S}^{n-1}$, we proceed as follows.

If $\lambda \geq 1$, we set $\varphi_2 = \fint_{\mathbb{S}^n} \varphi$, so that $\nabla_{\mathbb{S}^{n-1}}\varphi_2 = 0$ on $\mathbb{S}^{n-1}$, then

$$\varphi_1 = \varphi - \fint_{\mathbb{S}^n} \varphi,$$

and the estimate for φ_1 follows from the classical Morrey–Sobolev estimate.

If $0 < \lambda < 1$, we pick a non-negative radial cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$, with $\eta = 1$ in a neighborhood of 0, and define $\eta_\lambda(x) = \eta(\lambda^{-1}x)$ for $x \in \mathbb{R}^n$. We then consider the function

$$x \in \mathbb{R}^n \mapsto \int_{\mathbb{S}^{n-1}} \eta_\lambda(x-y)\,\mathrm{d}\sigma(y).$$

When restricted to $x \in \mathbb{S}^{n-1}$, this function is a constant independent of the choice of $x \in \mathbb{S}^{n-1}$, by rotation invariance of the integral. We then write c_λ for this constant, i.e.,

$$c_\lambda := \int_{\mathbb{S}^{n-1}} \eta_\lambda(x-y)\,\mathrm{d}\sigma(y), \quad |x| = 1. \tag{2.4}$$

Note that by our choice of η, when $0 < \lambda < 1$,

$$c_\lambda \simeq \lambda^{n-1}. \tag{2.5}$$

Now we define, for $x \in \mathbb{S}^{n-1}$, that

$$\varphi_2(x) = c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \eta_\lambda(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \quad \varphi_1(x) = \varphi(x) - \varphi_2(x).$$

Then for $x \in \mathbb{S}^{n-1}$, we have

$$\varphi_1(x) = c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \eta_\lambda(x-y)[\varphi(x) - \varphi(y)]\,\mathrm{d}\sigma(y)$$

by definition of c_λ. But for $x, y \in \mathbb{S}^{n-1}$, we have, by Morrey's embedding, that

$$|\varphi(x) - \varphi(y)| \leq C|x-y|^{\frac{1}{n}} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})}.$$

It follows that

$$|\varphi_1(x)| \leq C\|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \eta_\lambda(x-y)|x-y|^{\frac{1}{n}}\,\mathrm{d}\sigma(y).$$

Letting $\widetilde{\eta}(x) = |x|^{\frac{1}{n}}\eta(x)$ and $\widetilde{\eta}_\lambda(x) = \widetilde{\eta}(\lambda^{-1}x)$, we see that the right-hand side above is just

$$C\lambda^{\frac{1}{n}} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \widetilde{\eta}_\lambda(x-y)\,\mathrm{d}\sigma(y).$$

But this last integral can be estimated by

$$\int_{\mathbb{S}^{n-1}} \widetilde{\eta}_\lambda(x-y)\,\mathrm{d}\sigma(y) \lesssim \lambda^{n-1},$$

by the support and L^∞ bound of $\widetilde{\eta}_\lambda$. Hence using also (2.5), we see that

$$\|\varphi_1\|_{L^\infty(\mathbb{S}^{n-1})} \leq C\lambda^{\frac{1}{n}} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})}$$

as desired.

Next, suppose that $x \in \mathbb{S}^{n-1}$, and v is a unit tangent vector to $\mathbb{S}^{n-1}$ at x. Then

$$\partial_v \phi_2(x) = \lambda^{-1} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \langle v, \nabla\eta\rangle(\lambda^{-1}(x-y))\varphi(y)\,\mathrm{d}\sigma(y). \tag{2.6}$$

But if we differentiate both sides of the definition (2.4) of c_λ with respect to ∂_v, we see that

$$0 = \int_{\mathbb{S}^{n-1}} \langle v, \nabla\eta\rangle(\lambda^{-1}(x-y))\,\mathrm{d}\sigma(y). \tag{2.7}$$

Multiplying (2.7) by $\lambda^{-1}c_\lambda^{-1}\phi(x)$, and subtracting that from (2.6), we get

$$\partial_v \phi_2(x) = \lambda^{-1} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \langle v, \nabla\eta\rangle(\lambda^{-1}(x-y))[\varphi(y) - \varphi(x)]\,\mathrm{d}\sigma(y).$$

Using Morrey's embedding again, we see that

$$|\partial_v \phi_2(x)| \leq C\lambda^{-1}\|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} |\langle v, \nabla\eta\rangle|(\lambda^{-1}(x-y))|x-y|^{\frac{1}{n}}\,\mathrm{d}\sigma(y).$$

Letting $\overline{\eta}(x) = |x|^{\frac{1}{n}}|\langle v, \nabla\eta\rangle|(x)$ and $\overline{\eta}_\lambda(x) = \overline{\eta}(\lambda^{-1}x)$, we see that the right-hand side above is just

$$C\lambda^{\frac{1}{n}}\lambda^{-1} \|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})} c_\lambda^{-1} \int_{\mathbb{S}^{n-1}} \overline{\eta}_\lambda(x-y)\,\mathrm{d}\sigma(y).$$

But this last integral can be estimated by

$$\int_{\mathbb{S}^{n-1}} \overline{\eta}_\lambda(x-y)\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(y) \lesssim \lambda^{n-1},$$

by the support and L^∞ bound of $\overline{\eta}_\lambda$. Hence using also (2.5), we see that

$$\|\nabla_{\mathbb{S}^{n-1}}\varphi_2\|_{L^\infty(\mathbb{S}^{n-1})} \leq C\lambda^{\frac{1}{n}-1}\|\nabla_{\mathbb{S}^{n-1}}\varphi\|_{L^n(\mathbb{S}^{n-1})}$$

as desired.

