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Flat Solutions of Some Non-Lipschitz Autonomous Semilinear Equations May be Stable for $N \geq 3$ *

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(*Dedicated to a master, Haïm Brezis, with admiration*)

Abstract The authors prove that flat ground state solutions (i.e. minimizing the energy and with gradient vanishing on the boundary of the domain) of the Dirichlet problem associated to some semilinear autonomous elliptic equations with a strong absorption term given by a non-Lipschitz function are unstable for dimensions $N = 1, 2$ and they can be stable for $N \geq 3$ for suitable values of the involved exponents.

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1 Introduction and Main Results

Let $N \geq 1$, and let Ω be a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a $\mathcal{C}^1$-manifold. We consider the following semi-linear parabolic problem:

$$\mathrm{PP}(\alpha,\beta,\lambda,v_0)\quad \begin{cases} v_t - \Delta v + |v|^{\alpha-1}v = \lambda|v|^{\beta-1}v & \text{in } (0,+\infty)\times\Omega, \\ v = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ v(0,x) = v_0(x) & \text{on } \Omega. \end{cases} \tag{1.1}$$

Here λ is a positive parameter and $0 < \alpha < \beta \leq 1$. Our main goal is to give some stability criteria on solutions of the associated stationary problem

$$\mathrm{SP}(\alpha,\beta,\lambda)\quad \begin{cases} -\Delta u + |u|^{\alpha-1}u = \lambda|u|^{\beta-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Notice that since the diffusion-reaction balance involves the non-linear reaction term

$$f(\lambda,u) := \lambda|u|^{\beta-1}u - |u|^{\alpha-1}u$$

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and it is a non-Lipschitz function at zero (since $\alpha < 1$ and $\beta \le 1$), important peculiar behaviors of solutions of both problems arise. For instance, that may lead to the violation of the Hopf maximum principle on the boundary and the existence of compactly supported solutions as well as the so-called flat solutions (sometimes also called free boundary solutions) which correspond to weak solutions u such that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where ν denotes the unit outward normal to $\partial\Omega$. Solutions of this kind for stationary equations with non-Lipschitz nonlinearity have been investigated in a number of papers. The pioneering paper in which it was proved that the solution gives rise to a free boundary defined as the boundary of its support was due to Haïm Brezis [9] concerning multivalued non-autonomous semilinear equations. The semilinear case with non-Lipschitz perturbations was considered later in [4] (see also [6, 11–12]). For the case of semilinear autonomous elliptic equations, see e.g. [16–17, 25, 27, 29, 42, 44–45, 51], to mention only a few. For (1.2), the existence of radial flat solutions was first proved by Kaper and Kwong [44]. In this paper, applying shooting methods, they showed that there exists $R_0 > 0$ such that (1.2) considered in the ball $B_{R_0} = \{x \in \mathbb{R}^N : \ |x| \le R_0\} = \Omega$ has a radial compactly supported positive solution. Furthermore, by the moving-plane method, it was proved in [45] that any classical solution $u \in \mathcal{C}^2(\Omega)$ of (1.2) is necessarily radially symmetric if Ω is a ball. Observe that from this it follows that the Dirichlet boundary value problem (1.2) has a compactly supported solution if $B_{R_0} \subseteq \Omega$.

In this paper, we study the stability of solutions of the stationary problem $\mathrm{SP}(\alpha,\beta,\lambda)$. We point out that a direct analysis of the stability of the stationary solutions $u_\infty \in [0,+\infty)$ of the associated ODE

$$\mathrm{ODE}(\alpha,\beta,\lambda,v_0) \quad \begin{cases} v_t + |v|^{\alpha-1}v = \lambda |v|^{\beta-1}v & \text{in } (0,+\infty), \\ v(0) = v_0 \end{cases} \tag{1.4}$$

shows that the trivial solution $u_\infty \equiv 0$ is asymptotically stable, and that the nontrivial stationary solution $u_\infty := \lambda^{-\frac{1}{\beta-\alpha}}$ is unstable (see Figure 1).

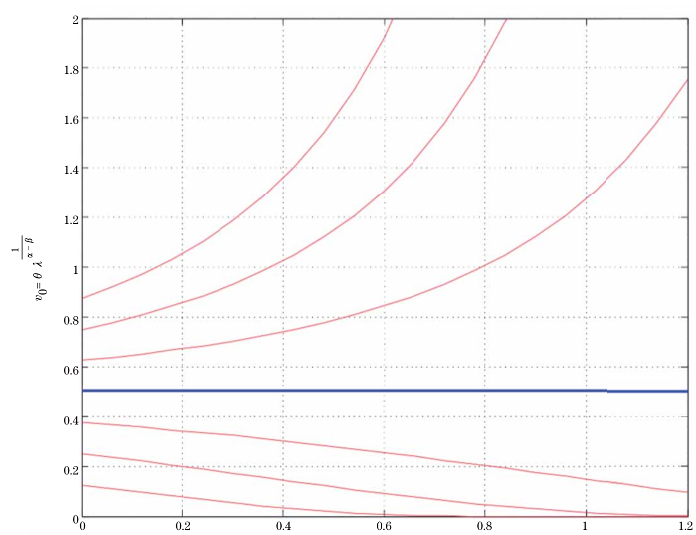

Figure 1 Paths for $\mathrm{ODE}\left(\frac{1}{2}, \frac{3}{2}, \lambda, v_0\right)$

Obviously, the same criteria hold for the case of the semilinear problem with Neumann boundary conditions. Nevertheless, unexpectedly, the situation is not similar for the case of Dirichlet boundary conditions, and so, as the main result of this paper will show, for dimensions $N \geq 3$, the nontrivial flat solution of $\mathrm{SP}(\alpha,\beta,\lambda)$ becomes stable in a certain range of the exponents $\alpha < \beta < 1$. To be more precise, our stability study will concern ground states solutions (also called simply ground state) of $\mathrm{SP}(\alpha,\beta,\lambda)$. By it, we mean a nonzero weak solution u_λ of $\mathrm{SP}(\alpha,\beta,\lambda)$ which satisfies

$$E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$

for any nonzero weak solution w_λ of $\mathrm{SP}(\alpha,\beta,\lambda)$. Here $E_\lambda(u)$ is the energy functional corresponding to $\mathrm{SP}(\alpha,\beta,\lambda)$ which is defined on the Sobolev space $\mathrm{H}_0^1(\Omega)$ as follows:

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1}\,\mathrm{d}x - \lambda\frac{1}{\beta+1}\int_\Omega |u|^{\beta+1}\,\mathrm{d}x.$$

For simplicity, we shall assume the initial value such that $v_0 \in \mathrm{L}^\infty(\Omega),\ v_0 \geq 0$. As we shall show in Section 2, then there exists a weak solution $v \in \mathcal{C}([0,+\infty),\mathrm{L}^2(\Omega))$ of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ satisfying $\lambda|v|^{\beta-1}v - |v|^{\alpha-1}v \in \mathrm{L}^\infty((0,+\infty)\times\Omega)$ and

$$v(t) = T(t)v_0 + \int_0^t T(t-s)(\lambda|v|^{\beta-1}v - |v|^{\alpha-1}v)\mathrm{d}s, \tag{1.5}$$

with $(T(t))_{t\geq 0}$ the heat semigroup with homogeneous Dirichlet boundary conditions, i.e., $T(t) = \mathrm{e}^{t(-\Delta)}$. Among some additional regularity properties of v, we mention that

$$v - T(t)v_0 \in \mathrm{L}^p(\tau,T;\mathrm{W}^{2,p}(\Omega)\cap \mathrm{W}_0^{1,p}(\Omega)) \cap \mathrm{W}^{1,p}(\tau,T;\mathrm{L}^p(\Omega)) \tag{1.6}$$

for every $p \in (1,\infty)$ and for any $0 < \tau < T$ (in fact, $\tau = 0$ if we also assume that $v_0 \in \mathrm{W}_0^{1,p}(\Omega)$). In particular, v satisfies the equation $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ for a.e. $t \in (0,+\infty)$. Moreover, if $v(0) \in \mathrm{H}_0^1(\Omega)$, then for any $t > 0$,

$$\int_0^t \|v_t(s)\|_{L^2}^2\mathrm{d}s + E_\lambda(v(t)) \leq E_\lambda(v(0)). \tag{1.7}$$

We shall show in Section 2 that there is uniqueness of solutions of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ in the class of solutions v, such that

$$v(t,x) \geq Cd(x)^{\frac{2}{1-\alpha}} \quad \text{in } \Omega,\ \text{for } t > 0 \tag{1.8}$$

for some constant $C > 0$, where $d(x) := \mathrm{dist}(x,\partial\Omega)$ (which we shall also denote simply as δ_Ω). Sufficient conditions implying this non-degeneracy property (1.8) will be given. We also prove that if $\lambda \in [0,\lambda_1)$, then the finite extinction time property is satisfied for solutions of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ (as in the pioneering paper [13] on multivalued semilinear parabolic problems; see also the survey [22]). Moreover, we shall show in Section 2 that there is a certain resemblance between the set of solutions of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ and the corresponding one of the ODE problem $\mathrm{ODE}(\alpha,\beta,\lambda,v_0)$, since

(a) for any $\lambda > 0$, the trivial solution $u \equiv 0$ of the stationary problem $\mathrm{SP}(\alpha,\beta,\lambda)$ is asymptotically stable in the sense that it attracts solutions of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ for small initial data v_0 (see Proposition 2.1);

(b) if v_0 is "large enough" the trajectory of the solution of $\mathrm{PP}(\alpha,\beta,\lambda,v_0)$ is not non-uniformly bounded when $t \nearrow +\infty$ (see Proposition 2.4).

Concerning the stationary problem $\mathrm{SP}(\alpha,\beta,\lambda)$ we recall that if $u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a weak stationary solution of $\mathrm{SP}(\alpha,\beta,\lambda)$, then, by standard regularity results, $u \in \mathrm{W}^{2,p}(\Omega)$ for any $p \in (1,\infty)$ and then $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$ for any γ.

In our stability study, we shall use some fibrering techniques. For given $u \in \mathrm{H}_0^1(\Omega)$, the fibrering mappings are defined by $\Phi_u(r) = E_\lambda(ru)$, so that from the variational formulation of $\mathrm{SP}(\alpha,\beta,\lambda)$, we know that $\Phi'_u(r) = 0$, where we use the notation

$$\Phi'_u(r) = \frac{\partial}{\partial r} E_\lambda(ru).$$

If we also define $\Phi''_u(r) = \frac{\partial^2}{\partial r^2} E_\lambda(ru)$, then, in case $\beta < 1$, the equation $\Phi'_u(r) = 0$ may have at most two nonzero roots $r_{\min} > 0$ and $r_{\max} > 0$ such that $\Phi''_u(r_{\max}) \geq 0$, $\Phi''_u(r_{\min}) \leq 0$ and $0 < r_{\max} \leq r_{\min}$ (see Figure 2), whereas, in case $\beta = 1$ the equation $\Phi'_u(r) = 0$ for any $\lambda > 0$ has precisely one nonzero root $r_{\max} > 0$ such that $\Phi''_u(r_{\max}) \leq 0$. This implies that any weak solution of $\mathrm{SP}(\alpha,\beta,\lambda)$ (any critical point of $E_\lambda(u)$) corresponds to one of the cases $r_{\min} = 1$ or $r_{\max} = 1$. However, it was discovered in [42] (see also [41]) that in case when we study compactly supported solutions this correspondence essentially depends on the relation between α, β and N.

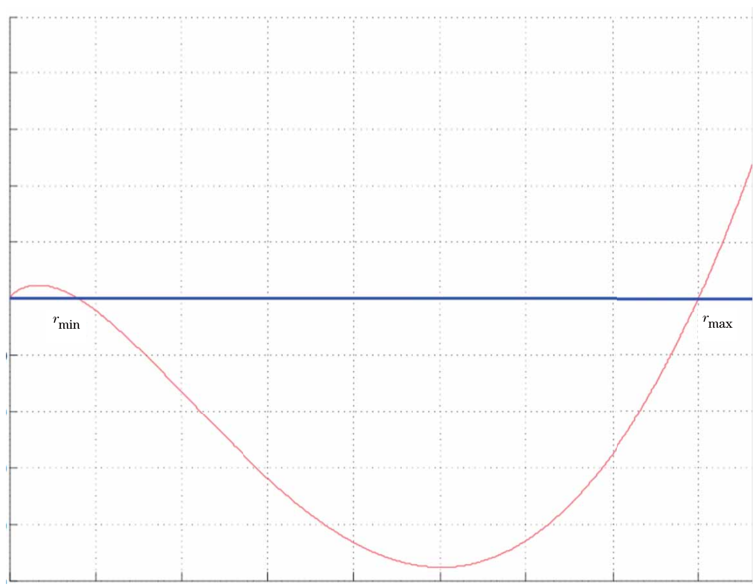

Figure 2 $r_{\min}$ and $r_{\max}$

In this paper, developing [42], we introduce in the set of relevant exponents $\mathcal{E} := \{(\alpha,\beta) : 0 < \alpha < \beta \leq 1\}$ the following critical exponents curve depending on the dimension N:

$$\mathcal{C}(N) := \{(\alpha,\beta) \in \mathcal{E} :\ 2(1+\alpha)(1+\beta) - N(1-\alpha)(1-\beta) = 0\}. \tag{1.9}$$

This curve exists if and only if $N \geq 3$ and it separates two sets of exponents in $\mathcal{E}$ (see Figure 3)

$$\mathcal{E}_s(N) := \{(\alpha,\beta) \in \mathcal{E} :\ 2(1+\alpha)(1+\beta) - N(1-\alpha)(1-\beta) < 0\},$$
$$\mathcal{E}_u(N) := \{(\alpha,\beta) \in \mathcal{E} :\ 2(1+\alpha)(1+\beta) - N(1-\alpha)(1-\beta) > 0\},$$

whereas in the cases $N = 1, 2$, one has $\mathcal{E} = \mathcal{E}_u(N)$.

