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New Identities for Weak KAM Theory*

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(*For Haim Brezis, in continuing admiration*)

Abstract This paper records for the Hamiltonian $H = \frac{1}{2}|p|^2 + W(x)$ some old and new identities relevant for the PDE/variational approach to weak KAM theory.

Keywords Weak KAM theory, Effective Hamiltonian, Hamiltonian dynamics

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1 Introduction

1.1 Weak KAM for a model Hamiltonian

This is a follow-up to two of my earlier papers [2–3] that propose a PDE/variational approach to weak KAM theory, originating with Mather and Fathi (see [5–6, 10–11], etc.). In this paper, we specialize to the classical Hamiltonian

$$H(p,x) = \frac{1}{2}|p|^2 + W(x), \tag{1.1}$$

where the potential W is smooth and $\mathbb{T}^n$-periodic, where $\mathbb{T}^n = [0,1]^n$ denotes the unit cube with opposite faces identified. Given a vector $P = (P_1, \cdots, P_n) \in \mathbb{R}^n$, the corresponding cell PDE reads

$$\frac{1}{2}|P + Dv|^2 + W = \overline{H}(P) \quad \text{in } \mathbb{T}^n, \tag{1.2}$$

where $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is the effective Hamiltonian corresponding to H, as introduced in the important, but unpublished, paper of Lions-Papanicolaou-Varadhan [9]. Here $v = v(P, x)$ denotes a $\mathbb{T}^n$-periodic viscosity solution. As shown for instance in [2], there exists also a Radon probability measure σ on $\mathbb{T}^n$ solving the transport PDE

$$\operatorname{div}(\sigma(P + Dv)) = 0 \quad \text{in } \mathbb{T}^n \tag{1.3}$$

in an appropriate weak sense.

A central goal of weak KAM theory is developing a nonperturbative methods to identify "integrable structures" within the otherwise possibly chaotic dynamics generated by a given

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Hamiltonian $H = H(p, x)$, and in particular to understand if and how the effective Hamiltonian $\overline{H}$ encodes such information. The PDE approach to weak KAM aims at extracting such information from the two coupled PDE (1.2)–(1.3).

This paper extends previous work by discovering for the particular case of the Hamiltonian (1.1) several new integral identities, especially for the variational approximations introduced below. We also record how some previously derived general formulas simplify in this case, and provide in Section 4 some applications.

1.2 Variational approximation

We consider for fixed $\varepsilon > 0$ the problem of minimizing the functional

$$I_\varepsilon[v] := \int_{\mathbb{T}^n} \mathrm{e}^{\frac{H(P+Dv,x)}{\varepsilon}}\,\mathrm{d}x,$$

among Lipschitz continuous functions $v : \mathbb{T}^n \to \mathbb{R}$ with mean zero: $\int_{\mathbb{T}^n} v\,\mathrm{d}x = 0$. We write $D = D_x$ to denote the gradient in the variables x. PDE and calculus of variations theory (see [2]) implies that this problem has a unique, smooth minimizer $v^\varepsilon = v^\varepsilon(P, x)$, which satisfies therefore the Euler-Lagrange PDE

$$\operatorname{div}(\mathrm{e}^{\frac{H(P+Dv^\varepsilon,x)}{\varepsilon}}(P + Dv^\varepsilon)) = 0 \quad \text{in } \mathbb{T}^n. \tag{1.4}$$

Standard regularity theory shows that v^ε is a smooth function of x and also of the parameters ε and $P = (P_1, \cdots, P_n)$.

It is convenient to change notation, writing

$$\begin{cases} H^\varepsilon := H(P + Dv^\varepsilon, x) = \dfrac{1}{2}|P + Dv^\varepsilon|^2 + W, \\ \overline{H}^\varepsilon(P) := \varepsilon \log\Big(\displaystyle\int_{\mathbb{T}^n} \mathrm{e}^{\frac{H^\varepsilon}{\varepsilon}}\,\mathrm{d}x\Big), \\ \sigma^\varepsilon := \mathrm{e}^{\frac{H^\varepsilon - \overline{H}^\varepsilon(P)}{\varepsilon}}. \end{cases} \tag{1.5}$$

Theorem 1.1 (i) *We have $\sigma^\varepsilon \geq 0$,*

$$\int_{\mathbb{T}^n} \sigma^\varepsilon\,\mathrm{d}x = 1, \tag{1.6}$$

$$\operatorname{div}(\sigma^\varepsilon(P + Dv^\varepsilon)) = 0 \tag{1.7}$$

and

$$(P + Dv^\varepsilon) \cdot DH^\varepsilon + \varepsilon\Delta v^\varepsilon = (P_i + v^\varepsilon_{x_i})(P_j + v^\varepsilon_{x_j})v^\varepsilon_{x_i x_j} + (P_i + v^\varepsilon_{x_i})W_{x_i} + \varepsilon\Delta v^\varepsilon = 0. \tag{1.8}$$

(ii) *Furthermore,*

$$\overline{H}^\varepsilon(P) \leq \overline{H}(P), \quad P \in \mathbb{R}^n \tag{1.9}$$

for each $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0} \overline{H}^\varepsilon(P) = \overline{H}(P), \quad P \in \mathbb{R}^n. \tag{1.10}$$

As above, $D = D_x$ means the first derivatives in x, and $D^2 = D_x^2$ the second derivatives in x. Likewise, $\Delta = \Delta_x$ means the Laplacian in the x-variables.

Proof The term $\overline{H}^\varepsilon(P)$ is introduced to achieve the normalization (1.6). The PDEs (1.7) and (1.8) are the Euler-Lagrange equation (1.4) rewritten in respective divergence and non-divergence forms respectively.

The assertion (1.9) follows upon our using a solution of (1.2) in the variational principle, and the limit (1.10) is demonstrated in [2].

Remark 1.1 As shown in [2], we have the uniform estimates

$$\max_{\mathbb{T}^n}\{|D_x v^\varepsilon|, |v^\varepsilon|\} \le C$$

for a constant C independent of ε. Hence we can extract a subsequence, such that

$$v^{\varepsilon_k} \to v \text{ uniformly on } \mathbb{T}^n, \quad \sigma^{\varepsilon_k} \rightharpoonup \sigma \text{ weakly as measures.}$$

A main assertion of [2] is that v, σ solve the transport equation (1.3) and the cell PDE (1.2) on the support of σ. In particular, Dv makes sense σ-almost everywhere, even if σ has a singular part with respect to Lebesgue measure.

See also Bernardi-Cardin-Guzzo [1], Gomes-Sanchez Morgado [8], Gomes-Iturriaga-Sanchez Morgado-Yu [7], etc. for more on this variational method.

