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# Quantitative Stability of the Brunn-Minkowski Inequality for Sets of Equal Volume*

Alessio FIGALLI[1] David JERISON[2]

(*Dedicated to Professor Haim Brezis on the occasion of his 70th birthday*)

**Abstract** The authors prove a quantitative stability result for the Brunn-Minkowski inequality on sets of equal volume: If $|A| = |B| > 0$ and $|A + B|^{\frac{1}{n}} = (2 + \delta)|A|^{\frac{1}{n}}$ for some small $\delta$, then, up to a translation, both $A$ and $B$ are close (in terms of $\delta$) to a convex set $\mathcal{K}$. Although this result was already proved by the authors in a previous paper, the present paper provides a more elementary proof that the authors believe has its own interest. Also, the result here provides a stronger estimate for the stability exponent than the previous result of the authors.

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## 1 Introduction

The Brunn-Minkowski inequality is a very classical and powerful inequality in convex geometry that has found important applications in analysis, statistics, and information theory. We refer the reader to [14] for an extended exposition on the Brunn-Minkowski inequality and its relation to several other famous inequalities (see also [6–7]).

To state the inequality, we first need some basic notation. Given two subset $A, B \subset \mathbb{R}^n$, and $c > 0$, we define the set sum and scalar multiple by

$$A + B := \{a + b : a \in A,\ b \in B\}, \quad cA := \{ca : a \in A\}. \tag{1.1}$$

We shall use $|E|$ to denote the Lebesgue measure of a set $E$. (If $E$ is not measurable, $|E|$ denotes the outer Lebesgue measure of $E$.) The Brunn-Minkowski inequality says that, given $A, B \subset \mathbb{R}^n$ measurable sets,

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}. \tag{1.2}$$

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[1]The University of Texas at Austin, Mathematics Dept. RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712-1202, USA. E-mail: figalli@math.utexas.edu
[2]Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307, USA. E-mail: jerison@math.mit.edu
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In addition, if $|A|, |B| > 0$, then equality holds if and only if there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$, $\lambda_A, \lambda_B > 0$, and $v_A, v_B \in \mathbb{R}^n$, such that

$$A \subset \lambda_A \mathcal{K} + v_A, \quad B \subset \lambda_B \mathcal{K} + v_B, \quad |(\lambda_A \mathcal{K} + v_A) \setminus A| = |(\lambda_B \mathcal{K} + v_B) \setminus B| = 0.$$

In other words, if equality holds in (1.2), then $A$ and $B$ are subsets of full measure in homothetic convex sets.

Because of the variety of applications of (1.2) as well as the fact the one can characterize the case of equality, a natural stability question that one would like to address is the following.

Let $A, B$ be two sets for which equality in (1.2) almost holds. Is it true that, up to translations and dilations, $A$ and $B$ are close to the same convex set?

This question has a long history. First of all, when $n = 1$ and $A = B$, inequality (1.2) reduces to $|A + A| \geq 2|A|$. If one approximates sets in $\mathbb{R}$ with finite unions of intervals, then one can translate the problem to $\mathbb{Z}$, and in the discrete setting the question becomes a well studied problem in additive combinatorics. There are many results on this topic, usually called Freiman-type theorems. The precise statement in one dimension is the following.

**Theorem 1.1** *Let $A \subset \mathbb{R}$ be a measurable set, and denote by* $\mathrm{co}(A)$ *its convex hull. Then*

$$|A + A| - 2|A| \geq \min\{|\,\mathrm{co}(A) \setminus A|, |A|\},$$

*or, equivalently, if $|A| > 0$, then*

$$\delta(A) \geq \frac{1}{2} \min\Big\{\frac{|\,\mathrm{co}(A) \setminus A|}{|A|}, 1\Big\}.$$

This theorem can be obtained as a corollary of a result of Freiman [12] about the structure of additive subsets of $\mathbb{Z}$ (see [13] or [17, Theorem 5.11] for a statement and a proof). However, it turns out that to prove Theorem 1.1, one only needs weaker results, and one can find an elementary self-contained proof of Theorem 1.1 in [8, Section 2].

In the case $n = 1$ but $A \neq B$, the following sharp stability result holds again as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman's theorem in [3–4].

**Theorem 1.2** *Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A + B| < |A| + |B| + \delta$ for some $\delta \leq \min\{|A|, |B|\}$, then $|\,\mathrm{co}(A) \setminus A| \leq \delta$ and $|\,\mathrm{co}(B) \setminus B| \leq \delta$.*

Concerning the higher dimensional case, in [1–2], Christ proved a qualitative stability result for (1.2), giving a positive answer to the stability question raised above. However, his results do not provide any quantitative control.

On the quantitative side, Diskant [5] and Groemer [15] obtained some stability results for convex sets in terms of the Hausdorff distance. More recently, in [10–11], the first author together with Maggi and Pratelli obtained a sharp stability result in terms of the $L^1$ distance, still on convex sets. Since this last result will be used later in our proofs, we state it in detail. (Here and from now on, $E \Delta F$ denotes the symmetric difference between sets $E$ and $F$, that is, $E \Delta F = (E \setminus F) \cup (F \setminus E)$.)

**Theorem 1.3** *Let $A, B \subset \mathbb{R}^n$ be convex sets, and define*

$$\mathscr{A}(A,B) := \inf_{x_0 \in \mathbb{R}^n} \Big\{ \frac{|A\Delta(x_0+\tau B)|}{|A|} \,:\, \tau = \Big(\frac{|A|}{|B|}\Big)^{\frac{1}{n}} \Big\}, \quad \sigma(A,B) := \max\Big\{\frac{|A|}{|B|}, \frac{|B|}{|A|}\Big\}.$$

*There exists a computable dimensional constant $C_0(n)$ such that*

$$|A+B|^{\frac{1}{n}} \geq (|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}})\Big\{1 + \frac{\mathscr{A}(A,B)^2}{C_0(n)\,\sigma(A,B)^{\frac{1}{n}}}\Big\}.$$

More recently, in [8, Theorem 1.2 and Remark 3.2], the present authors proved a quantitative stability result when $A = B$: Given a measurable set $A \subset \mathbb{R}^n$ with $|A| > 0$, set

$$\delta(A) := \frac{\left|\frac{1}{2}(A+A)\right|}{|A|} - 1 = \frac{|A+A|}{|2A|} - 1. \tag{1.3}$$

Then, a power of $\delta(A)$ dominates the measure of the difference between $A$ and its convex hull $\mathrm{co}(A)$.

**Theorem 1.4** *Let $A \subset \mathbb{R}^n$ be a measurable set of positive measure. There exist computable dimensional constants $\delta_n, c_n > 0$, such that if $\delta(A) \leq \delta_n$, then*

$$\delta(A)^{\alpha_n} \geq c_n \frac{|\operatorname{co}(A)\setminus A|}{|A|}, \quad \alpha_n := \frac{1}{8^{n-1} n! [(n-1)!]^2}.$$

*In addition, there exists a convex set $K \subset \mathbb{R}^n$ such that*

$$\delta(A)^{n\alpha_n} \geq c_n \frac{|K\Delta A|}{|A|}.$$

After that, we investigated the general case $A \neq B$. Notice that, after a dilation, one can always assume $|A| = |B| = 1$ while replacing the sum $A+B$ by a convex combination $S_t := tA + (1-t)B$. It follows by (1.2) that $|S_t| = 1 + \delta$ for some $\delta \geq 0$. The main theorem in [9] is a quantitative version of Christ's result. Since the proof is by induction on the dimension, it is convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of $\delta$. Here is the main result of that paper.

