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Recovery of Immersions from Their Metric Tensors and Nonlinear Korn Inequalities: A Brief Survey*

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(*Dedicated to Haïm Brezis on the occasion of his* 70*th birthday*)

Abstract The authors discuss the existence and uniqueness up to isometries of $\mathbb{E}^n$ of immersions $\boldsymbol{\phi} : \Omega \subset \mathbb{R}^n \to \mathbb{E}^n$ with prescribed metric tensor field $(g_{ij}) : \Omega \to \mathbb{S}^n_>$, and discuss the continuity of the mapping $(g_{ij}) \to \boldsymbol{\phi}$ defined in this fashion with respect to various topologies. In particular, the case where the function spaces have little regularity is considered. How, in some cases, the continuity of the mapping $(g_{ij}) \to \boldsymbol{\phi}$ can be obtained by means of nonlinear Korn inequalities is shown.

Keywords Isometric immersions, Nonlinear Korn inequalities, Metric tensor
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1 Introduction

Throughout this paper, n designates an integer ≥ 2. Then $\mathbb{E}^n$ denotes the n-dimensional real Euclidean space; $\mathbb{M}^n$ denotes the space of all real matrices of order n; $\mathbb{S}^n$ denotes the subspace of all symmetric matrices in $\mathbb{M}^n$; $\mathbb{S}^n_>$ denotes the subset of all positive-definite matrices in $\mathbb{S}^n$; $\mathbb{O}^n$ denotes the subset of all orthogonal matrices in $\mathbb{M}^n$; and $\mathbb{O}^n_+ := \{P \in \mathbb{O}^n;\ \det P = 1\}$. The Euclidean inner product in $\mathbb{E}^n$ is denoted by $\cdot$, and the Euclidean norm in $\mathbb{E}^n$ and the Frobenius norm in $\mathbb{M}^n$ are both denoted by $|\cdot|$.

In all that follows, Latin indices and exponents range in the set $\{1, \cdots, n\}$ (save when otherwise indicated, as for instance when they are used for indexing sequences) and the summation convention with respect to repeated indices and exponents are used. A generic point in an open set $\Omega \subset \mathbb{R}^n$ is denoted by $x = (x_i)$, and partial derivatives of the first and second order, in the classical or distributional sense, are denoted by $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$, respectively.

If $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ is a sufficiently smooth immersion from an open subset $\Omega \subset \mathbb{R}^n$ into $\mathbb{E}^n$, then the positive-definite symmetric tensor field $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ defined by

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \text{in } \Omega$$

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is a Riemannian metric in Ω, called the metric tensor field induced by the immersion $\boldsymbol{\phi}$, and the functions $g_{ij} : \Omega \to \mathbb{E}$ are called its covariant components.

The first objective of this paper is to review what can be said about the converse property: If $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ is a sufficiently smooth field of positive-definite symmetric matrices of order n defined over an open set $\Omega \subset \mathbb{R}^n$, then does there exist an immersion $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ such that

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega\,?$$

If the answer is yes, is such an immersion unique?

We will provide successive answers to these questions, for matrix fields $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ with components in the spaces (in this order)

$$\mathcal{C}^2(\Omega),\, \mathcal{C}^2(\overline{\Omega}),\, \mathcal{C}^1(\Omega),\, \mathcal{C}^1(\overline{\Omega}),\, W^{1,p}_{\mathrm{loc}}(\Omega) \text{ with } p > n \quad \text{and} \quad W^{1,p}(\Omega) \text{ with } p > n$$

(see the first theorem in Sections 2–7). In each case, the answer relies on the following two simple, yet crucial, observations.

Assume that there exists a sufficiently smooth immersion $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ such that

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega.$$

By definition of an immersion, the vector fields

$$\boldsymbol{g}_i := \partial_i \boldsymbol{\phi}$$

are thus linearly independent at each $x \in \Omega$. Hence there exist functions $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ such that

$$\partial_i \boldsymbol{g}_j = \Gamma^k_{ij} \boldsymbol{g}_k \quad \text{in } \Omega. \tag{1.1}$$

The first observation is that the relations $\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij}$ imply (after a series of straightforward computations) that the components Γ^k_{ij} are of the form

$$\Gamma^k_{ij} = \Gamma^k_{ji} = g^{k\ell}\Gamma_{ij\ell} \quad \text{in } \Omega,$$

where

$$(g^{k\ell}) := (g_{ij})^{-1} \quad \text{and} \quad \Gamma_{ij\ell} := \frac{1}{2}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

The functions $\Gamma_{ij\ell}$ (resp. Γ^k_{ij}) are the Christoffel symbols of the first kind (resp. of the second kind) associated with matrix field (g_{ij}).

The vector fields $\boldsymbol{g}_i$ thus satisfy a Pfaff system of linear partial differential equations (the equations (1.1) above), whose coefficients Γ^k_{ij} depend only on the matrix field (g_{ij}).

The second observation is that the relations $\partial_{ij}\boldsymbol{\phi} = \partial_i \boldsymbol{g}_j = \Gamma^k_{ij}\boldsymbol{g}_k$ imply that

$$\begin{aligned}\mathbf{0} &= \partial_j(\partial_{ik}\boldsymbol{\phi}) - \partial_k(\partial_{ij}\boldsymbol{\phi}) = \partial_j(\Gamma^\ell_{ik}\boldsymbol{g}_\ell) - \partial_k(\Gamma^\ell_{ij}\boldsymbol{g}_\ell)\\ &= (\partial_j\Gamma^\ell_{ik} - \partial_k\Gamma^\ell_{ij} + \Gamma^q_{ik}\Gamma^\ell_{jq} - \Gamma^q_{ij}\Gamma^\ell_{kq})\boldsymbol{g}_\ell \quad \text{in } \Omega,\end{aligned}$$

which in turn implies that (since the vector fields $\boldsymbol{g}_\ell$ are linearly independent)

$$R^{\ell}_{\cdot ijk} := \partial_j\Gamma^\ell_{ik} - \partial_k\Gamma^\ell_{ij} + \Gamma^q_{ik}\Gamma^\ell_{jq} - \Gamma^q_{ij}\Gamma^\ell_{kq} = 0 \quad \text{in } \Omega,$$

or equivalently,

$$R_{qijk} := g_{\ell q} R^{\ell}_{\cdot ijk} = \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + g^{\ell r}(\Gamma_{ij\ell}\Gamma_{kqr} - \Gamma_{ik\ell}\Gamma_{jqr}) = 0 \quad \text{in } \Omega.$$

The functions R_{qijk} (resp. $R^{\ell}_{\cdot ijk}$) are the covariant components (resp. mixed components) of the Riemann curvature tensor field associated with the matrix field (g_{ij}). The compatibility conditions $R_{qijk} = 0$ in Ω are thus necessary for the existence of the immersion $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ such that $\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij}$ in Ω.

We will then show that, under the crucial assumption that the open set Ω is simply-connected (additional conditions may be imposed in some cases on the boundary of Ω), the compatibility conditions

$$R_{qijk} = 0 \quad \text{in } \Omega,$$

possibly interpreted in the sense of distributions if need be, become also sufficient for the existence of immersions $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ satisfying

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega,$$

and that such immersions $\boldsymbol{\phi}$ are only defined up to composition by isometries of $\mathbb{E}^n$ (such isometries are defined in Section 2) (see Theorems 2.1, 3.1, 4.1, 5.1, 6.1 and 7.1).

The second objective of this paper is to provide a natural complement to these existence and uniqueness theorems of immersions $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ with prescribed metric tensor fields $(g_{ij}) : \Omega \to \mathbb{S}^n_>$, i.e., to establish continuity theorems, showing that the immersion $\boldsymbol{\phi}$ depends continuously on their metric tensor fields (g_{ij}) with respect to specific topologies (see Theorems 2.2, 3.3, 4.2, 5.3 and 7.2).

In fact, if the metric tensors fields (g_{ij}) have components in one of the spaces $\mathcal{C}^2(\overline{\Omega})$, $\mathcal{C}^1(\overline{\Omega})$, $W^{1,p}(\Omega)$, or $L^q(\Omega)$, these continuity theorems will be established as consequences of specific nonlinear Korn inequalities (see Theorems 3.2, 5.2, 6.2 and 8.1). More specifically, we will show that, given any immersion $\boldsymbol{\phi}^0 \in \mathcal{C}^{k+1}(\overline{\Omega}; \mathbb{E}^n)$, $k = 1, 2$, there exist two constants $C = C(\boldsymbol{\phi}^0) > 0$ and $\delta = \delta(\boldsymbol{\phi}^0) > 0$ such that

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\mathcal{C}^{k+1}(\overline{\Omega}; \mathbb{E}^n)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathcal{C}^k(\overline{\Omega}; \mathbb{S}^n)},$$

(the set $\mathrm{Isom}(\mathbb{E}^n)$ is defined in Section 2) for all immersions $\boldsymbol{\phi} \in \mathcal{C}^{k+1}(\overline{\Omega}; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^{k+1}(\overline{\Omega}; \mathbb{E}^n)$ that satisfy

$$\|(g_{ij}) - (g^0_{ij})\|_{\mathcal{C}^k(\overline{\Omega}; \mathbb{S}^n)} < \delta \quad \text{and} \quad \|(\widetilde{g}_{ij}) - (g^0_{ij})\|_{\mathcal{C}^k(\overline{\Omega}; \mathbb{S}^n)} < \delta,$$

where

$$g^0_{ij} := \partial_i \boldsymbol{\phi}^0 \cdot \partial_j \boldsymbol{\phi}^0, \quad g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \text{and} \quad \widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}}.$$

We will also show that, given any immersion $\boldsymbol{\phi}^0 \in W^{2,p}(\Omega; \mathbb{E}^n)$, $p > n$, there exist two constants $C = C(\boldsymbol{\phi}^0) > 0$ and $\delta = \delta(\boldsymbol{\phi}^0) > 0$ such that

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{W^{2,p}(\Omega; \mathbb{E}^n)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|_{W^{1,p}(\Omega; \mathbb{S}^n)}$$

for all immersions $\boldsymbol{\phi} \in W^{2,p}(\Omega; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{\phi}} \in W^{2,p}(\Omega; \mathbb{E}^n)$ that satisfy

$$\|(g_{ij}) - (g^0_{ij})\|_{W^{1,p}(\Omega; \mathbb{S}^n)} < \delta \quad \text{and} \quad \|(\widetilde{g}_{ij}) - (g^0_{ij})\|_{W^{1,p}(\Omega; \mathbb{S}^n)} < \delta.$$

Finally, we will show that, given any immersion $\phi \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^n)$ satisfying $\det \nabla\phi > 0$ in $\overline{\Omega}$, any $1 \le q < \infty$ and any $1 < p < \infty$ such that $q \le p \le 2q$, there exists a constant $C > 0$ such that, for all immersions $\widetilde{\phi} \in W^{1,2q}(\Omega; \mathbb{E}^n)$ that satisfy $\det \nabla\widetilde{\phi} > 0$ almost everywhere in Ω, we have

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\phi} - \phi\|_{W^{1,p}(\Omega;\, \mathbb{E}^n)} \le C \|(\widetilde{g}_{ij}) - (g_{ij})\|^{\frac{q}{p}}_{L^q(\Omega;\, \mathbb{S}^n)}$$

(the set $\mathrm{Isom}_+(\mathbb{E}^n)$ is defined in Section 8).

2 The Classical Case: Metric Tensor Fields in $\mathcal{C}^2(\Omega)$

We begin with the classical case, where the matrix field $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ has components $g_{ij} \in \mathcal{C}^2(\Omega)$. Note that this is the minimal regularity assumption that ensures that all the partial derivatives appearing in the proof of the existence and uniqueness theorem below are classical ones.

An isometry of $\mathbb{E}^n$ is a mapping $\boldsymbol{r} : \mathbb{E}^n \to \mathbb{E}^n$ of class $\mathcal{C}^1$ that preserves the Euclidean metric of $\mathbb{E}^n$, i.e., a mapping that satisfies

$$\partial_i \boldsymbol{r} \cdot \partial_j \boldsymbol{r} = \delta_{ij} \quad \text{in } \mathbb{E}^n,$$

where δ_{ij} denotes the Kronecker symbol. It is well known that $\boldsymbol{r}$ is an isometry of $\mathbb{E}^n$ if and only if there exists a vector $\boldsymbol{a} \in \mathbb{E}^n$ and an orthogonal matrix $\boldsymbol{R} \in \mathbb{O}^n$ such that

$$\boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}x \quad \text{for all } x \in \mathbb{E}^n,$$

and that the set of all isometries of $\mathbb{E}^n$, henceforth denoted by

$$\mathrm{Isom}(\mathbb{E}^n) := \{\boldsymbol{r} : \mathbb{E}^n \to \mathbb{E}^n;\ \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}x \text{ for all } x \in \mathbb{E}^n, \boldsymbol{a} \in \mathbb{E}^n,\ \boldsymbol{R} \in \mathbb{O}^n\},$$

is a smooth finite-dimensional manifold, since $\mathbb{O}^n$ is a smooth manifold of dimension $\frac{n(n-1)}{2}$ (see, e.g., [1]).

Given a smooth enough mapping $\phi : \Omega \to \mathbb{E}^n$ and a point $x \in \Omega$, the notation $\nabla\phi(x)$ designates the $n \times n$ matrix whose j-th column vector is the vector $\boldsymbol{g}_j(x) := \partial_j \phi(x)$.