2 Identities and Estimates

The ideas are to extract useful information from the two forms (1.7)–(1.8) of the Euler-Lagrange PDE. This section records various relevant integral identities, mostly derived by differentiating with respect to different variables. Some of the resulting formulas are special cases of those in [2–3] and some are new.

2.1 Differentiations in x

We start by differentiating with respect to x_k for $k = 1, \cdots, n$.

Theorem 2.1 *We have the identities*

$$\int_{\mathbb{T}^n} DW\sigma^\varepsilon \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^n} \Delta v^\varepsilon \sigma^\varepsilon \,\mathrm{d}x = 0 \tag{2.1}$$

and

$$\int_{\mathbb{T}^n} \Big[(|D^2 v^\varepsilon|^2 + \Delta W)\sigma^\varepsilon + \varepsilon\frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon}\Big] \,\mathrm{d}x = 0. \tag{2.2}$$

Proof In view of (1.1) and (1.5), we have

$$\frac{1}{2}|P + Dv^\varepsilon|^2 + W = \overline{H}^\varepsilon(P) + \varepsilon \log \sigma^\varepsilon. \tag{2.3}$$

Differentiating in x_k once, and then twice, we learn that

$$(P_i + v^\varepsilon_{x_i})v^\varepsilon_{x_i x_k} + W_{x_k} = \varepsilon\frac{\sigma^\varepsilon_{x_k}}{\sigma^\varepsilon} \tag{2.4}$$

and

$$(P+Dv^\varepsilon)\cdot D(\Delta v^\varepsilon)+|D^2v^\varepsilon|^2+\Delta W=\varepsilon\,\mathrm{div}_x\left(\frac{D\sigma^\varepsilon}{\sigma^\varepsilon}\right). \tag{2.5}$$

Multiply (2.4) by σ^ε, integrate by parts and recall (1.7) to derive the first identity in (2.1). The second follows upon our multiplying (1.8) by σ^ε and integrating. To get (2.2), multiply (2.5) by σ^ε and integrate.

Remark 2.1 As $D\sigma^\varepsilon=\frac{1}{\varepsilon}DH^\varepsilon\sigma^\varepsilon$, we obtain from (2.2) the estimate

$$\int_{\mathbb{T}^n}\Big(|D^2v^\varepsilon|^2+\frac{1}{\varepsilon}|DH^\varepsilon|^2\Big)\sigma^\varepsilon\,\mathrm{d}x\le C \tag{2.6}$$

for a constant C independent of ε. Recall that we write $DH^\varepsilon=D_xH^\varepsilon$.

We next generalize Theorem 2.1.

Theorem 2.2 *For each smooth function $\phi:\mathbb{R}\to\mathbb{R}$, we have the identity*

$$\begin{aligned}&\int_{\mathbb{T}^n}[|D^2v^\varepsilon|^2\phi(\sigma^\varepsilon)\sigma^\varepsilon+(\Delta v^\varepsilon)^2\phi'(\sigma^\varepsilon)(\sigma^\varepsilon)^2]\,\mathrm{d}x\\&+\frac{1}{\varepsilon}\int_{\mathbb{T}^n}|DH^\varepsilon|^2[\phi(\sigma^\varepsilon)\sigma^\varepsilon+\phi'(\sigma^\varepsilon)(\sigma^\varepsilon)^2]\,\mathrm{d}x\\&=-\int_{\mathbb{T}^n}\Delta W\phi(\sigma^\varepsilon)\sigma^\varepsilon\,\mathrm{d}x.\end{aligned} \tag{2.7}$$

Proof (1) We multiply the Euler-Lagrange equation (1.7) by $\mathrm{div}(\phi(\sigma^\varepsilon)(P+Dv^\varepsilon))$ and integrate by parts over $\mathbb{T}^n$ as follows:

$$\begin{aligned}0&=\int_{\mathbb{T}^n}(\sigma^\varepsilon(P+v^\varepsilon_{x_i}))_{x_i}(\phi(\sigma^\varepsilon)(P+v^\varepsilon_{x_j}))_{x_j}\,\mathrm{d}x\\&=\int_{\mathbb{T}^n}(\sigma^\varepsilon(P+v^\varepsilon_{x_i}))_{x_j}(\phi(\sigma^\varepsilon)(P+v^\varepsilon_{x_j}))_{x_i}\,\mathrm{d}x\\&=\int_{\mathbb{T}^n}(\sigma^\varepsilon_{x_j}(P+v^\varepsilon_{x_i})+\sigma^\varepsilon v^\varepsilon_{x_ix_j})(\phi'(\sigma^\varepsilon)\sigma^\varepsilon_{x_i}(P+v^\varepsilon_{x_j})+\phi(\sigma^\varepsilon)v^\varepsilon_{x_ix_j})\,\mathrm{d}x\\&=\int_{\mathbb{T}^n}\{|D^2v^\varepsilon|^2\phi(\sigma^\varepsilon)\sigma^\varepsilon+\phi'(\sigma^\varepsilon)|D\sigma^\varepsilon\cdot(P+Dv^\varepsilon)|^2\\&\quad+[\phi(\sigma^\varepsilon)+\sigma^\varepsilon\phi'(\sigma^\varepsilon)]v^\varepsilon_{x_ix_j}(P+v^\varepsilon_{x_j})\sigma^\varepsilon_{x_i}\}\,\mathrm{d}x.\end{aligned}$$

Now $D\sigma^\varepsilon\cdot(P+Dv^\varepsilon)=-\sigma^\varepsilon\Delta v^\varepsilon$, according to (1.7), and furthermore $v^\varepsilon_{x_jx_i}(P+v^\varepsilon_{x_j})$ $=H^\varepsilon_{x_i}-W_{x_i}$. We can therefore simplify, obtaining the identity

$$0=\int_{\mathbb{T}^n}\{|D^2v^\varepsilon|^2\phi(\sigma^\varepsilon)\sigma^\varepsilon+\phi'(\sigma^\varepsilon)(\sigma^\varepsilon)^2(\Delta v^\varepsilon)^2+[\phi(\sigma^\varepsilon)+\sigma^\varepsilon\phi'(\sigma^\varepsilon)](DH^\varepsilon-DW)\cdot D\sigma^\varepsilon\}\,\mathrm{d}x.$$

Since $D\sigma^\varepsilon=\frac{1}{\varepsilon}DH^\varepsilon\sigma^\varepsilon$, it follows that

$$\begin{aligned}&\int_{\mathbb{T}^n}[|D^2v^\varepsilon|^2\phi(\sigma^\varepsilon)\sigma^\varepsilon+(\Delta v^\varepsilon)^2\phi'(\sigma^\varepsilon)(\sigma^\varepsilon)^2]\,\mathrm{d}x+\frac{1}{\varepsilon}\int_{\mathbb{T}^n}|DH^\varepsilon|^2[\phi(\sigma^\varepsilon)\sigma^\varepsilon+\phi'(\sigma^\varepsilon)(\sigma^\varepsilon)^2]\,\mathrm{d}x\\&=\int_{\mathbb{T}^n}[\phi(\sigma^\varepsilon)+\sigma^\varepsilon\phi'(\sigma^\varepsilon)]DW\cdot D\sigma^\varepsilon\,\mathrm{d}x=-\int_{\mathbb{T}^n}\Delta W\sigma^\varepsilon\phi(\sigma^\varepsilon)\,\mathrm{d}x.\end{aligned}$$