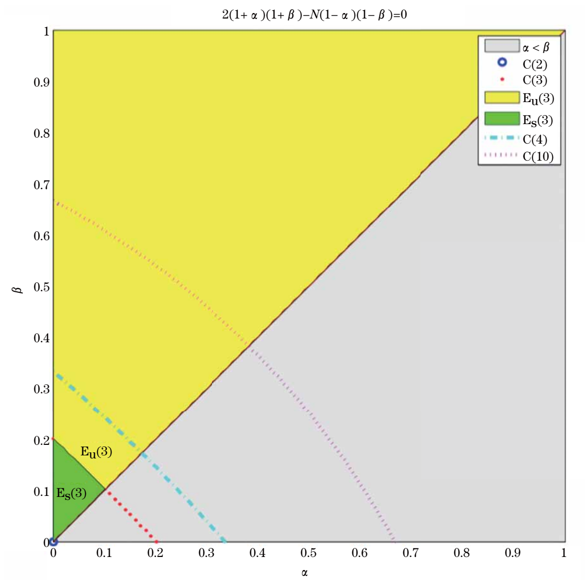

Figure 3 Sets $\mathcal{E}_s(N)$ and $\mathcal{E}_u(N)$ for $N = 3, 4$ and 10

The main property of $\mathcal{C}(N)$ is contained in the following lemma.

Lemma 1.1 *Let $N \geq 1$ and let Ω be a bounded and star-shaped domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a $\mathcal{C}^1$-manifold.*

(1) *Assume $(\alpha, \beta) \in \mathcal{C}(N)$. Then any flat ground state solution u of* (1.2) *satisfies $\Phi_u''(r)|_{r=1} = 0$.*

(2) *Assume $(\alpha, \beta) \in \mathcal{E}_u(N)$. Then any flat ground state solution u of* (1.2) *satisfies $\Phi_u''(r)|_{r=1} < 0$.*

(3) *Assume $(\alpha, \beta) \in \mathcal{E}_s(N)$. Then any ground state solution u of* (1.2) *satisfies $\Phi_u''(r)|_{r=1} > 0$.*

The existence of flat (or compactly supported) ground state solutions of (1.2) in the case $\beta < 1$, $N \geq 3$ and $(\alpha, \beta) \in \mathcal{E}_s(N)$ was obtained in [42]. Furthermore, the existence of flat solutions of (1.2) (not necessary ground states) in case $N \geq 1$, $0 < \alpha < \beta \leq 1$ was proved in [25, 27, 44–45].

As already mentioned, one of the main goals of this paper is to study the H_0^1-stability of flat ground state solutions of $\mathrm{SP}(\alpha, \beta, \lambda)$. We recall that, if $v(t; v_0)$ is a weak solution to $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$, we shall say that $v(t; v_0)$ is H_0^1-stable if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|v(t; v_0) - v(t; w_0)\|_1 < \varepsilon \quad \text{for any } w_0 \text{ such that } \|v_0 - w_0\|_1 < \delta,\ \forall t > 0, \tag{1.10}$$

where we use the $\mathrm{H}_0^1(\Omega)$-norm

$$\|u\|_1 = \Big(\int_\Omega |\nabla u|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Conversely, we say that a solution $v(t; v_0)$ of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ is H_0^1-unstable if there is $\varepsilon > 0$ such that for any $\delta > 0$ and $T > 0$, there exists

$$w_0 \in U_\delta(v_0) := \{w \in H_0^1(\Omega) : \ \|v_0 - w\|_1 < \delta\}$$

and there exists $T > 0$ such that for any $t > T$,

$$\|v(t; v_0) - v(t; w_0)\|_1 > \varepsilon, \tag{1.11}$$

where $v(t; w_0)$ is any weak solution of $\mathrm{PP}(\alpha, \beta, \lambda, w_0)$. Furthermore, we will use also the following definition: A solution u_λ of $\mathrm{SP}(\alpha, \beta, \lambda)$ is said to be linearly unstable stationary solution if $\lambda_1(-\Delta + \alpha u_\lambda^{\alpha-1} - \lambda\beta u_\lambda^{\beta-1}) < 0$.

In what follows, we will also use the following definition (see [5, 38]): A solution $v(t; v_0)$ of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ is said to be globally $\mathrm{H}_0^1(\Omega)$-unstable, if for any $\delta > 0$ there exists

$$w_0 \in U_\delta(v_0) := \{w \in H_0^1(\Omega) : \ \|v_0 - w\|_1 < \delta\},$$

such that

$$\|v(t; v_0) - v(t; w_0)\|_1 \to \infty \quad \text{as } t \to \infty. \tag{1.12}$$

Motivated by the uniqueness results for the $\mathrm{PP}(\alpha, \beta, \lambda, w_0)$, we shall assume later the following "isolation assumption":

(U) Given u_λ non-negative ground state solution of $\mathrm{SP}(\alpha, \beta, \lambda)$, there exists a "positive-neighborhood"

$$U_\delta(u_\lambda) := \{v \in H_0^1(\Omega), \ v \geq 0 \text{ on } \Omega, \ \text{such that } \ \|u_\lambda - v\|_1 < \delta\}$$

with $\delta > 0$ such that $\mathrm{SP}(\alpha, \beta, \lambda)$ has no other non-negative weak solution in $U_\delta(u_\lambda) \setminus u_\lambda$.

Our first two results concern the existence and (un-)stability of ground states of (1.2). In case $0 < \alpha < \beta < 1$, we have the following theorem.

Theorem 1.1 *Let $N \geq 1$, $0 < \alpha < \beta < 1$, Ω be a bounded domain in $\mathbb{R}^N$, with a smooth boundary.*

(1) *There exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$,* (1.2) *has a ground state u_λ which is non-negative in Ω and $u_\lambda \in \mathcal{C}^{1,\kappa}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\kappa \in (0, 1)$.*

(2) *Assume* (**U**)*, then the ground state u_λ is an $\mathrm{H}_0^1(\Omega)$-stable stationary solution of the parabolic problem* (1.1).

In case $\beta = 1$ we have following theorem.

Theorem 1.2 *Let $N \geq 1$, $\beta = 1$, $0 < \alpha < 1$, Ω be a bounded star-shaped domain in $\mathbb{R}^N$, with a smooth boundary.*

(1) *There exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$,* (1.2) *has a ground state u_λ which is non-negative in Ω and $u \in \mathcal{C}^{1,\kappa}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\kappa \in (0, 1)$.*

(2) *Assume* (**U**)*, the ground state u_λ is a globally $\mathrm{H}_0^1(\Omega)$-unstable stationary solution of the parabolic problem* (1.1).

Our main result on the $\mathrm{H}_0^1(\Omega)$-stability and $\mathrm{H}_0^1(\Omega)$-unstability of flat ground state solutions for $0 < \alpha < \beta < 1$ is as follows.

Theorem 1.3 *Let $N \geq 1$, Ω be a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a $\mathcal{C}^1$-manifold.*

(I) *Assume $N = 1, 2$. Then for every $(\alpha, \beta) \in \mathcal{E}$ (i.e., $0 < \alpha < \beta$) any flat ground state solution u_λ of* (1.2) *is a linearized unstable stationary solution of the parabolic problem* (1.1).

(II) *Assume* (**U**), *$N \geq 3$ and $(\alpha, \beta) \in \mathcal{E}_u(N)$. Then any flat ground state solution u_λ of* (1.2) *is a linearized unstable stationary solution of the parabolic problem* (1.1).

(III) *Assume $N \geq 3$, $(\alpha, \beta) \in \mathcal{E}_s(N)$ and Ω is a strictly star-shaped domain with respect to the origin.*

(1) *There exists $\lambda^* > 0$ such that* (1.2) *has a flat ground state u_{λ^*}, where $u_{\lambda^*} \geq 0$ and $u_{\lambda^*} \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0, 1)$.*

(2) *If in addition,* (**U**) *holds, then the flat ground state solution u_{λ^*} is an $\mathrm{H}_0^1(\Omega)$-stable stationary solution of the parabolic problem* (1.1).

In the case $\beta = 1$ we have following theorem.

Theorem 1.4 *Assume $N \geq 1$, $0 < \alpha < 1$, $\beta = 1$ and Ω be a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a $\mathcal{C}^1$-manifold.*

(1) *There exists $\lambda^* > 0$ such that* (1.2) *has a ground state u_{λ^*} which is a flat solution in Ω where $u_{\lambda^*} \geq 0$ and $u_{\lambda^*} \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ for some $\alpha \in (0, 1)$.*

(2) *If in addition* (**U**) *holds, the flat ground state solution u_{λ^*} is globally $\mathrm{H}_0^1(\Omega)$-unstable stationary solution of the parabolic problem* (1.1).

The limit case $\alpha = 0$ can be also considered. In particular, this shows that the first "compressed mode" function (solution of $\mathrm{SP}(0, 1, \lambda)$ (see [46–47])) of great relevance in signal processing is globally $\mathrm{H}_0^1(\Omega)$-unstable.

2 Parabolic Problem. Existence, Uniqueness and Boundedness on Non-negative Solutions

Given $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \geq 0$, we shall say that $v \in \mathcal{C}([0, +\infty), \mathrm{L}^2(\Omega))$ is a weak solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ if $v \geq 0$, $\lambda v^\beta - v^\alpha \in \mathrm{L}^\infty((0, T) \times \Omega)$ for any $T > 0$ and

$$v(t) = T(t)v_0 + \int_0^t T(t-s)(\lambda v^\beta(s) - v^\alpha(s))\mathrm{d}s. \tag{2.1}$$

Here $(T(t))_{t \geq 0}$ is the heat semigroup with homogeneous Dirichlet boundary conditions, i.e., $T(t) = \mathrm{e}^{t(-\Delta)}$. The existence of weak solutions is an easy variation of previous results in the literature (see, e.g., [3, 14] and the works [19–20] dealing with the more difficult case of singular equations $\alpha \in (-1, 0)$). For the reader convenience, we shall collect here some additional regularity information on weak solutions of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$.

Proposition 2.1 *For any $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \geq 0$ there exists a non-negative weak solution $v \in \mathcal{C}([0, +\infty), \mathrm{L}^2(\Omega))$ of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$. In fact, for every $p \in [1, \infty]$, $v \in \mathcal{C}([0, +\infty); \mathrm{L}^p(\Omega))$, and if $p < \infty$,*

$$v - T(\cdot)v_0 \in \mathrm{L}^p(\tau, T; \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega)) \cap \mathrm{W}^{1,p}(\tau, T; \mathrm{L}^p(\Omega)) \tag{2.2}$$

for any $0 < \tau < T$. In particular, v satisfies the equation $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ for a.e. $t \in (0, +\infty)$. Moreover, if we also assume that $v_0 \in \mathrm{H}_0^1(\Omega)$, then $\frac{\partial}{\partial t} E_\lambda(v(\cdot)) \in \mathrm{L}^1(\tau, T)$, and function $E_\lambda(v(\cdot))$

is absolutely continuous for a.e. $t \in (\tau, T)$ *and*

$$\frac{\partial}{\partial t} E_\lambda(v(t)) = \int_\Omega (\lambda v^\beta + v^\alpha) v_t(t) \mathrm{d}x - \int_\Omega v_t(t)^2 \mathrm{d}x. \tag{2.3}$$

Proof Among many possible methods to prove the existence of weak solutions, we shall follow here the one based on a fixed point argument as in [32] (see also [31], where the case $\beta = 0$ was considered on a Riemannian manifold). For every $h \in \mathrm{L}^\infty((0,T) \times \Omega)$, we consider the problem (P_h)

$$(P_h) \quad \begin{cases} v_t - \Delta v + |v|^{\alpha-1} v = h & \text{in } (0, +\infty) \times \Omega, \\ v = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ v(0, x) = v_0(x) & \text{on } \Omega, \end{cases}$$

which we can reformulate in terms of an abstract Cauchy problem on the Hilbert space $\mathrm{H} = \mathrm{L}^2(\Omega)$ as

$$(P_h) \quad \begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t}(t) + \mathcal{A}v(t) = h(t), \quad t \in (0, T), \text{ in H}, \\ v(0) = v_0, \end{cases}$$

where $\mathcal{A} = \partial\varphi$ denotes the subdifferential of the convex function

$$\varphi(v) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \dfrac{1}{\alpha+1} \int_\Omega |v|^{\alpha+1} \,\mathrm{d}x, & \text{if } v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^{\alpha+1}(\Omega), \\ +\infty, & \text{otherwise} \end{cases}$$

(see, e.g., [7–8, 21]). As in [31–32], we define the operator $\mathcal{T} : h \to g$, where $g = \lambda |v_h|^{\beta-1} v_h$ and v_h is the solution of (P_h). It is easy to see that every fixed point of $\mathcal{T}$ is a solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$. Then $\mathcal{T}$ satisfies the hypotheses of Kakutani Fixed Point Theorem (see, e.g., Vrabie [54]), since if $X = \mathrm{L}^2((0,T), \mathrm{L}^2(\Omega))$ then

(i) $K = \{h \in \mathrm{L}^2(0, T, \mathrm{L}^\infty(\Omega)) : \|h(t)\|_{\mathrm{L}^\infty(\Omega)} \le C_0 \text{ a.e. } t \in (0, T)\}$ is a nonempty, convex and weakly compact set of X.

(ii) $\mathcal{T} : K \mapsto 2^X$ with nonempty, convex and closed values such that $\mathcal{T}(g) \subset K$, $\forall g \in K$.

(iii) Graph($\mathcal{T}$) is weakly×weakly sequentially closed.

Consequently, $\mathcal{T}$ has at least one fixed point in K which is a local (in time) solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$. The final key point is to show that there is no blow-up phenomenon. This holds by the a priori estimate

$$0 \le v(t, x) \le z(t, x) \quad \text{for any } t \in [0, +\infty) \times \Omega,$$

where $v(t, x)$ is any weak solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$, and $z(t, x)$ is the solution of the corresponding auxiliary problem

$$\begin{cases} z_t - \Delta z = \lambda z^\beta & \text{in } (0, +\infty) \times \Omega, \\ z = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ z(0, x) = v_0(x) & \text{on } \Omega. \end{cases} \tag{2.4}$$

This implies that there is no finite blow-up (and thus the maximal existence time is $T_{\max} = +\infty$). In particular, if $\beta \in (0, 1)$, we have the estimate

$$\|v(t)\|_{\mathrm{L}^\infty(\Omega)} \le \left(\|v_0\|_{\mathrm{L}^\infty(\Omega)}^{1-\beta} + (1-\beta)t \right)^{\frac{1}{1-\beta}}.$$

If $\beta = 1$, then the function $w(t,x) = v(t,x)\mathrm{e}^{-\lambda t}$ satisfies

$$\begin{cases} w_t - \Delta w + \mathrm{e}^{-\lambda(1-\alpha)t} w^\alpha = 0 & \text{in } (0,+\infty)\times\Omega, \\ w = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ w(0,x) = v_0(x) & \text{on } \Omega, \end{cases} \tag{2.5}$$

which is uniformly (pointwise) bounded by the solution of the linear heat equation with the same initial datum. Since the operator $A = \overline{\partial\varphi}^{\mathrm{L}^p(\Omega)\times\mathrm{L}^p(\Omega)}$ is m-accretive in $\mathrm{L}^p(\Omega)$ for every $p \in [1,\infty]$ (see, e.g., the presentation made in [21]), by the regularity results for semilinear accretive operators we conclude the first part of the additional regularity of the statement (2.2). Finally, by [8, Theorem 3.6] we know that $\frac{\partial}{\partial t}\varphi(v_h) \in \mathrm{L}^1(\tau,T)$, $\varphi(v_h)$ is absolutely continuous and for a.e. $t \in (\tau,T)$,

$$\frac{\partial}{\partial t}\varphi(v_h) = \int_\Omega (h(t))(v_h)_t(t)\mathrm{d}x - \int_\Omega [(v_h)_t(t)]^2\mathrm{d}x.$$

Then (2.3) holds by taking $h = \lambda|v_h|^{\beta-1}v_h$ (the fixed point of $\mathcal{T}$).