The next theorem is classical (see, e.g., [15, 4]; for a detailed and self-contained proof, see [6] or [5, Theorems 8.6-1, 8.7-1]).

Theorem 2.1 *Let Ω be a simply-connected open set in $\mathbb{R}^n$, and let $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in Ω, i.e.,*

$$\partial_j \Gamma_{ik\ell} - \partial_k \Gamma_{ij\ell} + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q}) = 0 \quad \text{in } \Omega,$$

where

$$\Gamma_{ijk} := \frac{1}{2}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}) \quad \text{and} \quad (g^{rq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\phi \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ such that

$$\partial_i \phi \cdot \partial_j \phi = g_{ij} \quad \text{in } \Omega.$$

In addition, an immersion $\psi \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ satisfies

$$\partial_i \psi \cdot \partial_j \psi = g_{ij} \quad \text{in } \Omega$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \textit{in } \Omega.$$

Sketch of the Proof As already mentioned, the proof relies of the observation that, if an immersion $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ satisfies

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega,$$

then the n vector fields

$$\boldsymbol{g}_j := \partial_j \boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$$

satisfy the Pfaff system of partial differential equations

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \quad \text{in } \Omega,$$

where

$$\Gamma_{ij}^k := g^{k\ell} \Gamma_{ij\ell} = \frac{1}{2} g^{k\ell} (\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \in \mathcal{C}^1(\Omega).$$

Thus the idea of the proof of existence of a solution is to construct explicitly an immersion $\boldsymbol{\phi}$ by using these equations "in reverse order". In other words, the proof consists first in showing that there exist n vector fields $\boldsymbol{g}_j \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ that satisfy the Pfaff system

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \quad \text{in } \Omega, \tag{2.1}$$

second in showing that there exists a mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ that satisfies the Poincaré system

$$\partial_j \boldsymbol{\phi} = \boldsymbol{g}_j \quad \text{in } \Omega, \tag{2.2}$$

and third in showing that the mapping $\boldsymbol{\phi}$ obtained in this fashion is an immersion and satisfies the equations

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega. \tag{2.3}$$

(i) That the Pfaff system (2.1) possesses solutions is proved as follows. Pick any point x^0 in Ω and any n vectors $\boldsymbol{g}_j^0 \in \mathbb{E}^n$ that satisfy

$$\boldsymbol{g}_i^0 \cdot \boldsymbol{g}_j^0 = g_{ij}(x_0)$$

(it is easy to see that such vectors exist).

Next, given any point $x \in \Omega$, let $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^1([0,1]; \mathbb{E}^n)$ be a mapping that satisfies $\boldsymbol{\gamma}(0) = x_0$, $\boldsymbol{\gamma}(1) = x$, and $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ (the image of $[0,1]$ under the mapping $\boldsymbol{\gamma}$ is thus a curve contained in Ω that joins the points x_0 and x) and, for each j, let $\boldsymbol{f}_j \in \mathcal{C}^1([0,1]; \mathbb{E}^n)$ be the unique solution to the system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{f}_j}{\mathrm{d}t}(t) &= \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t) \Gamma_{ij}^k(\boldsymbol{\gamma}(t)) \boldsymbol{f}_k(t), \quad t \in [0,1], \\ \boldsymbol{f}_j(0) &= \boldsymbol{g}_j^0. \end{aligned}$$

The assumptions that Ω is simply-connected and that the matrix field (g_{ij}) satisfies the equations

$$\partial_j \Gamma_{ik\ell} - \partial_k \Gamma_{ij\ell} + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q}) = 0 \quad \text{in } \Omega$$

together imply that the value of $\boldsymbol{f}_j$ at $t = 1$ is independent of the choice of the path $\boldsymbol{\gamma}$ joining x_0 to x (the regularity assumption $g_{ij} \in \mathcal{C}^2(\Omega)$ is needed here), so that the n vector fields $\boldsymbol{g}_j : \Omega \to \mathbb{E}^n$ given by

$$\boldsymbol{g}_j(x) := \boldsymbol{f}_j(1)$$

are unambiguously defined. Then one proves that these vector fields belong to the space $\mathcal{C}^2(\Omega; \mathbb{E}^n)$, that they satisfy the system (2.1), and that

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} \quad \text{in } \Omega.$$

(ii) That the Poincaré system (2.2) possesses solutions is proved in a similar fashion.

More specifically, let x^0 be any point in Ω and let $\boldsymbol{f}^0$ be any vector in $\mathbb{E}^n$. Given any point $x \in \Omega$, let $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^3([0,1]; \mathbb{E}^n)$ be any mapping that satisfies $\boldsymbol{\gamma}(0) = x_0$, $\boldsymbol{\gamma}(1) = x$, and $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$, and let $\boldsymbol{f} \in \mathcal{C}^1([0,1]; \mathbb{E}^n)$ be the unique solution to the system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t}(t) &= \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}(t)\boldsymbol{g}_j(\boldsymbol{\gamma}(t)), \quad t \in [0,1], \\ \boldsymbol{f}(0) &= \boldsymbol{f}^0. \end{aligned}$$

The assumptions that Ω is simply-connected and the relations

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k = \Gamma_{ji}^k \boldsymbol{g}_k = \partial_j \boldsymbol{g}_i \quad \text{in } \Omega$$

(that $\Gamma_{ij}^k = \Gamma_{ji}^k$ is a simple consequence of the definition of Γ_{ij}^k (see the introduction)), together imply that the value of $\boldsymbol{f}$ at $t = 1$ is independent of the choice of the path $\boldsymbol{\gamma}$ joining x_0 to x, so that the mapping $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ given by

$$\boldsymbol{\phi}(x) := \boldsymbol{f}(1)$$

is unambiguously defined. Then one proves that this mapping belongs to the space $\mathcal{C}^3(\Omega; \mathbb{E}^n)$ and satisfies the Poincaré system (2.2).

(iii) That the mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ is an immersion that satisfies $\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij}$ is a simple consequence of its definition, since

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} \quad \text{in } \Omega,$$

which in turn implies that

$$(\det \boldsymbol{\nabla}\boldsymbol{\phi})^2 = \det(\boldsymbol{\nabla}\boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\nabla}\boldsymbol{\phi}) = \det(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) = \det(g_{ij}) > 0 \quad \text{in } \Omega.$$

(iv) Finally, in order to prove that such a mapping is unique up to isometries of $\mathbb{E}^n$, let $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ and $\boldsymbol{\psi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ be two immersions that satisfy

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{and} \quad \partial_i \boldsymbol{\psi} \cdot \partial_j \boldsymbol{\psi} = g_{ij} \quad \text{in } \Omega,$$

respectively. At each point $x \in \Omega$, let $\boldsymbol{C}^{\frac{1}{2}}(x) \in \mathbb{S}^n_>$ denote the square root of the matrix $\boldsymbol{C}(x) := (g_{ij}(x)) \in \mathbb{S}^n_>$, let $\boldsymbol{C}^{-\frac{1}{2}}(x) := (\boldsymbol{C}^{\frac{1}{2}}(x))^{-1}$, and let

$$\boldsymbol{P} := (\boldsymbol{\nabla}\boldsymbol{\phi})\boldsymbol{C}^{-\frac{1}{2}}, \quad \boldsymbol{Q} := (\boldsymbol{\nabla}\boldsymbol{\psi})\boldsymbol{C}^{-\frac{1}{2}} \quad \text{in } \Omega.$$

Then $\boldsymbol{P} \in \mathcal{C}^2(\Omega; \mathbb{O}^n)$, $\boldsymbol{Q} \in \mathcal{C}^2(\Omega; \mathbb{O}^n)$, and

$$\boldsymbol{\nabla}\boldsymbol{\phi} = \boldsymbol{P}\boldsymbol{C}^{\frac{1}{2}}, \quad \boldsymbol{\nabla}\boldsymbol{\psi} = \boldsymbol{Q}\boldsymbol{C}^{\frac{1}{2}} \quad \text{in } \Omega.$$

Given any point $x \in \Omega$, there exists a connected open set $U_x \subset \Omega$ containing x such that the restriction $\boldsymbol{\phi}|_{U_x} \in \mathcal{C}^3(U_x; \mathbb{E}^n)$ is injective, since $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ is an immersion. Let $U \subset \Omega$ denote any connected and open subset such that $\boldsymbol{\phi}|_U \in \mathcal{C}^3(U; \mathbb{E}^n)$ is injective, let $\widehat{U} := \boldsymbol{\phi}(U)$, and let $\boldsymbol{r} := \boldsymbol{\psi} \circ (\boldsymbol{\phi}|_U)^{-1}$. Then $\widehat{U}$ is a connected open subset of $\mathbb{E}^n$, $\boldsymbol{r} \in \mathcal{C}^3(\widehat{U}; \mathbb{E}^n)$, and

$$(\boldsymbol{\nabla}\boldsymbol{r}) \circ \boldsymbol{\phi} = \boldsymbol{\nabla}\boldsymbol{\psi}(\boldsymbol{\nabla}\boldsymbol{\phi})^{-1} = \boldsymbol{Q}\boldsymbol{P}^{\mathrm{T}} \quad \text{in } U.$$

This relation implies that

$$\boldsymbol{\nabla}\boldsymbol{r}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{r} = \boldsymbol{I} \quad \text{in } \widehat{U},$$

which means that $\boldsymbol{r}$ is the restriction to $\widehat{U}$ of an isometry of $\mathbb{E}^n$.

It thus follows that, given any $x \in \Omega$, there exists a connected open set $U_x \subset \Omega$ containing x and an isometry $\boldsymbol{r}_x \in \mathrm{Isom}(\mathbb{E}^n)$ such that

$$\boldsymbol{\psi} = \boldsymbol{r}_x \circ \boldsymbol{\phi} \quad \text{in } U_x.$$

This relation implies that $\boldsymbol{r}_x = \boldsymbol{r}_y$ whenever $U_x \cap U_y \neq \emptyset$. Combined with the assumption that Ω is in particular connected (as a simply-connected set), this implies that there exists a unique isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ such that

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \text{in } \Omega.$$

As a complement to Theorem 2.1, we now show that, up to an isometry of $\mathbb{E}^n$, an immersion $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ depends continuously on its metric tensor field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n)$ when both spaces are equipped with their respective Fréchet topologies.

More specifically, recall that a sequence $(f_m)_{m=1}^{\infty}$ of functions $f_m \in \mathcal{C}^k(\Omega)$, $k \in \mathbb{N}$, converges to $f \in \mathcal{C}^k(\Omega)$ with respect to the Fréchet topology of $\mathcal{C}^k(\Omega)$ if, for each compact subset $K \subset \Omega$,

$$\lim_{m \to \infty} \|f_m - f\|_{\mathcal{C}^k(K)} = 0$$

and, if this is the case, we write

$$f_m \to f \quad \text{in } \mathcal{C}^k(\Omega) \text{ as } m \to \infty.$$

Such notions can then be clearly extended to the spaces $\mathcal{C}^k(\Omega; \mathbb{E}^n)$ and $\mathcal{C}^k(\Omega; \mathbb{S}^n)$, $k \in \mathbb{N}$.

Theorem 2.2 *Let Ω be a connected and open subset of $\mathbb{R}^n$, and let*

$$\boldsymbol{\phi}^m \in \mathcal{C}^3(\Omega; \mathbb{E}^n), \quad m \geq 1 \quad \textit{and} \quad \boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$$

be immersions that satisfy

$$(g_{ij}^m) \to (g_{ij}) \quad \textit{in } \mathcal{C}^2(\Omega; \mathbb{S}^n) \textit{ as } m \to \infty,$$

where

$$g_{ij}^m := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m \in \mathcal{C}^2(\Omega), \quad m \geq 1 \quad \textit{and} \quad g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \in \mathcal{C}^2(\Omega),$$

respectively denote the covariant components of the metric tensor fields induced by the immersions $\boldsymbol{\phi}^m$, $m \geq 1$ *and* $\boldsymbol{\phi}$.