3 A Borderline Sobolev Embedding on the Real Hyperbolic Space $\mathbb{H}^n$

We now turn to a corresponding result on the real hyperbolic space $\mathbb{H}^n$. We will first give a direct proof in this current section, in the spirit of the earlier proof of Theorem 1.1 by using spherical averages. In the appendix, we give a less direct proof, using a variant of Theorem 1.1 on $\mathbb{R}^n$.

First we need some notations. We will use the upper half space model for the hyperbolic space. In other words, we take $\mathbb{H}^n$ to be

$$\mathbb{H}^n = \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon x_n > 0\},$$

and the metric on $\mathbb{H}^n$ to be

$$g := \frac{|\,\mathrm{d}x|^2}{x_n^2}.$$

We will use the following orthonormal frame of vector fields:

$$e_i := x_n \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n$$

at every point of $\mathbb{H}^n$. Note that if $j \neq n$, then

$$\nabla_{e_n} e_j = 0. \tag{3.1}$$

(Here $\nabla = \nabla_g$ is the Levi-Civita connection with respect to the hyperbolic metric g.) In fact, since $\{e_1, \cdots, e_n\}$ is an orthonormal basis, for any $k = 1, \cdots, n$, we have

$$\begin{aligned}\langle \nabla_{e_n} e_j, e_k\rangle_g &= \frac{1}{2}(\langle [e_n, e_j], e_k\rangle_g - \langle [e_n, e_k], e_j\rangle_g - \langle [e_j, e_k], e_n\rangle_g) \\ &= \frac{1}{2}(\langle e_j, e_k\rangle_g - \langle (1-\delta_{kn})e_k, e_j\rangle_g - \langle -\delta_{kn} e_j, e_n\rangle_g) = 0.\end{aligned}$$

Also, we have

$$\nabla_{e_n} e_n = 0. \tag{3.2}$$

This is because if $j \neq n$, then

$$\langle \nabla_{e_n} e_n, e_j\rangle_g = -\langle e_n, \nabla_{e_n} e_j\rangle_g = 0$$

by (3.1), and

$$\langle \nabla_{e_n} e_n, e_n\rangle_g = \frac{1}{2} e_n(\langle e_n, e_n\rangle_g) = 0.$$

To prove Theorem 1.2, note that we only need to consider the case $n \geq 2$, since when $n = 1$,

$$\|\phi\|_{L^\infty(\mathbb{H}^1)} \leq \int_0^\infty |\partial_y \phi(y)| \mathrm{d}y = \|\nabla_g \phi\|_{L^1(\mathbb{H}^1)},$$

and (1.2) follows trivially. Hence from now on we assume $n \geq 2$.

We will deduce Theorem 1.2 from the following proposition.

Proposition 3.1 *Assume $n \geq 2$. Let f, ϕ be as in Theorem* 1.2*. Write S for the copy of $(n-1)$-dimensional hyperbolic space inside $\mathbb{H}^n$, given by*

$$S = \{x \in \mathbb{H}^n : x_1 = 0\},$$

and X for the half-space

$$X = \{x \in \mathbb{H}^n : x_1 > 0\},$$

so that S is the boundary of X. Also write $\mathrm{d}V'_g$ for the volume measure on S with respect to the hyperbolic metric on S, and $\nu = e_1$ for the unit normal to S. Then

$$\Big| \int_S \langle f, \nu \rangle_g \langle \phi, \nu \rangle_g \, \mathrm{d}V'_g \Big| \leq C \|f\|_{L^1(X)}^{\frac{1}{n}} \|f\|_{L^1(S)}^{1-\frac{1}{n}} \|\phi\|_{W^{1,n}(S)}. \tag{3.3}$$

Here $\|\phi\|_{W^{1,n}(S)} = \|\phi\|_{L^n(S)} + \|\nabla_g \phi\|_{L^n(S)}$, and all integrals on S on the right-hand side are with respect to $\mathrm{d}V'_g$.

The S will be called a vertical hyperplane in $\mathbb{H}^n$. It is a totally geodesic submanifold of $\mathbb{H}^n$. We will consider all hyperbolic hyperplanes in $\mathbb{H}^n$, that is the image of S under all isometries of $\mathbb{H}^n$. The set of all such hypersurfaces in $\mathbb{H}^n$ will be denoted by $\mathcal{S}$; it will consist of all Euclidean parallel translates of S in the x'-direction, and all Euclidean northern hemispheres whose centers lie on the plane $\{x_n = 0\}$.

The proof of Proposition 3.1 in turn depends on the following two lemmas. The first one is a simple lemma about integration by parts, which is the counterpart of Lemma 2.1.

Lemma 3.1 *Assume $n \geq 2$. Let f, S, X, ν be as in Proposition* 3.1*. Then for any compactly supported smooth function ψ on $\mathbb{H}^n$, we have*

$$\int_S \langle f, \nu \rangle_g \psi \, \mathrm{d}V'_g = -\int_X \langle f, \nabla_g \psi \rangle_g \, \mathrm{d}V_g.$$

The second one is a decomposition lemma for functions on S, which is the counterpart of Lemma 2.2.