2.2 Differentiations in P

We next differentiate with respect to the parameters P_k, for $k = 1, \cdots, n$. In the following expressions, we write

$$V^\varepsilon := D\overline{H}^\varepsilon(P). \tag{2.8}$$

Hereafter D_P means the gradient in P, and $D^2_{x,P}$ means the mixed second derivatives in x and P. To minimize notational clutter, we can safely write $D\overline{H}^\varepsilon = D_P\overline{H}^\varepsilon$ and $D^2\overline{H}^\varepsilon = D^2_P\overline{H}^\varepsilon$, since $\overline{H}^\varepsilon = \overline{H}^\varepsilon(P)$ does not depend upon x.

Theorem 2.3 *These following further identities hold:*

$$D\overline{H}^\varepsilon(P) = V^\varepsilon = \int_{\mathbb{T}^n} (P + Dv^\varepsilon)\sigma^\varepsilon \,\mathrm{d}x, \tag{2.9}$$

$$D^2\overline{H}^\varepsilon(P) = \int_{\mathbb{T}^n} \Big[(I + D^2_{xP}v^\varepsilon) \otimes (I + D^2_{xP}v^\varepsilon)\sigma^\varepsilon + \varepsilon\frac{D_P\sigma^\varepsilon \otimes D_P\sigma^\varepsilon}{\sigma^\varepsilon}\Big] \,\mathrm{d}x. \tag{2.10}$$

Remark 2.2 Formula (2.10) means that for $k, l = 1, \cdots, n$,

$$\overline{H}^\varepsilon_{P_kP_l}(P) = \int_{\mathbb{T}^n} \Big[(\delta_{ik} + v^\varepsilon_{x_iP_k})(\delta_{il} + v^\varepsilon_{x_iP_l})\sigma^\varepsilon + \varepsilon\frac{\sigma^\varepsilon_{P_k}\sigma^\varepsilon_{P_l}}{\sigma^\varepsilon}\Big] \,\mathrm{d}x.$$

Therefore, $\xi \cdot D^2\overline{H}(P)\xi = \overline{H}^\varepsilon_{P_kP_l}(P)\xi_k\xi_l \geq 0$ for all $\xi = (\xi_1, \cdots, \xi_n)$, and hence

$$P \mapsto \overline{H}^\varepsilon(P) \text{ is convex.} \tag{2.11}$$

Proof (1) Differentiating (2.3) in P_k, and then in P_l, we find

$$(P_i + v^\varepsilon_{x_i})(\delta_{ik} + v^\varepsilon_{x_iP_k}) = \overline{H}^\varepsilon_{P_k} + \varepsilon\frac{\sigma^\varepsilon_{P_k}}{\sigma^\varepsilon}, \tag{2.12}$$

$$(P_i + v^\varepsilon_{x_i})v^\varepsilon_{x_iP_kP_l} + (\delta_{il} + v^\varepsilon_{x_iP_l})(\delta_{ik} + v^\varepsilon_{x_iP_k}) = \overline{H}^\varepsilon_{P_kP_l} + \varepsilon\Big(\frac{\sigma^\varepsilon_{P_k}}{\sigma^\varepsilon}\Big)_{P_l}. \tag{2.13}$$

(2) Since $\int_{\mathbb{T}^n} \sigma^\varepsilon \,\mathrm{d}x = 1$, we have

$$0 = \int_{\mathbb{T}^n} \sigma^\varepsilon_{P_k} \,\mathrm{d}x = \int_{\mathbb{T}^n} \frac{\sigma^\varepsilon_{P_k}}{\sigma^\varepsilon}\sigma^\varepsilon \,\mathrm{d}x. \tag{2.14}$$

We now multiply (2.12) by σ^ε and integrate, using (1.6)–(1.7) and (2.14) to derive (2.9).

In addition, (2.14) implies

$$\int_{\mathbb{T}^n} \Big(\frac{\sigma^\varepsilon_{P_k}}{\sigma^\varepsilon}\Big)_{P_l}\sigma^\varepsilon \,\mathrm{d}x = -\int_{\mathbb{T}^n} \frac{\sigma^\varepsilon_{P_k}\sigma^\varepsilon_{P_l}}{\sigma^\varepsilon} \,\mathrm{d}x.$$

So the identity (2.10) follows, if we multiply (2.13) by σ^ε and integrate.

Remark 2.3 It follows from (2.10) that

$$\begin{aligned}\operatorname{tr}(D^2\overline{H}^\varepsilon(P)) &= \int_{\mathbb{T}^n} \Big(|I + D^2_{x,P}v^\varepsilon|^2\sigma^\varepsilon + \varepsilon\frac{|D_P\sigma^\varepsilon|^2}{\sigma^\varepsilon}\Big)\mathrm{d}x \\ &= \int_{\mathbb{T}^n} \Big(|I + D^2_{x,P}v^\varepsilon|^2 + \frac{1}{\varepsilon}|(P + Dv^\varepsilon)(I + D^2_{x,P}v^\varepsilon) - V^\varepsilon|^2\Big)\sigma^\varepsilon \,\mathrm{d}x,\end{aligned} \tag{2.15}$$

where "tr" means trace.

2.3 Differentiations in ε

In the following, subscripts ε denote derivatives with respect to ε.