Then there exist isometries $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $m \geq 1$, *such that*

$$\boldsymbol{r}^m \circ \boldsymbol{\phi}^m \to \boldsymbol{\phi} \quad \textit{in } \mathcal{C}^3(\Omega; \mathbb{E}^n) \textit{ as } m \to \infty.$$

Sketch of the Proof The details of the proof below can be found in [7]. An argument similar to that used in the proof of the uniqueness part of Theorem 2.1 shows that it suffices to prove Theorem 2.2 in the particular case where $\boldsymbol{\phi} = \mathbf{id}$, where $\mathbf{id}$ denotes the identity mapping of the set Ω. So let a sequence of immersions

$$\boldsymbol{\phi}^m \in \mathcal{C}^3(\Omega; \mathbb{E}^n), \quad m \geq 1,$$

be given that satisfies

$$g_{ij}^m := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m \to \delta_{ij} \quad \text{in } \mathcal{C}^2(\Omega) \text{ as } m \to \infty.$$

For each $m \geq 1$, let $(g^{k\ell,m}(x))$ denote the inverse of the matrix $(g_{ij}^m(x))$, $x \in \Omega$, let

$$\Gamma_{ij}^{k,m} := \frac{1}{2} g^{k\ell,m}(\partial_i g_{j\ell}^m + \partial_j g_{i\ell}^m - \partial_\ell g_{ij}^m) \in \mathcal{C}^1(\Omega)$$

denote the Christoffel symbols of the second kind associated with the matrix field (g_{ij}^m), and let

$$\boldsymbol{g}_i^m := \partial_i \boldsymbol{\phi}^m \in \mathcal{C}^2(\Omega; \mathbb{E}^n).$$

The definition of the Christoffel symbols $\Gamma_{ij}^{k,m}$ implies on the one hand that

$$\Gamma_{ij}^{k,m} \to 0 \quad \text{in } \mathcal{C}^1(\Omega) \text{ as } m \to \infty.$$

On the other hand,

$$|\boldsymbol{g}_i^m|^2 := \boldsymbol{g}_i^m \cdot \boldsymbol{g}_i^m = g_{ii}^m \to 1 \quad \text{in } \mathcal{C}^2(\Omega) \text{ as } m \to \infty \text{ (no summation)}.$$

Combined with the relations (see the proof of Theorem 2.1)

$$\partial_i \boldsymbol{g}_j^m = \Gamma_{ij}^{k,m} \boldsymbol{g}_k^m \quad \text{in } \mathcal{C}^1(\Omega; \mathbb{E}^n),$$

the above convergences imply that

$$\partial_{ij} \boldsymbol{\phi}^m = \partial_i \boldsymbol{g}_j^m \to \mathbf{0} \quad \text{in } \mathcal{C}^1(\Omega; \mathbb{E}^n).$$

Let x_0 be a point in Ω, and, for each $m \geq 1$, let

$$\boldsymbol{R}^m := \boldsymbol{\nabla} \boldsymbol{\phi}^m(x_0)(g_{ij}^m(x_0))^{-\frac{1}{2}},$$

where $(g_{ij}^m(x_0))^{-\frac{1}{2}}$ denotes the inverse of the square root of the matrix $(g_{ij}(x_0)) \in \mathbb{S}_>^n$. Note that the matrix $\boldsymbol{R}^m$ is orthogonal since $g_{ij}^m(x_0) := \partial_i \boldsymbol{\phi}^m(x_0) \cdot \partial_j \boldsymbol{\phi}^m(x_0)$.

Let the functions $\boldsymbol{r}^m : \mathbb{E}^n \to \mathbb{E}^n$ and $\boldsymbol{\psi}^m : \Omega \to \mathbb{E}^n$ be defined by

$$\boldsymbol{r}^m(x) = x_0 + (\boldsymbol{R}^m)^{\mathrm{T}}(x - \boldsymbol{\phi}^m(x_0)) \quad \text{for all } x \in \mathbb{E}^n$$

and

$$\boldsymbol{\psi}^m := \boldsymbol{r}^m \circ \boldsymbol{\phi}^m,$$

respectively. Then $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $\boldsymbol{\psi}^m \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ and

$$\boldsymbol{\psi}^m(x_0) = x_0, \quad \boldsymbol{\nabla}\boldsymbol{\psi}^m(x_0) = (g_{ij}^m(x_0))^{\frac{1}{2}}, \quad \partial_{ij}\boldsymbol{\psi}^m = (\boldsymbol{R}^m)^{\mathrm{T}}\partial_{ij}\boldsymbol{\phi}^m.$$

Therefore,

$$\begin{aligned} \boldsymbol{\psi}^m(x_0) &\to \mathbf{id}(x_0) \quad \text{ in } \mathbb{E}^n, \\ \boldsymbol{\nabla}\boldsymbol{\psi}^m(x_0) &\to \boldsymbol{\nabla}\mathbf{id}(x_0) \quad \text{ in } \mathbb{M}^n, \\ \partial_{ij}\boldsymbol{\psi}^m &\to \partial_{ij}\mathbf{id} \quad \text{ in } \mathcal{C}^1(\Omega; \mathbb{E}^n) \end{aligned}$$

as $m \to \infty$, which in turn implies that

$$\boldsymbol{\psi}^m \to \mathbf{id} \quad \text{in } \mathcal{C}^3(\Omega; \mathbb{E}^n) \text{ as } m \to \infty.$$

3 Metric Tensor Fields in $\mathcal{C}^2(\overline{\Omega})$

A domain in $\mathbb{R}^n$ is a bounded and connected open subset of $\mathbb{R}^n$ whose boundary is Lipschitz-continuous, the set Ω being locally on one side of its boundary (see [19, 2]).

The space $\mathcal{C}^k(\overline{\Omega})$, where Ω denotes an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, is defined as the space of all functions $f \in \mathcal{C}^k(\Omega)$ that, together with all their derivatives up to order k, possess continuous extensions to the closure $\overline{\Omega}$ of Ω. In the particular case where Ω is a domain, we have

$$\mathcal{C}^k(\overline{\Omega}) := \{f|_\Omega;\ f \in \mathcal{C}^k(\mathbb{R}^n)\}$$

(see [20, 8]). The space $\mathcal{C}^k(\overline{\Omega})$ is equipped with the norm defined by

$$\|f\|_{\mathcal{C}^k(\overline{\Omega})} = \max_{|\boldsymbol{\alpha}| \le k} \sup_{x \in \overline{\Omega}} |\partial^{\boldsymbol{\alpha}} f(x)|,$$

where $\partial^{\boldsymbol{\alpha}}$ is the usual multi-index notation for partial derivatives operators.

We establish in this section results similar to those in the previous section, but now for matrix fields (g_{ij}) with components $g_{ij} \in \mathcal{C}^2(\overline{\Omega})$, instead of $g_{ij} \in \mathcal{C}^2(\Omega)$. While no boundedness or regularity assumption on the boundary of Ω was needed in the previous section, here we assume that Ω is a domain.

Theorem 3.1 *Let Ω be a simply-connected domain in $\mathbb{R}^n$, and let $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in Ω, i.e.,*

$$\partial_j\Gamma_{ik\ell} - \partial_k\Gamma_{ij\ell} + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q}) = 0 \quad \text{ in } \Omega,$$

where

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \quad \text{and} \quad (g^{rq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ such that

$$\partial_i\boldsymbol{\phi} \cdot \partial_j\boldsymbol{\phi} = g_{ij} \quad \text{in } \overline{\Omega}.$$

In addition, an immersion $\boldsymbol{\psi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ *satisfies*

$$\partial_i \boldsymbol{\psi} \cdot \partial_j \boldsymbol{\psi} = g_{ij} \quad \text{in } \overline{\Omega}$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \text{in } \overline{\Omega}.$$

Sketch of the Proof The details of the proof below can be found in [8]. By Theorem 2.1, there exists an immersion $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$ such that

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega.$$

It remains to prove that $\boldsymbol{\phi}$, together with its partial derivatives up to order three, possess continuous extensions to $\overline{\Omega}$.

The first step is to prove that each vector field $\boldsymbol{g}_i := \partial_i \boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$, together with its partial derivatives up to order two, possess continuous extensions to $\overline{\Omega}$. Let $B \subset \mathbb{R}^n$ be any open ball such that $\omega := B \cap \Omega \neq \emptyset$, let x and y be any two points in the set ω, let $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^1([0,1]; \omega)$ be any mapping that satisfies $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$, and let

$$\boldsymbol{f}_i := \boldsymbol{g}_i \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{E}^n).$$

Since the vector fields $\boldsymbol{g}_i$ satisfy the Pfaff system

$$\partial_i \boldsymbol{g}_j = \Gamma^k_{ij} \boldsymbol{g}_k \quad \text{in } \Omega, \tag{3.1}$$

where

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell} (\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \in \mathcal{C}^1(\overline{\Omega}) \quad \text{and} \quad (g^{k\ell}) := (g_{ij})^{-1} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>),$$

the vector fields $\boldsymbol{f}_i$ satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{f}_j}{\mathrm{d}t}(t) = \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t) \Gamma^k_{ij}(\boldsymbol{\gamma}(t)) \boldsymbol{f}_k(t), \quad t \in [0,1].$$

Since $\Gamma^k_{ij} \in \mathcal{C}^1(\overline{\Omega})$ and $|\boldsymbol{g}_i|^2 = \boldsymbol{g}_i \cdot \boldsymbol{g}_i = g_{ii} \in \mathcal{C}^2(\overline{\Omega})$ (no summation), and since $\overline{\omega}$ is a compact subset of $\overline{\Omega}$, there exists a constant $C_1 = C_1(\omega)$ such that

$$\sup_{x \in \overline{\omega}} |\Gamma^k_{ij}(x)| \leq C_1 \quad \text{and} \quad \max_{1 \leq i \leq n} \sup_{x \in \overline{\omega}} |\boldsymbol{g}_i(x)| \leq C_1.$$

We then infer from the inspection of the above system of ordinary differential equations that there exists a constant $C_2 = C_2(\omega)$ such that

$$|\boldsymbol{g}_i(y) - \boldsymbol{g}_i(x)| = |\boldsymbol{f}_i(1) - \boldsymbol{f}_i(0)| \leq C_2 \int_0^1 \left| \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}(t) \right| \mathrm{d}t.$$

The assumption that the boundary of Ω is Lipschitz continuous implies that the mapping $\boldsymbol{\gamma}$ can be chosen in such a way that the right-hand side of the above inequality is bounded by a constant times the diameter of the ball B. This implies that the vector fields $\boldsymbol{g}_i$ are uniformy continuous in Ω, hence that they can be extended by continuity up to the boundary of Ω. Since in addition the vector fields $\boldsymbol{g}_i$ satisfy the Pfaff system (3.1), whose coefficients Γ_{ij}^k belong to $\mathcal{C}^1(\overline{\Omega})$, these extensions of $\boldsymbol{g}_i$ belong in fact to the space $\mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$.

The second step consists in showing that the mapping $\boldsymbol{\phi} \in \mathcal{C}^3(\Omega; \mathbb{E}^n)$, together with its partial derivatives up to order three, possesses continuous extensions to $\overline{\Omega}$. This is done as in the first step, but this time using the Poincaré system

$$\partial_i \boldsymbol{\phi} = \boldsymbol{g}_i \quad \text{in } \Omega$$

satisfied by $\boldsymbol{\phi}$, instead of the Pfaff system (3.1) satisfied by $\boldsymbol{g}_i$.

Finally, the uniqueness part of the theorem is a simple consequence of the uniqueness part of Theorem 2.1.

Note that, as shown in [8], the assumption that Ω is a domain in Theorem 3.1, as well as in Theorem 3.2 below, can be replaced by the weaker, but a bit too technical to be reproduced here, assumption that Ω is connected and satisfies the "geodesic property".

The next theorem establishes a nonlinear Korn inequality in $\mathcal{C}^3(\overline{\Omega})$, which implies in particular (see Theorem 3.3) that, up to an isometry of $\mathbb{E}^n$, an immersion $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ depends continuously on its metric tensor field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$.

Theorem 3.2 *Let Ω be a domain in $\mathbb{R}^n$, and let $\boldsymbol{\phi}^0 \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ be an immersion. Then there exist two constants $C = C(\boldsymbol{\phi}^0) > 0$ and $\delta = \delta(\boldsymbol{\phi}^0) > 0$ such that*

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)}$$

for all immersions $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ that satisfy

$$\|(g_{ij}) - (g_{ij}^0)\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)} < \delta \quad \textit{and} \quad \|(\widetilde{g}_{ij}) - (g_{ij}^0)\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)} < \delta,$$

where

$$g_{ij}^0 := \partial_i \boldsymbol{\phi}^0 \cdot \partial_j \boldsymbol{\phi}^0, \;\; g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \textit{and} \quad \widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}}$$

denote the covariant components of the metric tensor fields induced by the immersions $\boldsymbol{\phi}^0$, $\boldsymbol{\phi}$, and $\widetilde{\boldsymbol{\phi}}$, respectively.

Sketch of the Proof The details of the proof below can be found in [8].

To begin with, one proves that the mappings

$$\boldsymbol{A} \in \mathbb{S}^n_> \to \boldsymbol{A}^{\frac{1}{2}} \in \mathbb{S}^n_>$$

and

$$(a_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \frac{1}{2} a^{k\ell} (\partial_i a_{j\ell} + \partial_j a_{i\ell} - \partial_\ell a_{ij}) \in \mathcal{C}^1(\overline{\Omega}),$$

where the functions $a^{k\ell} \in \mathcal{C}^2(\overline{\Omega})$ are defined at each point $x \in \overline{\Omega}$ by $(a^{k\ell}(x)) := (a_{ij}(x))^{-1}$, are of class $\mathcal{C}^\infty$.