Lemma 3.2 *Assume $n \geq 2$. Let φ be a smooth function with compact support on S. For any $\lambda > 0$, there exists a decomposition*

$$\varphi = \varphi_1 + \varphi_2 \quad \textit{on } S,$$

and an extension $\widetilde{\varphi}_2$ of φ_2 to $\mathbb{H}^n$, such that $\widetilde{\varphi}_2$ is smooth with compact support on $\mathbb{H}^n$, and

$$\|\varphi_1\|_{L^\infty(S)} \leq C \lambda^{\frac{1}{n}} \|\nabla_g \varphi\|_{L^n(S)}$$

with

$$\|\nabla_g \widetilde{\varphi}_2\|_{L^\infty(\mathbb{H}^n)} \leq C \lambda^{\frac{1}{n}-1} \|\nabla_g \varphi\|_{L^n(S)}.$$

We postpone the proofs of Lemmas 3.1–3.2 to the end of this section.

Now we are ready for the proof of Proposition 3.1.

Proof of Proposition 3.1 Let f,ϕ be as in the statement of Theorem 1.2. Apply Lemma 3.2 to $\varphi = \langle \phi, \nu\rangle_g$, where $\lambda > 0$ is to be chosen. Then since

$$\|\nabla_g \langle \phi, \nu\rangle_g\|_{L^n(S)} \leq C\|\phi\|_{W^{1,n}(S)},$$

(this follows since $|e_k\langle \phi, \nu\rangle| = |\langle \nabla_{e_k}\phi, \nu\rangle + \langle \phi, \nabla_{e_k}\nu\rangle| \leq |\nabla_g \phi|_g + |\phi|_g$ for all k), there exists a decomposition

$$\langle \phi, \nu\rangle_g = \varphi_1 + \varphi_2 \quad \text{on } S,$$

and an extension $\widetilde{\varphi}_2$ of φ_2 to $\mathbb{H}^n$, such that $\widetilde{\varphi}_2 \in C_c^\infty(\mathbb{H}^n)$,

$$\|\varphi_1\|_{L^\infty(S)} \leq C\lambda^{\frac{1}{n}}\|\phi\|_{W^{1,n}(S)}$$

and

$$\|\nabla_g \widetilde{\varphi}_2\|_{L^\infty(\mathbb{H}^n)} \leq C\lambda^{\frac{1}{n}-1}\|\phi\|_{W^{1,n}(S)}.$$

Now

$$\begin{aligned}\int_S \langle f, \nu\rangle_g \langle \phi, \nu\rangle_g \, \mathrm{d}V_g' &= \int_S \langle f, \nu\rangle_g \varphi_1 \, \mathrm{d}V_g' + \int_S \langle f, \nu\rangle_g \varphi_2 \, \mathrm{d}V_g' \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

In the first term, we estimate trivially

$$|\mathrm{I}| \leq \|f\|_{L^1(S)}\|\varphi_1\|_{L^\infty(S)} \leq C\lambda^{\frac{1}{n}}\|f\|_{L^1(S)}\|\phi\|_{W^{1,n}(S)}.$$

In the second term, we first integrate by parts using Lemma 3.1, with $\psi = \widetilde{\varphi}_2$, and obtain

$$\mathrm{II} = -\int_X \langle f, \nabla_g \widetilde{\varphi}_2\rangle_g \, \mathrm{d}V_g,$$

so

$$\begin{aligned}|\mathrm{II}| &\leq \|f\|_{L^1(X)}\|\nabla_g \widetilde{\varphi}_2\|_{L^\infty(\mathbb{H}^n)} \\ &\leq C\lambda^{\frac{1}{n}-1}\|f\|_{L^1(X)}\|\phi\|_{W^{1,n}(S)}.\end{aligned}$$

Together, by choosing $\lambda = \frac{\|f\|_{L^1(X)}}{\|f\|_{L^1(S)}}$, we get

$$\left|\int_S \langle f, \nu\rangle_g \langle \phi, \nu\rangle_g \, \mathrm{d}V_g'\right| \leq C\|f\|_{L^1(X)}^{\frac{1}{n}}\|f\|_{L^1(S)}^{1-\frac{1}{n}}\|\phi\|_{W^{1,n}(S)}$$

as desired.

We now deduce Theorem 1.2 from Proposition 3.1. The idea is to average (3.3) over all images of S under isometries in $\mathbb{H}^n$ (i.e., all hypersurfaces in the collection $\mathcal{S}$).

Proof of Theorem 1.2 First, for each fixed $x = (x', x_n) \in \mathbb{H}^n$, we have the following analogue of the identity (2.2), used in the proof of Theorem 1.1:

$$\langle f(x), \phi(x)\rangle_g = c\int_{\mathbb{S}^{n-1}} \langle f(x), x_n\omega\rangle_g \langle \phi(x), x_n\omega\rangle_g \, \mathrm{d}\sigma(\omega).$$

Here we are identifying $\omega \in \mathbb{S}^{n-1}$ with the corresponding tangent vector to $\mathbb{H}^n$ based at the point x. (Note then $x_n\omega$ has length 1 with respect to the metric g at x, so $x_n\omega$ belongs to the unit sphere bundle at x.) Furthermore, since the above integrand is even in ω, we may replace the integral over $\mathbb{S}^{n-1}$ by the integral only over the northern hemisphere $\mathbb{S}^{n-1}_+ := \{\omega \in \mathbb{S}^{n-1} \colon \omega_n > 0\}$. Hence to estimate $\int_{\mathbb{H}^n} \langle f(x), \phi(x)\rangle_g \, \mathrm{d}V_g$, it suffices to estimate

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{S}^{n-1}_+} \langle f(x), x_n\omega\rangle_g \langle \phi(x), x_n\omega\rangle_g \; \mathrm{d}\sigma(\omega) \frac{\mathrm{d}x}{x_n^n}. \tag{3.4}$$

We will compute this integral by making a suitable change of variables.