Corollary 2.1 *Assume $\beta = 1$. Then the weak solution is unique.*

Proof Thanks to the change of variable $w(t,x) = v(t,x)\mathrm{e}^{-\lambda t}$, the problem becomes (2.5) and the result follows from the semigroup theory since it is well-known that the operator $Aw := -\Delta w + \mathrm{e}^{-\lambda(1-\alpha)t}|w|^{\alpha-1}w$ is a T-accretive operator in $\mathrm{L}^p(\Omega)$ for any $p \in [1,+\infty]$ (see, e.g., [25, Chapter 4]).

A more delicate question deals with the proof of the uniqueness of weak solutions for $\beta \in (0,1)$. We point out that some previous results in the literature dealing with the case $\beta \in (0,1)$ (see [14] and its references) are not applicable to our framework due to the presence of the absorption term $|v|^{\alpha-1}v$.

We define the following class of functions:

$$\begin{aligned} \mathcal{M}(\nu,T) := \{v \in \mathcal{C}([0,T];\mathrm{L}^2(\Omega))\big|,\ \forall T' \in (0,T), \text{ there exists } C(T') > 0, \text{ such that} \\ \forall t \in (0,T'),\ v(t,x) \geq C(T')d(x)^\nu \text{ in } \Omega\}, \end{aligned} \tag{2.6}$$

where $\delta(x) := \mathrm{dist}(x,\partial\Omega)$ (which we shall denote simply as δ) and

$$\nu \in \Big(0, \frac{2}{1-\alpha}\Big]. \tag{2.7}$$

The following result collects some useful estimates leading to the uniqueness of non-degenerate weak solutions.

Theorem 2.1 *Let w (resp. v) be a weak subsolution* $\mathrm{PP}(\alpha,\beta,\lambda,w_0)$*, i.e.,*

$$\begin{cases} w_t - \Delta w + |w|^{\alpha-1}w \leq \lambda|w|^{\beta-1}w & \text{in } (0,+\infty)\times\Omega, \\ w = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ w(0,x) = w_0(x) & \text{on } \Omega \end{cases}$$

with $w \in \mathcal{C}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty((0,T)\times\Omega) \cap \mathrm{L}^2_{\mathrm{loc}}(0,T:\mathrm{H}^1_0(\Omega))$, $w \in \mathrm{H}^1_{\mathrm{loc}}(0,T:\mathrm{H}^{-1}(\Omega))$ (resp. similar conditions for v but with the reversed inequalities).

(i) *If $v \in \mathcal{M}(\nu,T)$ for some $\nu \in \big(0,\frac{2}{1-\alpha}\big]$, there exists a constant $C > 0$, such that for any $t \in [0,T)$, we have*

$$\|[w(t)-v(t)]_+\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{e}^{\lambda Ct}\|[w_0-v_0]_+\|_{\mathrm{L}^2(\Omega)}. \tag{2.8}$$

(ii) *If $w \in \mathcal{M}(\nu, T)$ for some $\nu \in \left(0, \frac{2}{1-\alpha}\right]$, there exists a constant $C > 0$, such that for any $t \in [0, T)$, we have*

$$\|[w(t) - v(t)]_-\|_{L^2(\Omega)} \le \mathrm{e}^{\lambda C t} \|[w_0 - v_0]_-\|_{L^2(\Omega)}. \tag{2.9}$$

(iii) *Assume $w_0 \le v_0$, and $v \in \mathcal{M}(\nu, T)$ or $w \in \mathcal{M}(\nu, T)$. Then, for any $t \in [0, T]$, $w(t, \cdot) \le v(t, \cdot)$ a.e. in Ω.*

(iv) *There is uniqueness of weak solutions in the class $\mathcal{M}(\nu, T)$. Moreover, if $v, w \in \mathcal{M}(\nu, T)$ are weak solutions of $\mathrm{PP}(\alpha, \beta, \lambda, w_0)$ and $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$, respectively, then there exists a constant $C > 0$, such that for any $t \in [0, T)$, we have*

$$\|w(t) - v(t)\|_{\mathrm{L}^2(\Omega)} \le \mathrm{e}^{\lambda C t} \|w_0 - v_0\|_{L^2(\Omega)}. \tag{2.10}$$

We shall get later some sufficient conditions on the initial datum v_0 ensuring that there exists some weak solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ belonging to the class $\mathcal{M}(\nu, T)$.

Proof of Theorem 2.1 Multiplying by $(w(t) - v(t))_+$ the difference of the inequalities satisfied by w and v, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega [w(t) - v(t)]_+^2 \mathrm{d}x + \int_\Omega |\nabla [w(t) - v(t)]_+|^2 \mathrm{d}x + \int_\Omega (w(t)^\alpha - v(t)^\alpha)[w(t) - v(t)]_+ \mathrm{d}x \\ &\le \lambda \int_{\{w > v\}} (w(t)^\beta - v(t)^\beta)[w(t) - v(t)] \mathrm{d}x. \end{aligned}$$

But, since $\beta \in (0, 1)$,

$$w^\beta - v^\beta \le \frac{\beta}{v^{1-\beta}}(w - v) \quad \text{for any } 0 < v < w \le M$$

for some $M > 0$. On the other hand, since $v \in \mathcal{M}(\nu, T)$, and $\alpha < \beta$, by applying Young's inequality, we get

$$v^{\beta - 1} \le \frac{1}{C^{(1-\beta)} d(x)^{\nu(1-\beta)}} \le \frac{\varepsilon}{d(x)^2} + C_\varepsilon$$

for any $\varepsilon > 0$ and for some $C_\varepsilon > 0$. Then, from the monotonicity of the function $w \to w^\alpha$, taking $M = \max(\|w\|_{\mathrm{L}^\infty((0,T)\times\Omega)}, \|v\|_{\mathrm{L}^\infty((0,T)\times\Omega)})$, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega [w(t) - v(t)]_+^2 \mathrm{d}x + \int_\Omega |\nabla [w(t) - v(t)]_+|^2 \mathrm{d}x \\ &\le \lambda \varepsilon \int_\Omega \frac{[w(t) - v(t)]_+^2}{d(x)^2} \mathrm{d}x + \lambda C_\varepsilon \int_\Omega [w(t) - v(t)]_+^2 \mathrm{d}x. \end{aligned}$$

Applying Hardy's inequality,

$$\int_\Omega \frac{z^2}{d(x)^2} \mathrm{d}x \le C \int_\Omega |\nabla z|^2 \mathrm{d}x$$

for any $z \in H_0^1(\Omega)$, choosing $\varepsilon > 0$ sufficiently small and using Gronwall's inequality, we get the conclusion (i). The proof of (ii) is similar, but this time we multiply by $(v(t) - w(t))_-$ the difference of the inequalities satisfied by v and w and use the fact that, since $\beta \in (0, 1)$,

$$(w^\beta - v^\beta)[w(t) - v(t)]_- \le \frac{\beta}{w^{1-\beta}}[w(t) - v(t)]_-^2 \quad \text{for any } 0 < v, w \le M$$

for some $M > 0$. Again, since $v \in \mathcal{M}(\nu, T)$, and $\alpha < \beta$, by applying Young's inequality, we get

$$w^{\beta-1} \leq \frac{1}{C^{(1-\beta)} d(x)^{\nu(1-\beta)}} \leq \frac{\varepsilon}{d(x)^2} + C_\varepsilon$$

for any $\varepsilon > 0$ and for some $C_\varepsilon > 0$ and the proof ends as in the case (i). The proofs of (iii) and (iv) are easy consequences of (i) and (ii).

Proposition 2.2 *Assume*

$$v_0(x) \geq K_0 d(x)^{\frac{2}{1-\alpha}} \quad \text{for any } x \in \overline{\Omega}, \tag{2.11}$$

for some constant $K_0 > 0$. Let v be a weak solution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$.

(a) *Given $T > 0$ for any $K_0 > 0$, there is a $T_0 = T_0(K_0) \in (0, T]$ such that $v \in \mathcal{M}(\nu, T_0)$ for $\nu = \frac{2}{1-\alpha}$.*

(b) *If K_0 and λ are large enough, then $v \in \mathcal{M}(\nu, T)$ for $\nu = \frac{2}{1-\alpha}$, for any $T > 0$.*

Proof By (iii) of the above theorem, it is enough to construct a (local) subsolution satisfying the required boundary behavior. We shall carry out such construction by adapting the techniques presented in [24] (see also some related local subsolutions in [1, 23, 30]). From the assumption (2.11) for any $x_0 \in \partial\Omega$, there exist $\epsilon > 0$, $\delta \geq 1$, $C_0 > 0$ and $x_1 \in \Omega$ with $B_{\delta\epsilon}(x_1) \subset \Omega$ such that

$$v_0(x) \geq C_0 |x - x_0|^\nu, \quad \text{a.e. } x \in B_\epsilon(x_1). \tag{2.12}$$

Let us take $x_1 \in \Omega$ such that $\delta\epsilon > |x_1 - x_0| \geq \left(\frac{\delta+1}{2}\right)\epsilon$, and define

$$\underline{U}(x) = \begin{cases} K_1\epsilon^\nu - K_2|x - x_1|^\nu, & \text{if } \ 0 \leq |x - x_1| \leq \epsilon, \\ K_3(\delta\epsilon - |x - x_1|)^\nu, & \text{if } \ \epsilon \leq |x - x_1| \leq \delta\epsilon, \end{cases}$$

and for $x \in B_{\delta\epsilon}(x_1)$, $t \in (0, T]$,

$$\underline{V}(t, x) = \varphi(t)\underline{U}(x).$$

We shall show that it is possible to choose all the above constants and function $\varphi(t)$, such that $\underline{V}$ is a weak subsolution of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ with the desired growth near $\partial B_{\delta\epsilon}(x_1)$ for suitable time interval $[0, T_0(K_0))$ in case (a) or on the whole interval $[0, T]$ in case (b). Since $\underline{U}(x) = \eta(|x-x_1|)$ on $B_{\delta\epsilon}(x_1)$, the Laplacian operator can be written as

$$\Delta\eta(r) = \eta''(r) + \frac{N-1}{r}\eta'(r)$$

with $r \in (0, \delta\epsilon)$. By defining $\eta_1(r) = K_1\epsilon^\nu - K_2 r^\nu$ and $\eta_2(r) = K_3(\delta\epsilon - r)^\nu$, we have

$$\eta(r) = \begin{cases} \eta_1(r), & 0 \leq r \leq \epsilon, \\ \eta_2(r), & \epsilon \leq r \leq \delta\epsilon. \end{cases}$$

The list of conditions which we must check to ensure that $\underline{V}(t, x)$ is a local-weak-subsolution is the following:

(1) $\underline{V} \in \mathcal{C}([0, T]; \mathrm{L}^2(B_{\delta\epsilon}(x_1))) \cap \mathrm{L}^\infty((0, T) \times B_{\delta\epsilon}(x_1)) \cap \mathrm{L}^2_{\mathrm{loc}}(0, T : \mathrm{H}^1_0(B_{\delta\epsilon}(x_1)))$, $\underline{V} \in \mathrm{H}^1_{\mathrm{loc}}(0, T : \mathrm{H}^{-1}(B_{\delta\epsilon}(x_1)))$. This is guaranteed if we take $\varphi \in \mathrm{H}^1(0, T)$ and $\underline{U} \in \mathcal{C}^1(B_{\delta\epsilon}(x_1))$ (since by construction $\underline{U} = 0$ on $\partial B_{\delta\epsilon}(x_1)$). In particular, we must have

$$(K_1 - K_2)\epsilon^\nu = K_3(\epsilon(\delta - 1))^\nu, \tag{2.13}$$

$$\nu K_2 \epsilon^{\nu-1} = -\nu K_3(\epsilon(\delta - 1))^{\nu-1}. \tag{2.14}$$

(2) $\underline{V}(0,x) \le v_0(x)$ a.e. on $B_{\delta\epsilon}(x_1)$. Thanks to (2.12), since $\eta_1(r)$ is concave and $C_0 r^\nu$ is convex, it is enough to have

$$\varphi(0)K_3(\epsilon(\delta-1))^\nu \le C_0\epsilon^\nu \quad \text{on } B_{\delta\epsilon}(x_1).$$

(3) $\underline{V}_t - \Delta\underline{V} + \underline{V}^\alpha \le \lambda\underline{V}^\beta$ (in a weak form) on $[0, T_0(K_0)) \times B_{\delta\epsilon}(x_1)$. For $\mu > 0$, let us introduce $\mathcal{L}(\eta : \mu) = -\Delta\eta + \mu\eta^\alpha$. Then, if we write $r = \epsilon s$,

$$\begin{aligned}\mathcal{L}(\eta_1) &\le \nu(\nu-1)K_2 r^{\nu-2} + \nu(\mathrm{N}-1)K_2 r^{\nu-2} + \mu[K_1\epsilon^\nu - K_2 r^\nu]^\alpha \\ &= [\nu(\nu-1)K_2 s^{\nu\alpha} + \nu(N-1)K_2 s^{\nu\alpha} + \mu(K_1 - K_2 s^\nu)^\alpha]\epsilon^{\alpha\nu} \\ &\le K_4\epsilon^{\nu\alpha},\end{aligned}$$

where

$$K_4 = K_4(\mu) := \nu[(\nu-1) + (N-1)K_2] + \mu K_1. \tag{2.15}$$

On the other hand,

$$\begin{aligned}\mathcal{L}(\eta_2) &\le -\lambda\nu(\nu-1)K_3(\delta\epsilon - r)^{\nu-2} + (N-1)\nu K_3\frac{(\delta\epsilon-r)^{\nu-1}}{r} + \mu K_3^\alpha(\delta\epsilon)^{\nu\alpha} \\ &\le \nu K_3(\delta\epsilon - r)^{\nu\alpha}\Big(-(\nu-1) + (N-1)\frac{(\delta\epsilon - r)}{r} + \mu K_3^{\alpha-1}\nu^{-1}\Big).\end{aligned}$$

Now $\frac{(\delta\epsilon-r)}{r} \le \delta - 1$ when $\epsilon \le r \le \delta\epsilon$, and thus if

$$1 \le \delta < 1 + \frac{\nu\alpha+1}{N-1}, \tag{2.16}$$

so, if we choose K_3 as

$$K_3 = K_3(\mu,\delta) := \Big(\frac{\mu\nu^{-1}}{(\nu\alpha+1)-(N-1)(\delta-1)}\Big)^{\frac{1}{1-\alpha}}, \tag{2.17}$$

we obtain that $-\Lambda\eta_2 + \mu\eta_2^\alpha \le 0$.