**Theorem 1.5** *Let $n \geq 2$, let $A, B \subset \mathbb{R}^n$ be measurable sets, and define $S_t := tA + (1-t)B$ for some $t \in [\tau, 1-\tau]$, $0 < \tau \leq \frac{1}{2}$. There are computable dimensional constants $N_n$ and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$, such that if*

$$||A|-1| + ||B|-1| + ||S_t|-1| \leq \delta \tag{1.4}$$

*for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,*

$$A, B \subset \mathcal{K} \quad \text{and} \quad |\mathcal{K}\setminus A| + |\mathcal{K}\setminus B| \leq \tau^{-N_n}\delta^{\varepsilon_n(\tau)}.$$

*Explicitly, we may take*

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log\tau|^{3^n}}{\tau^{3^n}}, \quad \varepsilon_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log\tau|^{3^n}}.$$

In particular, the measure of the difference between the sets $A$ and $B$ and their convex hull is bounded by a power $\delta^\epsilon$, confirming a conjecture of Christ [1].

The result above provides a general quantitative stability for the Brunn-Minkowski inequality in arbitrary dimension. However, the exponent degenerates very quickly as the dimension increases (much faster than in Theorem 1.4), and, in addition, the argument in [9] is very long and involved. The aim of this paper is to provide a shorter and more elementary proof when $|A| = |B| > 0$, that we believe to be of independent interest.

After a dilation, one can assume with no loss of generality that $|A| = |B| = 1$. In this case, it follows by (1.2) that $|\frac{1}{2}(A+B)| = 1+\delta$ for some $\delta \geq 0$, and we want to show that a power of $\delta$ controls the closeness of $A$ and $B$ to the same convex set $\mathcal{K}$. Again, as in the previous theorem, it will be convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of $\delta$.

Here is the main result of this paper.

**Theorem 1.6** *Let $A, B \subset \mathbb{R}^n$ be measurable sets, and define their semi-sum $S := \frac{1}{2}(A+B)$. There exist computable dimensional constants $\delta_n, C_n > 0$, such that if*

$$\big||A|-1\big| + \big||B|-1\big| + \big||S|-1\big| \leq \delta \tag{1.5}$$

*for some $\delta \leq \delta_n$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,*

$$A, B \subset \mathcal{K} \quad \textit{and} \quad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq C_n \delta^{\beta_n},$$

*where*

$$\beta_1 := 1, \quad \beta_n := \frac{1}{2^{6n-5}3^{n-1}n!(n-1)!}\prod_{k=1}^n \alpha_k^2 \quad \forall\, n \geq 2,$$

*and $\alpha_k$ is given by Theorem* 1.4. (*Recall that $|S|$ is the outer measure of $S$ if $S$ is not measurable.*)

The proof of this theorem is specific to the case $|A|$ near $|B|$. It uses a symmetrization and other techniques introduced by Christ [2–3], Theorems 1.3–1.4, and two propositions of independent interest, Propositions 2.1–2.2 below. See Section 3 for further discussion of the strategy of the proof.

## 2 Notation and Preliminary Results

Let $\mathcal{H}^k$ denote the $k$-dimensional Hausdorff measure on $\mathbb{R}^n$. Denote by $x = (y,t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ a point in $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\overline{\pi} : \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections, i.e.,

$$\pi(y,t) := y \quad \text{and} \quad \overline{\pi}(y,t) := t.$$

Given a compact set $E \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, and $\lambda > 0$, we use the notation

$$E_y := E \cap \pi^{-1}(y) \subset \{y\}\times\mathbb{R}, \quad E(t) := E \cap \overline{\pi}^{-1}(t) \subset \mathbb{R}^{n-1}\times\{t\}, \tag{2.1}$$

$$\mathcal{E}(\lambda) := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_y) > \lambda\}. \tag{2.2}$$

Following Christ [2], we consider two symmetrizations and combine them. For our purposes (see the proof of Proposition 2.1), it is convenient to use a definition of Schwarz symmetrization that is slightly different from the classical one. (In the usual definition of Schwarz symmetrization, $E^*(t) = \emptyset$ whenever $\mathcal{H}^{d-1}(E(t)) = 0$.)

**Definition 2.1** *Let $E \subset \mathbb{R}^n$ be a compact set. We define the Schwarz symmetrization $E^*$ of $E$ as follows. For each $t \in \mathbb{R}$,*

*(1) If $\mathcal{H}^{d-1}(E(t)) > 0$, then $E^*(t)$ is the closed disk centered at $0 \in \mathbb{R}^{n-1}$ with the same measure.*

*(2) If $\mathcal{H}^{d-1}(E(t)) = 0$ but $E(t)$ is non-empty, then $E^*(t) = \{0\}$.*

*(3) If $E(t)$ is empty, then $E^*(t)$ is empty as well.*

*We define the Steiner symmetrization $E^\star$ of $E$ so that for each $y \in \mathbb{R}^{n-1}$, the set $E^\star_y$ is empty if $\mathcal{H}^1(E_y) = 0$; otherwise it is the closed interval of length $\mathcal{H}^1(E_y)$ centered at $0 \in \mathbb{R}$. Finally, we define $E^\natural := (E^\star)^*$.*

As for instance in [2, Section 2], both the Schwarz and the Steiner symmetrization preserve the measure of sets, and the $\natural$-symmetrization preserves the measure of the sets $\mathcal{E}(\lambda)$. The following statement collects all these results.

**Lemma 2.1** *Let $A, B \subset \mathbb{R}^n$ be compact sets. Then $|A| = |A^*| = |A^\star| = |A^\natural|$,*

$$|A^* + B^*| \leq |A + B|, \quad |A^\star + B^\star| \leq |A + B|, \quad |A^\natural + B^\natural| \leq |A + B|,$$

*and, for almost every $\lambda > 0$,*

$$|A \setminus \pi^{-1}(\mathcal{A}(\lambda))| = |A^\natural \setminus \pi^{-1}(\mathcal{A}^\natural(\lambda))|, \quad \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) = \mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)),$$

*where $\mathcal{A}(\lambda) := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A_y) > \lambda\}$, $\mathcal{A}^\natural(\lambda) := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A^\natural_y) > \lambda\}$.*

Another important fact is that a bound on the measure of $A + B$ in terms of the measures of $A$ and $B$ gives bounds relating the sizes of

$$\sup_y \mathcal{H}^1(A_y), \quad \sup_y \mathcal{H}^1(B_y), \quad \mathcal{H}^{n-1}(\pi(A)), \quad \mathcal{H}^{n-1}(\pi(B)).$$

We refer to [9, Lemma 3.2] for a proof.