Let $x_0 \in \overline{\Omega}$. Using that the above mappings are in particular of class $\mathcal{C}^1$, one proves that there exists two constants $\delta = \delta(\boldsymbol{\phi}^0) > 0$ and $D_1 = D_1(\boldsymbol{\phi}^0) > 0$ such that, for all immersions $\boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^n)$ that satisfies

$$\|(g_{ij}) - (g_{ij}^0)\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)} < \delta \quad \text{and} \quad \|(\widetilde{g}_{ij}) - (g_{ij}^0)\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)} < \delta,$$

where $g^0_{ij} := \partial_i \boldsymbol{\phi}^0 \cdot \partial_j \boldsymbol{\phi}^0 \in \mathcal{C}^2(\overline{\Omega})$, $g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega})$ and $\widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}} \in \mathcal{C}^2(\overline{\Omega})$, then

$$|(\widetilde{g}_{ij}(x_0))^{\frac{1}{2}} - (g_{ij}(x_0))^{\frac{1}{2}}| \leq D_1 \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathcal{C}^2(\overline{\Omega};\mathbb{S}^n)} \tag{3.2}$$

and

$$\sum_{i,j,k} \|\widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij}\|_{\mathcal{C}^1(\overline{\Omega})} \leq D_1 \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathcal{C}^2(\overline{\Omega};\mathbb{S}^n)}, \tag{3.3}$$

where

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \quad \text{with } (g^{k\ell}) = (g_{ij})^{-1},$$
$$\widetilde{\Gamma}^k_{ij} := \frac{1}{2} \widetilde{g}^{k\ell}(\partial_i \widetilde{g}_{j\ell} + \partial_j \widetilde{g}_{i\ell} - \partial_\ell \widetilde{g}_{ij}) \quad \text{with } (\widetilde{g}^{k\ell}) = (\widetilde{g}_{ij})^{-1}.$$

Next, define an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ by letting

$$\boldsymbol{r}(x) := \boldsymbol{\phi}(x_0) + \boldsymbol{R}\widetilde{\boldsymbol{R}}^{\mathrm{T}}(x - \widetilde{\boldsymbol{\phi}}(x_0)) \quad \text{for all } x \in \mathbb{E}^n,$$

where

$$\boldsymbol{R} := \boldsymbol{\nabla}\boldsymbol{\phi}(x_0)(g_{ij}(x_0))^{-\frac{1}{2}} \quad \text{and} \quad \widetilde{\boldsymbol{R}} := \boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}(x_0)(\widetilde{g}_{ij}(x_0))^{-\frac{1}{2}}.$$

Note that the matrices $\boldsymbol{R}$ and $\widetilde{\boldsymbol{R}}$ are orthogonal, since

$$\partial_i \boldsymbol{\phi}(x_0) \cdot \partial_j \boldsymbol{\phi}(x_0) = g_{ij}(x_0) \quad \text{and} \quad \partial_i \widetilde{\boldsymbol{\phi}}(x_0) \cdot \partial_j \widetilde{\boldsymbol{\phi}}(x_0) = \widetilde{g}_{ij}(x_0).$$

Let

$$\boldsymbol{\psi} := \boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} \in \mathcal{C}^3(\overline{\Omega};\mathbb{E}^n),$$

and define the vector fields

$$\boldsymbol{g}_i := \partial_i \boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega};\mathbb{E}^n) \quad \text{and} \quad \boldsymbol{h}_i := \partial_i \boldsymbol{\psi} \in \mathcal{C}^2(\overline{\Omega};\mathbb{E}^n).$$

Then, thanks to the definition of the isometry $\boldsymbol{r}$,

$$\boldsymbol{\psi}(x_0) = \boldsymbol{\phi}(x_0) \quad \text{and} \quad \boldsymbol{\nabla}\boldsymbol{\psi}(x_0) = \boldsymbol{\nabla}\boldsymbol{\phi}(x_0)(g_{ij}(x_0))^{-\frac{1}{2}}(\widetilde{g}_{ij}(x_0))^{\frac{1}{2}}.$$

Noting that the mapping $\boldsymbol{\psi} - \boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega};\mathbb{E}^n)$ satisfies the Poincaré system

$$\partial_i(\boldsymbol{\psi} - \boldsymbol{\phi}) = \boldsymbol{h}_i - \boldsymbol{g}_i \quad \text{in } \overline{\Omega},$$
$$(\boldsymbol{\psi} - \boldsymbol{\phi})(x_0) = \boldsymbol{0},$$

one deduces (as in the proof of Theorem 3.1, by integrating along paths $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\Omega)$ joining x_0 to a generic point $x \in \Omega$) that

$$\|\boldsymbol{\psi} - \boldsymbol{\phi}\|_{\mathcal{C}^3(\overline{\Omega};\mathbb{E}^n)} \leq D_2 \sum_{i=1}^n \|\boldsymbol{h}_i - \boldsymbol{g}_i\|_{\mathcal{C}^2(\overline{\Omega};\mathbb{E}^n)}, \tag{3.4}$$

where the constant D_2 depends only on Ω.

Since the isometries $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ satisfy

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \ \text{ in } \overline{\Omega} \quad \text{and} \quad \partial_i \boldsymbol{\psi} \cdot \partial_j \boldsymbol{\psi} = \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}} = \widetilde{g}_{ij} \ \text{ in } \overline{\Omega},$$

the vector fields $\boldsymbol{g}_i$ and $\boldsymbol{h}_i$ satisfy the Pffaf systems (see the proof of Theorem 2.1)

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \ \text{ in } \overline{\Omega} \quad \text{and} \quad \partial_i \boldsymbol{h}_j = \widetilde{\Gamma}_{ij}^k \boldsymbol{h}_k \ \text{ in } \overline{\Omega},$$

respectively. Hence

$$\partial_i(\boldsymbol{h}_j - \boldsymbol{g}_j) = \widetilde{\Gamma}_{ij}^k(\boldsymbol{h}_k - \boldsymbol{g}_k) + (\widetilde{\Gamma}_{ij}^k - \Gamma_{ij}^k)\boldsymbol{g}_k \quad \text{in } \overline{\Omega},$$

which, combined with the estimate (3.3), yields (again by integrating along paths $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\Omega)$ joining x_0 to a generic point $x \in \Omega$, but this time using in addition Gronwall's inequality):

$$\sum_{i=1}^n \|\boldsymbol{h}_i - \boldsymbol{g}_i\|_{\mathcal{C}^2(\overline{\Omega};\mathbb{E}^n)} \le D_3\Big(\sum_{i=1}^n |(\boldsymbol{h}_i - \boldsymbol{g}_i)(x_0)| + \sum_{i,j,k} \|\widetilde{\Gamma}_{ij}^k - \Gamma_{ij}^k\|_{\mathcal{C}^1(\overline{\Omega})}\Big), \tag{3.5}$$

where the constant D_3 depends on Ω and $\boldsymbol{\phi}^0$.

Finally, since the vectors $(\boldsymbol{h}_i - \boldsymbol{g}_i)(x_0) \in \mathbb{E}^n$ are the column vectors of the matrix

$$\boldsymbol{\nabla}(\boldsymbol{\psi} - \boldsymbol{\phi})(x_0) = \boldsymbol{\nabla}\boldsymbol{\phi}(x_0)(g_{ij}(x_0))^{-\frac{1}{2}}((\widetilde{g}_{ij}(x_0))^{\frac{1}{2}} - (g_{ij}(x_0))^{\frac{1}{2}}) \in \mathbb{M}^n,$$

it follows that

$$\sum_{i=1}^n |(\boldsymbol{h}_i - \boldsymbol{g}_i)(x_0)| \le D_4|(\widetilde{g}_{ij}(x_0))^{\frac{1}{2}} - (g_{ij}(x_0))^{\frac{1}{2}}|, \tag{3.6}$$

where the constant D_4 depends only on $\boldsymbol{\phi}^0$.

Combining inequalities (3.2) to (3.6) yields the announced nonlinear Korn inequality.

An immediate consequence of the nonlinear Korn inequality of Theorem 3.2 is the following convergence result, similar to that of Theorem 2.2.

Theorem 3.3 *Let Ω be a domain in $\mathbb{R}^n$, and let*

$$\boldsymbol{\phi}^m \in \mathcal{C}^3(\overline{\Omega};\mathbb{E}^n), \quad m \ge 1 \quad \text{and} \quad \boldsymbol{\phi} \in \mathcal{C}^3(\overline{\Omega};\mathbb{E}^n)$$

be immersions that satisfy

$$(g_{ij}^m) \to (g_{ij}) \quad \text{in } \mathcal{C}^2(\overline{\Omega};\mathbb{S}^n) \text{ as } m \to \infty,$$

where

$$g_{ij} := \partial_i\boldsymbol{\phi} \cdot \partial_j\boldsymbol{\phi} \quad \text{and} \quad g_{ij}^m := \partial_i\boldsymbol{\phi}^m \cdot \partial_j\boldsymbol{\phi}^m, \quad m \ge 1.$$

Then there exist isometries $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $m \ge 1$, such that

$$\boldsymbol{r}^m \circ \boldsymbol{\phi}^m \to \boldsymbol{\phi} \quad \text{in } \mathcal{C}^3(\overline{\Omega};\mathbb{E}^n) \text{ as } m \to \infty.$$

4 Metric Tensor Fields in $\mathcal{C}^1(\Omega)$

Beginning with this section, we consider regularity assumptions that are weaker than those classically made, like in Section 2, for establishing the existence and uniqueness of an immersion with prescribed metric tensor field.

To begin with, we show that the theorems established in Section 2 still hold for matrix fields with coefficients $g_{ij} \in \mathcal{C}^1(\Omega)$, instead of $g_{ij} \in \mathcal{C}^2(\Omega)$. The notation $\mathcal{D}(\Omega)$ designates the space of infinitely differentiable functions with compact support in Ω.

Theorem 4.1 *Let Ω be a simply-connected open set in $\mathbb{R}^n$, and let $(g_{ij}) \in \mathcal{C}^1(\Omega; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in the sense of distributions, i.e.,*

$$\int_\Omega \big(\Gamma_{ij\ell}\partial_k\varphi - \Gamma_{ik\ell}\partial_j\varphi + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q})\varphi\big)\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$

where

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \quad \textit{and} \quad (g^{rq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ such that

$$\partial_i\boldsymbol{\phi} \cdot \partial_j\boldsymbol{\phi} = g_{ij} \quad \textit{in } \Omega.$$

In addition, an immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ satisfies

$$\partial_i\boldsymbol{\psi} \cdot \partial_j\boldsymbol{\psi} = g_{ij} \quad \textit{in } \Omega$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ such that

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \textit{in } \Omega.$$

Sketch of the Proof The details of the proof sketched below can be found in [16]. Like in the proof of Theorem 2.1, we first show that there exist n vector fields $\boldsymbol{g}_j \in \mathcal{C}^1(\Omega; \mathbb{E}^n)$ that satisfy the Pfaff system

$$\partial_i\boldsymbol{g}_j = \Gamma^k_{ij}\boldsymbol{g}_k \quad \text{in } \Omega, \quad \text{where } \Gamma^k_{ij} := g^{k\ell}\Gamma_{ij\ell}, \tag{4.1}$$

we then prove that there exists a mapping $\boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ that satisfies the Poincaré system

$$\partial_j\boldsymbol{\phi} = \boldsymbol{g}_j \quad \text{in } \Omega, \tag{4.2}$$

and finally, we prove that the mapping $\boldsymbol{\phi}$ obtained in this fashion is an immersion and satisfies the equations

$$\partial_i\boldsymbol{\phi} \cdot \partial_j\boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega. \tag{4.3}$$

The difference, however, is that we now have to prove the existence of a solution to the Pfaff system (4.1) under the weaker assumption that $(g_{ij}) \in \mathcal{C}^1(\Omega; \mathbb{S}^n_>)$. As a result, the coefficients

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij})$$

are only continuous in Ω, and they satisfy the relations

$$\partial_j\Gamma_{ik\ell} - \partial_k\Gamma_{ij\ell} + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q}) = 0 \tag{4.4}$$

only in the distributional sense (while in Theorem 2.1 the relations (4.4) were satisfied in the classical sense, i.e., pointwise). The rest of the proof, which consists in showing that the Poincaré system (4.2) has a solution $\boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ and that this solution satisfies the relations (4.3), uses the same arguments as those used in the proof of Theorem 2.1 (save for the regularity assumptions), so we do not discuss this issue here.

We only briefly sketch the proof of the existence of a solution to the Pfaff system (4.1) under the assumptions that $(g_{ij}) \in \mathcal{C}^1(\Omega; \mathbb{S}^n_>)$ and that the functions $\Gamma_{ij\ell} \in \mathcal{C}^0(\Omega)$ satisfy (4.4) in the distributional sense.

First, one shows that, given any cube $\omega \subset \Omega$ whose edges are parallel to the coordinate axes of $\mathbb{R}^n$ and any n vectors $\boldsymbol{v}_i \in \mathbb{E}^n$, there exist n vector fields $\boldsymbol{g}_j^\omega \in \mathcal{C}^1(\Omega; \mathbb{E}^n)$ that satisfy the Pfaff system

$$\partial_i \boldsymbol{g}_j^\omega = \Gamma_{ij}^k \boldsymbol{g}_k^\omega \quad \text{in } \omega, \tag{4.5}$$

$$\boldsymbol{g}_j^\omega(x^\omega) = \boldsymbol{v}_j, \tag{4.6}$$

where $x^\omega = (x_i^\omega)$ denotes the center of the cube ω.