Moreover,

$$\underline{V}_t - \Delta\underline{V} + \underline{V}^\alpha = \varphi'\eta - \varphi\Big(\eta'' + \frac{N-1}{r}\eta'\Big) + \varphi^\alpha\eta^\alpha.$$

Then, if we have $\varphi \in \mathcal{C}^1(0,T)$, such that

$$\varphi'(t) \le 0, \tag{2.18}$$

then we have

$$\varphi(0) \le 1. \tag{2.19}$$

Given $\varepsilon_1 \in (0,1)$, we always can find $T_0(\varepsilon_1) \le T$, such that

$$\varepsilon_1 \le \varphi(t) \le 1 \quad \text{for any } t \in [0, T_0(\varepsilon_1)]$$

and hence, if

$$\mu = \frac{1}{(\varepsilon_1)^{1-\alpha}}, \tag{2.20}$$

we have

$$\Delta \underline{V} + \underline{V}^{\alpha} \leq (\varepsilon_1)^{1-\alpha}\varphi(t)^{\alpha}(-\Delta\eta(r) + \mu\eta^{\alpha}) \leq 0.$$

This implies that $\underline{V}_t - \Delta\underline{V} + \underline{V}^{\alpha} \leq \lambda \underline{V}^{\beta}$ (in a weak form) on $[0, T_0(\varepsilon_1)) \times (B_{\delta\epsilon}(x_1) \setminus B_{\epsilon}(x_1))$. The remaining condition is to have the above inequality also on $B_{\epsilon}(x_1)$. This will be an easy consequence, if we take any subsolution of the associated ODE as function φ, more precisely, such that

$$\varphi'(t) + \frac{(\max \eta_1)^{\alpha}}{\min \eta_1}\varphi(t)^{\alpha} \leq \frac{\lambda}{(\min \eta_1)^{1-\beta}}\varphi(t)^{\beta}.$$

By taking $\varphi(0)$ and ε_1 small enough, it is easy to see that it is possible to choose the rest of constants, such that all the above conditions follow and this ends the proof of case (a). In case (b) the arguments are very similar, but in this case, it is possible to take as the function $\varphi(t)$ given by

$$\varphi(t) = (\varepsilon_2 + \mathrm{e}^{-kt})$$

for suitable $\varepsilon_2 > 0$ and $k > 0$ small enough.

Corollary 2.2 *Assume v_0 as in Proposition* 2.2 *and let v be a weak solution of* $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$*, such that the non-degeneracy constant C in* (2.6) *is independent of T for any $T > 0$. Let $u \in \mathrm{L}^{\infty}(\Omega)$ be a solution of the stationary problem* $\mathrm{SP}(\alpha, \beta, \lambda)$*, such that $v(t) \to u$ in* $\mathrm{L}^2(\Omega)$ *a.e. $t \nearrow +\infty$. Then u satisfies the nondeneracy property $u(x) \geq K d(x)^{\frac{2}{1-\alpha}}$ for some $K > 0$.*

The stability of the trivial solution $u \equiv 0$ of $\mathrm{SP}(\alpha, \beta, \lambda)$ for λ small is very well illustrated by means of the following "extinction in finite time" property of solutions of the associated parabolic problem $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ assumed λ small enough.

Theorem 2.2 *Assume*

$$\lambda \in [0, \lambda_1). \tag{2.21}$$

Let $v_0 \in \mathrm{L}^{\infty}(\Omega)$, $v_0 \geq 0$. Assume $\beta = 1$ or (2.11)*. Then there exists $T_0 > 0$, such that the solution v of* $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ *satisfies $v(t) \equiv 0$ on Ω for any $t \geq T_0$.*

Proof We shall use an energy method in the spirit of [2] (see also [33]). By multiplying by $v(t)$ and integrating by parts (as in the proof of uniqueness), we arrive to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} v(t)^2 \mathrm{d}x + \int_{\Omega} |\nabla v(t)|^2 \mathrm{d}x + \int_{\Omega} v(t)^{\alpha+1} \mathrm{d}x = \lambda \int_{\Omega} v(t)^{\beta+1} \mathrm{d}x.$$

Assume now that $\beta = 1$. Then, by using the Poincaré inequality

$$\lambda_1 \int_{\Omega} v(t)^2 \mathrm{d}x \leq \int_{\Omega} |\nabla v(t)|^2 \mathrm{d}x, \tag{2.22}$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} v(t)^2 \mathrm{d}x + \Big(1 - \frac{\lambda}{\lambda_1}\Big)\int_{\Omega} |\nabla v(t)|^2 \mathrm{d}x + \int_{\Omega} v(t)^{\alpha+1} \mathrm{d}x \leq 0$$

and the result holds exactly as in [2, Proposition 1.1, Chapter 2]. Indeed, by applying the Gagliardo-Nirenberg inequality,

$$\Big[\int_\Omega v^r \mathrm{d}x\Big]^{\frac{1}{r}} \le C\Big[\int_\Omega |\nabla v|^2 \mathrm{d}x\Big]^{\frac{\theta}{2}}\Big[\int_\Omega v \mathrm{d}x\Big]$$

for any $r \in [1, +\infty)$ if $N \le 2$ and $r \in \big[1, \frac{2N}{N-2}\big]$ if $N > 2$ $\big($with $\theta = \frac{2N(r-1)}{r+2N} \in (0,1)\big)$, we have that the function

$$y(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v(t)^2 \mathrm{d}x$$

satisfies the inequality

$$y'(t) + Cy^{\upsilon}(t) \le 0$$

for some $C > 0$ and $\upsilon \in (0,1)$. If $\beta \in (0,1)$, then we introduce the change of unknown $v = \mu\widehat{v}$ getting

$$\mu\widehat{v}_t - \mu\Delta\widehat{v} + \mu^\alpha\widehat{v}^\alpha = \lambda\mu^\beta\widehat{v}^\beta.$$

By choosing μ such that

$$\mu < \frac{1}{\lambda_1^{\frac{1}{\beta-\alpha}}},$$

we can assume without loss of generality that $\lambda < \min(\lambda_1, 1)$. Moreover, since

$$\lambda\int_\Omega v(t)^{\beta+1}\mathrm{d}x \le \lambda\int_\Omega v(t)^2\mathrm{d}x + \lambda\int_\Omega v(t)^{\alpha+1}\mathrm{d}x,$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v(t)^2\mathrm{d}x + \Big(1 - \frac{\lambda}{\lambda_1}\Big)\int_\Omega |\nabla v(t)|^2\mathrm{d}x + (1-\lambda)\int_\Omega v(t)^{\alpha+1}\mathrm{d}x \le 0,$$

and the proof ends as in the precedent case.

Remark 2.1 The assumption (2.21) is optimal if $\beta = 1$. Indeed, by the results of [26], we know that for any $\lambda > \lambda_1$, there exists a non-negative nontrivial solution u of the associated stationary problem $\mathrm{SP}(\alpha, 1, \lambda)$.

In fact, for any $\lambda > 0$, the trivial solution $u \equiv 0$ of the stationary problem $\mathrm{SP}(\alpha, \beta, \lambda)$ is asymptotically $\mathrm{L}^\infty(\Omega)$-stable in the sense that it attracts solutions of $\mathrm{PP}(\alpha, \beta, \lambda, v_0)$ in $\mathrm{L}^\infty(\Omega)$ for small initial data v_0.

Proposition 2.3 *Let $v_0 \in \mathrm{L}^\infty(\Omega)$, $v_0 \ge 0$. Assume $\beta = 1$ or (2.11). Given $\lambda > 0$, assume that*

$$\|v_0\|_{\mathrm{L}^\infty(\Omega)} < \lambda^{-\frac{1}{\beta-\alpha}}.$$

Then $v(t) \to 0$ in $\mathrm{L}^\infty(\Omega)$ as $t \to +\infty$.

Proof Use the solution of the associated ODE (with $\|v_0\|_{\mathrm{L}^\infty(\Omega)}$ as initial datum) as super-solution.

Concerning non-uniformly bounded trajectories we have the following proposition.

Proposition 2.4 *Let $v_0 \in L^\infty(\Omega)$, $v_0 \geq 0$, such that*

$$0 < u_\lambda(x) + \varepsilon_0 \leq v_0(x) \quad a.e.\ x \in \Omega \tag{2.23}$$

for some $\varepsilon_0 > 0$ and u_λ being the solution of the associated stationary problem $\mathrm{SP}(\alpha, \beta, \lambda)$ *such that*

$$meas\{x \in \Omega : u_\lambda(x) + \varepsilon_0 > \lambda^{-\frac{1}{\beta-\alpha}}\} > 0.$$

Assume $\beta = 1$ or (2.11)*. Then $\|v(t)\|_{\mathrm{L}^\infty(\Omega)} \nearrow +\infty$ as $t \to +\infty$.*

Proof Since obviously u_λ is a solution of $\mathrm{PP}(\alpha, \beta, \lambda, u_\lambda)$, we first get, by Theorem 2.1, that $u_\lambda(x) \leq v(t, x)$ for any $t \in [0, +\infty)$ and a.e. $x \in \Omega$. Moreover, $u_\lambda(x) > \lambda^{-\frac{1}{\beta-\alpha}} > 0$ on a positively measured subset Ω_λ of Ω, where we can apply the strong maximum principle to conclude that $u_\lambda(x) < v(t, x)$ for any $t \in [0, +\infty)$ and a.e. $x \in \Omega_\lambda$. Since $u_\lambda \in \mathcal{C}(\overline{\Omega})$, there exists $x_\lambda \in \overline{\Omega}_\lambda$ such that

$$u_\lambda(x_\lambda) = \min_{\overline{\Omega}_\lambda} u_\lambda.$$

Taking now $U(t)$ as the solution of the ODE

$$\mathrm{ODE}(\alpha, \beta, \lambda, u_\lambda(x_\lambda) + \varepsilon_0) \quad \begin{cases} U_t + U^\alpha = \lambda U^\beta, & \text{in } (0, +\infty), \\ U(0) = u_\lambda(x_\lambda) + \varepsilon_0, \end{cases} \tag{2.24}$$

by the standard comparison principle (noticing that now the involved nonlinearities are Lipschitz continuous on this set of values), we get that for any $t \in [0, +\infty)$,

$$U(t) \leq v(t, x) \quad \text{a.e. } x \in \Omega_\lambda.$$

Finally, since we know that $U(t) \nearrow +\infty$ as $t \to +\infty$, we get the result.

3 Critical Exponents Curve on the Plane (α, β)

In this section, using Pohozaev's identity (see [49]) and developing the spectral analysis with respect to the fibering procedure [39], we introduce the critical exponents curve $\mathcal{C}(N)$ on the plane (α, β) and study its main properties.

From now on, we will use the notations

$$T(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \quad A(u) = \int_\Omega |u|^{\alpha+1} \, \mathrm{d}x, \quad B(u) = \int_\Omega |u|^{\beta+1} \, \mathrm{d}x.$$

Then

$$E_\lambda(u) = \frac{1}{2} T(u) + \frac{1}{\alpha+1} A(u) - \lambda \frac{1}{\beta+1} B(u). \tag{3.1}$$

Case $0 < \alpha < \beta < 1$ Assume that $0 < \alpha < \beta < 1$. Then for any fixed $u \in \mathrm{H}_0^1(\Omega) \setminus \{0\}$, the equation

$$E'_\lambda(ru) = 0 \tag{3.2}$$

may have at most two roots $r_{\max}(u), r_{\min}(u) \in \mathbb{R}^+$ such that $r_{\max}(u) \le r_{\min}(u)$. Furthermore, $r_{\max}(u) < r_{\min}(u)$ if and only if

$$E''_\lambda(r_{\max}(v)\cdot v)(v,v) < 0 \quad \text{and} \quad E''_\lambda(r_{\min}(v)\cdot v)(v,v) > 0,$$

and $r_{\max}(v) = r_{\min}(v) =: r_s(v)$ if and only if $E''_\lambda(r_s(v)\cdot v) = 0$ (see Figure 2).

In [42], it was introduced the following characteristic (nonlinear fibering eigenvalue):

$$\Lambda_0 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \lambda_0(u), \tag{3.3}$$

where

$$\lambda_0(u) = c_0^{\alpha,\beta}\lambda(u),$$
$$c_0^{\alpha,\beta} = \frac{(1-\alpha)(1+\beta)}{(1-\beta)(1+\alpha)}\Big(\frac{(1+\alpha)(1-\beta)}{2(\beta-\alpha)}\Big)^{\frac{\beta-\alpha}{1-\alpha}}$$

and

$$\lambda(u) = \frac{A(u)^{\frac{1-\beta}{1-\alpha}}T(u)^{\frac{\beta-\alpha}{1-\alpha}}}{B(u)}. \tag{3.4}$$

Note that by the Gagliardo-Nirenberg inequality (see [42, Proposition 2]) it follows that $0 < \Lambda_0 < +\infty$. In [42], the following proposition was proved.

Proposition 3.1 *If $\lambda \ge \Lambda_0$, then there exists $u \in H_0^1(\Omega)\setminus\{0\}$, such that $E'_\lambda(u) = 0$ and $E_\lambda(u) \le 0$, $E''_\lambda(u) > 0$.*

We also need the following characteristic value from [42]:

$$\Lambda_1 = \inf_{u\in \mathrm{H}_0^1\setminus\{0\}} \lambda_1(u), \tag{3.5}$$

where

$$\lambda_1(u) = c_1^{\alpha,\beta}\lambda(u), \tag{3.6}$$

where

$$c_1^{\alpha,\beta} = \frac{1-\alpha}{1-\beta}\Big(\frac{1-\beta}{\beta-\alpha}\Big)^{\frac{\beta-\alpha}{1-\alpha}}. \tag{3.7}$$

As before, we have $0 < \Lambda_1 < +\infty$. Furthermore, $0 < \Lambda_1 < \Lambda_0 < +\infty$ (see [42, Claim 2]) and we have the following proposition (see also [42]).