**Lemma 2.2** *Let $A, B \subset \mathbb{R}^n$ be compact sets such that $|A|, |B| \geq \frac{1}{2}$ and $|\frac{1}{2}(A + B)| \leq 2$. There exists a dimensional constant $M > 1$, such that*

$$\frac{\sup_y \mathcal{H}^1(A_y)}{\sup_y \mathcal{H}^1(B_y)} \in \Big(\frac{1}{M}, M\Big), \quad \frac{\mathcal{H}^{n-1}(\pi(A))}{\mathcal{H}^{n-1}(\pi(B))} \in \Big(\frac{1}{M}, M\Big),$$
$$\Big(\sup_y \mathcal{H}^1(A_y)\Big)\mathcal{H}^{n-1}(\pi(A)) \in \Big(\frac{1}{M}, M\Big), \quad \Big(\sup_y \mathcal{H}^1(B_y)\Big)\mathcal{H}^{n-1}(\pi(B)) \in \Big(\frac{1}{M}, M\Big).$$

*Thus, up a measure preserving affine transformation of the form $(y, t) \mapsto (\tau y, \tau^{1-n} t)$ with $\tau > 0$, all the quantities $\sup_y \mathcal{H}^1(A_y)$, $\sup_y \mathcal{H}^1(B_y)$, $\mathcal{H}^{n-1}(\pi(A))$, $\mathcal{H}^{n-1}(\pi(B))$ are of order one.*

*In particular,*

$$\mathcal{H}^{n-1}(\pi(A)) + \mathcal{H}^{n-1}(\pi(B)) + \sup_y \mathcal{H}^1(A_y) + \sup_y \mathcal{H}^1(B_y) \le M. \tag{2.3}$$

*In this case, we say that $A$ and $B$ are $M$-normalized.*

The following result of Christ [1, Lemma 4.1] shows that $\sup_t \mathcal{H}^{n-1}(A(t))$ and $\sup_t \mathcal{H}^{n-1}(B(t))$ are close in terms of $\delta$.

**Lemma 2.3** *Let $A, B \subset \mathbb{R}^n$ be compact sets, define $S := \frac{1}{2}(A+B)$, and assume that* (1.5) *holds for some $\delta \le \frac{1}{2}$. Also, suppose that $A$ and $B$ are $M$-normalized as defined in Lemma* 2.2. *Then, there exists a dimensional constant $C > 0$ such that*

$$\frac{\sup_t \mathcal{H}^{n-1}(A(t))}{\sup_t \mathcal{H}^{n-1}(B(t))} \in (1 - C\delta^{\frac{1}{2}}, 1 + C\delta^{\frac{1}{2}}).$$

Two other key ingredients in our proof of Theorem 1.6 are the following propositions, whose proofs are postponed to Section 4.

**Proposition 2.1** *Let $A, B \subset \mathbb{R}^n$ be compact sets, define $S := \frac{1}{2}(A+B)$, and assume that* (1.5) *holds for some $\delta \le \frac{1}{2}$. Also, suppose that we can find a convex set $K \subset \mathbb{R}^n$ such that*

$$|S \Delta K| \le C\delta^{\alpha}$$

*for some $\alpha > 0$, where $C > 0$ is a dimensional constant. Then there exists a dimensional constant $C' > 0$ such that*

$$|\operatorname{co}(S) \setminus S| \le C'\delta^{\frac{\alpha}{2n}}.$$

**Proposition 2.2** *Let $A, B \subset \mathbb{R}^n$ be compact sets, define $S := \frac{1}{2}(A+B)$, and assume that* (1.5) *holds for some $\delta \le \frac{1}{2}$. Also, suppose that*

$$|\operatorname{co}(S) \setminus S| \le C\delta^{\beta} \tag{2.4}$$

*for some $\beta > 0$, where $C > 0$ is a dimensional constant. Then, up to a translation,*

$$|A \Delta B| \le C'\delta^{\frac{\beta}{2}},$$

*and there exists a convex set $\mathcal{K}$ containing both $A$ and $B$ such that*

$$|\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \le C'\delta^{\frac{\beta}{2n}}$$

*for some dimensional constant $C' > 0$.*

## 3 Proof of Theorem 1.6

As explained in [8], by inner approximation[1] it suffices to prove the result when $A, B$ are compact sets. Hence, let $A$ and $B$ be compact, define $S := \frac{1}{2}(A+B)$, and assume that (1.5)

[1]The approximation of $A$ (and analogously for $B$) is by a sequence of compact sets $A_k \subset A$ such that $|A_k| \to |A|$ and $|\operatorname{co}(A_k)| \to |\operatorname{co}(A)|$. One way to construct such sets is to define $A_k := A'_k \cup V_k$, where $A'_k \subset A$ are compact sets satisfying $|A'_k| \to |A|$, and $V_k \subset V_{k+1} \subset A$ are finite sets satisfying $|\operatorname{co}(V_k)| \to |\operatorname{co}(A)|$.

holds. We want to prove that there exists a convex set $\mathcal{K}$ such that, up to a translation,

$$A, B \subset \mathcal{K}, \quad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \le C_n \delta^{\beta_n}.$$

Moreover, since the statement and the conclusions are invariant under measure preserving affine transformations, by Lemma 2.2, we can assume that $A$ and $B$ are $M$-normalized (see (2.3)).

Ultimately, we wish to show that, up to translation, each of $A$, $B$, and $S$ is of nearly full measure in the same convex set. The strategy of the proof is to show first that $S$ is close to a convex set, and then apply Propositions 2.1–2.2. To obtain the closeness of $S$ to a convex set, we would like prove that $|\frac{1}{2}(S+S)|$ is close to $|S|$ and then apply Theorem 1.4. It is simpler, however, to construct a subset $\overline{S} \subset S$, such that $|S \setminus \overline{S}|$ is small and $|\frac{1}{2}(\overline{S}+\overline{S})|$ is close to $|\overline{S}|$.

To carry out our argument, one important ingredient will be to use the inductive hypothesis on the level sets $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ defined in (2.2). However, two difficulties arise here: First of all, to apply the inductive hypothesis, we need to know that $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ are close. In addition, the Brunn-Minkowski inequality does not have a natural proof by induction unless the measures of all the level sets $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ are the nearly same (see (3.11) below). Hence, it is important for us to have a preliminary quantitative estimate on the difference between $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ for most $\lambda > 0$. For this, we follow an approach used first in [2] and readapted in [9], in which we begin by showing our theorem in the special case of symmetrized sets $A = A^\natural$ and $B = B^\natural$ (recall Definition 2.1). Thanks to Lemma 2.1, this will give us the desired closeness between $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ for most $\lambda > 0$, which allows us to apply the strategy described above and prove the theorem in the general case.

Throughout the proof, $C$ will denote a generic constant depending only on the dimension, which may change from line to line.