Theorem 2.4 *We have*

$$\overline{H}^{\varepsilon}_{\varepsilon}(P) = -\int_{\mathbb{T}^n} \sigma^{\varepsilon} \log \sigma^{\varepsilon}\, \mathrm{d}x = -\int_{\mathbb{T}^n} \frac{H^{\varepsilon} - \overline{H}^{\varepsilon}(P)}{\varepsilon} \sigma^{\varepsilon}\, \mathrm{d}x, \tag{2.16}$$

and so

$$\int_{\mathbb{T}^n} H^{\varepsilon} \sigma^{\varepsilon}\, \mathrm{d}x = \overline{H}^{\varepsilon}(P) - \varepsilon \overline{H}^{\varepsilon}_{\varepsilon}(P). \tag{2.17}$$

In addition,

$$\begin{aligned} \overline{H}^{\varepsilon}_{\varepsilon\varepsilon}(P) &= \int_{\mathbb{T}^n} |Dv^{\varepsilon}_{\varepsilon}|^2 \sigma^{\varepsilon} + \varepsilon \frac{|\sigma^{\varepsilon}_{\varepsilon}|^2}{\sigma^{\varepsilon}}\, \mathrm{d}x \\ &= \int_{\mathbb{T}^n} \Big(|Dv^{\varepsilon}_{\varepsilon}|^2 + \frac{1}{\varepsilon} \Big| (P + Dv^{\varepsilon}) \cdot Dv^{\varepsilon}_{\varepsilon} - \frac{H^{\varepsilon} - \overline{H}^{\varepsilon}(P) + \varepsilon \overline{H}^{\varepsilon}_{\varepsilon}(P)}{\varepsilon} \Big|^2 \Big) \sigma^{\varepsilon}\, \mathrm{d}x. \end{aligned} \tag{2.18}$$

Remark 2.4 The identity (2.18) implies that

$$\varepsilon \mapsto \overline{H}^{\varepsilon}(P) \text{ is convex.} \tag{2.19}$$

Differentiating in x and then in P, we can likewise show that

$$(P, \varepsilon) \mapsto \overline{H}^{\varepsilon}(P) \text{ is jointly convex.} \tag{2.20}$$

Proof (1) We differentiate (2.3) twice in ε, to learn that

$$(P + Dv^{\varepsilon}) \cdot Dv^{\varepsilon}_{\varepsilon} = \overline{H}^{\varepsilon}_{\varepsilon}(P) + \log \sigma^{\varepsilon} + \varepsilon \frac{\sigma^{\varepsilon}_{\varepsilon}}{\sigma^{\varepsilon}}, \tag{2.21}$$

and then

$$(P + Dv^{\varepsilon}) \cdot Dv^{\varepsilon}_{\varepsilon\varepsilon} + |Dv^{\varepsilon}_{\varepsilon}|^2 = \overline{H}^{\varepsilon}_{\varepsilon\varepsilon}(P) + \frac{\sigma^{\varepsilon}_{\varepsilon}}{\sigma^{\varepsilon}} + \varepsilon \Big(\frac{\sigma^{\varepsilon}_{\varepsilon}}{\sigma^{\varepsilon}} \Big)_{\varepsilon}. \tag{2.22}$$

Multiply (2.21) by σ^{ε} and recall (1.7), to derive (2.16).

(2) Next multiply (2.22) by σ^{ε}. We observe that

$$0 = \int_{\mathbb{T}^n} \sigma^{\varepsilon}_{\varepsilon}\, \mathrm{d}x = \int_{\mathbb{T}^n} \frac{\sigma^{\varepsilon}_{\varepsilon}}{\sigma^{\varepsilon}} \sigma^{\varepsilon}\, \mathrm{d}x,$$

and thus

$$\int_{\mathbb{T}^n} \Big(\frac{\sigma^{\varepsilon}_{\varepsilon}}{\sigma^{\varepsilon}} \Big)_{\varepsilon} \sigma^{\varepsilon}\, \mathrm{d}x = -\int_{\mathbb{T}^n} \frac{|\sigma^{\varepsilon}_{\varepsilon}|^2}{\sigma^{\varepsilon}}\, \mathrm{d}x.$$

This gives the first equality in (2.18), and the second follows when we explicitly calculate $\sigma^{\varepsilon}_{\varepsilon}$.

2.4 Estimates for $Du^{\varepsilon} - Du$

A key question is how well v^{ε} and σ^{ε} approximate as $\varepsilon \to 0$ particular solutions v, σ of the weak KAM PDE (1.2)–(1.3).

Now let v be a viscosity solution of (1.2) and σ a corresponding weak solution of (1.3). To allow for changes in P we also assume, for this subsection only, that v^{ε} solves the variational problem for the vector P^{ε}. Consequently, we have

$$\frac{1}{2} |P^{\varepsilon} + Dv^{\varepsilon}|^2 + W = \overline{H}^{\varepsilon}(P^{\varepsilon}) + \varepsilon \log \sigma^{\varepsilon}.$$

Theorem 2.5 *These following identities hold:*

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\sigma^\varepsilon\,\mathrm{d}x\\&=\overline{H}(P)-\overline{H}^\varepsilon(P^\varepsilon)+\varepsilon\overline{H}^\varepsilon_\varepsilon(P^\varepsilon)+D\overline{H}^\varepsilon(P^\varepsilon)\cdot(P^\varepsilon-P)\end{aligned}\tag{2.23}$$

and

$$\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\,\mathrm{d}\sigma=\int_{\mathbb{T}^n}H^\varepsilon\,\mathrm{d}\sigma-\overline{H}(P)+D\overline{H}(P)\cdot(P-P^\varepsilon).\tag{2.24}$$

Observe that right-hand sides involve Taylor expansions of $\overline{H}^\varepsilon$ and $\overline{H}$. Thus if $\overline{H}^\varepsilon$ approximates $\overline{H}$ sufficiently well for small ε and if P^ε is close to P, then $|Dv^\varepsilon-Dv|$ is small on the support of σ^ε.

Proof (1) We have

$$\frac{1}{2}|P^\varepsilon+Dv^\varepsilon|^2+W=\overline{H}^\varepsilon(P^\varepsilon)+\varepsilon\log\sigma^\varepsilon,\quad \frac{1}{2}|P+Dv|^2+W=\overline{H}(P),$$

and so

$$\frac{1}{2}(|P^\varepsilon+Dv^\varepsilon|^2-|P+Dv|^2)=\overline{H}^\varepsilon(P^\varepsilon)-\overline{H}(P)+\varepsilon\log\sigma^\varepsilon.\tag{2.25}$$

Since $|a|^2-|b|^2=-|a-b|^2+2a\cdot(a-b)$, we calculate that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}^n}(|P^\varepsilon+Dv^\varepsilon|^2-|P+Dv|^2)\,\sigma^\varepsilon\,\mathrm{d}x\\&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\sigma^\varepsilon\,\mathrm{d}x\\&\quad+\int_{\mathbb{T}^n}(P^\varepsilon+Dv^\varepsilon)\cdot((P^\varepsilon+Dv^\varepsilon)-(P+Dv))\sigma^\varepsilon\,\mathrm{d}x\\&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\sigma^\varepsilon\,\mathrm{d}x+(P^\varepsilon-P)\cdot\int_{\mathbb{T}^n}(P^\varepsilon+Dv^\varepsilon)\sigma^\varepsilon\,\mathrm{d}x\\&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\sigma^\varepsilon\,\mathrm{d}x+(P^\varepsilon-P)\cdot D\overline{H}^\varepsilon(P^\varepsilon),\end{aligned}$$

the second equality resulting from (1.7). Consequently, (2.25) implies

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\sigma^\varepsilon\,\mathrm{d}x&=\overline{H}(P)-\overline{H}^\varepsilon(P^\varepsilon)+(P^\varepsilon-P)\cdot D\overline{H}^\varepsilon(P^\varepsilon)\\&\quad-\varepsilon\int_{\mathbb{T}^n}\log\sigma^\varepsilon\sigma^\varepsilon\,\mathrm{d}x.\end{aligned}$$

The identity (2.23) follows, since $\overline{H}^\varepsilon_\varepsilon(P^\varepsilon)=-\int_{\mathbb{T}^n}\log\sigma^\varepsilon\sigma^\varepsilon\,\mathrm{d}x$ according to (2.16).