To do so, given $x \in \mathbb{R}^n_+$ and $\omega \in \mathbb{S}^{n-1}_+$, let $S(x,\omega)$ be the hyperbolic hypersurface in $\mathcal{S}$ passing through x with normal vector ω at x. In other words, $S(x,\omega)$ would be a Euclidean hemisphere, with center on the plane $\{x_n = 0\}$; we denote the center of this Euclidean hemisphere by $(z,0)$, where $z = z(x,\omega)$.

For each fixed $x \in \mathbb{R}^n_+$, the map $\omega \mapsto z(x,\omega)$ provides an invertible change of variables from $\mathbb{S}^{n-1}_+$ to $\mathbb{R}^{n-1}$. Thus we are led to parametrize the integral in (3.4) by z instead of ω. In order to do that, we observe that the vectors $x - (z,0)$ and ω are collinear. This implies that if $z = z(x,\omega)$, then

$$\omega = \frac{x-(z,0)}{|x-(z,0)|}.$$

(Here $|x-(z,0)|$ is the Euclidean norm of $x-(z,0)$.) Write $\Phi_x(z)$ for the right-hand side of the above equation. We view Φ_x as a map $\Phi_x : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}_+ \subset \mathbb{R}^n$, and compute the Jacobian of the map. We have

$$(D\Phi_x)^t(z) = \frac{1}{|x-(z,0)|}\Big((-I,0) + \frac{(x'-z)\otimes(x-(z,0))^t}{|x-(z,0)|^2}\Big).$$

(Here we think of x, z as column vectors, and $D\Phi_x$ as an $(n-1)\times n$ matrix.) Thus

$$(D\Phi_x)^t D\Phi_x(z) = \frac{1}{|x-(z,0)|^2}\Big(I - \frac{(x'-z)\otimes(x'-z)^t}{|x-(z,0)|^2}\Big).$$

By computing the determinant in a basis that contains $x'-z$, we get

$$\begin{aligned}
\operatorname{Jac}\Phi_x(z) &= \sqrt{\det[(D\Phi_x)^t D\Phi_x(z)]} \\
&= \frac{1}{|x-(z,0)|^{n-1}}\sqrt{1 - \frac{|x'-z|^2}{|x-(z,0)|^2}} \\
&= \frac{x_n}{|x-(z,0)|^n}.
\end{aligned}$$

By a change of variable $\omega = \Phi_x(z)$, and using Fubini's theorem, we see that (3.4) is equal to

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_+} \frac{\langle f(x), x_n\Phi_x(z)\rangle_g \langle \phi(x), x_n\Phi_x(z)\rangle_g}{|x-(z,0)|^n} \, \frac{\mathrm{d}x}{x_n^{n-1}} \, \mathrm{d}z. \tag{3.5}$$

Now we fix $z \in \mathbb{R}^{n-1}$, and compute the inner integral over x by integrating over successive hemispheres of radius r centered at $(z,0)$. More precisely, let $S(z,r)$ be the Euclidean northern hemisphere with center $(z,0)$ and of radius $r>0$. Then $S(z,r)\in\mathcal{S}$, and for any $z\in\mathbb{R}^{n-1}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\frac{\langle f(x),x_n\Phi_x(z)\rangle_g\langle\phi(x),x_n\Phi_x(z)\rangle_g}{|x-(z,0)|^n}\frac{\mathrm{d}x}{x_n^{n-1}}\\&=\int_0^\infty\int_{x\in S(z,r)}\langle f(x),x_n\Phi_x(z)\rangle_g\langle\phi(x),x_n\Phi_x(z)\rangle_g\,\frac{\mathrm{d}\sigma(x)}{x_n^{n-1}}\frac{\mathrm{d}r}{r^n},\end{aligned}$$

where $\mathrm{d}\sigma(x)$ is the Euclidean surface measure on $S(z,r)$. However, if $x\in S(z,r)$, then $x_n\Phi_x(z)$ is precisely the upward unit normal to $S(z,r)$ at x. Also, if $\mathrm{d}V_g'$ is the induced surface measure on $S(z,r)$ from the hyperbolic metric on $\mathbb{H}^n$, then

$$\mathrm{d}V_g'=\frac{\mathrm{d}\sigma(x)}{x_n^{n-1}r};$$

indeed if we write $\omega=\frac{x-(z,0)}{|x-(z,0)|}$, then at $x\in S(z,r)$, we have

$$\mathrm{d}V_g'=i_{x_n\omega}\,\mathrm{d}V_g=i_{x_n\omega}\frac{\mathrm{d}x}{x_n^n}=\frac{i_{r\omega}\,\mathrm{d}x}{x_n^{n-1}r}=\frac{\mathrm{d}\sigma(x)}{x_n^{n-1}r}.$$

(Here i denotes the interior product of a vector with a differential form.) Hence the integral (3.5) is just equal to

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{S(z,r)}\langle f,\nu\rangle_g\langle\phi,\nu\rangle_g\,\mathrm{d}V_g'\frac{\mathrm{d}r}{r^{n-1}}\,\mathrm{d}z. \tag{3.6}$$

By Proposition 3.1 and its invariance under isometries of the hyperbolic space $\mathbb{H}^n$, we have

$$\begin{aligned}&\Big|\int_{S(z,r)}\langle f,\nu\rangle_g\langle\phi,\nu\rangle_g\,\mathrm{d}V_g'\Big|\\&\le C\|f\|_{L^1(\mathbb{H}^n)}^{\frac1n}\Big(\int_{S(z,r)}|f|_g\,\mathrm{d}V_g'\Big)^{1-\frac1n}\Big(\int_{S(z,r)}(|\nabla_g\phi|_g^n+|\phi|_g^n)\,\mathrm{d}V_g'\Big)^{\frac1n}.\end{aligned}$$