### 3.1 The case $A = A^\natural$ and $B = B^\natural$

Let $A, B \subset \mathbb{R}^n$ be compact sets satisfying $A = A^\natural$, $B = B^\natural$. Since

$$\pi(A(t)) \subset \pi(A(0)) = \pi(A) \text{ and } \pi(B(t)) \subset \pi(B(0)) = \pi(B) \text{ are disks centered at the origin,}$$

applying Lemma 2.3, we deduce that

$$\mathcal{H}^{n-1}(\pi(A)\Delta\pi(B)) \le C\delta^{\frac{1}{2}}. \tag{3.1}$$

Hence, if we define

$$\overline{S} := \bigcup_{y\in\pi(A)\cap\pi(B)} \frac{A_y + B_y}{2},$$

then $\overline{S}_y \subset S_y$ for all $y \in \mathbb{R}^{n-1}$. In addition, using (1.5), (2.3), and (3.1), we have

$$\begin{aligned}
1 + \delta \ge |S| &= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(S_y)\,\mathrm{d}y \ge \int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(S_y)\,\mathrm{d}y \ge \int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(\overline{S}_y)\,\mathrm{d}y \\
&= |\overline{S}| \ge \frac{1}{2}\int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(A_y)\,\mathrm{d}y + \frac{1}{2}\int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(B_y)\,\mathrm{d}y \\
&\ge \frac{|A|+|B|}{2} - M\mathcal{H}^{n-1}(\pi(A)\Delta\pi(B)) \ge 1 - C\delta^{\frac{1}{2}},
\end{aligned}$$

which implies (since $\overline{S} \subset S$)

$$|S \setminus \overline{S}| \leq C\delta^{\frac{1}{2}}. \tag{3.2}$$

Furthermore, since each section $S_y$ is an interval centered at $0 \in \mathbb{R}$, for all $y', y'' \in \pi(A) \cap \pi(B)$ such that $\frac{y'+y''}{2} = y$,

$$\overline{S}_{y'} + \overline{S}_{y''} = \frac{A_{y'} + B_{y'}}{2} + \frac{A_{y''} + B_{y''}}{2} = \frac{A_{y'} + B_{y''}}{2} + \frac{A_{y''} + B_{y'}}{2} \subset S_y + S_y = 2S_y,$$

which gives

$$\frac{\overline{S} + \overline{S}}{2} \subset S. \tag{3.3}$$

Recalling (1.3), by (3.2)–(3.3), we obtain that $\delta(\overline{S}) \leq C\delta^{\frac{1}{2}}$. Hence, we can apply Theorem 1.4 to $\overline{S}$ to find a convex set $\overline{K}$ such that

$$|\overline{S} \Delta \overline{K}| \leq C\delta^{\frac{n\alpha_n}{2}}.$$

Hence, by (3.3),

$$|S \Delta \overline{K}| \leq C\delta^{\frac{n\alpha_n}{2}},$$

and using Propositions 2.1–2.2, we deduce that, up to a translation, there exists a convex set $K$ such that $A \cup B \subset K$ and

$$|A \Delta B| \leq C\delta^{\frac{\alpha_n}{8}}, \quad |K \setminus A| + |K \setminus B| \leq C\delta^{\frac{\alpha_n}{8n}}. \tag{3.4}$$

Notice that, because $A = A^\natural$ and $B = B^\natural$, it is easy to check that the above properties still hold with $K^\natural$ in place of $K$. Hence, in this case, without loss of generality, one can assume that $K = K^\natural$.

### 3.2 The general case

Since, by Theorem 1.2, the result is true when $n = 1$, we may assume that we already proved Theorem 1.6 through $n - 1$, and we want to show its validity for $n$.

**Step 1** There exist a dimensional constant $\zeta > 0$ and $\overline{\lambda} \sim \delta^\zeta$ such that we can apply the inductive hypothesis to $\mathcal{A}(\overline{\lambda})$ and $\mathcal{B}(\overline{\lambda})$.

Let $A^\natural$ and $B^\natural$ be as in Definition 2.1 and denote

$$\overline{\alpha} := \frac{\alpha_n}{8}. \tag{3.5}$$

Thanks to Lemma 2.1, $A^\natural$ and $B^\natural$ still satisfy (1.5), so we can apply the result proved in Section 3.1 above to get (see (3.4))

$$\int_{\mathbb{R}^{n-1}} |\mathcal{H}^1(A^\natural_y) - \mathcal{H}^1(B^\natural_y)| \,\mathrm{d}y \leq \int_{\mathbb{R}^{n-1}} |\mathcal{H}^1(A^\natural_y \Delta B^\natural_y)| \,\mathrm{d}y = |A^\natural \Delta B^\natural| \leq C\delta^{\overline{\alpha}} \tag{3.6}$$

and

$$K \supset A^\natural \cup B^\natural, \quad |K \setminus A^\natural| + |K \setminus B^\natural| \leq C\delta^{\frac{\overline{\alpha}}{n}} \tag{3.7}$$

for some convex set $K = K^\natural$.

In addition, because $A$ and $B$ are $M$-normalized (see (2.3)), so are $A^\natural$ and $B^\natural$, and by (3.7) we deduce that there exists a dimensional constant $R_n > 0$ such that

$$K \subset B_{R_n}. \tag{3.8}$$

Also, by (3.6) and Chebyshev's inequality, we obtain that, except for a set of measure $\le C\delta^{\frac{\overline{\alpha}}{2}}$,

$$|\mathcal{H}^1(A^\natural_y) - \mathcal{H}^1(B^\natural_y)| \le \delta^{\frac{\overline{\alpha}}{2}}.$$

Thus, recalling Lemma 2.1, for almost every $\lambda > 0$,

$$\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) = \mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)) \le \mathcal{H}^{n-1}(\mathcal{B}^\natural(\lambda - \delta^{\frac{\overline{\alpha}}{2}})) + C\delta^{\frac{\overline{\alpha}}{2}} = \mathcal{H}^{n-1}(\mathcal{B}(\lambda - \delta^{\frac{\overline{\alpha}}{2}})) + C\delta^{\frac{\overline{\alpha}}{2}}.$$

Since, by (2.3),

$$\int_0^M \left(\mathcal{H}^{n-1}(\mathcal{B}(\lambda)) - \mathcal{H}^{n-1}(\mathcal{B}(\lambda + \delta^{\frac{\overline{\alpha}}{2}}))\right) \mathrm{d}\lambda = \int_0^{\delta^{\frac{\overline{\alpha}}{2}}} \mathcal{H}^{n-1}(\mathcal{B}(\lambda))\, \mathrm{d}\lambda \le M\delta^{\frac{\overline{\alpha}}{2}},$$

by Chebyshev's inequality, we deduce that

$$\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \le \mathcal{H}^{n-1}(\mathcal{B}(\lambda)) + C\delta^{\frac{\overline{\alpha}}{4}}$$

for all $\lambda$ outside a set of measure $\le C\delta^{\frac{\overline{\alpha}}{4}}$. Exchanging the roles of $A$ and $B$, we obtain that there exists a set $F \subset [0, M]$, such that

$$\mathcal{H}^1(F) \le C\delta^{\frac{\overline{\alpha}}{4}}, \quad |\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) - \mathcal{H}^{n-1}(\mathcal{B}(\lambda))| \le C\delta^{\frac{\overline{\alpha}}{4}}, \quad \forall\, \lambda \in [0, \infty] \setminus F. \tag{3.9}$$