(2) To prove (2.24), we next integrate (2.25) with respect to the measure σ (recall (1.3)),

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^n}(|P+Dv|^2-|P^\varepsilon+Dv^\varepsilon|^2)\,\mathrm{d}\sigma\\
&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\,\mathrm{d}\sigma\\
&\quad+\int_{\mathbb{T}^n}(P+Dv)\cdot((P+Dv)-(P^\varepsilon+Dv^\varepsilon))\,\mathrm{d}\sigma\\
&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\,\mathrm{d}\sigma+(P-P^\varepsilon)\cdot\int_{\mathbb{T}^n}(P+Dv)\,\mathrm{d}\sigma\\
&=-\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\,\mathrm{d}\sigma+(P^\varepsilon-P)\cdot D\overline{H}(P).
\end{aligned}$$

Hence, (2.25) gives

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^n}|(P^\varepsilon+Dv^\varepsilon)-(P+Dv)|^2\,\mathrm{d}\sigma\\
&=\overline{H}^\varepsilon(P^\varepsilon)-\overline{H}(P)+(P^\varepsilon-P)\cdot D\overline{H}(P)+\varepsilon\int_{\mathbb{T}^n}\log\sigma^\varepsilon\,\mathrm{d}\sigma\\
&=\int_{\mathbb{T}^n}H^\varepsilon\,\mathrm{d}\sigma-\overline{H}(P)+(P^\varepsilon-P)\cdot D\overline{H}(P).
\end{aligned}$$

3 Linearizations and Adjoints

3.1 Linearizing the PDE

The linearization about v^ε of the Euler-Lagrange equation (1.8) is the operator

$$L_\varepsilon[w]:=-(P_i+v^\varepsilon_{x_i})(P_j+v^\varepsilon_{x_j})w_{x_ix_j}-(2(P_j+v^\varepsilon_{x_j})v^\varepsilon_{x_ix_j}+W_{x_i})w_{x_i}-\varepsilon\Delta w, \tag{3.1}$$

defined for smooth, periodic functions $w:\mathbb{T}^n\to\mathbb{R}$.

Lemma 3.1 *We have the alternative formulas*

$$L_\varepsilon[w]=-(P_i+v^\varepsilon_{x_i})((P_j+v^\varepsilon_{x_j})w_{x_j})_{x_i}-H^\varepsilon_{x_i}w_{x_i}-\varepsilon\Delta w \tag{3.2}$$

and

$$L_\varepsilon[w]=-\frac{1}{\sigma^\varepsilon}([(P_i+v^\varepsilon_{x_i})(P_j+v^\varepsilon_{x_j})+\varepsilon\delta_{ij}]\sigma^\varepsilon w_{x_j})_{x_i}. \tag{3.3}$$

Proof (1) Formula (3.2) follows immediately from (3.1).

(2) Recall from (1.7) that $\operatorname{div}((P+Dv^\varepsilon)\sigma^\varepsilon)=0$. Consequently, the expression on the right-hand side of (3.3) equals

$$-(P_i+v^\varepsilon_{x_i})((P_j+v^\varepsilon_{x_j})w_{x_j})_{x_i}-\varepsilon\frac{\sigma^\varepsilon_{x_i}}{\sigma^\varepsilon}w_{x_i}-\varepsilon\Delta w.$$

But this is the formula (3.2) for $L_\varepsilon[w]$, since $\sigma^\varepsilon=\mathrm{e}^{\frac{H^\varepsilon-\overline{H}^\varepsilon}{\varepsilon}}$.

The linearization L_ε is useful, as it appears when we differentiate the nonlinear PDE (1.8).

Theorem 3.1 *These following identities hold:*

$$L_\varepsilon[v^\varepsilon] = -2(P_i + v^\varepsilon_{x_i})v^\varepsilon_{x_j}v^\varepsilon_{x_ix_j} + P_iW_{x_i}, \tag{3.4}$$

$$L_\varepsilon[v^\varepsilon_{x_k}] = (P_i + v^\varepsilon_{x_i})W_{x_ix_k}, \quad k = 1, \cdots, n, \tag{3.5}$$

$$L_\varepsilon[v^\varepsilon_{P_k}] = 2(P_i + v^\varepsilon_{x_i})v^\varepsilon_{x_ix_k} + W_{x_k}, \quad k = 1, \cdots, n, \tag{3.6}$$

$$L_\varepsilon[x_k] = -2(P_i + v^\varepsilon_{x_i})v^\varepsilon_{x_ix_k} - W_{x_k}, \quad k = 1, \cdots, n, \tag{3.7}$$

$$L_\varepsilon[v^\varepsilon_\varepsilon] = \Delta v^\varepsilon, \tag{3.8}$$

$$L_\varepsilon[H^\varepsilon] = -|DH^\varepsilon|^2 - \varepsilon|D^2v^\varepsilon|^2 - \varepsilon\Delta W. \tag{3.9}$$

Proof (1) According to (3.1) and (1.8),

$$\begin{aligned}
L_\varepsilon[v^\varepsilon] &= -(P_i + v^\varepsilon_{x_i})(P_j + v^\varepsilon_{x_j})v^\varepsilon_{x_ix_j} - \varepsilon\Delta v^\varepsilon - (2(P_j + v^\varepsilon_{x_j})v^\varepsilon_{x_ix_j} + W_{x_i})v^\varepsilon_{x_i} \\
&= (P_i + v^\varepsilon_{x_i})W_{x_i} - (2(P_j + v^\varepsilon_{x_j})v^\varepsilon_{x_ix_j} + W_{x_i})v^\varepsilon_{x_i} \\
&= P_iW_{x_i} - 2(P_j + v^\varepsilon_{x_j})v^\varepsilon_{x_ix_j}v^\varepsilon_{x_i}.
\end{aligned}$$

This is (3.4).

(2) We obtain (3.5) upon differentiating (1.8) with respect to x_k: The left-hand side appears when the differentiation falls upon v^ε and the right-hand side appears when the differentiation falls upon the term involving the potential W.

Similarly, (3.6) results from our differentiating (1.8) with respect to P_k, and (3.8) from our differentiating in ε. We directly compute from the definition (3.1) that (3.9) is also valid.