Hence by Hölder's inequality, (3.6) is bounded by

$$\begin{aligned}&C\|f\|_{L^1(\mathbb{H}^n)}^{\frac1n}\Big(\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{S(z,r)}|f|_g\,\mathrm{d}V_g'\frac{\mathrm{d}r}{r^{n-1}}\,\mathrm{d}z\Big)^{1-\frac1n}\\&\cdot\Big(\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{S(z,r)}(|\nabla_g\phi|_g^n+|\phi|_g^n)\,\mathrm{d}V_g'\frac{\mathrm{d}r}{r^{n-1}}\,\mathrm{d}z\Big)^{\frac1n}.\end{aligned}$$

But undoing our earlier changes of variable, we see that

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{S(z,r)}|f|_g\,\mathrm{d}V_g'\frac{\mathrm{d}r}{r^{n-1}}\,\mathrm{d}z\\&=\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{x\in S(z,r)}|f(x)|_g\,\frac{\mathrm{d}\sigma(x)}{x_n^{n-1}}\frac{\mathrm{d}r}{r^n}\,\mathrm{d}z\\&=\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n_+}\frac{|f(x)|_g}{|x-(z,0)|^n}\frac{\mathrm{d}x}{x_n^{n-1}}\,\mathrm{d}z\\&=\int_{\mathbb{R}^n_+}|f(x)|_g\Big(\int_{\mathbb{R}^{n-1}}\frac{x_n}{|x-(z,0)|^n}\,\mathrm{d}z\Big)\frac{\mathrm{d}x}{x_n^n}\\&=C\|f\|_{L^1(\mathbb{H}^n)}.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{S(z,r)} (|\nabla_g \phi|_g^n + |\phi|_g^n) \, \mathrm{d}V_g' \frac{\mathrm{d}r}{r^{n-1}} \, \mathrm{d}z = C(\|\nabla_g \phi\|_{L^n(\mathbb{H}^n)}^n + \|\phi\|_{L^n(\mathbb{H}^n)}^n).$$

Altogether, (3.6) (and hence (3.4)) is bounded by

$$C\|f\|_{L^1(\mathbb{H}^n)}(\|\nabla_g \phi\|_{L^n(\mathbb{H}^n)} + \|\phi\|_{L^n(\mathbb{H}^n)}).$$

This is almost what we want, except that on the right-hand side we have an additional $\|\phi\|_{L^n(\mathbb{H}^n)}$. To fix this, one applies Lemma 3.3 below, with $p = n$, and the desired conclusion of Theorem 1.2 follows.

Lemma 3.3 *Assume $n \geq 2$. For any compactly supported smooth vector field ϕ on $\mathbb{H}^n$, and any $1 \leq p < \infty$, we have*

$$\|\phi\|_{L^p(\mathbb{H}^n)} \leq C\|\nabla_g \phi\|_{L^p(\mathbb{H}^n)}.$$

Proof In fact, for any function $\Phi \in C_c^\infty(\mathbb{H}^n)$, and any exponent $1 \leq p < \infty$, we have, from Hardy's inequality, that

$$\|\Phi\|_{L^p(\mathbb{H}^n)} \leq C\|e_n \Phi\|_{L^p(\mathbb{H}^n)}. \tag{3.7}$$

This is because

$$\int_0^\infty |\Phi(x)|^p \frac{\mathrm{d}x_n}{x_n^n} \leq \left(\frac{p}{n-1}\right)^p \int_0^\infty |e_n \Phi|^p(x) \frac{\mathrm{d}x_n}{x_n^n}$$

by Hardy's inequality. (3.7) then follows by integrating over all $x' \in \mathbb{R}^{n-1}$ with respect to $\mathrm{d}x'$. Now we apply (3.7) to $\Phi = \langle \phi, e_j \rangle$, $1 \leq j \leq n-1$. In view of (3.1), we have

$$e_n \Phi = \langle \nabla_{e_n} \phi, e_j \rangle + \langle \phi, \nabla_{e_n} e_j \rangle = \langle \nabla_{e_n} \phi, e_j \rangle,$$

so we get

$$\|\langle \phi, e_j \rangle\|_{L^p(\mathbb{H}^n)} \leq C\|\nabla_g \phi\|_{L^p(\mathbb{H}^n)}.$$

Similarly, we can apply (3.7) to $\Phi = \langle \phi, e_n \rangle$, and use (3.2) in place of (3.1). Altogether, we see that

$$\|\phi\|_{L^p(\mathbb{H}^n)} \leq C\|\nabla_g \phi\|_{L^p(\mathbb{H}^n)},$$

as desired.

We now turn to the proofs of Lemmas 3.1–3.2.

Proof of Lemma 3.1 Note that $\langle f, \nu \rangle_g \psi = \langle \psi f, \nu \rangle_g$, and ν is the inward unit normal to ∂X. Also $\mathrm{d}V_g'$ agrees with the induced surface measure on S from $\mathbb{H}^n$. So by the divergence theorem on $\mathbb{H}^n$, we have

$$\int_S \langle f, \nu \rangle_g \psi \, \mathrm{d}V_g' = -\int_X \mathrm{div}_g \, (\psi f) \, \mathrm{d}V_g.$$

But since $\mathrm{div}_g f = 0$, we have

$$\mathrm{div}_g(\psi f) = \langle f, \nabla_g \psi \rangle_g + \psi \, \mathrm{div}_g f = \langle f, \nabla_g \psi \rangle_g,$$

and the desired equality follows.

The proof of Lemma 3.2 will be easy, once we establish the following lemma.