Using the elementary inequality

$$\left(\frac{a+b}{2}\right)^{n-1} \ge \frac{a^{n-1} + b^{n-1}}{2} - C|a-b|^2, \quad \forall\, 0 \le a, b \le M,$$

and replacing $a$ and $b$ with $a^{\frac{1}{n-1}}$ and $b^{\frac{1}{n-1}}$, respectively, we get

$$\left(\frac{a^{\frac{1}{n-1}} + b^{\frac{1}{n-1}}}{2}\right)^{n-1} \ge \frac{a+b}{2} - C|a-b|^{\frac{2}{n-1}}, \quad \forall\, 0 \le a, b \le M \tag{3.10}$$

(notice that $|a^{\frac{1}{n-1}} - b^{\frac{1}{n-1}}| \le |a-b|^{\frac{1}{n-1}}$). Finally, it is easy to check that

$$\frac{\mathcal{A}(\lambda) + \mathcal{B}(\lambda)}{2} \subset \mathcal{S}(\lambda), \quad \forall\, \lambda > 0.$$

Hence, by the Brunn-Minkowski inequality (1.2) applied to $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, using (1.5), (2.3) and (3.9)–(3.10), we get

$$
\begin{aligned}
1+\delta \geq |S| &= \int_0^M \mathcal{H}^{n-1}(\mathcal{S}(\lambda))\,\mathrm{d}\lambda \\
&\geq \frac{1}{2^{n-1}} \int_0^M (\mathcal{H}^{n-1}(\mathcal{A}(\lambda))^{\frac{1}{n-1}} + \mathcal{H}^{n-1}(\mathcal{B}(\lambda))^{\frac{1}{n-1}})^{n-1}\,\mathrm{d}\lambda \\
&\geq \frac{1}{2}\int_0^M (\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) + \mathcal{H}^{n-1}(\mathcal{B}(\lambda)))\,\mathrm{d}\lambda \\
&\quad - C\int_0^M |\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) - \mathcal{H}^{n-1}(\mathcal{B}(\lambda))|^{\frac{2}{n-1}}\,\mathrm{d}\lambda \\
&= \frac{|A|+|B|}{2} - C\delta^{\frac{\overline{\alpha}}{2(n-1)}} \\
&\geq 1 - C\delta^{\frac{\overline{\alpha}}{2(n-1)}}.
\end{aligned}
\tag{3.11}
$$

We also observe that, since $K = K^\natural$, by Lemma 2.1, (3.8), and [2, Lemma 4.3], for almost every $\lambda > 0$, we have

$$
\begin{aligned}
|A \setminus \pi^{-1}(\mathcal{A}(\lambda))| &= |A^\natural \setminus \pi^{-1}(\mathcal{A}^\natural(\lambda))| \\
&\leq |K \setminus \pi^{-1}(\mathcal{K}(\lambda))| + M\,\mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)) \\
&\leq C\lambda^2 + M\,\mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)),
\end{aligned}
\tag{3.12}
$$

and analogously for $B$. Also, by (3.7),

$$
\int_0^M (\mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)) + \mathcal{H}^{n-1}(\mathcal{B}^\natural(\lambda)\Delta\mathcal{K}(\lambda)))\,\mathrm{d}\lambda \leq |K\setminus A^\natural| + |K \setminus B^\natural| \leq C\delta^{\frac{\overline{\alpha}}{n}}. \tag{3.13}
$$

Define

$$
\eta := \min\Big\{\frac{\overline{\alpha}}{2(n-1)}, \frac{\overline{\alpha}}{4}\Big\}, \tag{3.14}
$$

and note that $\eta \leq \frac{\overline{\alpha}}{n}$. Let $\zeta \in (0,\eta)$ to be fixed later. Then by (3.9), (3.11)–(3.13), and by Chebyshev's inequality, we can find a level

$$
\overline{\lambda} \in [10\delta^\zeta, 20\delta^\zeta], \tag{3.15}
$$

such that

$$
|\mathcal{H}^{n-1}(\mathcal{A}(\overline{\lambda})) - \mathcal{H}^{n-1}(\mathcal{B}(\overline{\lambda}))| \leq C\delta^\eta, \tag{3.16}
$$

$$
2^{n-1}\mathcal{H}^{n-1}(\mathcal{S}(\overline{\lambda})) \leq (\mathcal{H}^{n-1}(\mathcal{A}(\overline{\lambda}))^{\frac{1}{n-1}} + \mathcal{H}^{n-1}(\mathcal{B}(\overline{\lambda}))^{\frac{1}{n-1}})^{n-1} + C\delta^{\eta-\zeta}, \tag{3.17}
$$

$$
|A \setminus \pi^{-1}(\mathcal{A}(\overline{\lambda}))| + |B \setminus \pi^{-1}(\mathcal{B}(\overline{\lambda}))| \leq C(\delta^{2\zeta} + \delta^{\eta-\zeta}). \tag{3.18}
$$

In addition, from the properties $\mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \leq M$ for any $\lambda > 0$ (see (2.3)), $\int_0^M \mathcal{H}^{n-1}(\mathcal{A}(\lambda))\,\mathrm{d}\lambda = |A| \geq 1-\delta$, and $s \mapsto \mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ is a decreasing function, we deduce that

$$
\frac{1}{2M} \leq \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \leq M, \quad \forall\,\lambda \in (0,(2M)^{-1}).
$$

The same holds for $B$ and $S$, hence

$$
\mathcal{H}^{n-1}(\mathcal{S}(\overline{\lambda})),\ \mathcal{H}^{n-1}(\mathcal{A}(\overline{\lambda})),\ \mathcal{H}^{n-1}(\mathcal{B}(\overline{\lambda})) \in [(2M)^{-1}, M]
$$

provided that $\delta$ is small enough. Set $\rho := \frac{1}{\mathcal{H}^{n-1}(\mathcal{A}(\overline{\lambda}))^{\frac{1}{n-1}}} \in \big[\frac{1}{M^{\frac{1}{n-1}}}, (2M)^{\frac{1}{n-1}}\big]$, and define

$$A' := \rho\mathcal{A}(\overline{\lambda}), \quad B' := \rho\mathcal{B}(\overline{\lambda}), \quad S' := \rho\mathcal{S}(\overline{\lambda}).$$

By (3.16)–(3.17), we get

$$\mathcal{H}^{n-1}(A') = 1, \quad |\mathcal{H}^{n-1}(B') - 1| \leq C\delta^{\eta}, \quad \mathcal{H}^{n-1}(S') \leq 1 + C\delta^{\eta-\zeta},$$

while, by (1.2),

$$\mathcal{H}^{n-1}(S')^{\frac{1}{n-1}} \geq \frac{\mathcal{H}^{n-1}(A')^{\frac{1}{n-1}} + \mathcal{H}^{n-1}(B')^{\frac{1}{n-1}}}{2} \geq 1 - C\delta^{\eta},$$

therefore

$$|\mathcal{H}^{n-1}(A') - 1| + |\mathcal{H}^{n-1}(B') - 1| + |\mathcal{H}^{n-1}(S') - 1| \leq C\delta^{\eta-\zeta}.$$