Remark 3.1 We observe from (3.6)–(3.7) that

$$L_\varepsilon[x_k + v^\varepsilon_{P_k}] = 0, \quad k = 1, \cdots, n. \tag{3.10}$$

But note also that $x + D_Pv^\varepsilon$ is not $\mathbb{T}^n$-periodic. We will return to this point in Subsection 4.2.

3.2 The adjoint operator

We introduce next the adjoint L^*_ε of L_ε with respect to the standard inner product in $L^2(\mathbb{T}^n)$, so that

$$\int_{\mathbb{T}^n} L_\varepsilon[f]g\,\mathrm{d}x = \int_{\mathbb{T}^n} fL^*_\varepsilon[g]\,\mathrm{d}x \tag{3.11}$$

for all smooth, $\mathbb{T}^n$-periodic functions f and g.

Theorem 3.2 (i) *We have*

$$L^*_\varepsilon[w] = -\Big([(P_i + v^\varepsilon_{x_i})(P_j + v^\varepsilon_{x_j}) + \varepsilon\delta_{ij}]\sigma^\varepsilon\Big(\frac{w}{\sigma^\varepsilon}\Big)_{x_i}\Big)_{x_j}. \tag{3.12}$$

(ii) *Therefore*

$$L^*_\varepsilon[\sigma^\varepsilon w] = \sigma^\varepsilon L_\varepsilon[w] \tag{3.13}$$

and

$$L^*_\varepsilon[\sigma^\varepsilon] = 0. \tag{3.14}$$

Proof The identity (3.12) follows from (3.3) and an integration by parts.

Remark 3.2 It follows from (3.13) (or (3.3)) that the operator L_ε, acting on smooth functions, is symmetric with respect to the L^2 inner product weighted by σ^ε:

$$\int_{\mathbb{T}^n} L_\varepsilon[f] g \, \sigma^\varepsilon \, \mathrm{d}x = \int_{\mathbb{T}^n} f L_\varepsilon[g] \, \sigma^\varepsilon \, \mathrm{d}x \tag{3.15}$$

for smooth, $\mathbb{T}^n$-periodic functions f, g. Perhaps the spectrum of L_ε contains useful dynamical information in the limit $\varepsilon \to 0$.

3.3 More identities

We can employ the foregoing formulas to rewrite some of the expressions from Section 2.

Theorem 3.3 *We have the identity*

$$\begin{aligned}\varepsilon \operatorname{tr}(D^2 \overline{H}^\varepsilon(P)) = \varepsilon n + \int_{\mathbb{T}^n} (&|P + Dv^\varepsilon - V^\varepsilon|^2 - |(P + Dv^\varepsilon) \cdot D^2_{xP} v^\varepsilon|^2 \\ &- \varepsilon |D^2_{xP} v^\varepsilon|^2) \sigma^\varepsilon \, \mathrm{d}x\end{aligned} \tag{3.16}$$

and consequently the estimate

$$\int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D^2_{xP} v^\varepsilon|^2 + \varepsilon |D^2_{xP} v^\varepsilon|^2) \sigma^\varepsilon \, \mathrm{d}x \leq \varepsilon n + \int_{\mathbb{T}^n} |P + Dv^\varepsilon - V^\varepsilon|^2 \sigma^\varepsilon \, \mathrm{d}x. \tag{3.17}$$

Proof (1) Owing to (3.6), we have

$$L_\varepsilon \left[\frac{1}{2} |D_P v^\varepsilon|^2 \right] = v^\varepsilon_{P_k} (2(H^\varepsilon)_{x_k} - W_{x_k}) - |(P + Dv^\varepsilon) \cdot D^2_{xP} v^\varepsilon|^2 - \varepsilon |D^2_{xP} v^\varepsilon|^2.$$

We multiply by σ^ε and integrate, recalling from (3.14) that $L^*_\varepsilon[\sigma^\varepsilon] = 0$,

$$\begin{aligned}&\int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D^2_{xP} v^\varepsilon|^2 + \varepsilon |D^2_{xP} v^\varepsilon|^2) \sigma^\varepsilon \, \mathrm{d}x \\ &= \int_{\mathbb{T}^n} v^\varepsilon_{P_k} (2(P_i + v^\varepsilon_{x_i}) v^\varepsilon_{x_i x_k} + W_{x_k}) \sigma^\varepsilon \, \mathrm{d}x \\ &= \int_{\mathbb{T}^n} v^\varepsilon_{P_k} ((P_i + v^\varepsilon_{x_i}) v^\varepsilon_{x_i x_k} + H^\varepsilon_{x_k}) \sigma^\varepsilon \, \mathrm{d}x \\ &= \int_{\mathbb{T}^n} v^\varepsilon_{P_k} (P_i + v^\varepsilon_{x_i}) v^\varepsilon_{x_i x_k} \sigma^\varepsilon \, \mathrm{d}x + \varepsilon \int_{\mathbb{T}^n} v^\varepsilon_{P_k} \sigma^\varepsilon_{x_k} \, \mathrm{d}x \\ &= -\int_{\mathbb{T}^n} v^\varepsilon_{P_k x_i} (P_i + v^\varepsilon_{x_i}) v^\varepsilon_{x_k} \sigma^\varepsilon \, \mathrm{d}x - \varepsilon \int_{\mathbb{T}^n} v^\varepsilon_{P_k x_k} \sigma^\varepsilon \, \mathrm{d}x \\ &= -\int_{\mathbb{T}^n} v^\varepsilon_{P_k x_i} (P_i + v^\varepsilon_{x_i})(P_k + v^\varepsilon_{x_k} - V^\varepsilon_k) \sigma^\varepsilon \, \mathrm{d}x - \varepsilon \int_{\mathbb{T}^n} v^\varepsilon_{P_k x_k} \sigma^\varepsilon \, \mathrm{d}x.\end{aligned} \tag{3.18}$$

The last equality follows from (1.7).