Proposition 3.2 *If $\lambda > \Lambda_1$, then there exists $u \in \mathrm{H}_0^1(\Omega)\setminus\{0\}$, such that $E'_\lambda(u) = 0$, whereas if $\lambda < \Lambda_1$, then $E'_\lambda(u) > 0$ for any $u \in \mathrm{H}_0^1(\Omega)\setminus\{0\}$.*

Let $u \in \mathrm{H}_0^1(\Omega)$ be a weak solution of (1.2). Standard regularity arguments show that $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0,1)$. Note that by the assumption, $\partial\Omega$ is a $\mathcal{C}^1$-manifold. Therefore, Pohozaev's identity holds (see [43, 49]), namely,

$$P_\lambda(u) + \frac{1}{2N}\int_{\partial\Omega}\Big|\frac{\partial u}{\partial \nu}\Big|^2 x\cdot\nu \,\mathrm{d}s = 0, \tag{3.8}$$

where

$$P_\lambda(u) := \frac{N-2}{2N}T(u) + \frac{1}{\alpha+1}A(u) - \lambda\frac{1}{\beta+1}B(u), \quad u \in \mathrm{H}_0^1(\Omega).$$

Note that if Ω is a star-shaped (strictly star-shaped) domain with respect to the origin of $\mathbb{R}^N$, then $x \cdot \nu \geq 0$ ($x \cdot \nu > 0$) for all $x \in \partial\Omega$. Thus we have the result as follows.

Proposition 3.3 *Assume that Ω is a star-shaped domain with respect to the origin of $\mathbb{R}^N$, then $P_\lambda(u) \leq 0$ ($P_\lambda(u) = 0$) for any weak (flat or compactly supported) solution u of* (1.2). *If, in addition, Ω is strictly star-shaped, then a weak solution u of* (1.2) *is flat or it has compact support if and only if $P_\lambda(u) = 0$.*

Let us study the critical exponent curve $\mathcal{C}(N)$ (see (1.9)) and prove Lemma 1.1. Consider the system (see [42])

$$\begin{cases} E'_\lambda(u) := \; T(u) + A(u) - \lambda B(u) = 0, \\ P_\lambda(u) := \dfrac{N-2}{2N}T(u) + \dfrac{1}{\alpha+1}A(u) - \lambda\dfrac{1}{\beta+1}B(u) = 0, \\ E''_\lambda(u) := T(u) + \alpha A(u) - \lambda\beta B(u) = 0. \end{cases} \tag{3.9}$$

This system is solvable with respect to the variables $T(u), A(u), B(u)$, if the corresponding determinant

$$D = \frac{(\beta-\alpha)(2(1+\alpha)(1+\beta) - N(1-\alpha)(1-\beta))}{2N(1+\alpha)(1+\beta)} \tag{3.10}$$

is non-zero.

On the other hand $D = 0$ if and only if $(\alpha, \beta) \in \mathcal{C}(N)$.

Proof of Lemma 1.1 Let Ω be a star-shaped domain with respect to the origin of $\mathbb{R}^N$. Then by Proposition 3.3, we have $P_\lambda(u) = 0$ for any flat or compactly supported solution u of (1.2). Note also that $E'_\lambda(u) = 0$. Thus, in case $(\alpha, \beta) \in \mathcal{C}(N)$, i.e., when the determinant of system (3.9) is equal to zero one has $E''_\lambda(u) = 0$ and we get the proof of the statement (1) of Lemma 1.1. Observe that

$$\begin{aligned} &D \cdot \frac{2N(1+\alpha)}{(1-\alpha)[-2(1+\alpha) - \mathrm{N}(1-\alpha)]}B(u) \\ &= \frac{1}{1-\alpha}(E''_\lambda(u) - E'_\lambda(u)) - \frac{2N(1+\alpha)}{(N-2)(1+\alpha) - 2\mathrm{N}}\Big(P_\lambda(u) - \frac{N-2}{2N}E'_\lambda(u)\Big). \end{aligned}$$

Thus if $(\alpha, \beta) \in \mathcal{E}_u(N)$ and $P_\lambda(u) = 0$, $E'_\lambda(u) = 0$, then

$$E''_\lambda(u) = -D \cdot \frac{2N(1+\alpha)}{(1-\alpha)[2(1+\alpha) + N(1-\alpha)]}B(u) < 0,$$

and we obtain the proof of the statement (2) of Lemma 1.1.

Under the assumption (3) of Lemma 1.1, for a weak solution u of (1.2), we have $P_\lambda(u) \leq 0$ (see Proposition 3.3) and therefore (3) yields

$$E''_\lambda(u) \geq -D \cdot \frac{2N(1+\alpha)}{(1-\alpha)[-2(1+\alpha) - N(1-\alpha)]}B(u) > 0,$$

since $D > 0$ for $(\alpha, \beta) \in \mathcal{E}_s(N)$. This completes the proof of Lemma 1.1.

Case $\boldsymbol{\beta = 1}$ Recall some results from [27]. In what follows, (λ_1, φ_1) denotes the first eigenpair of the operator $-\Delta$ in Ω with zero boundary conditions. Let $u \in \mathrm{H}_0^1(\Omega)$. The fibering mapping in this case is defined by

$$\Phi_u(r) = E_\lambda(ru) = \frac{r^2}{2}\mathrm{H}_\lambda(u) + \frac{r^{1+\alpha}}{1+\alpha}A(u),$$

where we denote

$$\mathrm{H}_\lambda(u) := \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega |u|^2 \,\mathrm{d}x.$$

Then

$$\Phi'_u(r) = E'_\lambda(ru) = r\mathrm{H}_\lambda(u) + r^\alpha A(u)$$

and the equation $\Phi'_u(r) = 0$ has a positive solution only, if both terms in $\Phi'_u(r)$ have opposite sign, that is, if and only if $\mathrm{H}_\lambda(u) < 0$. Note that there is $u \in \mathrm{H}_0^1(\Omega)$ such that $\mathrm{H}_\lambda(u) < 0$ if and only if $\lambda > \lambda_1$. It turns out that the only point $r(u)$, where $\Phi'_u(r) = 0$ is given by

$$r(u) = \Big(\frac{A(u)}{-\mathrm{H}_\lambda(u)}\Big)^{\frac{1}{1-\alpha}}. \tag{3.11}$$

Furthermore, $E''_\lambda(r(u)u)(u,u) < 0$ and

$$E_\lambda(r(u)u) = \max_{r>0} E_\lambda(ru). \tag{3.12}$$

Substituting (3.11) into $E_\lambda(ru)$, we obtain

$$J_\lambda(u) := E_\lambda(r_\lambda(u)u) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{A(u)^{\frac{2}{1-\alpha}}}{(-\mathrm{H}_\lambda(u))^{\frac{1+\alpha}{1-\alpha}}}. \tag{3.13}$$

Consider

$$\widehat{E}_\lambda = \min\{J_\lambda(u) : \; u \in \mathrm{H}_0^1(\Omega) \setminus \{0\}, \; \mathrm{H}_\lambda(u) < 0\}. \tag{3.14}$$

It follows directly

Proposition 3.4 *A point $u \in \mathrm{H}_0^1(\Omega)$ is a minimizer of* (3.14) *if and only if $\widetilde{u} = r(u)u$ is a ground state of* (5.1).

Remark 3.1 We point out that in both cases, $\beta < 1$ and $\beta = 1$, the above results can be extended to the case in which the ground solution of $\mathrm{SP}(\alpha, \beta, \lambda)$ minimizes the energy on the closed convex cone

$$K = \{v \in \mathrm{H}_0^1(\Omega), v \geq 0 \text{ on } \Omega\}.$$

Indeed, we introduce the modified energy functional

$$E^+_\lambda(u) = E_\lambda(u) + \int_\Omega j(u)\mathrm{d}x,$$

where

$$j(u)=\begin{cases}0, & \text{if } u\in K,\\ +\infty, & \text{otherwise.}\end{cases}$$

Notice that $j(ru)=j(u)$ for any $r>0$. Obviously, $E_\lambda^+(u)=E_\lambda(u)$ if $u\in K$. Moreover the additional term arising in the associated Euler-Lagrange equation, given by the subdifferential of the convex function $\int_\Omega j(u)\mathrm{d}x$, vanishes when the ground state solution of $\mathrm{SP}(\alpha,\beta,\lambda)$ is non-negative.

4 Existence of Ground State

In this section, we prove the first parts of Theorems 1.1–1.2.

Proof of Theorem 1.1(1) Assume $\beta<1$. In this case, the existence of a ground state of (1.2) when $(\alpha,\beta)\in\mathcal{E}_s(N)$ was proved in [42]. The proof for the points $(\alpha,\beta)\in\mathcal{E}\setminus\mathcal{E}_s(N)$ can be obtained in a similar way. However, for the sake of completeness, we present a summary of the proof.

Consider the constrained minimization problem of $E_\lambda(u)$ on the associated Nehari manifold

$$\begin{cases}E_\lambda(u)\to\min,\\ E'_\lambda(u)(u)=0.\end{cases}\tag{4.1}$$

We denote by

$$\mathcal{N}_\lambda:=\{u\in H_0^1(\Omega):\ E'_\lambda(u)=0\}$$

the admissible set of (4.1), i.e., the corresponding Nehari manifold. Denote also

$$\widehat{E_\lambda}:=\min\{E_\lambda(u):\ u\in\mathcal{N}_\lambda\}$$

the minimum value in this problem. Note that by Proposition 3.2, $\mathcal{N}_\lambda\neq\emptyset$ for any $\lambda>\Lambda_1$. Furthermore, by Sobolev's inequalities, we have

$$E_\lambda(u)\geq\frac{1}{2}\|u\|_1^2-c_1\|u\|_1^{1+\beta}\to\infty$$

as $\|u\|_1\to\infty$, since $2>1+\beta$. Thus $E_\lambda(u)$ is a coercive functional on $\mathrm{H}_0^1(\Omega)$. Using this it is not hard to prove the following proposition (see also [42, Lemma 9]).

Proposition 4.1 *Let $(\alpha,\beta)\in\mathcal{E}$. Then for any $\lambda\geq\Lambda_1$, (4.1) has a minimizer $u_\lambda\in\mathrm{H}_0^1(\Omega)\setminus\{0\}$, i.e., $E_\lambda(u_\lambda)=\widehat{E_\lambda}$ and $u_\lambda\in\mathcal{N}_\lambda$.*

Let $\lambda\geq\Lambda_1$ and $u_\lambda\in\mathrm{H}_0^1(\Omega)\setminus\{0\}$ be a minimizer of (4.1). Then by the Lagrange multipliers rule, there exist μ_1, μ_2 such that

$$\mu_1DE_\lambda(u_\lambda)=\mu_2DE'_\lambda(u_\lambda)(u_\lambda),\tag{4.2}$$

and $|\mu_1|+|\mu_2|\neq0$. Thus, if $\mu_2=0$, then u_λ is a weak solution of (1.2).

This condition is satisfied under the assumptions of the following result.

Proposition 4.2 *Let $(\alpha,\beta)\in\mathcal{E}$. Then for any $\lambda\geq\Lambda_0$, (1.2) has a ground state u_λ which is non-negative, $u\in\mathcal{C}^{1,\gamma}(\overline{\Omega})\cap\mathcal{C}^2(\Omega)$ for some $\gamma\in(0,1)$ and $E''_\lambda(u_\lambda)(u_\lambda,u_\lambda)>0$.*

Proof Since $0 < \Lambda_1 < \Lambda_0$, then by Proposition 4.1, for any$\lambda \ge \Lambda_0$, there exists a minimizer $u_\lambda \in \mathrm{H}_0^1(\Omega) \setminus \{0\}$ of (4.1). Lemma 3.1 implies that there is $u \in \mathcal{N}_\lambda$ such that $E_\lambda(u) \le 0$, and therefore $E_\lambda(u_\lambda) \le E_\lambda(u) \le 0$. This implies that $E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$. Let us test (4.2) by u_λ. Then

$$\mu_1 E'_\lambda(u_\lambda)(u_\lambda) = \mu_2(E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) + E'_\lambda(u_\lambda)(u_\lambda)).$$

Since $E'_\lambda(u_\lambda)(u_\lambda) = 0$, this yields that $\mu_2 E''_\lambda(u_\lambda) = 0$. But $E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) \ne 0$, and therefore $\mu_2 = 0$. Thus, by (4.2), we obtain $DE_\lambda(u_\lambda) = 0$, i.e., u_λ is a weak solution of (1.2). Since any weak solution w_λ of (1.2) belongs to $\mathcal{N}_\lambda$, (4.1) yields that u_λ is a ground state. The rest of the lemma is proved in a standard way.

From this proposition arguing by contradiction, it is not hard to show that there is an interval $(\Lambda_0 - \varepsilon, +\infty)$ for some $\varepsilon > 0$, such that for any $\lambda \in (\Lambda_0 - \varepsilon, +\infty)$ the minimizer u_λ of (4.1) satisfies $E''_\lambda(u_\lambda) > 0$. From this, as in the proof of Proposition 4.2, it follows that u_λ is a ground state of (1.2) which is non-negative and $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0, 1)$.

Thus we have a proof that there exists $\lambda^* \in (\Lambda_1, \Lambda_0)$, such that for all $\lambda > \lambda^*$ (1.2) has a ground state u_λ, which is non-negative in Ω, $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0, 1)$ and $E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$. This completes the proof of the statement (1) of Theorem 1.1.

Proof of Theorem 1.2(1) The existence of a ground state is obtained from the constrained minimization problem (3.14) and then using Proposition 3.4. The implementation of this proof was done in [27, Theorem 2.1, p. 6].