Thus, by the inductive hypothesis of Theorem 1.6, up to a translation there exists an $(n-1)$-dimensional convex set $\Omega'$, such that

$$\Omega' \supset A' \cup B', \quad \mathcal{H}^{n-1}(\Omega' \setminus A') + \mathcal{H}^{n-1}(\Omega' \setminus B') \leq C\delta^{(\eta-\zeta)\beta_{n-1}},$$

and defining $\Omega := \frac{\Omega'}{\rho}$ we obtain (recall that $\frac{1}{\rho} \leq M^{\frac{1}{n-1}}$)

$$\Omega \supset \mathcal{A}(\overline{\lambda}) \cup \mathcal{B}(\overline{\lambda}), \quad \mathcal{H}^{n-1}(\Omega \setminus \mathcal{A}(\overline{\lambda})) + \mathcal{H}^{n-1}(\Omega \setminus \mathcal{B}(\overline{\lambda})) \leq C\delta^{(\eta-\zeta)\beta_{n-1}}. \tag{3.19}$$

**Step 2** We apply Theorem 1.2 to the sets $A_y$ and $B_y$ for most $y \in \mathcal{A}(\overline{\lambda}) \cap \mathcal{B}(\overline{\lambda})$.

Define $\mathcal{C} := \mathcal{A}(\overline{\lambda}) \cap \mathcal{B}(\overline{\lambda}) \subset \mathcal{S}(\overline{\lambda})$. By (3.18)–(3.19) and (2.3), we have

$$\begin{aligned}
|A \setminus \pi^{-1}(\mathcal{C})| + |B \setminus \pi^{-1}(\mathcal{C})| &\leq |A \setminus \pi^{-1}(\mathcal{A}(\overline{\lambda}))| + |B \setminus \pi^{-1}(\mathcal{B}(\overline{\lambda}))| \\
&\quad + \int_{(\mathcal{A}(\overline{\lambda}))\setminus(\mathcal{B}(\overline{\lambda}))} \mathcal{H}^1(A_y)\,\mathrm{d}y + \int_{(\mathcal{B}(\overline{\lambda}))\setminus(\mathcal{A}(\overline{\lambda})} \mathcal{H}^1(B_y)\,\mathrm{d}y \\
&\leq C(\delta^{2\zeta} + \delta^{\eta-\zeta}) + M(\mathcal{H}^{n-1}(\Omega \setminus \mathcal{A}(\overline{\lambda})) + \mathcal{H}^{n-1}(\Omega \setminus \mathcal{B}(\overline{\lambda}))) \\
&\leq C(\delta^{2\zeta} + \delta^{\eta-\zeta} + \delta^{(\eta-\zeta)\beta_{n-1}}) \leq C\delta^{2\zeta},
\end{aligned} \tag{3.20}$$

provided that we choose

$$\zeta := \frac{\eta\beta_{n-1}}{3} \tag{3.21}$$

(recall that $\beta_{n-1} \leq 1$). Hence, by (1.5) and (3.20),

$$\begin{aligned}
\int_{\mathcal{C}} \mathcal{H}^1\Big(S_y \Big\backslash \frac{A_y + B_y}{2}\Big)\,\mathrm{d}y &\leq \int_{\mathcal{C}} [\mathcal{H}^1(S_y) - \frac{1}{2}(\mathcal{H}^1(A_y) + \mathcal{H}^1(B_y))]\,\mathrm{d}y \\
&= |S \cap \pi^{-1}(\mathcal{C})| - \frac{|A \cap \pi^{-1}(\mathcal{C})| + |B \cap \pi^{-1}(\mathcal{C})|}{2} \\
&\leq |S| - \frac{|A| + |B|}{2} + \frac{|A \setminus \pi^{-1}(\mathcal{C})| + |B \setminus \pi^{-1}(\mathcal{C})|}{2} \\
&\leq C\delta^{2\zeta}.
\end{aligned} \tag{3.22}$$

Write $\mathcal{C}$ as $\mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \Big\{ y \in \mathcal{C} : 2\mathcal{H}^1(S_y) - \mathcal{H}^1(A_y) - \mathcal{H}^1(B_y) \le \frac{\delta^\zeta}{2} \Big\}, \quad \mathcal{C}_2 := \mathcal{C} \setminus \mathcal{C}_1.$$

By Chebyshev's inequality and (3.22),

$$\mathcal{H}^{n-1}(\mathcal{C}_2) \le C\delta^\zeta, \tag{3.23}$$

while, recalling (3.15),

$$\min\{\mathcal{H}^1(A_y), \mathcal{H}^1(B_y)\} \ge \overline{\lambda} \ge 10\delta^\zeta > \frac{\delta^\zeta}{2}, \quad \forall\, y \in \mathcal{C}_1.$$

Hence, by Theorem 1.2 applied to $A_y, B_y \subset \mathbb{R}$ for $y \in \mathcal{C}_1$, we deduce that

$$\mathcal{H}^1(\mathrm{co}(A_y) \setminus A_y) + \mathcal{H}^1(\mathrm{co}(B_y) \setminus B_y) \le \delta^\zeta. \tag{3.24}$$

Let $\widehat{\mathcal{C}}_1 \subset \mathcal{C}_1$ denote the set of $y \in \mathcal{C}_1$ such that

$$\mathcal{H}^1\Big( S_y \Big\backslash \frac{A_y + B_y}{2} \Big) \le \delta^\zeta, \tag{3.25}$$

and notice that, by (3.22) and Chebyshev's inequality, $\mathcal{H}^{n-1}(\mathcal{C}_1 \setminus \widehat{\mathcal{C}}_1) \le C\delta^\zeta$. Then choose a compact set $\mathcal{C}_1' \subset \widehat{\mathcal{C}}_1$ such that $\mathcal{H}^{n-1}(\widehat{\mathcal{C}}_1 \setminus \mathcal{C}_1') \le \delta^\zeta$ to obtain

$$\mathcal{H}^{n-1}(\mathcal{C}_1 \setminus \mathcal{C}_1') \le C\delta^\zeta. \tag{3.26}$$

**Step 3** We find $\overline{S} \subset S$, so that $|S \setminus \overline{S}|$ and $\delta(\overline{S})$ are small.

Define the compact set

$$\overline{S} := \bigcup_{y \in \mathcal{C}_1'} \frac{A_y + B_y}{2} \subset \mathbb{R}^n.$$

Observe, thanks to (3.20), (3.23), (3.26), (2.3) and (1.5),

$$\begin{aligned} 2|\overline{S}| &\ge \int_{\mathcal{C}_1'} \mathcal{H}^1(A_y)\,\mathrm{d}y + \int_{\mathcal{C}_1'} \mathcal{H}^1(B_y)\,\mathrm{d}y \\ &\ge |A| + |B| - |A \setminus \pi^{-1}(\mathcal{C})| - |B \setminus \pi^{-1}(\mathcal{C})| - M\,\mathcal{H}^{n-1}(\mathcal{C} \setminus \mathcal{C}_1') \\ &\ge 2|S| - C\delta^\zeta. \end{aligned}$$