Lemma 3.4 *Let φ be a smooth function with compact support on $\mathbb{H}^m$, $m \geq 1$. For any $p > m$ and $\lambda > 0$, there exists a decomposition*

$$\varphi = \varphi_1 + \varphi_2 \quad \text{on } \mathbb{H}^m,$$

such that φ_2 is smooth with compact support on $\mathbb{H}^m$, and

$$\|\varphi_1\|_{L^\infty(\mathbb{H}^m)} \leq C\lambda^{1-\frac{m}{p}} \|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}$$

with

$$\|\nabla_g \varphi_2\|_{L^\infty(\mathbb{H}^m)} \leq C\lambda^{-\frac{m}{p}} \|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}.$$

We postpone the proof of this lemma to the end of this section.

Proof of Lemma 3.2 Suppose that φ and λ are as in Lemma 3.2. We identify S with $\mathbb{H}^m$, where $m = n-1$. (This is possible because the restriction of the metric of $\mathbb{H}^n$ to S induces a metric on S that is isometric to that of $\mathbb{H}^m$.) Using Lemma 3.4, with $p = n$, we obtain a decomposition $\varphi = \varphi_1 + \varphi_2$ on S, such that

$$\|\varphi_1\|_{L^\infty(S)} \leq C\lambda^{\frac{1}{n}} \|\nabla_g \varphi\|_{L^n(S)}$$

with

$$\|\nabla_g \varphi_2\|_{L^\infty(S)} \leq C\lambda^{\frac{1}{n}-1} \|\nabla_g \varphi\|_{L^n(S)}. \tag{3.8}$$

We then extend φ_2 to $\mathbb{H}^n$ by setting for $(x_1, x'', x_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}_+$,

$$\widetilde{\varphi}_2(x_1, x'', x_n) = \varphi_2\big(0, x'', \sqrt{x_1^2 + x_n^2}\big).$$

One immediately checks that $\widetilde{\varphi}_2$ is smooth with compact support on $\mathbb{H}^n$, with

$$\|\nabla_g \widetilde{\varphi}_2\|_{L^\infty(\mathbb{H}^n)} \leq \|\nabla_g \varphi_2\|_{L^\infty(S)}.$$

In view of (3.8), we obtain the desired bound for $\nabla_g \widetilde{\varphi}_2$.

It remains to prove Lemma 3.4.

Proof of Lemma 3.4 When $m = 1$, the 1-dimensional hyperbolic space $\mathbb{H}^1$ is isometric to $\mathbb{R}$, and Lemma 3.4 follows from its counterpart on $\mathbb{R}$ (see, e.g., [9]). Alternatively, it will follow from our treatment in the case $m \geq 2$, $0 < \lambda < 1$ below.

So assume from now on $m \geq 2$. Suppose that φ is smooth with compact support on $\mathbb{H}^m$, $p > m$ and $\lambda > 0$. We will construct our desired decomposition $\varphi = \varphi_1 + \varphi_2$. Recall that since $p > m$, the Morrey inequality on $\mathbb{H}^m$ implies

$$\|\varphi\|_{L^\infty(\mathbb{H}^m)} \leq C\|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}. \tag{3.9}$$

To see this, let $\zeta \in C_c^\infty(\mathbb{R})$ be a cut-off function, such that $\zeta(s) = 1$ if $|s| \leq \frac{1}{2}$, and $\zeta(s) = 0$ if $|s| \geq 1$. Let $x_0 := (0, 1) \in \mathbb{H}^m$, and let $\zeta_{x_0}(x) := \zeta(d(x, x_0))$, where d is the hyperbolic distance on $\mathbb{H}^m$. Consider the localization $\zeta_{x_0}\varphi$ of φ, to the unit ball centered at x_0. It satisfies

$$\|\zeta_{x_0}\varphi\|_{W^{1,p}(\mathbb{R}^m)} \leq C\|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}, \tag{3.10}$$

where the left-hand side is a shorthand for

$$\|\zeta_{x_0}\varphi\|_{L^p(\mathbb{R}^m)} + \|\nabla_e(\zeta_{x_0}\varphi)\|_{L^p(\mathbb{R}^m)}.$$

Here ∇_e denotes the Euclidean gradient of a function. (3.10) holds because by the support of ζ_{x_0}, we have

$$\|\zeta_{x_0}\varphi\|_{L^p(\mathbb{R}^m)} \simeq \|\zeta_{x_0}\varphi\|_{L^p(\mathbb{H}^m)} \leq \|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)}$$

and

$$\begin{aligned}\|\nabla_e(\zeta_{x_0}\varphi)\|_{L^p(\mathbb{R}^m)} &\leq \|(\nabla_e\zeta_{x_0})\varphi\|_{L^p(\mathbb{R}^m)} + \|\zeta_{x_0}(\nabla_e\varphi)\|_{L^p(\mathbb{R}^m)} \\ &\leq C(\|\varphi\|_{L^p(\mathbb{H}^m)} + \|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)}) \\ &\leq C\|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)},\end{aligned}$$

where we have used Lemma 3.3 in the last inequalities (note that Lemma 3.3 applies now, since $m \geq 2$). In particular, by Morrey's inequality on $\mathbb{R}^m$, from (3.10), we get

$$|\zeta_{x_0}(x)\varphi(x) - \zeta_{x_0}(y)\varphi(y)| \leq C\|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)}|x-y|^{1-\frac{m}{p}}$$

for all $x, y \in \mathbb{R}^m$. Taking $x = x_0$ and $y \in \mathbb{H}^m$ such that $d(y, x_0) = 2$, we get

$$|\varphi(x_0)| \leq C\|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)}.$$

Since the isometry group of $\mathbb{H}^m$ acts transitively on $\mathbb{H}^m$, and since the right-hand side of the above inequality is invariant under isometries, we obtain

$$|\varphi(x)| \leq C\|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)}$$

for all $x \in \mathbb{H}^m$, and hence (3.9).