5 Existence of Ground State Flat Solutions in Case $\beta = 1$

In this section, we prove the statement (1) in Theorem 1.4. Consider now the following auxiliary problem on the whole space $\mathbb{R}^N$:

$$\begin{cases} -\Delta u + u^\alpha = u & \text{in } \mathbb{R}^N, \\ u \ge 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{5.1}$$

Here and subsequently, $\mathrm{H}^1(\mathbb{R}^N)$ denotes the standard Sobolev space with the norm

$$\|u\|_1 = \Big(\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}.$$

Then (5.1) has a variational form with the Euler-Lagrange functional

$$E(u) = \frac{1}{2}\mathrm{H}(u) + \frac{1}{\alpha+1} A(u), \quad u \in \mathrm{W}^{1,2}(\mathbb{R}^N),$$

where

$$\mathrm{H}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x, \quad A(u) = \int_{\mathbb{R}^N} |u|^{\alpha+1} \,\mathrm{d}x.$$

As above, we call a nonzero weak solution u_λ of (5.1) a ground state of (5.1), if it holds

$$E(u_\lambda) \le E(w_\lambda)$$

for any nonzero weak solution w_λ of (5.1). The fibering map in this case is given as follows:

$$\Phi_u(r) := E(ru) = \frac{r^2}{2}\mathrm{H}(u) + \frac{r^{1+\alpha}}{\alpha+1} A(u), \quad u \in \mathrm{H}^1(\mathbb{R}^N),\ t \in \mathbb{R}^+,$$

and for fix $u \in \mathrm{H}^1(\mathbb{R}^N)$, the equation

$$\Phi'_u(r) \equiv r\mathrm{H}(u) + r^\alpha A(u) = 0, \quad r \in \mathbb{R}^+$$

has only one root

$$r(u) = \Big(\frac{A(u)}{-\mathrm{H}_\lambda(u)}\Big)^{\frac{1}{1-\alpha}}, \tag{5.2}$$

which exists if and only if $\mathrm{H}(u) < 0$.

As above, substituting this root into $E_\lambda(ru)$, we obtain a zero-homogeneous functional

$$J(u) := E(r(u)u) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{A(u)^{\frac{2}{1-\alpha}}}{(-\mathrm{H}(u))^{\frac{1+\alpha}{1-\alpha}}}, \tag{5.3}$$

and we consider

$$\widehat{E}^\infty = \min\{J(u) : \ u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus \{0\}, \ \mathrm{H}(u) < 0\}. \tag{5.4}$$

As above, it follows directly the proposition below.

Proposition 5.1 *We have that u is a minimizer of* (5.4) *if and only if $\widetilde{u} = r(u)u$ is a ground state of* (5.1).

In Appendix below, using (5.4), we prove the following lemma.

Lemma 5.1 *Assume $0 < \alpha < 1$. Then* (5.1) *has a classical non-negative solution $u \in \mathrm{H}^1(\mathbb{R}^N)$ which is a ground state.*

The following result can be found in [51].

Lemma 5.2 *Assume $0 < \alpha < 1$. Then any classical solution u of* (5.1) *has a compact support. Furthermore, if we define*

$$\Theta := \{x \in \mathbb{R}^N : u(x) > 0\},$$

then for every connected component Ξ of Θ, we have that

(1) *Ξ is a ball;*

(2) *u is radially symmetric with respect to the centre of the ball Ξ.*

Lemmas 5.1–5.2 yield the following corollary.

Corollary 5.1 *Assume $0 < \alpha < 1$. Then there is a radius $R^* > 0$, such that* (5.1) *has a ground state u^* which is a flat classical radial solution and*

$$\mathrm{supp}(u^*) = B_{R^*}.$$

Let us return to (1.2). From Corollary 5.1, we have the following result.

Corollary 5.2 *Assume that $B_{R^*} \subset \Omega$. Then the ground state u_λ of* (1.2) *with $\lambda = 1$ coincides with the ground state u^* of* (5.1)*, that is, $u_\lambda|_{\lambda=1}$ is a compact support classical radial solution and*

$$\mathrm{supp}(u_\lambda)|_{\lambda=1} \equiv \overline{\Theta} = B_{R^*}.$$

Proof Any function w from $\mathrm{H}_0^1(\Omega)$ can be extended to $\mathbb{R}^N$ as

$$\begin{cases} \widetilde{w} = w & \text{in } \Omega, \\ \widetilde{w} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.5}$$

Then $\widetilde{w} \in \mathrm{H}^1(\mathbb{R}^N)$, and in this sense, we may assume that $\mathrm{H}_0^1(\Omega) \subset \mathrm{H}^1(\mathbb{R}^N)$. Therefore,

$$\widehat{E}^\infty \le \widehat{E}_1 \equiv \min\{J_1(v) : \ v \in \mathrm{H}_0^1(\Omega) \setminus \{0\}, \ v \ge 0, \ \mathrm{H}_1(v) < 0\}.$$

Note that $u^* \in K \subset \mathrm{H}_0^1(B_{R^*}) \subset \mathrm{H}_0^1(\Omega)$. This yields $\widehat{E}^\infty = E(u^*) = \widehat{E}_1$ and we get the proof.

Assume now that Ω is a star-shaped domain in $\mathbb{R}^N$, with respect to some point $z \in \mathbb{R}^N$, which without loss of generality, we may assume coincides with the origin $0 \in \mathbb{R}^N$.

Let u_λ be a ground state of (1.2). By making a change of variable $v_{\lambda(\kappa)}(y) = \kappa^{-\frac{2}{1-\alpha}} u_\lambda(\kappa y)$, $y \in \Omega_\kappa$, with $\kappa > 0$, we get

$$\begin{cases} -\Delta v_{\lambda(\kappa)} = \lambda(\kappa) v_{\lambda(\kappa)} - v_{\lambda(\kappa)}^\alpha & \text{in } \Omega_\kappa, \\ v_{\lambda(\kappa)} = 0 & \text{on } \Omega_\kappa, \end{cases} \tag{5.6}$$

where $\lambda(\kappa) = \lambda\kappa^2$, $\Omega_\kappa = \{y \in \mathbb{R}^N : \ y = \frac{x}{\kappa}, \ x \in \Omega\}$. Since u_λ is a ground state of (1.2), it is easy to see that $v_{\lambda(\kappa)}$ is also a ground state of (5.6). Note that if $\kappa = \sqrt{\frac{1}{\lambda}}$ then $\lambda(\kappa) = 1$. On the other hand, if κ is sufficiently small then $B_{R^*} \subset \Omega_\kappa$. Hence, by Corollary 5.1, there is a sufficiently large λ^*, such that for any $\lambda > \lambda^*$ the ground state $v_{\lambda(\kappa)}$ with $\lambda(\kappa) = \lambda \cdot (\kappa)^2$, $\kappa = \sqrt{\frac{1}{\lambda}}$ is a flat or compactly supported classical radial solution of (5.6) which coincides with the ground state u^* of (5.1). Thus we complete the proof.

Corollary 5.3 *Assume* $0 < \alpha < 1$. *Then there exists* $\lambda^* > 0$, *such that for any* $\lambda \ge \lambda^*$, (1.2) *has a ground state* u_λ *which is a flat classical radial solution. Furthermore,* $u_{\lambda^*}(x) = \kappa^{\frac{2}{1-\alpha}} u^*\left(\frac{x}{\kappa}\right)$, *where* $\kappa = \sqrt{\frac{1}{\lambda}}$ *and* u^* *is a flat classical radial ground state of* (5.1).

Note that by [27, Lemma 3.3],

$$\lambda^* > \lambda^c = \left(1 + \frac{2(1+\alpha)}{\mathrm{N}(1-\alpha)}\right) \cdot \lambda_1(\Omega).$$

Furthermore, for any $\lambda \in (\lambda_1(\Omega), \lambda^c)$, (1.2) cannot have flat solutions in $C^1(\overline{\Omega})$.

6 Lyapunov Stability of Flat Ground States

In this section, first we prove the statement (2) of Theorem 1.1 and then prove Theorem 1.3(III).

To prove the stability, we will use the Lyapunov function method. Let u_λ be a ground state of (1.2), such that $E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$. For $\delta > 0$, denote

$$U_\delta(u_\lambda) := \{v \in \mathrm{H}_0^1(\Omega) : \ \|u_\lambda - v\| < \delta\}.$$

Observe that $E_\lambda, E''_\lambda : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ are continuous maps. Hence there exists $\delta_0 > 0$, such that $E''_\lambda(u)(u, u) > 0$ for all $u \in U_\delta(u_\lambda)$ if $0 < \delta < \delta_0$.

In the next two lemmas, we show that E_λ is a Lyapunov function in the neighborhood $U_\delta(u_\lambda)$ if $0 < \delta < \delta_0$.

Lemma 6.1 *Assume* (**U**)*. Let* $\lambda > \lambda^*$ *and* u_λ *be a ground state of* (1.2)*, such that* $E''_\lambda(u_\lambda) > 0$*. Then for any* $\delta \in (0, \delta_0)$*, it satisfies*

$$E_\lambda(u) > E_\lambda(u_\lambda) = \widehat{E}_\lambda, \quad \forall u \in U_\delta(u_\lambda) \setminus \{u_\lambda\}. \tag{6.1}$$

Proof Suppose contrary to our claim that for every $\delta \in (0, \delta_0)$ there exists $u^\delta \in U_\delta(u_\lambda) \setminus \{u_\lambda\}$, such that $E_\lambda(u^\delta) \le E_\lambda(u_\lambda)$. This implies that there exists a sequence $u^n \in U_{\delta_0}(u_\lambda)$, such that $u^n \to u_\lambda$ in $\mathrm{H}^1_0(\Omega)$ as $n \to \infty$ and

$$E_\lambda(u^n) \le E_\lambda(u_\lambda), \quad n = 1, 2, \cdots . \tag{6.2}$$

Note that by property (U), we may assume that the point u^n for any $n = 1, 2, \cdots$, is not a ground state of (1.2). Furthermore, $r_{\min}(u_\lambda) = 1$ since $E''_\lambda(u_\lambda) > 0$. Thus by (4.1), we have

$$E_\lambda(r_{\min}(u^n)u^n) > E_\lambda(u_\lambda), \quad n = 1, 2, \cdots .$$

Moreover, this and (6.2) yield that

$$1 < r_{\max}(u^n) < r_{\min}(u^n). \tag{6.3}$$

Note that $r_{\max}(\cdot), r_{\min}(\cdot) : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ are continuous maps. Hence

$$r_{\min}(u^n) \to r_{\min}(u_\lambda) = 1 \quad \text{as } n \to \infty,$$

since $u^n \to u_\lambda$ in $\mathrm{H}^1_0(\Omega)$ as $n \to \infty$. Then by (6.3), we have also

$$r_{\max}(u^n) \to r_{\min}(u_\lambda) = 1 \quad \text{as } n \to \infty.$$

From this and since $E''_\lambda(r_{\max}(u^n)u^n) \le 0$ and $E''_\lambda(r_{\min}(u^n)u^n) \ge 0$, we conclude that

$$E''_\lambda(u_\lambda) = 0.$$

But this is impossible by the assumption. This contradiction completes the proof.

Lemma 6.2 *Let* $v(t)$*,* $t \in [0, T)$ *be a weak solution of* (1.1)*. Then*

$$\frac{\partial}{\partial t} E_\lambda(v(t)) \le 0 \quad \textit{in } (0, T). \tag{6.4}$$

Proof By the additional regularity obtained in Section 2, there exists $\frac{\partial}{\partial t} E_\lambda(v(t))$ in $(0, T)$ and

$$\frac{\partial}{\partial t} E_\lambda(v(t)) = D_u E_\lambda(v(t))(v_t(t)) = \langle -\Delta v(t) - \lambda |v|^{\beta-1} v + |v|^{\alpha-1} v, v_t(t) \rangle = -\|v_t(t)\|^2_{L^2} \le 0.$$

Thus we get the result.

The proof of Theorem 1.1(2) will follow from the following lemma.

Lemma 6.3 *Assume* (U)*. Let* $\lambda > \lambda^*$ *and* u_λ *be a ground state of* (1.2) *such that* $E''_\lambda(u_\lambda) > 0$*. Then for any given* $\varepsilon > 0$*, there exists* $\delta \in (0, \delta_0)$ *such that*

$$\|u_\lambda - v(t; w_0)\|_1 < \varepsilon \quad \textit{for any } w_0 \ge 0 \quad \textit{such that } \|u_\lambda - w_0\|_1 < \delta, \quad \forall t > 0. \tag{6.5}$$

Proof Without loss of generality, we may assume that $\varepsilon \in (0, \delta_0)$. Consider

$$d_\varepsilon := \inf\{E_\lambda(w) : w \in \mathrm{H}_0^1(\Omega), \|u_\lambda - w\|_1 = \varepsilon\}. \tag{6.6}$$

Then $d_\varepsilon > \widehat{E}_\lambda$. Indeed, assume the opposite, that there is a sequence $w^n \in K$, $\|u_\lambda - w^n\|_1 = \varepsilon$ and $E_\lambda(w^n) \to \widehat{E}_\lambda$. Hence, (w^n) is bounded in $\mathrm{H}_0^1(\Omega)$, and therefore by the embedding theorem, there exists a subsequence (again denoted by (w^n)), such that $w^n \to w_0$ weakly in $\mathrm{H}_0^1(\Omega)$ and strongly in L_p, $1 < p < 2^*$ for some $w_0 \in \mathrm{H}_0^1(\Omega)$. Since $\|u\|_1^2$ is a weakly lower semi-continuous functional on $\mathrm{H}_0^1(\Omega)$, one has $\widehat{E}_\lambda \geq E_\lambda(w_0)$ and $\|u_\lambda - w_0\|_1 \leq \varepsilon$. By Lemma 6.1, this is possible only if w_0 is a ground state of (1.2), i.e., a minimizer of (4.1). But then $\widehat{E}_\lambda = E_\lambda(w_0)$ implies that $w^n \to w_0$ strongly in $\mathrm{H}_0^1(\Omega)$. From here we have $\varepsilon = \|u_\lambda - w^n\|_1 \to \|u_\lambda - w_0\|_1$. Thus $w_0 \in U_{\delta_0}(u_\lambda)$ and $u_\lambda \neq w_0$. Since by property (U), u_λ is the unique non-negative solution of (1.2) in $U_{\delta_0}(u_\lambda)$, we get a contradiction.