So, since $\overline{S} \subset S$,

$$|S \Delta \overline{S}| \le C\delta^\zeta. \tag{3.27}$$

Now, we want to estimate the measure of $\frac{1}{2}(\overline{S} + \overline{S})$. First of all, since

$$S_y = \bigcup_{2y = y' + y''} \frac{A_{y'} + B_{y''}}{2}, \tag{3.28}$$

by (3.25), we get

$$\mathcal{H}^1\Big( \Big( \bigcup_{2y = y' + y''} \frac{A_{y'} + B_{y''}}{2} \Big) \Big\backslash \frac{A_y + B_y}{2} \Big) \le \delta^\zeta, \quad \forall\, y \in \mathcal{C}_1'. \tag{3.29}$$

Also, if we define the characteristic functions

$$\chi_y^A(\lambda) := \begin{cases} 1, & \text{if } (y,\lambda) \in A_y, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi_y^{A,*}(\lambda) := \begin{cases} 1, & \text{if } (y,\lambda) \in \mathrm{co}(A_y), \\ 0, & \text{otherwise,} \end{cases}$$

and analogously for $B_y$, by (3.24) we have the following estimate on their convolutions:

$$\begin{aligned} \|\chi_{y'}^{A,*} * \chi_{y''}^{B,*} - \chi_{y'}^{A} * \chi_{y''}^{B}\|_{L^\infty(\mathbb{R})} &\le \|\chi_{y''}^{B,*} - \chi_{y''}^{B}\|_{L^1(\mathbb{R})} + \|\chi_{y'}^{A,*} - \chi_{y'}^{A}\|_{L^1(\mathbb{R})} \\ &= \mathcal{H}^1(\mathrm{co}(B_{y''}) \setminus B_{y''}) + \mathcal{H}^1(\mathrm{co}(A_{y'}) \setminus A_{y'}) \\ &\le \delta^\zeta < 3\delta^\zeta, \quad \forall\, y', y'' \in \mathcal{C}_1'. \end{aligned} \tag{3.30}$$

Recalling that $\overline{\pi} : \mathbb{R}^n \to \mathbb{R}$ is the orthogonal projection onto the last component (that is, $\overline{\pi}(y,t) = t$), we denote by $[a,b]$ the interval $\overline{\pi}(\mathrm{co}(A_{y'}) + \mathrm{co}(B_{y''}))$, and notice that, since by construction

$$\min\{\mathcal{H}^1(A_y), \mathcal{H}^1(B_y)\} \ge \overline{\lambda} \ge 10\delta^\zeta, \quad \forall\, y \in \mathcal{C}_1'$$

(see (3.15)), this interval has length greater than $20\delta^\zeta$. Also, it is easy to check that the function $\chi_{y'}^{A,*} * \chi_{y''}^{B,*}$ is supported on $[a,b]$, has slope equal to 1 (resp. $-1$) in $[a, a+3\delta^\zeta]$ (resp. $[b-3\delta^\zeta, b]$), and it is greater than $3\delta^\zeta$ in $[a+3\delta^\zeta, b-3\delta^\zeta]$. Hence, since $\overline{\pi}(A_{y'} + B_{y''})$ contains the set $\{\chi_{y'}^{A} * \chi_{y''}^{B} > 0\}$, by (3.30), we deduce that

$$\overline{\pi}(A_{y'} + B_{y''}) \supset [a+3\delta^\zeta, b-3\delta^\zeta], \tag{3.31}$$

which implies in particular that

$$\mathcal{H}^1(\mathrm{co}(A_{y'}) + \mathrm{co}(B_{y''})) \le \mathcal{H}^1(A_{y'} + B_{y''}) + 6\delta^\zeta, \quad \forall\, y', y'' \in \mathcal{C}_1'. \tag{3.32}$$

Also, by the same argument as in [8, Step 2-a], if we denote by

$$[\alpha_y, \beta_y] := \overline{\pi}(\mathrm{co}(A_y) + \mathrm{co}(B_y)),$$

using (3.25) and (3.31), we have

$$\overline{\pi}(\mathrm{co}(A_{y'}) + \mathrm{co}(B_{y''})) \subset [\alpha_y - 16\delta^\zeta, \beta_y + 16\delta^\zeta], \quad \forall\, y', y'', y = \frac{y'+y''}{2} \in C_1'. \tag{3.33}$$

(Compare with [8, (3.25)].)

We now estimate the size of $\left[\frac{1}{2}(\overline{S} + \overline{S})\right]_y$. Observe that, for all $y \in C_1'$,

$$\begin{aligned} \left[\frac{1}{2}(\overline{S} + \overline{S})\right]_y &= \bigcup_{2y = y'+y'',\, y', y'' \in C_1'} \Big( \frac{\frac{1}{2}(A_{y'} + B_{y'}) + \frac{1}{2}(A_{y''} + B_{y''})}{2} \Big) \\ &= \bigcup_{2y = y'+y'',\, y', y'' \in C_1'} \Big( \frac{\frac{1}{2}(A_{y'} + B_{y''}) + \frac{1}{2}(A_{y''} + B_{y'})}{2} \Big) \\ &\subset \frac{1}{2}\Big( \bigcup_{2y = y'+y'',\, y', y'' \in C_1'} \frac{1}{2}(A_{y'} + B_{y''}) + \bigcup_{2y = y'+y'',\, y', y'' \in C_1'} \frac{1}{2}(A_{y'} + B_{y''}) \Big). \end{aligned}$$

Hence, by (3.33), we deduce that each of the latter sets is contained inside the convex set $\{y\} \times [\alpha_y - 16\delta^\zeta, \beta_y + 16\delta^\zeta]$, so also their semi-sum is contained in the same set, and using (3.32) with $y' = y'' = y$, we get

$$\begin{aligned} \mathcal{H}^1\Big(\Big[\frac{\overline{S}+\overline{S}}{2}\Big]_y\Big) &\le \mathcal{H}^1\Big(\frac{\mathrm{co}(A_y)+\mathrm{co}(B_y)}{2}\Big) + 16\delta^\zeta \\ &\le \mathcal{H}^1\Big(\frac{A_y+B_y}{2}\Big) + 22\delta^\zeta \\ &= \mathcal{H}^1(\overline{S}_y) + 22\delta^\zeta, \quad \forall\, y \in C_1'. \end{aligned} \tag{3.34}$$

In order to estimate $\big[\frac{1}{2}(\overline{S}+\overline{S})\big]_y$ when $y \in \frac{C_1'+C_1'}{2} \setminus C_1'$, we argue as follows. By (3.33) and the fact that $\mathcal{H}^1(\mathrm{co}(A_y))$ and $\mathcal{H}^1(\mathrm{co}(B_y))$ are universally bounded (see (2.3) and (3.24)), the following holds: If we denote by $c^A(y)$ the barycenter of $\mathrm{co}(A_y)$ (and analogously for $B$ and $\overline{S}$), we have

$$|c^A(y') + c^B(y'') - 2c^{\overline{S}}(y)| \le C, \quad \forall\, y, y', y'' \in \mathcal{C}_1',\ y = \frac{y'+y''}{2}$$

(notice that $\mathrm{co}(\overline{S}_y) = \mathrm{co}(A_y) + \mathrm{co}(B_y)$). Exchanging the role of $A$ and $B$ and adding up the two inequalities, we deduce that

$$|c^{\overline{S}}(y') + c^{\overline{S}}(y'') - 2c^{\overline{S}}(y)| \le C, \quad \forall\, y, y', y'' \in \mathcal{C}_1',\ y = \frac{y'+y''}{2}.$$