In particular, in view of (3.9), when $\lambda \geq 1$, it suffices to take $\varphi_1 = \varphi$ and $\varphi_2 = 0$. We then get the desired estimates for φ_1 and φ_2 trivially.

On the other hand, suppose now $0 < \lambda < 1$. We fix a compactly supported smooth function $\eta \in C_c^\infty(\mathbb{R}^m)$, with

$$\int_{\mathbb{R}^m} \eta(v)\,\mathrm{d}v = 1.$$

For $x = (x', x_m) \in \mathbb{H}^m$, we define

$$\varphi_2(x) = \int_{\mathbb{R}^m} \varphi(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m})\lambda^{-m}\eta(\lambda^{-1}v)\,\mathrm{d}v,$$

where we wrote $v = (v', v_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, and define

$$\varphi_1(x) = \varphi(x) - \varphi_2(x).$$

Note that φ_2 is smooth with compact support on $\mathbb{H}^m$, and hence so is φ_1. Now for $i = 1, 2, \cdots, m-1$, we have

$$(e_i\varphi_2)(x, y) = \int_{\mathbb{R}^m} \mathrm{e}^{-v_m}(e_i\varphi)(x' + x_m u' \mathrm{e}^{v_m}, x_m \mathrm{e}^{v_m})\lambda^{-m}\eta(\lambda^{-1}v)\,\mathrm{d}v.$$

Since $v \mapsto \eta(\lambda^{-1}v)$ has support uniformly bounded with respect to $0 < \lambda < 1$, we have $\mathrm{e}^{-v_m} \leq C$ on the support of the integral, where C is independent of $0 < \lambda < 1$. Hence by Hölder's inequality, we have

$$\begin{aligned}
&\|e_i\varphi_2\|_{L^\infty(\mathbb{H}^m)} \\
&\leq C\Big(\int_{\mathbb{R}^m} |e_i\varphi|^p(x' + x_m\mathrm{e}^{v_m}v', x_m\mathrm{e}^{v_m})\,\mathrm{d}v\Big)^{\frac{1}{p}} \|\lambda^{-m}\eta(\lambda^{-1}v)\|_{L^{p'}(\mathrm{d}v)} \\
&= C\lambda^{-\frac{m}{p}}\Big(\int_{\mathbb{R}^m_+} |e_i\varphi|^p(z)\frac{\mathrm{d}z}{z_m^m}\Big)^{\frac{1}{p}},
\end{aligned}$$

the last line following from the changes of variables $z_m = \mathrm{e}^{v_m}$, and then $z' = x' + z_m v'$. We thus see that

$$\|e_i\varphi_2\|_{L^\infty(\mathbb{H}^m)} \leq C\lambda^{-\frac{m}{p}}\|e_i\varphi\|_{L^p(\mathbb{H}^m)},$$

as desired.

Furthermore, when $i = m$,

$$\begin{aligned}
(e_m\varphi_2)(x) = \int_{\mathbb{R}^m} \Big[&(e_m\varphi)(x' + x_m v'\mathrm{e}^{v_m}, x_m\mathrm{e}^{v_m}) \\
&+ \sum_{i=1}^{m-1} v_i(e_i\varphi)(x' + x_m v'\mathrm{e}^{v_m}, x_m\mathrm{e}^{v_m})\Big]\lambda^{-m}\eta(\lambda^{-1}v)\,\mathrm{d}v.
\end{aligned}$$

Using that $|v_i| \leq C$ on the support of the integrals (uniformly in $0 < \lambda < 1$), and Hölder's inequality as above, we see that

$$\|e_m\varphi_2\|_{L^\infty(\mathbb{H}^m)} \leq C\lambda^{-\frac{m}{p}}\|\nabla_g\varphi\|_{L^p(\mathbb{H}^m)},$$

as desired.

Finally, to estimate φ_1, note that

$$\varphi(x) = \lim_{\lambda\to 0^+}\int_{\mathbb{R}^m}\varphi(x' + x_m\mathrm{e}^{v_m}v', x_m\mathrm{e}^{v_m})\lambda^{-m}\eta(\lambda^{-1}v)\,\mathrm{d}v.$$

Hence

$$\begin{aligned}
\varphi_1(x) &= \varphi(x) - \varphi_2(x) \\
&= -\int_0^\lambda\int_{\mathbb{R}^m}\varphi(x' + x_m\mathrm{e}^{v_m}v', x_m\mathrm{e}^{v_m})\frac{\mathrm{d}}{\mathrm{d}s}[s^{-m}\eta(s^{-1}v)]\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

But

$$-\frac{\mathrm{d}}{\mathrm{d}s}[s^{-m}\eta(s^{-1}v)] = \sum_{i=1}^m\frac{\partial}{\partial v_i}[s^{-m}(v_i\eta)(s^{-1}v)],$$

so we can plug this back in the equation for φ_1, and integrate by parts in v. Now

$$\frac{\partial}{\partial v_i}[\varphi(x' + x_m\mathrm{e}^{v_m}v', x_m\mathrm{e}^{v_m})] = (e_i\varphi)(x' + x_m\mathrm{e}^{v_m}v', x_m\mathrm{e}^{v_m}), \quad i = 1, \cdots, m-1$$

and

$$\frac{\partial}{\partial v_m}[\varphi(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m})] = (e_m\varphi)(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m}) + \sum_{i=1}^{m-1} v_i (e_i\varphi)(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m}).$$