The vector fields $\boldsymbol{g}_j$ are defined at each $x \in \omega$ by integrating the above system along a broken line joining x^ω to x, the edges of which are parallel to the coordinate axes of $\mathbb{R}^n$. More specifically, let ε denote the half-length of the edge of the cube ω, let $\omega^1 := (x_1^\omega - \varepsilon, x_1^\omega + \varepsilon)$, let $\omega^2 := (x_1^\omega - \varepsilon, x_1^\omega + \varepsilon) \times (x_2^\omega - \varepsilon, x_2^\omega + \varepsilon)$, etc., so that $\omega = \omega^n$. Then, using relations (4.4), one shows that there exist n vector fields $\boldsymbol{h}_j^1 \in \mathcal{C}^1(\omega^1; \mathbb{E}^n)$ such that

$$\begin{aligned} &\partial_1 \boldsymbol{h}_j^1 = \Gamma_{1j}^k(\cdot, x_2^\omega, \cdots, x_n^\omega) \boldsymbol{h}_k^1 \quad \text{in } \omega^1, \\ &\boldsymbol{h}_j^1(x_1^\omega) = \boldsymbol{v}_j, \end{aligned}$$

then that there exist n vector fields $\boldsymbol{h}_j^2 \in \mathcal{C}^1(\omega^2; \mathbb{E}^n)$ such that

$$\begin{aligned} &\partial_i \boldsymbol{h}_j^2 = \Gamma_{ij}^k(\cdot, x_3^\omega, \cdots, x_n^\omega) \boldsymbol{h}_k^2 \text{ in } \omega^2, \quad i \in \{1, 2\}, \\ &\boldsymbol{h}_j^2(x_1^\omega, x_2^\omega) = \boldsymbol{v}_j, \end{aligned}$$

and finally, after n steps, that there exist n vector fields $\boldsymbol{h}_j^n \in \mathcal{C}^1(\omega^n; \mathbb{E}^n)$ such that

$$\begin{aligned} &\partial_i \boldsymbol{h}_j^n = \Gamma_{ij}^k \boldsymbol{h}_k^n \quad \text{in } \omega^n, \\ &\boldsymbol{h}_j^n(x^\omega) = \boldsymbol{v}_j. \end{aligned}$$

The vector fields $\boldsymbol{g}_j^\omega := \boldsymbol{h}_j^n$ then satisfy the Pfaff system (4.5).

Second, one shows that the vector fields $\boldsymbol{v}_i$ used in the previous step to define the vector fields $\boldsymbol{g}_i^\omega$ can be chosen in such a way that

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = g_{ij}(x^\omega),$$

then that

$$\boldsymbol{g}_i^\omega \cdot \boldsymbol{g}_j^\omega = g_{ij} \quad \text{in } \omega. \tag{4.7}$$

The last implication is established by showing that the Pfaff system

$$\begin{aligned} &\partial_\ell h_{ij} = \Gamma_{i\ell}^k h_{jk} + \Gamma_{j\ell}^k h_{ik} \quad \text{in } \omega, \\ &h_{ij}(x^\omega) = g_{ij}(x^\omega) \end{aligned}$$

has at most one solution $(h_{ij}) \in \mathcal{C}^1(\omega; \mathbb{S}^n)$ and that both matrix fields $(\boldsymbol{g}_i^\omega \cdot \boldsymbol{g}_j^\omega) \in \mathcal{C}^1(\omega; \mathbb{S}^n)$ and $(g_{ij}) \in \mathcal{C}^1(\omega; \mathbb{S}^n)$ satisfy the above Pfaff system.

Third, one shows that, if $\omega \subset \Omega$ and $\widetilde{\omega} \subset \Omega$ are two cubes such that $\omega \cap \widetilde{\omega} \neq \emptyset$, if $\boldsymbol{g}_i \in \mathcal{C}^1(\omega; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{g}}_i \in \mathcal{C}^1(\widetilde{\omega}; \mathbb{E}^n)$ are vector fields that satisfy respectively the Pfaff systems

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \ \text{ in } \omega \quad \text{and} \quad \partial_i \widetilde{\boldsymbol{g}}_j = \Gamma_{ij}^k \widetilde{\boldsymbol{g}}_k \ \text{ in } \widetilde{\omega},$$

and if there exists a point $x_0 \in \omega \cap \widetilde{\omega}$ such that

$$\boldsymbol{g}_i(x_0) = \widetilde{\boldsymbol{g}}_i(x_0),$$

then

$$\boldsymbol{g}_i = \widetilde{\boldsymbol{g}}_i \quad \text{in } \omega \cap \widetilde{\omega}.$$

To this end, it suffices to observe that $\omega \cap \widetilde{\omega}$ is connected, and that, for each point $x \in \omega \cap \widetilde{\omega}$ and for each mapping $\boldsymbol{\gamma} = (\gamma^i) \in \mathcal{C}^1([0,1]; \omega \cap \widetilde{\omega})$ such that $\boldsymbol{\gamma}(0) = x_0$ and $\boldsymbol{\gamma}(1) = x$, the vector fields

$$\boldsymbol{g}_i \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{E}^n) \quad \text{and} \quad \widetilde{\boldsymbol{g}}_i \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{E}^n)$$

satisfy the same Cauchy problem.

Finally, a solution to the Pfaff system (4.1) is constructed by "glueing together" the solutions of the Pfaff systems (4.5). This is possible thanks to the simple-connectedness of Ω and to the uniqueness result proved in the the third step above.

The proof that the immersion $\boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ obtained in this fashion is unique up to isometries in $\mathbb{E}^n$ is analogous to the part (iv) in the proof of Theorem 2.1 (it suffices to replace the spaces $\mathcal{C}^k(\Omega; \mathbb{E}^n)$ by the spaces $\mathcal{C}^{k-1}(\Omega; \mathbb{E}^n)$ at each one of their occurrences). This concludes the proof of Theorem 4.1.

The next theorem is a natural complement to Theorem 4.1, which shows that the immersion $\boldsymbol{\phi}$ is a continuous function of the matrix field (g_{ij}) for some ad hoc Fréchet topologies (the definition of a Fréchet topology is recalled in Section 2).

Theorem 4.2 *Let Ω be a connected open subset of $\mathbb{R}^n$, and let*

$$\boldsymbol{\phi}^m \in \mathcal{C}^2(\Omega; \mathbb{E}^n), \quad m \geq 1 \quad \textit{and} \quad \boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$$

be immersions that satisfy

$$(g_{ij}^m) \to (g_{ij}) \ \textit{ in } \mathcal{C}^1(\Omega; \mathbb{S}^n) \ \textit{as } m \to \infty,$$

where

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \in \mathcal{C}^1(\Omega) \quad \textit{and} \quad g_{ij}^m := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m \in \mathcal{C}^1(\Omega), \quad m \geq 1.$$

Then there exist isometries $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $m \geq 1$, such that

$$\boldsymbol{r}^m \circ \boldsymbol{\phi}^m \to \boldsymbol{\phi} \quad \textit{in } \mathcal{C}^2(\Omega; \mathbb{E}^n) \ \textit{as } m \to \infty.$$

Proof It suffices to replace in the proof of Theorem 2.2 the spaces $\mathcal{C}^k(\Omega)$, $k = 1, 2, 3$, by $\mathcal{C}^{k-1}(\Omega)$, at each one of their occurrences. More specifically, one first shows that it suffices to consider the particular case where $\boldsymbol{\phi} = \mathbf{id}$, so that the immersions

$$\boldsymbol{\phi}^m \in \mathcal{C}^2(\Omega; \mathbb{E}^n), \quad m \geq 1$$

satisfy

$$g_{ij}^m := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m \to \delta_{ij} \quad \text{in } \mathcal{C}^1(\Omega) \text{ as } m \to \infty.$$

Then (the notations used here are the same as in the proof of Theorem 2.2)

$$\Gamma_{ij}^{k,m} \to 0 \quad \text{in } \mathcal{C}^0(\Omega) \text{ as } m \to \infty$$

and

$$|\boldsymbol{g}_i^m|^2 = g_{ii}^m \to 1 \quad \text{in } \mathcal{C}^1(\Omega) \text{ as } m \to \infty \text{ (no summation)}.$$

Since the vector fields $\boldsymbol{g}_i^m := \partial_i \boldsymbol{\phi}^m \in \mathcal{C}^1(\Omega; \mathbb{E}^n)$ satisfy the Pfaff systems

$$\partial_i \boldsymbol{g}_j^m = \Gamma_{ij}^{k,m} \boldsymbol{g}_k^m \quad \text{in } \mathcal{C}^0(\Omega; \mathbb{E}^n),$$

the above convergences imply that

$$\partial_{ij} \boldsymbol{\phi}^m = \partial_i \boldsymbol{g}_j^m \to \mathbf{0} \quad \text{in } \mathcal{C}^0(\Omega; \mathbb{E}^n).$$

Let x_0 be a point in Ω, let

$$\boldsymbol{R}^m := \boldsymbol{\nabla} \boldsymbol{\phi}^m(x_0)(g_{ij}^m(x_0))^{-\frac{1}{2}} \in \mathbb{O}^n,$$

let $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$ be defined by

$$\boldsymbol{r}^m(x) := x_0 + (\boldsymbol{R}^m)^{\mathrm{T}}(x - \boldsymbol{\phi}^m(x_0)) \quad \text{for all } x \in \mathbb{E}^n,$$

and let the mapping $\boldsymbol{\psi}^m \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ be defined by

$$\boldsymbol{\psi}^m := \boldsymbol{r}^m \circ \boldsymbol{\phi}^m.$$

Then

$$\begin{aligned}
\boldsymbol{\psi}^m(x_0) &= \mathbf{id}(x_0) \quad \text{in } \mathbb{E}^n \text{ for each } m \geq 1,\\
\boldsymbol{\nabla}\boldsymbol{\psi}^m(x_0) &\to \boldsymbol{\nabla}\mathbf{id}(x_0) \quad \text{in } \mathbb{M}^n \text{ as } m \to \infty,\\
\partial_{ij}\boldsymbol{\psi}^m &\to \partial_{ij}\mathbf{id} \quad \text{in } \mathcal{C}^0(\Omega; \mathbb{E}^n) \text{ as } m \to \infty,
\end{aligned}$$

which in turn implies that

$$\boldsymbol{\psi}^m \to \mathbf{id} \quad \text{in } \mathcal{C}^2(\Omega; \mathbb{E}^n) \text{ as } m \to \infty.$$

5 Metric Tensor Fields in $\mathcal{C}^1(\overline{\Omega})$

In this section, we show that the theorems established in Section 3 still hold for matrix fields with coefficients $g_{ij} \in \mathcal{C}^1(\overline{\Omega})$, instead of $g_{ij} \in \mathcal{C}^2(\overline{\Omega})$. As in Theorem 3.1, the assumption that Ω is a domain can be replaced in theorem below by the weaker assumption that Ω is a connected open subset of $\mathbb{R}^n$ that satisfies the "geodesic property".

Theorem 5.1 *Let Ω be a simply-connected domain in $\mathbb{R}^n$, and let $(g_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in the sense of distributions, i.e.,*

$$\int_\Omega \big(\Gamma_{ij\ell}\partial_k \varphi - \Gamma_{ik\ell}\partial_j \varphi + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q})\varphi\big)\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$

where

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \quad \textit{and} \quad (g^{rq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ *such that*

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \textit{in } \overline{\Omega}.$$

In addition, an immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ *satisfies*

$$\partial_i \boldsymbol{\psi} \cdot \partial_j \boldsymbol{\psi} = g_{ij} \quad \textit{in } \overline{\Omega}$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \textit{in } \overline{\Omega}.$$

Proof Theorem 5.1 is deduced from Theorem 4.1 in the same way that Theorem 3.1 was deduced from Theorem 2.1; in other words, Theorem 5.1 is proved by showing that any immersion $\boldsymbol{\phi} \in \mathcal{C}^2(\Omega; \mathbb{E}^n)$ that satisfies

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega$$

(the existence of such an immersion is guaranteed by Theorem 4.1), as well as all of its partial derivatives up to order two, possess continuous extensions to $\overline{\Omega}$. To see that this is indeed the case, it suffices to replace in the proof of Theorem 3.1 the spaces $\mathcal{C}^k(\Omega)$ by the spaces $\mathcal{C}^{k-1}(\Omega)$ at each one of their occurrences.

The next theorem establishes a nonlinear Korn inequality in $\mathcal{C}^2(\overline{\Omega})$, similar to the nonlinear Korn inequality in $\mathcal{C}^3(\overline{\Omega})$ established in Theorem 3.2. This inequality implies in particular (see Theorem 5.3) that, up to an isometry of $\mathbb{E}^n$, an immersion $\boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ depends continuously on its metric tensor field $(g_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^n_>)$. As in Theorem 3.2, the assumption that Ω is a domain can be replaced in Theorem 5.2 by the weaker assumption that Ω is a connected open subset of $\mathbb{R}^n$ that satisfies the "geodesic property".

Theorem 5.2 *Let Ω be a domain in $\mathbb{R}^n$, and let $\boldsymbol{\phi}^0 \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ be an immersion. Then there exist two constants $C = C(\boldsymbol{\phi}^0) > 0$ and $\delta = \delta(\boldsymbol{\phi}^0) > 0$ such that*

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathcal{C}^1(\overline{\Omega}; \mathbb{S}^n)}$$

for all immersions $\boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ and $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{E}^n)$ that satisfy

$$\|(g_{ij}) - (g^0_{ij})\|_{\mathcal{C}^1(\overline{\Omega}; \mathbb{S}^n)} < \delta \quad \textit{and} \quad \|(\widetilde{g}_{ij}) - (g^0_{ij})\|_{\mathcal{C}^1(\overline{\Omega}; \mathbb{S}^n)} < \delta,$$

respectively, where

$$g^0_{ij} := \partial_i \boldsymbol{\phi}^0 \cdot \partial_j \boldsymbol{\phi}^0, \quad g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \textit{and} \quad \widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}}$$

denote the covariant components of the metric tensor fields induced by the immersions $\boldsymbol{\phi}^0$, $\boldsymbol{\phi}$, and $\widetilde{\boldsymbol{\phi}}$, respectively.