(2) In view of (2.15),

$$\begin{aligned}\varepsilon \operatorname{tr}(D^2 \overline{H}^\varepsilon(P)) = &\int_{\mathbb{T}^n} (\varepsilon |I + D^2_{x,P} v^\varepsilon|^2 + |(P + Dv^\varepsilon) D^2_{x,P} v^\varepsilon|^2 + |P + Dv^\varepsilon - V^\varepsilon|^2) \sigma^\varepsilon \, \mathrm{d}x \\ &+ 2 \int_{\mathbb{T}^n} (P + Dv^\varepsilon) D^2_{x,P} v^\varepsilon \cdot (P + Dv^\varepsilon - V^\varepsilon) \sigma^\varepsilon \, \mathrm{d}x.\end{aligned}$$

We use (3.18) to see that the second term equals

$$-2\int_{\mathbb{T}^n}(|(P+Dv^\varepsilon)\cdot D^2_{xP}v^\varepsilon|^2+\varepsilon|D^2_{xP}v^\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x-2\varepsilon\int_{\mathbb{T}^n}v^\varepsilon_{P_kx_k}\sigma^\varepsilon\,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\varepsilon\,\mathrm{tr}(D^2\overline{H}^\varepsilon(P))&=\int_{\mathbb{T}^n}(\varepsilon|I+D^2_{x,P}v^\varepsilon|^2+|(P+Dv^\varepsilon)D^2_{x,P}v^\varepsilon|^2+|P+Dv^\varepsilon-V^\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x\\&\quad-2\int_{\mathbb{T}^n}(|(P+Dv^\varepsilon)\cdot D^2_{xP}v^\varepsilon|^2+\varepsilon|D^2_{xP}v^\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x-2\varepsilon\int_{\mathbb{T}^n}v^\varepsilon_{P_kx_k}\sigma^\varepsilon\,\mathrm{d}x\\&=\int_{\mathbb{T}^n}(|P+Dv^\varepsilon-V^\varepsilon|^2-|(P+Dv^\varepsilon)D^2_{x,P}v^\varepsilon|^2+\varepsilon|I|^2-\varepsilon|D^2_{x,P}v^\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x.\end{aligned}$$

The formula (3.16) follows, as does the inequality (3.17), since $\mathrm{tr}(D^2\overline{H}^\varepsilon(P))\geq 0$.

Theorem 3.4 *We have*

$$\varepsilon\overline{H}^\varepsilon_{\varepsilon\varepsilon}(P)=\int_{\mathbb{T}^n}\Big(\frac{|H^\varepsilon-\overline{H}^\varepsilon(P)+\varepsilon\overline{H}^\varepsilon_\varepsilon(P)|^2}{\varepsilon^2}-|(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon|^2-\varepsilon|Dv^\varepsilon_\varepsilon|^2\Big)\sigma^\varepsilon\,\mathrm{d}x,\qquad(3.19)$$

and therefore

$$\int_{\mathbb{T}^n}(|(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon|^2+\varepsilon|Dv^\varepsilon_\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x\leq\int_{\mathbb{T}^n}\frac{|H^\varepsilon-\overline{H}^\varepsilon(P)+\varepsilon\overline{H}^\varepsilon_\varepsilon(P)|^2}{\varepsilon^2}\sigma^\varepsilon\,\mathrm{d}x.\qquad(3.20)$$

Proof (1) According to (3.8), we have

$$L_\varepsilon\Big[\frac{1}{2}(v^\varepsilon_\varepsilon)^2\Big]=v^\varepsilon_\varepsilon\Delta v^\varepsilon-|(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon|^2-\varepsilon|Dv^\varepsilon_\varepsilon|^2.$$

We multiply by σ^ε and integrate as follows:

$$\int_{\mathbb{T}^n}(|(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon|^2+\varepsilon|Dv^\varepsilon_\varepsilon|^2)\sigma^\varepsilon\,\mathrm{d}x=\int_{\mathbb{T}^n}v^\varepsilon_\varepsilon\Delta v^\varepsilon\sigma^\varepsilon\,\mathrm{d}x.\qquad(3.21)$$

The formula (2.18) implies

$$\begin{aligned}\varepsilon\overline{H}^\varepsilon_{\varepsilon\varepsilon}(P)&=\int_{\mathbb{T}^n}\Big(\varepsilon|Dv^\varepsilon_\varepsilon|^2+|(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon|^2+\frac{|H^\varepsilon-\overline{H}^\varepsilon(P)+\varepsilon\overline{H}^\varepsilon_\varepsilon(P)|^2}{\varepsilon^2}\Big)\sigma^\varepsilon\,\mathrm{d}x\\&\quad-2\int_{\mathbb{T}^n}(P+Dv^\varepsilon)\cdot Dv^\varepsilon_\varepsilon\frac{H^\varepsilon-\overline{H}^\varepsilon(P)+\varepsilon\overline{H}^\varepsilon_\varepsilon(P)}{\varepsilon}\sigma^\varepsilon\,\mathrm{d}x.\end{aligned}$$

Recalling yet again (1.7), we observe that the second integral term equals

$$2\int_{\mathbb{T}^n}v^\varepsilon_\varepsilon(P+Dv^\varepsilon)\cdot\frac{DH^\varepsilon}{\varepsilon}\sigma^\varepsilon\,\mathrm{d}x=-2\int_{\mathbb{T}^n}v^\varepsilon_\varepsilon\Delta v^\varepsilon\sigma^\varepsilon\,\mathrm{d}x,$$

the last equality following from (1.8). We substitute (3.21) and rewrite, obtaining (3.19).

4 Some Applications

We collect in the concluding section some applications of the foregoing formulas, of which those in Subsection 4.2 concerning nonresonance are the most interesting.

4.1 $\overline{H}^{\varepsilon}$ as $\varepsilon \to 0$

An overall goal is understanding how $\overline{H}$ and its approximations $\overline{H}^{\varepsilon}$ for small $\varepsilon > 0$ provide analytic control of v^{ε}, σ^{ε}, and thus in the limit of v, σ.

As an illustration, we show next that if $\overline{H}^{\varepsilon}(P)$ is nice enough as a function of ε near zero, then we can construct a limit measure σ that is absolutely continuous with respect to Lebesgue measure.

Theorem 4.1 *If*

$$\Big(\frac{\overline{H}^{\varepsilon}_{\varepsilon\varepsilon}(P)}{\varepsilon}\Big)^{\frac{1}{2}} \ \textit{is integrable near } \varepsilon = 0, \tag{4.1}$$

then

$$\sigma^{\varepsilon} \to \sigma \quad \textit{in } L^1(\mathbb{T}^n) \tag{4.2}$$

and $\sigma \in L^1(\mathbb{T}^n)$ *solves* (1.3).

Proof If $0 < \varepsilon_1 < \varepsilon_2$, we have

$$\begin{aligned}
\int_{\mathbb{T}^n} |\sigma^{\varepsilon_2} - \sigma^{\varepsilon_1}|\,\mathrm{d}x &\le \int_{\varepsilon_1}^{\varepsilon_2} \int_{\mathbb{T}^n} |\sigma^{\varepsilon}_{\varepsilon}|\,\mathrm{d}x\mathrm{d}\varepsilon \\
&\le \int_{\varepsilon_1}^{\varepsilon_2} \Big(\int_{\mathbb{T}^n} \frac{|\sigma^{\varepsilon}_{\varepsilon}|^2}{\sigma^{\varepsilon}}\,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{T}^n} \sigma^{\varepsilon}\,\mathrm{d}x\Big)^{\frac{1}{2}} \mathrm{d}\varepsilon \\
&\le \int_{\varepsilon_1}^{\varepsilon_2} \Big(\frac{\overline{H}^{\varepsilon}_{\varepsilon\varepsilon}}{\varepsilon}\Big)^{\frac{1}{2}} \mathrm{d}\varepsilon,
\end{aligned}$$

according to (2.18). Consequently, (4.1) implies that $\{\sigma^{\varepsilon}\}_{\varepsilon>0}$ is a Cauchy sequence in $L^1(\mathbb{T}^n)$ as $\varepsilon \to 0$.