Hence

$$\begin{aligned}\varphi_1(x) = &-\int_0^\lambda \sum_{i=1}^{m} \int_{\mathbb{R}^m} (e_i\varphi)(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m}) s^{-m} (v_i\eta)(s^{-1}v)\,\mathrm{d}v\,\mathrm{d}s \\ &-\int_0^\lambda \sum_{i=1}^{m-1} \int_{\mathbb{R}^m} v_i (e_i\varphi)(x' + x_m \mathrm{e}^{v_m} v', x_m \mathrm{e}^{v_m}) s^{-m} (v_i\eta)(s^{-1}v)\,\mathrm{d}v\,\mathrm{d}s.\end{aligned}$$

When $0 < \lambda < 1$, the integral in v in each term can now be estimated by Hölder's inequality, yielding

$$\begin{aligned}\|\varphi_1\|_{L^\infty(\mathbb{H}^n)} &\le \int_0^\lambda C s^{-\frac{m}{p}} \|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}\,\mathrm{d}s \\ &= C\lambda^{1-\frac{m}{p}} \|\nabla_g \varphi\|_{L^p(\mathbb{H}^m)}.\end{aligned}$$

This completes the proof of Lemma 3.4.

Appendix A Indirect Proof of Theorem 1.2

We will give an alternative proof of Theorem 1.2 from the following variant of Theorem 1.1, whose proof can be found, for instance, in [9] (it can also be deduced by a small modification of the proof we gave above of Theorem 1.1).

Proposition A.1 (see [9]) *Suppose that f is a smooth vector field on $\mathbb{R}^n$ (not necessarily* $\operatorname{div} f = 0$)*. Then for any compactly supported smooth vector field ϕ on $\mathbb{R}^n$, we have*

$$\Big|\int_{\mathbb{R}^n} \langle f, \phi\rangle\Big| \le C(\|f\|_{L^1}\|\nabla\phi\|_{L^n} + \|\operatorname{div} f\|_{L^1}\|\phi\|_{L^n}), \tag{A.1}$$

where $\langle\cdot,\cdot\rangle$ is the pointwise Euclidean inner product of two vector fields in $\mathbb{R}^n$.

To prove Theorem 1.2, we consider a function $\zeta \in C_c^\infty(\mathbb{R})$ and we define for $\alpha \in \mathbb{H}^n$ the function $\zeta_\alpha : \mathbb{H}^n \to \mathbb{R}$ by

$$\zeta_\alpha(x) = \zeta(d(x,\alpha)).$$

We assume that

$$\int_{\mathbb{H}^n} \zeta_\alpha^2(x)\,\mathrm{d}V_g(\alpha) = 1.$$

Now given vector fields f and ϕ as in Theorem 1.2, we write

$$\int_{\mathbb{H}^n} \langle f, \phi\rangle_g\,\mathrm{d}V_g = \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \langle \zeta_\alpha f(x), \zeta_\alpha\phi(x)\rangle_g\,\mathrm{d}V_g(z)\,\mathrm{d}V_g(\alpha).$$

If $\alpha = (0,1) \in \mathbb{H}^n$, then

$$\int_{\mathbb{H}^n} \langle \zeta_\alpha f(x), \zeta_\alpha\phi(x)\rangle_g\,\mathrm{d}V_g = \int_{\mathbb{R}^n_+} \Big\langle \zeta_\alpha f, \frac{\zeta_\alpha\phi}{x_n^{n+2}}\Big\rangle_{\mathrm{e}}\,\mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle_{\mathrm{e}}$ is the Euclidean inner product of two vectors. Hence by Proposition A.1, this last integral is bounded by

$$C\Big(\|\zeta_\alpha f\|_{L^1(\mathbb{R}^n)} \Big\| \nabla_{\mathrm{e}} \Big(\frac{\zeta_\alpha \phi}{x_n^{n+2}} \Big) \Big\|_{L^n(\mathbb{R}^n)} + \|\langle \nabla_{\mathrm{e}}(\zeta_\alpha), f\rangle_{\mathrm{e}}\|_{L^1(\mathbb{R}^n)} \Big\| \frac{\zeta_\alpha \phi}{x_n^{n+2}} \Big\|_{L^n(\mathbb{R}^n)} \Big),$$

where we write ∇_{e} to emphasize that the gradients are with respect to the Euclidean metric. Now on the support of ζ_α, we have $|x_n| \simeq 1$, so altogether, we get

$$\begin{aligned} &\Big| \int_{\mathbb{H}^n} \langle \zeta_\alpha f, \zeta_\alpha \phi \rangle_g \, \mathrm{d}V_g \Big| \\ &\le C(\|\zeta_\alpha f\|_{L^1(\mathbb{H}^n)} + \|\langle \nabla_g(\zeta_\alpha), f\rangle_g\|_{L^1(\mathbb{H}^n)})(\|\nabla_g \phi\|_{L^n(\mathbb{H}^n)} + \|\phi\|_{L^n(\mathbb{H}^n)}). \end{aligned} \tag{A.2}$$

This remains true even if $\alpha \neq (0,1)$, since there is an isometry mapping α to $(0,1)$, and since (A.2) is invariant under isometries of $\mathbb{H}^n$. By integrating with respect to $\alpha \in \mathbb{H}^n$, we see that

$$\Big| \int_{\mathbb{H}^n} \langle f, \phi \rangle_g \, \mathrm{d}V_g \Big| \le C \|f\|_{L^1(\mathbb{H}^n)} (\|\nabla_g \phi\|_{L^n(\mathbb{H}^n)} + \|\phi\|_{L^n(\mathbb{H}^n)}).$$

We now use Lemma 3.3 to bound $\|\phi\|_{L^n(\mathbb{H}^n)}$ by $\|\nabla_g \phi\|_{L^n(\mathbb{H}^n)}$. This concludes our alternative proof of Theorem 1.2.

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