Proof It suffices to replace in the proof of Theorem 3.2 the spaces $\mathcal{C}^k(\overline{\Omega})$ by the spaces $\mathcal{C}^{k-1}(\overline{\Omega})$ at each one of their occurrences.

An immediate consequence of the nonlinear Korn inequality of Theorem 5.2 is the following convergence result, similar to that of Theorem 3.3.

Theorem 5.3 *Let Ω be a domain in $\mathbb{R}^n$, and let*

$$\boldsymbol{\phi}^m \in \mathcal{C}^2(\overline{\Omega};\mathbb{E}^n),\quad m \geq 1 \quad \textit{and} \quad \boldsymbol{\phi} \in \mathcal{C}^2(\overline{\Omega};\mathbb{E}^n)$$

be immersions that satisfy

$$(g_{ij}^m) \to (g_{ij}) \quad \textit{in } \mathcal{C}^1(\overline{\Omega};\mathbb{S}^n) \textit{ as } m \to \infty,$$

where

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \textit{and} \quad g_{ij}^m := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m, \quad m \geq 1.$$

Then there exist isometries $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $m \geq 1$, such that

$$\boldsymbol{r}^m \circ \boldsymbol{\phi}^m \to \boldsymbol{\phi} \quad \textit{in } \mathcal{C}^2(\overline{\Omega};\mathbb{E}^n) \textit{ as } m \to \infty.$$

6 Metric Tensor Fields in $\boldsymbol{W^{1,p}(\Omega)}$, $\boldsymbol{p > n}$

Given an open subset Ω of $\mathbb{R}^n$ and $1 < p < \infty$, let

$$\begin{aligned}
L^p(\Omega;\mathbb{E}^n) &:= \{(f_i) : \Omega \to \mathbb{E}^n;\ f_i \in L^p(\Omega)\},\\
W^{m,p}(\Omega;\mathbb{E}^n) &:= \{(f_i) : \Omega \to \mathbb{E}^n;\ f_i \in W^{m,p}(\Omega)\},\\
L^p(\Omega;\mathbb{M}^n) &:= \{(g_{ij}) : \Omega \to \mathbb{M}^n;\ g_{ij} \in L^p(\Omega)\},\\
W^{1,p}(\Omega;\mathbb{M}^n) &:= \{(g_{ij}) : \Omega \to \mathbb{M}^n;\ g_{ij} \in W^{1,p}(\Omega)\},
\end{aligned}$$

where $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $m = 1, 2$, designate the usual Lebesgue and Sobolev spaces. These spaces are respectively equipped with the norms

$$\begin{aligned}
\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} &:= \Big(\int_\Omega |\boldsymbol{f}(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \quad \text{for all } \boldsymbol{f} \in L^p(\Omega;\mathbb{E}^n),\\
\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)} &:= \Big(\|\boldsymbol{f}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{\nabla f}\|^p_{\mathbb{L}^p(\Omega)}\Big)^{\frac{1}{p}} \quad \text{for all } \boldsymbol{f} \in W^{1,p}(\Omega;\mathbb{E}^n),\\
\|\boldsymbol{f}\|_{\boldsymbol{W}^{2,p}(\Omega)} &:= \Big(\|\boldsymbol{f}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \sum_{i,j} \|\partial_{ij}\boldsymbol{f}\|^p_{\boldsymbol{L}^p(\Omega)}\Big)^{\frac{1}{p}} \quad \text{for all } \boldsymbol{f} \in W^{2,p}(\Omega;\mathbb{E}^n),\\
\|\boldsymbol{C}\|_{\mathbb{L}^p(\Omega)} &:= \Big(\int_\Omega |\boldsymbol{C}(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \quad \text{for all } \boldsymbol{C} \in L^p(\Omega;\mathbb{M}^n),\\
\|\boldsymbol{C}\|_{\mathbb{W}^{1,p}(\Omega)} &:= \Big(\|\boldsymbol{C}\|^p_{\mathbb{L}^p(\Omega)} \mathrm{d}x + \sum_{i} \|\partial_i\boldsymbol{C}\|^p_{\mathbb{L}^p(\Omega)}\Big)^{\frac{1}{p}} \quad \text{for all } \boldsymbol{C} \in W^{1,p}(\Omega;\mathbb{M}^n).
\end{aligned}$$

It is well known that, if Ω is a domain in $\mathbb{R}^n$ and if $p > n$, then the inclusion

$$W^{1,p}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$$

holds and there exists a constant C_0 (which depends on Ω and p) such that

$$\sup_{x\in\overline{\Omega}} |f(x)| \leq C_0 \|f\|_{W^{1,p}(\Omega)} \quad \text{for all } f \in W^{1,p}(\Omega).$$

As is customary, a function $f \in W^{1,p}(\Omega)$, i.e., in effect an equivalence class, is identified in this case with its continuous representative.

It is also well known that, again if Ω is a domain in $\mathbb{R}^n$ and $p > n$, the space $W^{1,p}(\Omega)$ is a Banach algebra, in the sense that the product of two functions in $W^{1,p}(\Omega)$ is still in $W^{1,p}(\Omega)$ and there exists a constant C_1 (which depends on p) such that

$$\|fg\|_{W^{1,p}(\Omega)} \leq C_1 \|f\|_{W^{1,p}(\Omega)} \|g\|_{W^{1,p}(\Omega)} \quad \text{for all } f, g \in W^{1,p}(\Omega)$$

(particularly neat proofs of such results for domains with smooth boundaries are found in [3]).

Up to now, the metric fields $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ (resp. $(g_{ij}) : \overline{\Omega} \to \mathbb{S}^n_>$) were assumed to be at least of class $\mathcal{C}^1$ on Ω (resp. on $\overline{\Omega}$). We now consider the situation where the metric fields $(g_{ij}) : \Omega \to \mathbb{S}^n_>$ are in the space $W^{1,p}(\Omega; \mathbb{S}^n)$ for some $p > n$ and Ω is a domain in $\mathbb{R}^n$. It is remarkable that, under such a weak regularity assumption, an existence and uniqueness theorem analogous to the previous ones still holds as follows.

Theorem 6.1 *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let $p > n$, and let $(g_{ij}) \in W^{1,p}(\Omega; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in the sense of distributions, i.e.,*

$$\int_\Omega (\Gamma_{ij\ell}\partial_k\varphi - \Gamma_{ik\ell}\partial_j\varphi + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q})\varphi)\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$

where

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \in L^p(\Omega) \quad \textit{and} \quad (g^{rq}) := (g_{ij})^{-1} \in W^{1,p}(\Omega; \mathbb{S}^n_>).$$

Then there exists an immersion $\boldsymbol{\phi} \in W^{2,p}(\Omega; \mathbb{E}^n)$ such that

$$\partial_i\boldsymbol{\phi} \cdot \partial_j\boldsymbol{\phi} = g_{ij} \quad \textit{in } \overline{\Omega}.$$

In addition, an immersion $\boldsymbol{\psi} \in W^{2,p}(\Omega; \mathbb{E}^n)$ satisfies

$$\partial_i\boldsymbol{\psi} \cdot \partial_j\boldsymbol{\psi} = g_{ij} \quad \textit{in } \overline{\Omega}$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ such that

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \textit{in } \overline{\Omega}.$$

Idea of the Proof First, notice that the assumption that $(g_{ij}(x)) \in \mathbb{S}^n_>$ at each $x \in \overline{\Omega}$ together with the inclusion $W^{1,p}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$ implies that $\inf_{x\in\overline{\Omega}} \det(g_{ij}(x)) > 0$. This property, combined with the property that $W^{1,p}(\Omega)$ is an algebra in turn implies that $g^{rq} \in W^{1,p}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$. Hence products like $g^{rq}\Gamma_{ijr}\Gamma_{k\ell q}$ are well defined in the space $L^{\frac{p}{2}}(\Omega)$ (which is contained in the space $L^1(\Omega)$ since $p > n \geq 2$).

The existence proof follows the same pattern as before, i.e., it consists in first finding vector fields $\boldsymbol{g}_j \in W^{1,p}(\Omega;\mathbb{E}^n)$ that satisfy the Pfaff system

$$\partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \quad \text{in } \Omega, \quad \text{where } \Gamma_{ij}^k := g^{k\ell}\Gamma_{ij\ell} \in L^p(\Omega), \tag{6.1}$$

then in finding an immersion $\boldsymbol{\phi} \in W^{2,p}(\Omega;\mathbb{E}^n)$ that satisfies the Poincaré system

$$\partial_j \boldsymbol{\phi} = \boldsymbol{g}_j \quad \text{in } \Omega. \tag{6.2}$$

Otherwise the proof is much more delicate and technical than those of the previous existence theorems. It combines a key existence theorem for Pfaff systems "with coefficients in L^p" (see [17, Theorem 6.8]), with an approach due to Mardare (see [18, Theorem 4.1]) for solving the above Pfaff and Poincaré systems, which relies in particular on a careful "glueing" of local solutions.

The next theorem, which is due to [11] (see also [12]), establishes a nonlinear Korn inequality in $W^{2,p}(\Omega)$, with the same function spaces for the metric tensors and for the immersions as in the above existence theorem. Note, however, that the existence of the immersions is assumed here.

Theorem 6.2 *Let Ω be a domain in $\mathbb{R}^n$, and let $p > n$. Given any $\varepsilon > 0$, define the set*

$$\boldsymbol{\Phi}_\varepsilon := \Big\{\boldsymbol{\phi} \in W^{2,p}(\Omega;\mathbb{E}^n); \det(g_{ij}) \geq \varepsilon \text{ in } \Omega \text{ and } \|(g_{ij})\|_{\mathbb{W}^{1,p}(\Omega)} \leq \frac{1}{\varepsilon}\Big\},$$

where $g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi}$. Then there exists a constant $C_\varepsilon > 0$ such that

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C_\varepsilon \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathbb{W}^{1,p}(\Omega)} \quad \textit{for all } \boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}} \in \boldsymbol{\Phi}_\varepsilon,$$

where $\widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}}$.

Idea of the Proof Pick any point $x_0 \in \Omega$. Then one shows that, given any $\boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}} \in \boldsymbol{\Phi}_\varepsilon$, the matrix $\boldsymbol{Q}_0 := \boldsymbol{\nabla}\boldsymbol{\phi}(x_0)(g_{ij}(x_0))^{-\frac{1}{2}}(\widetilde{g}_{ij}(x_0))^{\frac{1}{2}}(\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}(x_0))^{-1}$ is orthogonal and the vector fields

$$\boldsymbol{g}_i := \partial_i \boldsymbol{\phi} \in W^{1,p}(\Omega;\mathbb{E}^n) \quad \text{and} \quad \widetilde{\boldsymbol{g}}_i^\sharp := \partial_i(\boldsymbol{Q}_0 \widetilde{\boldsymbol{\phi}}) \in W^{1,p}(\Omega;\mathbb{E}^n)$$

satisfy the Pfaff systems

$$\begin{cases} \partial_i \boldsymbol{g}_j = \Gamma_{ij}^k \boldsymbol{g}_k \quad \text{a.e. in } \Omega, \\ \boldsymbol{g}_i(x_0) = \partial_i \boldsymbol{\phi}(x_0), \end{cases} \qquad \begin{cases} \partial_i \widetilde{\boldsymbol{g}}_j^\sharp = \widetilde{\Gamma}_{ij}^k \widetilde{\boldsymbol{g}}_k^\sharp \quad \text{a.e. in } \Omega, \\ \widetilde{\boldsymbol{g}}_i^\sharp(x_0) = \boldsymbol{Q}_0 \partial_i \widetilde{\boldsymbol{\phi}}(x_0), \end{cases}$$

where

$$\Gamma_{ij}^k := g^{k\ell}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \quad \text{and} \quad \widetilde{\Gamma}_{ij}^k := \widetilde{g}^{k\ell}(\partial_j \widetilde{g}_{i\ell} + \partial_i \widetilde{g}_{j\ell} - \partial_\ell \widetilde{g}_{ij}).$$

The rest of the proof relies on a series of careful estimates, which eventually allow to use a key comparison theorem between the solutions of Pfaff systems "with coefficients in L^p" (see [17, Theorem 4.1]).

Note that the assumption that Ω is a domain can be replaced in Theorem 6.2 by the weaker assumption that Ω is a bounded and connected open subset of $\mathbb{R}^n$ that satisfies the "uniform interior cone property" (see [2]).

As in [11] (see also [12]), one can then establish the following local Lipschitz-continuity result as a corollary to the nonlinear Korn inequality of Theorem 6.2.