4.2 Nonresonance phenomena

We assume hereafter that we can select P^{ε} so that

$$V := D\overline{H}^{\varepsilon}(P^{\varepsilon}) \tag{4.3}$$

does not depend upon ε. Write $V = (V_1, \cdots, V_n)$. We suppose also the nonresonance condition that for some constant $c > 0$,

$$|V \cdot k| \ge \frac{c}{|k|^{\gamma}} \quad \text{for all vectors } k \in \mathbb{Z}^n,\ k \ne 0. \tag{4.4}$$

Next, take $g : \mathbb{T}^n \to \mathbb{R}$ to be smooth and have zero mean

$$\int_{\mathbb{T}^n} g(X)\,\mathrm{d}X = 0.$$

Then using a standard Fourier series representation and the nonresonance condition (4.4), we have the following lemma.

Lemma 4.1 *There exists a smooth $\mathbb{T}^n$-periodic solution $f = f(X)$ of the linear elliptic PDE*

$$-V_k V_l f_{X_k X_l} = g \quad \textit{in } \mathbb{T}^n. \tag{4.5}$$

Furthermore, we have for each $s \geq 0$ the estimate

$$\|f\|_{H^s(\mathbb{T}^n)} \leq C_s \|g\|_{H^{s+\gamma}(\mathbb{T}^n)} \tag{4.6}$$

for a constant C_s.

Theorem 4.2 *Assume that*

$$\varepsilon \ \mathrm{tr}(D^2\overline{H}^{\varepsilon}(P^\varepsilon)) = o(1) \quad \text{as } \varepsilon \to 0. \tag{4.7}$$

Then for each smooth function $g : \mathbb{T}^n \to \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{T}^n} g(x + D_P v^\varepsilon)\sigma^\varepsilon \,\mathrm{d}x = \int_{\mathbb{T}^n} g(X)\,\mathrm{d}X. \tag{4.8}$$

Remark 4.1 This is a variant of a theorem in [2]. The formal interpretation is that under the symplectic change of variable

$$(x, p) \to (X, P),$$

defined implicitly by the formulas $p = P + D_x v$, $X = x + D_P v$ the dynamics become linear: $X(t) = X_0 + tV$ for $t \geq 0$. Since $V \cdot k \neq 0$ for all $k \in \mathbb{Z}^n - \{0\}$, the flow is therefore asymptotically equidistributed with respect to Lebesgue measure. The rigorous assertion (4.8) is consistent with this picture.

Proof (1) Subtracting a constant if necessary, we may assume that the average of g is zero. Now let f solve the linear PDE (4.5), and define

$$\widetilde{f}(x) := f(x + D_P v^\varepsilon).$$

The function $\widetilde{f}$ is $\mathbb{T}^n$-periodic, although $x + D_P v^\varepsilon$ is not.

Recalling from (3.10) that $L_\varepsilon[x + D_P v^\varepsilon] = 0$, we compute

$$\begin{aligned} L_\varepsilon[\widetilde{f}] &= D_X f \cdot L_\varepsilon[x + D_P v^\varepsilon] \\ &\qquad - [(P_i + v^\varepsilon_{x_i})(P_j + v^\varepsilon_{x_j}) + \varepsilon\delta_{ij}](\delta_{ik} + v^\varepsilon_{x_i P_k})(\delta_{jl} + v^\varepsilon_{x_j P_l}) f_{X_k X_l} \\ &= -[(P_i + v^\varepsilon_{x_i})(P_j + v^\varepsilon_{x_j}) + \varepsilon\delta_{ij}](\delta_{ik} + v^\varepsilon_{x_i P_k})(\delta_{jl} + v^\varepsilon_{x_j P_l}) f_{X_k X_l}. \end{aligned}$$

Here and afterwards f is evaluated at $x + D_P v^\varepsilon$. It follows that

$$L_\varepsilon[\widetilde{f}] + g(x + D_P v^\varepsilon) = E_1 + E_2 \tag{4.9}$$

for

$$\begin{aligned} E_1 &:= [(P_i + v^\varepsilon_{x_i})(\delta_{ik} + v^\varepsilon_{x_i P_k})(P_j + v^\varepsilon_{x_j})(\delta_{jl} + v^\varepsilon_{x_j P_l}) - V_k V_l] f_{X_k X_l}, \\ E_2 &:= \varepsilon(\delta_{ik} + v^\varepsilon_{x_i P_k})(\delta_{il} + v^\varepsilon_{x_i P_l}) f_{X_k X_l}. \end{aligned}$$

(2) Selecting s large enough, we deduce from (4.6) that $\|f\|_{C^{1,1}}$ is bounded. Consequently, (2.15) implies the estimate

$$\begin{aligned} \int_{\mathbb{T}^n} |E_1|\sigma^\varepsilon \,\mathrm{d}x &\leq C \int_{\mathbb{T}^n} (1 + |I + D^2_{x,P} v^\varepsilon|)|(P + Dv^\varepsilon)(I + D^2_{x,P} v^\varepsilon) - V^\varepsilon|\sigma^\varepsilon \,\mathrm{d}x \\ &\leq C(\varepsilon \ \mathrm{tr}(D^2\overline{H}^{\varepsilon}(P^\varepsilon)))^{\frac{1}{2}} = o(1). \end{aligned}$$

Likewise,

$$\int_{\mathbb{T}^n} |E_2|\sigma^\varepsilon \, \mathrm{d}x \leq C\varepsilon \; \mathrm{tr}(D^2\overline{H}^\varepsilon(P^\varepsilon)) = o(1).$$

(3) It follows now from (4.9) and (3.14) that

$$\begin{aligned}\int_{\mathbb{T}^n} g(x + D_P v^\varepsilon)\sigma^\varepsilon \, \mathrm{d}x &= \int_{\mathbb{T}^n} (-L_\varepsilon[\widetilde{f}] - E_1 - E_2)\sigma^\varepsilon \, \mathrm{d}x \\ &= -\int_{\mathbb{T}^n} \widetilde{f} L_\varepsilon^*[\sigma^\varepsilon] \, \mathrm{d}x + o(1) \\ &= o(1)\end{aligned}$$

as $\varepsilon \to 0$.

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