Let $\sigma > 0$ be an arbitrary value such that $d_\varepsilon - \sigma > \widehat{E}_\lambda$. Then by continuity of $E_\lambda(w)$, one can find $\delta \in (0, \varepsilon)$, such that

$$E_\lambda(w) < d_\varepsilon - \sigma, \quad \forall w \in U_\delta(u_\lambda) \subset U_\varepsilon(u_\lambda). \tag{6.7}$$

We claim that for any $w_0 \in U_\delta(u_\lambda)$, the solution $v(t, w_0)$ belongs to $U_\varepsilon(u_\lambda)$ for all $t > 0$. Indeed, suppose the opposite, since $v(t, w_0) \in \mathcal{C}((0,T), \mathrm{H}_0^1(\Omega))$, there exists $t_0 > 0$ such that $\|u_\lambda - v(t_0, w_0)\|_1 = \varepsilon$. This implies that

$$d_\varepsilon \leq E_\lambda(v(t_0, w_0)).$$

On the other hand, by Lemma 6.3, we have $E_\lambda(v(t_0, w_0)) \leq E_\lambda(w_0)$. Thus by (6.7), one gets

$$d_\varepsilon \leq E_\lambda(v(t_0, w_0)) \leq E_\lambda(w_0) < d_\varepsilon - \sigma.$$

This contradiction proves the claim.

Proof of Theorem 1.3(III) Assume that $N \geq 3$, $(\alpha, \beta) \in \mathcal{E}_s(N)$ and Ω is a strictly star-shaped domain with respect to the origin. By [42, Corollary 15], it follows that there exists $\lambda^* > 0$ such that (1.2) has a flat ground state u_{λ^*} which $u_{\lambda^*} \geq 0$ and $u_{\lambda^*} \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $\gamma \in (0,1)$. Now applying Theorem 1.1(2), we conclude that u_{λ^*} is a stable non-negative stationary solution of the parabolic problem (1.1).

Remark 6.1 Related linearized stability results were obtained in [5] working in Sobolev spaces in the framework of degenerate parabolic equations of porous media type.

7 Linearized Unstability

In this section, we prove statements (I)–(II) of Theorem 1.3.

Lemma 7.1 *Let u_λ be a non-negative weak solution of* (1.2) *such that $E''(u_\lambda) < 0$. Then u_λ is unstable stationary solution of* (1.1) *in the sense that $\lambda_1(-\Delta - \lambda\beta u_\lambda^{\beta-1} + \alpha u_\lambda^{\alpha-1}) < 0$.*

Proof Let u_λ be a non-negative weak solution of $\mathrm{SP}(\alpha, \beta, \lambda)$. Then the corresponding linearized problem at u_λ is

$$\begin{cases} -\Delta\psi - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1})\psi = \mu\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

Then there is a first eigenvalue μ_1 to (7.1) with a positive eigenfunction $\psi_1 > 0$ such that $\psi_1 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0^1(\overline{\Omega})$. The existence of μ_1 is a particular case of the results in [28], using the estimates on the boundary behavior of u_λ obtained in [23–24], namely that

$$\underline{K} d(x)^{\frac{2}{1-\alpha}} \leq u_\lambda(x) \leq \overline{K} d(x)^{\frac{2}{1-\alpha}} \quad \text{for any } x \in \overline{\Omega} \tag{7.2}$$

for some constants $\overline{K} > \underline{K} > 0$. We shall sketch the argument for the reader's convenience. From this estimates, it follows that, roughly speaking, $u_\lambda(x)^{\alpha-1}$ "behaves like" $d(x)^{-2}$ and $u_\lambda(x)^{\beta-1}$ as $d(x)^{\frac{-2(1-\beta)}{1-\alpha}}$ with $\gamma := \frac{2(1-\beta)}{1-\alpha} < 2$ from $\alpha < \beta$. Then from the used monotonicity properties of eigenvalues, it is enough to show that a first eigenvalue of the problem

$$\begin{cases} -\Delta w + \dfrac{\alpha}{d(x)^2} w - \dfrac{\lambda\beta}{d(x)^\gamma} w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{7.3}$$

is well-defined and has the usual properties. This is carried by reducing the problem to an equivalent "fixed point" argument for an associated (linear) eigenvalue problem. Assume first that $\mu > 0$. Then (7.3) is equivalent to the existence of μ such that $r(\mu) = 1$, where $r(\mu)$ is the first eigenvalue for the associated problem

$$\begin{cases} -\Delta w + \dfrac{\alpha}{d(x)^2} w = r\Big(\dfrac{\lambda\beta}{d(x)^\gamma} w + \mu w\Big) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.4}$$

That $r(\mu) > 0$ is well-defined follows by showing that (7.4) is equivalently formulated as $Tw = rw$ with $T = i \circ P \circ F$, where $F : L^2(\Omega, d^\gamma) \to \mathrm{H}^{-1}(\Omega)$ defined by

$$F(w) = \frac{\lambda\beta}{d(x)^\gamma} w + \mu w,$$

$P : \mathrm{H}^{-1}(\Omega) \to \mathrm{H}_0^1(\Omega)$ is the solution operator for the linear problem

$$\begin{cases} -\Delta z + \dfrac{\alpha}{d(x)^2} z = h(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{7.5}$$

for $h \in \mathrm{H}^{-1}(\Omega)$, and $i : \mathrm{H}_0^1(\Omega) \to L^2(\Omega, d^\gamma)$ is the standard embedding. It is possible to prove that F and P are continuous, and i is compact by using Hardy's inequality and the Lax-Milgram lemma (see [5, 28]). Since T is an irreductible compact linear operator, by applying the weak maximum principle, it is possible to apply Krein-Rutman's theorem in the formulation in [18]. We have the variational formulation

$$r(\mu) = \inf_{w \in \mathrm{H}_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \Big(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2\Big)\,\mathrm{d}x}{\lambda\beta \displaystyle\int_\Omega \frac{w^2}{d(x)^\gamma}\,\mathrm{d}x + \mu \int_\Omega w^2 \mathrm{d}x}. \tag{7.6}$$

Hence a positive eigenvalue exists if and only if there is a $\mu > 0$ such that $r(\mu) = 1$. A completely analogous argument gives the formulation for $\mu < 0$, namely with

$$r_1(\mu) = \inf_{w \in \mathrm{H}_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \Big(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2 + \mu w^2\Big)\,\mathrm{d}x}{\lambda\beta \displaystyle\int_\Omega \frac{w^2}{d(x)^\gamma}\,\mathrm{d}x}. \tag{7.7}$$

Notice that $r(\mu)$ (resp. $r_1(\mu)$) is decreasing (resp. increasing) in μ. Then

$$r(0) = r_1(0) = \inf_{w\in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Big(|\nabla w|^2 + \frac{\alpha}{d(x)^2} w^2\Big)\,\mathrm{d}x}{\displaystyle\lambda\beta\int_\Omega \frac{w^2}{d(x)^\gamma}\,\mathrm{d}x},$$

and there exists a positive eigenvalue if $r(0) > 1$ and a negative one if $r(0) < 1$.

Coming back to our instability analysis, by Courant minimax principle, we have

$$\mu_1 = \inf_{\psi\in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla\psi|^2 - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1})\psi^2)\,\mathrm{d}x}{\displaystyle\int_\Omega |\psi|^2\,\mathrm{d}x}. \tag{7.8}$$

Let us put $\psi = u_\lambda$ in the minimizing functional of (7.8). Then we get

$$\frac{\displaystyle\int_\Omega (|\nabla u_\lambda|^2 - (\lambda\beta u_\lambda^{\beta-1} - \alpha u_\lambda^{\alpha-1})u_\lambda^2)\,\mathrm{d}x}{\displaystyle\int_\Omega |u_\lambda|^2\,\mathrm{d}x} = \frac{E''_\lambda(u_\lambda)}{\displaystyle\int_\Omega |u_\lambda|^2\,\mathrm{d}x} < 0,$$

by the assumption $E''(u_\lambda) < 0$. This yields by the definition (7.8) that $\lambda_1(-\Delta - \lambda\beta u_\lambda^{\beta-1} + \alpha u_\lambda^{\alpha-1}) := \mu_1 < 0$. Thus we get unstability.

Proof of Theorem 1.3(I)–(II)

(I) Assume $N = 1, 2$ and $(\alpha,\beta) \in \mathcal{E}$. Let u_λ be a free boundary solution of (1.2). Since $\mathcal{E} = \mathcal{E}_u(N)$, Lemma 1.1(2) implies that $E''_\lambda(u_\lambda) < 0$. However, this yields by Lemma 7.1 that u_λ is a linearized unstable stationary solution of the parabolic problem (1.1).

(II) Assume $N \geq 3$ and $(\alpha,\beta) \in \mathcal{E}_u(N)$. Let u_λ be a free boundary solution of (1.2). Then by Lemma 1.1(2), we have $E''_\lambda(u_\lambda) < 0$. This yields as above by Lemma 7.1 that u_λ is a linearized unstable stationary solution of the parabolic problem (1.1).

8 Globally Unstable Ground State of (1.1) in Case $\beta = 1$

In this section, we prove Theorem 1.4(2).

Let us introduce the so-called exterior potential well (see [48])

$$\mathcal{W} := \{u \in \mathrm{H}_0^1(\Omega) : E_\lambda(u) < \widehat{E}_\lambda,\ E'_\lambda(u) < 0\}. \tag{8.1}$$

The proof of the theorem will be obtained from the following lemma.

Lemma 8.1 *If $v_0 \in \mathcal{W}$, then $\|v(t, v_0)\|_{L^2(\Omega)} \to \infty$ as $t \to +\infty$.*

Proof First we show that $\mathcal{W}$ is invariant under the flow (1.1). Let $v(t, v_0)$ be a weak solution of (1.1). Then using the additional regularity obtained in Section 2, we have

$$E_\lambda(v(t)) \leq \int_0^t \|v_t\|_{L^2}^2 \mathrm{d}s + E_\lambda(v(t)) \leq E_\lambda(v_0) < \widehat{E}_\lambda$$

for all $t > 0$. Thus $v(t)$ may leave $\mathcal{W}$ only if there is a time $t_0 > 0$ such that $r_\lambda(v(t_0)) = 1$ (since, formally, $E'_\lambda(v(t_0)) = 0$). But then, by (3.12), we have

$$E_\lambda(v(t_0)) = \max_{r>0} E_\lambda(rv(t_0)) \geq \widehat{E}_\lambda.$$

Thus we get a contradiction and indeed

$$E_\lambda(v(t,v_0)) < \widehat{E}_\lambda, \quad E'_\lambda(v(t,v_0)) < 0, \quad \forall t > 0 \tag{8.2}$$

for any $v_0 \in \mathcal{W}$.

Furthermore, we have the following proposition.

Proposition 8.1 *Assume* $v \in \mathrm{L}^\infty(0,+\infty : \mathrm{H}^1_0(\Omega))$. *Then there exists* $c_0 < 0$, *which does not depend on* $t > 0$, *such that*

$$E'_\lambda(v(t)) \le c_0 < 0 \quad \textit{for a.e. } t > 0. \tag{8.3}$$

Proof By regularizing v_0, we can assume that $E'_\lambda(v(t))$ is continuous in t. Suppose, contrary to our claim, that there is (t_m), such that the sequence $v_m := v(t_m)$ $(m = 1, 2, \cdots)$ satisfies

$$E'_\lambda(v_m) \to 0 \quad \text{as } m \to \infty. \tag{8.4}$$

Note that by (8.2) we have

$$E_\lambda(v_m) < \widehat{E}_\lambda \quad \text{for } m = 1, 2, \cdots . \tag{8.5}$$

By assumption (v_m) is bounded in $\mathrm{H}^1_0(\Omega)$. Therefore, we have that there are the following convergences (up choosing a subsequence):

$$v_m \to \overline{v} \quad \text{as } m \to \infty \quad \text{in } \mathrm{L}^p,\ 1 < p < 2^*, \tag{8.6}$$

$$v_m \rightharpoonup \overline{v} \quad \text{as } m \to \infty \quad \text{weakly in } \mathrm{H}^1_0(\Omega), \tag{8.7}$$

$$\lim_{m\to\infty} E_\lambda(v_m) = a \tag{8.8}$$

for some $\overline{v} \in \mathrm{H}^1_0(\Omega)$ and $a \in \mathbb{R}$. Hence by the weakly lower semi-continuity of $T(u)$ in $\mathrm{H}^1_0(\Omega)$, we have

$$E_\lambda(\overline{v}) \le \lim_{m\to\infty} E_\lambda(v_m) = a, \tag{8.9}$$

$$E'_\lambda(\overline{v}) \le \lim_{m\to\infty} E'_\lambda(v_m) = 0. \tag{8.10}$$

Since $v \in \mathcal{C}([0,T] : \mathrm{H}^1_0(\Omega))$, by Proposition 2.1 we have

$$\int_0^t \|v_t\|^2_{L^2} \mathrm{d}s + E_\lambda(v(t)) \le E_\lambda(v(0)). \tag{8.11}$$

Hence,

$$a = \lim_{m\to\infty} E_\lambda(v_m) \le E_\lambda(v_0) < \widehat{E}_\lambda$$

for any $v_0 \in \mathcal{W}$, and therefore $E_\lambda(\overline{v}) < \widehat{E}_\lambda$. Observe that this implies a contradiction in case that the equality holds in (8.10). Indeed, if $E'_\lambda(\overline{v}) = 0$, then $r(\overline{v}) = 1$, and therefore (3.11) and (3.13)–(3.14) yield $E_\lambda(\overline{v}) \ge \widehat{E}_\lambda$.

Suppose that $E'_\lambda(\overline{v}) < 0$. Then there is $r \in (0,1)$ such that $E'_\lambda(r\overline{v}) = 0$. Observe that (8.6) and (8.8) imply

$$\frac{1}{2} \lim_{m\to\infty} \mathrm{H}_\lambda(v_m) = a - \frac{1}{1+\alpha} A(\overline{v}), \tag{8.12}$$

and (8.4) implies

$$\lim_{m\to\infty} \mathrm{H}_\lambda(v_m) = -A(\overline{v}). \tag{8.13}$$

From here, we obtain

$$\begin{aligned} E_\lambda(r\overline{v}) &= \frac{r^2}{2}\mathrm{H}_\lambda(\overline{v}) + \frac{r^{1+\alpha}}{1+\alpha}A(\overline{v}) \\ &\le \frac{r^2}{2}\lim_{m\to\infty}\mathrm{H}_\lambda(v_m) + \frac{r^{1+\alpha}}{1+\alpha}A(\overline{v}) \\ &= \frac{1}{2}\lim_{m\to\infty}\mathrm{H}_\lambda(v_m) + \frac{1}{2}(r^2-1)\lim_{m\to\infty}\mathrm{H}_\lambda(v_m) + \frac{r^{1+\alpha}}{1+\alpha}A(\overline{v}) \\ &= a - \frac{1}{1+\alpha}A(\overline{v}) - \frac{1}{2}(r^2-1)A(\overline{v}) + \frac{r^{1+\alpha}}{1+\alpha}A(\overline{v}) \\ &= a + \Big[-\frac{1}{1+\alpha} - \frac{1}{2}(r^2-1) + \frac{r^{1+\alpha}}{1+\alpha}\Big]A(\overline{v}). \end{aligned}$$

It is easy to see that

$$\max_{1\le r\le 1}\Big\{\Big[-\frac{1}{1+\alpha} - \frac{1}{2}(r^2-1) + \frac{r^{1+\alpha}}{1+\alpha}\Big]\Big\} = 0.$$

Thus we get $E_\lambda(r\overline{v}) \le a < \widehat{E}_\lambda$. However, this contradicts the definition of $\widehat{E}_\lambda$, since $E'_\lambda(r\overline{v}) = 0$. This completes the proof of the proposition.