Theorem 6.3 *Let Ω be a domain in $\mathbb{R}^n$, let $p > n$, and let $\boldsymbol{\phi}^0 \in W^{2,p}(\Omega; \mathbb{E}^n)$ be an immersion. Then there exist constants $\delta = \delta(\boldsymbol{\phi}^0) > 0$ and $C = C(\boldsymbol{\phi}_0) > 0$ such that*

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathbb{W}^{1,p}(\Omega)}$$

for all $\boldsymbol{\phi} \in W^{2,p}(\Omega; \mathbb{E}^n)$ and all $\widetilde{\boldsymbol{\phi}} \in W^{2,p}(\Omega; \mathbb{E}^n)$ that satisfy

$$\|(g_{ij}) - (g^0_{ij})\|_{\mathbb{W}^{1,p}(\Omega)} < \delta \quad \textit{and} \quad \|(\widetilde{g}_{ij}) - (g^0_{ij})\|_{\mathbb{W}^{1,p}(\Omega)} < \delta,$$

where

$$g^0_{ij} := \partial_i \boldsymbol{\phi}^0 \cdot \partial_j \boldsymbol{\phi}^0, \quad g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \textit{and} \quad \widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}}.$$

7 Metric Tensor Fields in $\boldsymbol{W^{1,p}_{\mathrm{loc}}(\Omega)}$, $\boldsymbol{p > n}$

The existence and uniqueness theorems and the nonlinear Korn inequalities of Sections 3 and 5 have been extended in Section 6 from continuously-differentiable immersions in the closure of a domain to immersions in Sobolev spaces.

In this section, we show that the existence and uniqueness theorems and the continuity theorems of Sections 2 and 4 can be likewise extended from continuously-differentiable immersions in an open set to immersions that belong locally to Sobolev spaces. Note that, by contrast with Section 6, no regularity assumptions on the boundary of Ω are needed in this section.

To begin with, we show that the existence and uniqueness theorems established in Sections 2 and 6 still hold for matrix fields with coefficients $g_{ij} \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > n$, instead of $g_{ij} \in \mathcal{C}^k(\Omega)$, $k = 1, 2$. Recall that a function $f : \Omega \to \mathbb{E}$ belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > n$, if, for each point x_0 in Ω, there exists an open ball $B \subset \Omega$ containing x_0 such that $f|_B \in W^{1,p}(B)$.

Theorem 7.1 *Let Ω be a simply-connected open set in $\mathbb{R}^n$, let $p > n$, and let $(g_{ij}) \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{S}^n_>)$ be a matrix field whose Riemann curvature tensor field vanishes in the sense of distributions, i.e.,*

$$\int_\Omega (\Gamma_{ij\ell} \partial_k \varphi - \Gamma_{ik\ell} \partial_j \varphi + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q})\varphi) \mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\Omega),$$

where

$$\Gamma_{ij\ell} := \frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}) \quad \textit{and} \quad (g^{rq}) := (g_{ij})^{-1}.$$

Then there exists an immersion $\boldsymbol{\phi} \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$ such that

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \textit{in } \Omega.$$

In addition, an immersion $\boldsymbol{\psi} \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$ satisfies

$$\partial_i \boldsymbol{\psi} \cdot \partial_j \boldsymbol{\psi} = g_{ij} \quad \textit{in } \Omega$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)$ such that

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\phi} \quad \textit{in } \Omega.$$

Proof Pick any point x_0 in Ω. Given any point $x \in \Omega$, let $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\Omega)$ be any mapping that satisfies $\boldsymbol{\gamma}(0) = x_0$ and $\boldsymbol{\gamma}(1) = x$ (such a mapping exists since Ω is connected). Let $\varepsilon > 0$ denote the half-distance from the compact set $\boldsymbol{\gamma}([0,1])$ to the boundary of Ω, let N be the smallest integer that is $> \left(\frac{1}{\varepsilon}\right) \sup\limits_{t\in[0,1]} \left|\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}(t)\right|$, and, for each $m = 0, 1, \cdots, N$, let $t_m := \frac{m}{N}$ and let B_m denote the open ball in $\mathbb{R}^n$ centered at $x_m := \boldsymbol{\gamma}(t_m)$ with radius ε. Then $x = x_N$ and

$$\begin{aligned}
&\overline{B}_m \subset \Omega \quad \text{for all } 0 \leq m \leq N,\\
&x_m \in B_{m-1} \quad \text{for all } 1 \leq m \leq N.
\end{aligned}$$

Since B_m is a simply-connected domain in $\mathbb{R}^n$, since $(g_{ij}|_{B_m}) \in W^{1,p}(B_m;\mathbb{S}^n_>)$, and since the relations

$$\int_\Omega \big(\Gamma_{ij\ell}\partial_k\varphi - \Gamma_{ik\ell}\partial_j\varphi + g^{rq}(\Gamma_{ijr}\Gamma_{k\ell q} - \Gamma_{ikr}\Gamma_{j\ell q})\varphi\big)\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega)$$

imply in particular that the Riemann curvature tensor field of the matrix field $(g_{ij}|_{B_m})$ vanishes in the sense of distributions on B_m, Theorem 6.1 implies that there exists an immersion $\boldsymbol{\phi}_m \in W^{2,p}(B_m;\mathbb{E}^n)$ such that

$$\partial_i\boldsymbol{\phi}_m \cdot \partial_j\boldsymbol{\phi}_m = g_{ij} \quad \text{in } \overline{B}_m.$$

Let the immersions

$$\boldsymbol{\psi}_m := \boldsymbol{r}_m \circ \boldsymbol{\phi}_m : B_m \to \mathbb{E}^n, \quad m = 0, 1, \cdots, N,$$

be defined for $m = 0$ by

$$\boldsymbol{r}_0(x) := \boldsymbol{R}_0(x - \boldsymbol{\phi}_0(x_0)) \quad \text{for all } x \in \mathbb{E}^n,$$

where

$$\boldsymbol{R}_0 := (g_{ij}(x_0))^{-\frac{1}{2}}(\boldsymbol{\nabla}\boldsymbol{\phi}_0(x_0))^{\mathrm{T}} \in \mathbb{O}^n,$$

and, for $m = 1, 2, \cdots, N$ by

$$\boldsymbol{r}_m(x) := \boldsymbol{\psi}_{m-1}(x_m) + \boldsymbol{R}_m(x - \boldsymbol{\phi}_m(x_m)) \quad \text{for all } x \in \mathbb{E}^n,$$

where

$$\boldsymbol{R}_m := \boldsymbol{\nabla}\boldsymbol{\psi}_{m-1}(x_m)(g_{ij}(x_m))^{-1}(\boldsymbol{\nabla}\boldsymbol{\phi}_m(x_m))^{\mathrm{T}} \in \mathbb{O}^n.$$

The above definition of the immersions $\boldsymbol{\psi}_m \in W^{2,p}(B_m;\mathbb{E}^n)$ implies that

$$\partial_i\boldsymbol{\psi}_m \cdot \partial_j\boldsymbol{\psi}_m = g_{ij} \quad \text{in } \overline{B}_m,\ m = 0, 1, \cdots, N,$$

and that

$$\boldsymbol{\psi}_m(x_m) = \boldsymbol{\psi}_{m-1}(x_m) \quad \text{and} \quad \boldsymbol{\nabla}\boldsymbol{\psi}_m(x_m) = \boldsymbol{\nabla}\boldsymbol{\psi}_{m-1}(x_m), \quad m = 1, 2 \cdots, N.$$

Then the simple connectedness of Ω and the uniqueness part of Theorem 6.1 imply that the point $\boldsymbol{\psi}_N(x) \in \mathbb{E}^n$ only depends on the matrix field (g_{ij}) and on x_0 (in particular, it is independent of the path $\boldsymbol{\gamma}$ joining x_0 to x). It follows that the mapping $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$, where

$$\boldsymbol{\phi}(x) := \boldsymbol{\psi}_N(x) \in \mathbb{E}^n \quad \text{for all } x \in \Omega,$$

is unambiguously defined, that

$$\boldsymbol{\phi}|_{B_N} = \boldsymbol{\psi}_N \in W^{2,p}(B_N; \mathbb{S}^n_>),$$

and that

$$\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} = g_{ij} \quad \text{in } \Omega.$$

The proof of the uniqueness part of the theorem is similar to the proof of the uniqueness part of Theorem 2.1, where it suffices to replace the spaces $\mathcal{C}^k$ by $W^{k-1,p}_{\mathrm{loc}}$ at each one of their occurrences.

We now show that the convergence theorems established in Sections 2 and 4 still hold for immersions in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$, $p > n$, instead of immersions in $\mathcal{C}^k(\Omega)$, $k = 2, 3$.

Recall that a sequence $(f_m)_{m=1}^\infty$ of functions $f_m \in W^{k,p}_{\mathrm{loc}}(\Omega)$, $k = 1, 2$, $p > n$, converges to $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$ with respect to the Fréchet topology of $W^{k,p}_{\mathrm{loc}}(\Omega)$ if and only if

$$\lim_{m\to\infty} \|f_m - f\|_{W^{k,p}(B)} = 0$$

for each open ball B in $\mathbb{R}^n$ such that $\overline{B} \subset \Omega$. If this is the case, we write

$$f_m \to f \quad \text{in } W^{k,p}_{\mathrm{loc}}(\Omega) \text{ as } m \to \infty.$$

Such notions can then be clearly extended to the spaces $W^{k,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$ and $W^{k,p}_{\mathrm{loc}}(\Omega; \mathbb{S}^n)$.

Theorem 7.2 *Let Ω be a connected open subset of $\mathbb{R}^n$, let $p > n$, and let*

$$\boldsymbol{\phi}^m \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n), \quad m \geq 1 \quad \textit{and} \quad \boldsymbol{\phi} \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$$

be immersions that satisfy

$$(g^m_{ij}) \to (g_{ij}) \quad \textit{in } W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{S}^n) \textit{ as } m \to \infty,$$

where

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \in W^{1,p}_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad g^m_{ij} := \partial_i \boldsymbol{\phi}^m \cdot \partial_j \boldsymbol{\phi}^m \in W^{1,p}_{\mathrm{loc}}(\Omega), \quad m \geq 1.$$

Then there exist isometries $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$, $m \geq 1$, such that

$$\boldsymbol{r}^m \circ \boldsymbol{\phi}^m \to \boldsymbol{\phi} \quad \textit{in } W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n) \textit{ as } m \to \infty.$$

Proof Pick any point x_0 in Ω. For each $m \geq 1$, let

$$\boldsymbol{R}^m := \boldsymbol{\nabla}\boldsymbol{\phi}(x_0)(g_{ij}(x_0))^{-\frac{1}{2}}(g^m_{ij}(x_0))^{-\frac{1}{2}}(\boldsymbol{\nabla}\boldsymbol{\phi}^m(x_0))^{\mathrm{T}} \in \mathbb{O}^n,$$

let $\boldsymbol{r}^m \in \mathrm{Isom}(\mathbb{E}^n)$ be defined by

$$\boldsymbol{r}^m(x) := \boldsymbol{\phi}(x_0) + \boldsymbol{R}^m(x - \boldsymbol{\phi}^m(x_0)) \quad \text{for all } x \in \mathbb{E}^n,$$

and let

$$\boldsymbol{\psi}^m := \boldsymbol{r}^m \circ \boldsymbol{\phi}^m \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n).$$

Note that the immersions $\boldsymbol{\psi}^m$ satisfy in particular the relations

$$\begin{aligned} \boldsymbol{\psi}^m(x_0) &= \boldsymbol{\phi}(x_0) \quad \text{for all } m \geq 1, \\ \boldsymbol{\nabla}\boldsymbol{\psi}^m(x_0) &\to \boldsymbol{\nabla}\boldsymbol{\phi}(x_0) \quad \text{in } \mathbb{M}^n \text{ as } m \to \infty. \end{aligned} \tag{7.1}$$

Let B be any open ball in $\mathbb{R}^n$ such that $\overline{B} \subset \Omega$. Since the open set Ω is connected, there exists a domain ω in $\mathbb{R}^n$ such that $x_0 \in \omega$, $B \subset \omega$, and $\overline{\omega} \subset \Omega$.

On the one hand, the assumptions of the theorem imply that $\boldsymbol{\phi}|_\omega \in W^{2,p}(\omega; \mathbb{E}^n)$, $\boldsymbol{\phi}^m|_\omega \in W^{2,p}(\omega; \mathbb{E}^n)$, $m \geq 1$, and

$$(g_{ij}^m|_\omega) \to (g_{ij}|_\omega) \quad \text{in } W^{1,p}(\omega; \mathbb{S}^n) \text{ as } m \to \infty.$$

On the other hand, Theorem 6.3 implies that there exist constants $\delta = \delta(\boldsymbol{\phi}|_\omega) > 0$ and $C = C(\boldsymbol{\phi}|_\omega) > 0$ such that

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\omega)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathbb{W}^{1,p}(\omega)}$$

for all $\widetilde{\boldsymbol{\phi}} \in W^{2,p}(\omega; \mathbb{E}^n)$ that satisfy

$$\|(\widetilde{g}_{ij}) - (g_{ij})\|_{\mathbb{W}^{1,p}(\omega)} < \delta.$$

The two properties above imply that there exist isometries $\boldsymbol{r}_\omega^m \in \mathrm{Isom}(\mathbb{E}^n)$ such that the immersions $\boldsymbol{\psi}_\omega^m := \boldsymbol{r}_\omega^m \circ \boldsymbol{\phi}^m \in W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n)$ satisfy

$$\|\boldsymbol{\psi}_\omega^m - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\omega)} \to 0 \quad \text{as } m \to \infty. \tag{7.2}$$

Note that the immersions $\boldsymbol{\psi}_\omega^m$ satisfy in particular the relations

$$\begin{aligned} \boldsymbol{\psi}_\omega^m(x_0) &\to \boldsymbol{\phi}(x_0) \quad \text{as } m \to \infty, \\ \boldsymbol{\nabla}\boldsymbol{\psi}_\omega^m(x_0) &\to \boldsymbol{\nabla}\boldsymbol{\phi}(x_0) \quad \text{as } m \to \infty. \end{aligned} \tag{7.3}$$

The immersions $\boldsymbol{\psi}^m$ and $\boldsymbol{\psi}_\omega^m$ defined above are related by

$$\boldsymbol{\psi}^m = \boldsymbol{\rho}_\omega^m \circ \boldsymbol{\psi}_\omega^m \quad \text{in } \Omega,$$

where $\boldsymbol{\rho}_\omega^m := \boldsymbol{r}^m \circ (\boldsymbol{r}_\omega^m)^{-1} \in \mathrm{Isom}(\mathbb{E}^n)$. Since $\boldsymbol{\rho}_\omega^m$ are isometries of $\mathbb{E}^n$, there exist $\boldsymbol{a}_\omega^m \in \mathbb{E}^n$ and $\boldsymbol{Q}_\omega^m \in \mathbb{O}^n$ such that

$$\boldsymbol{\rho}_\omega^m(x) = \boldsymbol{a}_\omega^m + \boldsymbol{Q}_\omega^m x \quad \text{for all } x \in \mathbb{E}^n.$$

Noting that the matrix $\boldsymbol{\nabla}\boldsymbol{\phi}(x_0)$ is invertible and using relations (7.1) and (7.3), we easily infer that

$$\boldsymbol{a}_\omega^m \to \boldsymbol{0} \quad \text{and} \quad \boldsymbol{Q}_\omega^m \to \boldsymbol{I} \quad \text{as } m \to \infty.$$

That

$$\boldsymbol{\psi}^m \to \boldsymbol{\phi} \quad \text{in } W^{2,p}_{\mathrm{loc}}(\Omega; \mathbb{E}^n) \text{ as } m \to \infty,$$

then follows by using these convergences and the convergence (7.2) in the right-hand side of the following inequality:

$$\begin{aligned} \|\boldsymbol{\psi}^m - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\omega)} &\leq \|(\boldsymbol{\rho}_\omega^m - \mathbf{id}) \circ \boldsymbol{\psi}_\omega^m\|_{\boldsymbol{W}^{2,p}(\omega)} + \|\boldsymbol{\psi}_\omega^m - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\omega)} \\ &\leq \|\boldsymbol{a}_\omega^m\|_{L^p(\omega)} + |\boldsymbol{Q}_\omega^m - \boldsymbol{I}| \|\boldsymbol{\psi}_\omega^m\|_{\boldsymbol{W}^{2,p}(\omega)} + \|\boldsymbol{\psi}_\omega^m - \boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p}(\omega)}. \end{aligned}$$

8 Metric Tensor Fields in $L^q(\Omega)$, $1 \leq q < \infty$

In the previous sections, we considered immersions $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ defined over a connected open subset Ω of $\mathbb{R}^n$ and whose gradient field $\boldsymbol{\nabla\phi}$ is at least continuous in Ω. Hence $\det \boldsymbol{\nabla\phi}$ is either > 0, or < 0, in Ω.

In this section, we consider immersions $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ whose gradient field $\boldsymbol{\nabla\phi}$ is only in $L^p(\Omega; \mathbb{M}^n)$, which only means that $\det \boldsymbol{\nabla\phi} \neq 0$ almost everywhere in Ω, so that $\det \boldsymbol{\nabla\phi}$ may change sign in Ω. This is why we will assume that all the immersions $\boldsymbol{\phi} : \Omega \to \mathbb{E}^n$ considered in this section preserve the orientation, in the sense that they satisfy $\det \boldsymbol{\nabla\phi} > 0$ almost everywhere in Ω. This assumption naturally leads to using "proper" isometries of $\mathbb{E}^n$ in this section (instead of isometries of $\mathbb{E}^n$ as in the previous sections), according to the following definition.

A proper isometry of $\mathbb{E}^n$ is an isometry of $\mathbb{E}^n$ that preserves the orientation of $\mathbb{E}^n$. In other words, a proper isometry of $\mathbb{E}^n$ is an element of the set

$$\mathrm{Isom}_+(\mathbb{E}^n) := \{\boldsymbol{r} : \mathbb{E}^n \to \mathbb{E}^n;\ \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}x \text{ for all } x \in \mathbb{E}^n, \boldsymbol{a} \in \mathbb{E}^n,\ \boldsymbol{R} \in \mathbb{O}^n_>\}.$$

The next theorem establishes a nonlinear Korn inequality in $W^{1,p}(\Omega)$ similar to those of Theorems 3.2, 5.2, or 6.2, but again with respect to Lebesgue and Sobolev norms like in the nonlinear Korn inequality in $W^{2,p}(\Omega)$ of Theorem 6.2.

Theorem 8.1 *Let Ω be a domain in $\mathbb{R}^n$, and let $\boldsymbol{\phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^n)$ be an immersion that satisfies* $\det \boldsymbol{\nabla\phi} > 0$ *in $\overline{\Omega}$.*

Then, given any $1 \leq q < \infty$ and any $1 < p < \infty$ such that $q \leq p \leq 2q$, there exists a constant $C = C(p, q, \boldsymbol{\phi}) > 0$ such that, for all mappings $\widetilde{\boldsymbol{\phi}} \in W^{1,2q}(\Omega; \mathbb{E}^n)$ that satisfy $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}} > 0$ *almost everywhere in Ω,*

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E}^n)} \|\boldsymbol{r} \circ \widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|(\widetilde{g}_{ij}) - (g_{ij})\|^{\frac{q}{p}}_{\mathbb{L}^q(\Omega)},$$

where

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \in \mathcal{C}^0(\overline{\Omega}) \quad \textit{and} \quad \widetilde{g}_{ij} := \partial_i \widetilde{\boldsymbol{\phi}} \cdot \partial_j \widetilde{\boldsymbol{\phi}} \in L^q(\Omega)$$

denote the covariant components of the metric tensor fields induced by the immersions $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$, respectively.

Sketch of the Proof The details of the proof below can be found in [9] (see also [10]).

(i) The Poincaré-Wirtinger inequality implies that there exists a constant D such that, for all vector fields $\boldsymbol{f} \in W^{1,p}(\Omega; \mathbb{E}^n)$,

$$\inf_{\boldsymbol{a} \in \mathbb{E}^n} \|\boldsymbol{f} + \boldsymbol{a}\|_{\boldsymbol{L}^p(\Omega)} \leq D \|\boldsymbol{\nabla f}\|_{\mathbb{L}^p(\Omega)}.$$

Therefore, for each $\boldsymbol{R} \in \mathbb{O}^n_+$ and each $\widetilde{\boldsymbol{\phi}} \in W^{1,p}(\Omega; \mathbb{E}^n)$,

$$\begin{aligned}\inf_{\boldsymbol{a} \in \mathbb{E}^n} \|\boldsymbol{a} + \boldsymbol{R}\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} &= \inf_{\boldsymbol{a} \in \mathbb{E}^n} \|\boldsymbol{a} + \boldsymbol{R}\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{\nabla}(\boldsymbol{R}\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi})\|^p_{\mathbb{L}^p(\Omega)} \\ &\leq (D^p + 1) \|\boldsymbol{\nabla}(\boldsymbol{R}\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi})\|^p_{\mathbb{L}^p(\Omega)},\end{aligned}$$

which next implies that

$$\begin{aligned}\inf_{\boldsymbol{r}\in \mathrm{Isom}_+(\mathbb{E}^n)} \|\boldsymbol{r}\circ\widetilde{\boldsymbol{\phi}}-\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,p}(\Omega)} &= \inf_{\boldsymbol{R}\in\mathbb{O}^n_+}\inf_{\boldsymbol{a}\in\mathbb{E}^n}\|\boldsymbol{a}+\boldsymbol{R}\widetilde{\boldsymbol{\phi}}-\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &\le (1+D^p)^{\frac{1}{p}}\inf_{\boldsymbol{R}\in\mathbb{O}^n_+}\|\boldsymbol{\nabla}(\boldsymbol{R}\widetilde{\boldsymbol{\phi}}-\boldsymbol{\phi})\|_{\mathbb{L}^p(\Omega)}.\end{aligned}$$

Thus it suffices to prove that there exists a constant $C_1 = C_1(p,q,\boldsymbol{\phi})$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^n_+}\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}-\boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\phi}\|_{\mathbb{L}^p(\Omega)} \le C_1\|(\widetilde{g}_{ij})-(g_{ij})\|^{\frac{q}{p}}_{\mathbb{L}^q(\Omega)}.$$

(ii) The polar factorization theorem for invertible matrices implies that the matrix fields $\boldsymbol{\nabla}\boldsymbol{\phi}\in\mathcal{C}^0(\overline{\Omega};\mathbb{M}^n)$ and $\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}\in L^{2q}(\Omega;\mathbb{M}^n)$ can be written as

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\phi} &= \boldsymbol{P}(\boldsymbol{\nabla}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\phi})^{\frac12} = \boldsymbol{P}\,(g_{ij})^{\frac12} \quad \text{in } \overline{\Omega},\\ \boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}} &= \boldsymbol{Q}\big(\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}^{\mathrm{T}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}\big)^{\frac12} = \boldsymbol{Q}\,(\widetilde{g}_{ij})^{\frac12} \quad \text{a.e. in } \Omega,\end{aligned}$$

respectively, where $\boldsymbol{P}(x)\in\mathbb{O}^n_+$ for all $x\in\overline{\Omega}$ and $\boldsymbol{Q}\in\mathbb{O}^n_+$ for almost all $x\in\Omega$ (the assumptions that $\det\boldsymbol{\nabla}\boldsymbol{\phi}>0$ in $\overline{\Omega}$ and $\det\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}>0$ a.e. in Ω are used here).

Since the Frobenius norm $|\cdot|$ is invariant under rotations, the above relations imply that, for almost all $x\in\Omega$,

$$\inf_{\boldsymbol{R}\in\mathbb{O}^n_+}|\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}(x)-\boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\phi}(x)| \le |(\widetilde{g}_{ij}(x))^{\frac12}-(g_{ij}(x))^{\frac12}|.$$

Besides, one can prove that, for each $1\le q<\infty$ and each $1<p<\infty$ such that $q\le p\le 2q$, there exists a constant $C_2 = C_2(p,q,\boldsymbol{\phi})<\infty$ (the constant C_2 depends on $\boldsymbol{\phi}$ via the norms $\|(g_{ij})\|_{L^\infty(\Omega;\mathbb{S}^n)}$ and $\|(g_{ij})^{-1}\|_{L^\infty(\Omega;\mathbb{S}^n)}$) such that, for almost all $x\in\Omega$,

$$|(\widetilde{g}_{ij}(x))^{\frac12}-(g_{ij}(x))^{\frac12}| \le C_2|(\widetilde{g}_{ij}(x))-(g_{ij}(x))|^{\frac{q}{p}}.$$

The last two inequalities combined with step (i) show that the announced nonlinear Korn inequality will follow if one can find a constant $C_3 = C_3(p,\boldsymbol{\phi})$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^n_+}\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}-\boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\phi}\|_{\mathbb{L}^p(\Omega)} \le C_3\Big\|\inf_{\boldsymbol{R}\in\mathbb{O}^n_+}|\boldsymbol{\nabla}\widetilde{\boldsymbol{\phi}}-\boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\phi}|\Big\|_{\mathbb{L}^p(\Omega)}.$$

(iii) The above inequality was established, first for $\boldsymbol{\phi}=\mathbf{id}$ and $p=2$ by Friesecke, James and Müller [14] under the name of "geometric rigidity lemma", then generalized to any $1<p<\infty$ by Conti [13], and finally generalized to any immersion $\boldsymbol{\phi}\in\mathcal{C}^1(\overline{\Omega};\mathbb{E}^n)$ and any $1<p<\infty$ by Ciarlet and Mardare [9].

The nonlinear Korn inequalities of Theorems 3.2, 5.2, and 6.2 are established by using that the vector fields $\boldsymbol{g}_i := \partial_i\boldsymbol{\phi}$ associated with a sufficiently smooth immersion $\boldsymbol{\phi}:\Omega\to\mathbb{E}^n$ satisfy the Pfaff system

$$\partial_i\boldsymbol{g}_j = \Gamma^k_{ij}\boldsymbol{g}_k \quad \text{in } \Omega,$$

whose coefficients are defined by

$$\Gamma^k_{ij} := \frac12 g^{k\ell}(\partial_i g_{j\ell}+\partial_j g_{i\ell}-\partial_\ell g_{ij}),$$

where

$$g_{ij} := \partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \quad \text{and} \quad (g^{k\ell}) := (g_{ij})^{-1}.$$

Note that, by contrast, the nonlinear Korn inequality of Theorem 8.1 cannot be established in the same way, since the immersions appearing there do not possess enough regularity to ensure that the above Pfaff system makes sense (the above definition of the coefficients Γ_{ij}^k does not make sense for immersions $\boldsymbol{\phi}$ that are only in the space $W^{1,p}(\Omega; \mathbb{E}^n)$).

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