Let us now conclude the proof of the lemma. Suppose, contrary to our claim, that the set $(v(t))$, $t > 0$ is bounded in $\mathrm{L}^2(\Omega)$. Then this set is also bounded in $\mathrm{H}^1_0(\Omega)$, since $\mathrm{H}_\lambda(v(t)) := T(v(t)) - \lambda G(v(t)) < 0$ for all $t > 0$.

Let us consider

$$y(t) := \|v(t)\|^2_{\mathrm{L}^2}, \quad t \ge 0,$$

where $v(t) := v(t, v_0)$. Observe that

$$\|v(t)\|^2_{\mathrm{L}^2} = \|v_0\|^2_{\mathrm{L}^2} + 2\int_0^t (v_t(s), v(s))\,\mathrm{d}s,$$

and by (1.1),

$$(v_t(s), v(s)) = (\Delta v(s) + \lambda v(s) - |v(s)|^{\alpha-1}v(s), v(s)) = -E'_\lambda(v(s)).$$

Therefore,

$$y(t) = \|v_0\|^2_{\mathrm{L}^2} - 2\int_0^t E'_\lambda(v(s))\mathrm{d}s \tag{8.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \equiv \dot{y}(t) = -2E'_\lambda(v(t)).$$

Hence, estimate (8.3) of Proposition 8.1 yields $\dot{y}(t) > -2c_0 > 0$ for all $t > 0$, and therefore $y(t) = \|v(t)\|^2_{\mathrm{L}^2} \to +\infty$ as $t \to \infty$. This completes the proof of Lemma 8.1.

Conclusion of the proof of Theorem 1.4(2) Let u_λ be a ground state of (1.1) and give any $\delta > 0$. Observe that for any $r > 1$,

$$E_\lambda(ru_\lambda) < \widehat{E}_\lambda, \quad E'_\lambda(ru_\lambda) < 0.$$

Thus $ru_\lambda \in \mathcal{W}$ for any $r > 1$, and by Lemma 8.1, $\|v(t;v_0)\|_{L^2} \to +\infty$ with $v_0 = ru_\lambda$. Therefore,

$$\|u_\lambda - v(t;v_0)\|_{\mathrm{L}^2} \to +\infty \quad \text{as } t \to \infty.$$

On the other hand, evidently $\|u_\lambda - ru_\lambda\|_{\mathrm{L}^2} < \delta$ for sufficiently small $|r-1|$. This concludes the proof of Theorem 1.4.

9 Appendix. Existence of a Ground State Solution of (5.1)

In this section, we prove Lemma 5.1.

Consider

$$\widehat{E}^\infty = \min\{J(v): \ v \in \mathrm{H}^1_0(\Omega) \setminus \{0\}, \ \mathrm{H}(v) < 0\}. \tag{9.1}$$

Lemma 9.1 *There exists a minimizer v of* (9.1).

Proof Let (v_m) be a minimizing sequence of (9.1). Since $J(u)$ is a zero-homogeneous functional, we may assume that $\|v_m\|_1 = 1$, $m = 1, 2, \cdots$. This implies that

$$|\mathrm{H}(v_m)| < C < \infty \quad \text{uniformly on } m = 1, 2, \cdots. \tag{9.2}$$

Observe that

$$\|v_m\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \equiv \int |v_m|^2 \mathrm{d}x > c_1 > 0 \tag{9.3}$$

uniformly on $m = 1, 2, \cdots$. Indeed, if we suppose the contrary $\int |v_m|^2 \mathrm{d}x \to 0$ as $m \to \infty$, then the assumption $\|v_m\|_1 = 1$ $(m = 1, 2, \cdots)$ implies that $\int |\nabla v_m|^2\, \mathrm{d}x \to 1$, and therefore $\mathrm{H}(v_m) = \int |\nabla v_m|^2\, \mathrm{d}x - \int |v_m|^2\, \mathrm{d}x \to 1$ as $m \to \infty$. But this is impossible, since by the construction $\mathrm{H}(v_m) < 0$.

Let us show that

$$A(v_m) > c_0 > 0 \quad \text{uniformly on } m = 1, 2, \cdots. \tag{9.4}$$

Assume the opposite, that $A(v_m) \to 0$ as $m \to \infty$. Then $\int |v_m|^2 \mathrm{d}x \to 0$ as $m \to \infty$, since by Hölder and Sobolev inequalities

$$\int |v_m|^2 \mathrm{d}x \le \Big(\int |v_m|^{\alpha+1} \mathrm{d}x \Big)^{\frac{\kappa}{\alpha+1}} \Big(\int |v_m|^{2^*} \mathrm{d}x \Big)^{\frac{\alpha+1-\kappa}{\alpha+1}} \le C_0 A(v_m)^{\frac{\kappa}{\alpha+1}} \|v_m\|_1^{2^* \frac{\alpha+1-\kappa}{\alpha+1}},$$

where $\kappa = \dfrac{(\alpha+1)(2^*-2)}{2^* - \alpha + 1}$. But this contradicts (9.3).

Observe that (5.3), (9.2) and (9.4) yield

$$\widehat{E}^\infty > 0, \tag{9.5}$$

and we have

$$0 < c_0 < \|v_m\|^{1+\alpha}_{\mathrm{L}^{1+\alpha}} \equiv A(v_m) < C_1 < +\infty \tag{9.6}$$

uniformly on $m = 1, 2, \cdots$.

We need the following lemma (see [34, Lemma I.1, p. 231]).

Lemma 9.2 *Let $1 \le q < +\infty$ with $q \le 2^*$ if $\mathrm{N} \ge 3$. Assume that (w_n) is bounded in $\mathrm{H}_0^1(\mathbb{R}^N)$ and $\mathrm{L}^q(\mathbb{R}^N)$, and*

$$\sup_{y\in\mathbb{R}^N}\int_{y+B_R}|w_n|^q\mathrm{d}x \to 0 \quad \text{as } n\to\infty, \text{ for some } R>0.$$

Then $\|w_n\|_{\mathrm{L}^\beta} \to 0$ for $\beta \in (q, 2^*)$.

Let $R > 0$. Observe that

$$\liminf_{m\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}|v_m|^{1+\alpha}\mathrm{d}x := \delta > 0. \tag{9.7}$$

Indeed, let us assume that

$$\liminf_{m\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}|v_m|^{1+\alpha}\mathrm{d}x = 0.$$

Then by Lemma 9.2, we have $\|v_m\|_{\mathrm{L}^2} \to 0$ as $m \to \infty$. But this contradicts (9.3).

Thus there is a sequence $\{y_m\} \subset \mathbb{R}^N$ such that

$$\int_{y_m+B_R}|v_m|^{1+\alpha}\mathrm{d}x > \frac{\delta}{2}, \quad m = 1, 2, \cdots.$$

Introduce $u_m := v_m(\cdot + y_m)$, $m = 1, 2, \cdots$. Then

$$\int_{B_R}|u_m|^{1+\alpha}\mathrm{d}x > \frac{\delta}{2}, \quad m = 1, 2, \cdots, \tag{9.8}$$

and $\{u_m\}$ is a minimizing sequence of (9.1).

Furthermore, by the zero-homogeneity of $J(u)$, now we may normalize the sequence $\{u_m\}$ (again denoted by $\{u_m\}$), such that

$$A(u_m) = 1, \quad m = 1, 2, \cdots. \tag{9.9}$$

Then (9.6) implies that the renormalized sequence $\{u_m\}$ will be again bounded in $\mathrm{H}^1(\mathbb{R}^N)$. Thus by Eberlein-Smulian theorem there is a subsequence of $\{u_m\}$ (again denoting $\{u_m\}$) and a limit point $\overline{u} \in \mathrm{H}_0^1(\Omega)$, such that

$$u_m \rightharpoonup \overline{u} \quad \text{weakly in } \ \mathrm{H}_0^1(\Omega), \text{ as } m \to \infty. \tag{9.10}$$

Furthermore,

$$u_m \to \overline{u} \quad \text{ a.e. on } \ \mathbb{R}^{\mathrm{N}}, \text{ as } m \to \infty, \tag{9.11}$$

and for $2 < q < 2^*$,

$$u_m \to \overline{u} \quad \text{in } \mathrm{L}^q_{\mathrm{loc}} \quad \text{as } m \to \infty, \tag{9.12}$$

since by Rellich-Kondrachov theorem, $\mathrm{H}_0^1(B_R)$ is compactly embedded in $\mathrm{L}^q(B_R)$ for $2 < q < 2^*$ and any $B_R := \{x \in \mathbb{R}^N : |x| \le R\}$, $R > 0$. Note that (9.8) implies that

$$\overline{u} \ne 0.$$

We need the Brezis-Lieb lemma (see [10]).

Lemma 9.3 *Let Ω be an open subset of $\mathbb{R}^N$ and let $\{w_n\} \subset \mathrm{L}^q(\Omega)$, $1 \leq q < \infty$. If*
(a) *$\{w_n\}$ bounded in $\mathrm{L}^q(\Omega)$,*
(b) *$w_n \to w$ a.e. on Ω,*
then

$$\lim_{n\to\infty} \left(\|w_n\|^q_{\mathrm{L}^q} - \|w_n - w\|^q_{\mathrm{L}^q}\right) = \|w\|^q_{\mathrm{L}^q}.$$

Let us denote $\omega_m := u_m - \overline{u}$. Then the Brezis-Lieb lemma yields

$$1 = A(\overline{u}) + \lim_{m\to\infty} A(\omega_m). \tag{9.13}$$

Observe

$$\mathrm{H}(\omega_m) = \mathrm{H}(\overline{u}) + \mathrm{H}(u_m) - \mathrm{H}'(u_m)(\overline{u}). \tag{9.14}$$

Note that due to the weak convergence (9.10), we have $H'(\omega_m)(\overline{u}) \to 0$ as $m \to \infty$. Therefore, $H(\omega_m) < 0$ for sufficiently large m, since $H(u) < 0$ and $H(u_m) < 0$ for $m = 1, 2, \cdots$. On the other hand,

$$\mathrm{H}(u_m) = \mathrm{H}(\overline{u}) + \mathrm{H}(\omega_m) + \mathrm{H}'(\omega_m)(\overline{u}),$$

and therefore

$$\lim_{m\to\infty} \mathrm{H}(u_m) = \mathrm{H}(\overline{u}) + \lim_{m\to\infty} \mathrm{H}(\omega_m). \tag{9.15}$$

Observe that (9.1) implies that for any $v \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$ such that $\mathrm{H}(v) < 0$, it holds

$$-\mathrm{H}(v) \leq k_\alpha \frac{A(v)^{\frac{2}{1+\alpha}}}{\widehat{E}^\infty}, \tag{9.16}$$

where

$$k_\alpha = \left(\frac{(1-\alpha)}{2(1+\alpha)}\right)^{\frac{1-\alpha}{1+\alpha}}.$$

Hence

$$-\mathrm{H}(\overline{u}) \leq k_\alpha \frac{A(\overline{u})^{\frac{2}{1+\alpha}}}{\widehat{E}^\infty}$$

and

$$-\mathrm{H}(\omega_m) \leq k_\alpha \frac{A(\omega_m)^{\frac{2}{1+\alpha}}}{\widehat{E}^\infty} \tag{9.17}$$

for sufficiently large m. Since $A(u_m) = 1$, we have

$$\lim_{m\to\infty} k_\alpha \frac{1}{(-\mathrm{H}(u_m))} = \widehat{E}^\infty.$$

Hence, we have

$$\begin{aligned} k_\alpha \frac{1}{\widehat{E}^\infty} &= \lim_{m\to\infty} (-\mathrm{H}(u_m)) \\ &= -\mathrm{H}(\overline{u}) + \lim_{m\to\infty} (-\mathrm{H}(\omega_m)) \\ &\leq k_\alpha \frac{A(\overline{u})^{\frac{2}{1+\alpha}}}{\widehat{E}^\infty} + \lim_{m\to\infty} k_\alpha \frac{A(\omega_m)^{\frac{2}{1+\alpha}}}{\widehat{E}^\infty} \\ &= k_\alpha \frac{1}{\widehat{E}^\infty} (A(\overline{u})^{\frac{2}{1+\alpha}} + (1 - A(\overline{u}))^{\frac{2}{1+\alpha}}). \end{aligned}$$

Note since $\frac{2}{1+\alpha} > 1$, we have that $f(r) := r^{\frac{2}{1+\alpha}} + (1-r)^{\frac{2}{1+\alpha}} \geq 1$ for $r \in [0,1]$ and that $f(r) = 1$ if and only if $r = 0$ or $r = 1$. Thus we have

$$A(\overline{u}) = 1 \quad \text{or} \quad A(\overline{u}) = 0.$$

Now taking into account that $\overline{u} \neq 0$, we get that $A(\overline{u}) = 1$. Hence by (9.13), we obtain $A(\omega_m) \to 0$ as $m \to \infty$, and consequently by (9.17), we have $(-\mathrm{H}(\omega_m)) \to 0$ as $m \to \infty$. From here, it is not hard to conclude that $u_m \to \overline{u}$ strongly in $\mathrm{H}^1(\mathbb{R}^{\mathrm{N}})$, and therefore $J(\overline{u}) = \widehat{E}^{\infty}$. Thus $\overline{u}$ is a minimizer of (9.1).

Proof of Lemma 5.1 By Lemma 9.1, there exists a minimizer $\overline{u}$ of (9.1). Since J is an even functional then $|\overline{u}|$ is also a minimizer of (9.1). Thus we may assume that $\overline{u}$ is non-negative function. By Proposition 3.4, it follows that $u = r(\overline{u})\overline{u}$ is a weak solution of (5.1) which is non-negative since $r(\overline{u}) > 0$. By regularity theory, we derive that $u \in \mathcal{C}^2(\mathbb{R}^N)$.

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