As shown in [8, Step 3], this estimate combined with the fact that $\mathcal{C}_1'$ is almost of full measure inside the convex set $\Omega$ (see (3.19), (3.23) and (3.26)) proves that, up to an affine transformation of the form

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (y,t) \mapsto (Ty, t - Ly) + (y_0, t_0) \tag{3.35}$$

with $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $L : \mathbb{R}^{n-1} \to \mathbb{R}$, $\det(T) = 1$, and $(y_0, t_0) \in \mathbb{R}^n$, the set $\overline{S}$ is universally bounded, say $\overline{S} \subset B_R$ for some dimensional constant $R$. This implies that $\big[\frac{1}{2}(\overline{S}+\overline{S})\big]_y \subset [-R, R]$, so $\mathcal{H}^1\big(\big[\frac{1}{2}(\overline{S}+\overline{S})\big]_y\big) \le 2R$.

Hence, since $\frac{1}{2}(\mathcal{C}_1' + \mathcal{C}_1') \subset \Omega$, by (3.34), (3.19) and (3.21),

$$\begin{aligned} \Big|\frac{\overline{S}+\overline{S}}{2} \setminus \overline{S}\Big| &= \int_{[\frac{1}{2}(\mathcal{C}_1'+\mathcal{C}_1')]\cap\mathcal{C}_1'} \mathcal{H}^1\Big(\Big[\frac{\overline{S}+\overline{S}}{2}\Big]_y\Big) - \mathcal{H}^1(\overline{S}_y)\,\mathrm{d}y \\ &\quad + \int_{[\frac{1}{2}(\mathcal{C}_1'+\mathcal{C}_1')]\setminus\mathcal{C}_1'} \mathcal{H}^1\Big(\Big[\frac{\overline{S}+\overline{S}}{2}\Big]_y\Big)\,\mathrm{d}y \\ &\le 22\delta^\zeta\,\mathcal{H}^{n-1}(\Omega) + 2R\,\mathcal{H}^{n-1}(\Omega \setminus \mathcal{C}_1') \le C\delta^\zeta, \end{aligned}$$

that is,

$$\delta(\overline{S}) \le C\delta^\zeta.$$

**Step 4** Conclusion

By the previous step, we have that $\delta(\overline{S}) \le C\delta^\zeta$. Hence, applying Theorem 1.4 to $\overline{S}$, we find a convex set $\overline{\mathcal{K}}$ such that

$$|\overline{S}\Delta\overline{\mathcal{K}}| \le C\delta^{n\alpha_n\zeta},$$

so, by (3.27),

$$|S\Delta\overline{\mathcal{K}}| \le C\delta^{n\alpha_n\zeta}.$$

Using this estimate together with Propositions 2.1–2.2, we deduce that, up to a translation, there exists a convex set $\mathcal{K}$ convex such that $A\cup B\subset\mathcal{K}$ and

$$|\mathcal{K}\setminus A| + |\mathcal{K}\setminus B| \le C\delta^{\frac{\alpha_n\zeta}{4n}}.$$

Recalling the definition of $\zeta$ (see (3.5), (3.14), (3.21)), we see that

$$\beta_n := \frac{\alpha_n\zeta}{4n} = \min\left\{\frac{1}{n-1}, \frac{1}{2}\right\}\frac{\alpha_n^2}{3\cdot 2^6 n}\,\beta_{n-1}.$$

Since $\beta_1 = 1$ (by Theorem 1.2), it is easy to check that

$$\beta_n = \frac{1}{2^{6n-5}3^{n-1}n!(n-1)!}\prod_{k=1}^{n}\alpha_k^2, \quad \forall\, n\ge 2,$$

concluding the proof.

## 4 Technical Results

As in the previous section, we use $C$ to denote a generic constant depending only on the dimension, which may change from line to line.

### 4.1 Proof of Proposition 2.1

Assume that

$$|S\Delta K| \le C\delta^{\alpha}$$

for some $\alpha\in(0,1]$. By John's lemma (see [16]), after a volume preserving affine transformation, we can assume that $B_{r_n}\subset K\subset B_{nr_n}$, with $r_n = r_n(K) > 0$ bounded above and below by positive dimensional constants. Note, however, that with this normalization, we will not be able to assume that $A$ and $B$ are $M$-normalized, since we have already chosen a different affine normalization.

We want to prove that

$$S\subset(1+C\delta^{\frac{\alpha}{2n}})K. \tag{4.1}$$

Let $\overline{x}_0\in S\setminus K$, and set $\rho := \mathrm{dist}(\overline{x}_0, K) = |\overline{x}_0 - \overline{x}_1|$ with $\overline{x}_1\in K$. Without loss of generality, we can assume that $\overline{x}_1 = \tau e_n$, for some $\tau>0$, $\overline{x}_0 = (\tau+\rho)e_n$, and $K\subset\{x_n\le\tau\}$. We need to prove that $\rho\le C\delta^{\frac{\alpha}{2n}}$.

Let us consider the sets $A^*$, $B^*$, $S^*$, $K^*$ obtained from $A$, $B$, $S$, $K$ performing a Schwarz symmetrization around the $e_n$-axis (see Definition 2.1). Set $S' := \frac{1}{2}(A^*+B^*)$. Since

$$|S^*\Delta K^*| \le |S\Delta K| \le C\delta^{\alpha},$$

and, by (1.5) (notice that $S'\subset S^*$ and that $|S'|\ge 1-C\delta$ by (1.2)),

$$|S^*\setminus S'| = |S^*| - |S'| = |S| - |S'| \le C\delta,$$

we get that $|S'\Delta K^*| \leq C\delta^\alpha$. In addition, $K^* \subset \{x_n \leq \tau\}$, $\overline{x}_1 \in K^*$ and $\overline{x}_0 \in S^*$. Hence, without loss of generality, we can assume from the beginning that $A = A^*$, $B = B^*$, $S = \frac{1}{2}(A^* + B^*)$ and $K = K^*$.

For a compact set $E \subset \mathbb{R}^n$, recall the notation $E(t) \subset \mathbb{R}^{n-1} \times \{t\}$ in (2.1), and define $E[s] \subset \mathbb{R}$ by

$$E[s] := \{t : \mathcal{H}^{n-1}(E(t)) \geq s\}. \tag{4.2}$$

Since $S = \frac{1}{2}(A + B)$, we have

$$\frac{A(t) + B(t)}{2} \subset S(t), \quad \forall\, t \in \mathbb{R},$$

so, by (1.2), we deduce that

$$S[s] \supset \frac{A[s] + B[s]}{2}, \quad \forall\, s > 0.$$

Hence

$$\mathcal{H}^1(A[s]) + \mathcal{H}^1(B[s]) \leq 2\mathcal{H}^1(S[s]), \quad \forall\, s > 0, \tag{4.3}$$

and integrating with respect to $s$, by (1.5), we get

$$4\delta \geq 2|S| - |A| - |B| = \int_0^\infty (2\mathcal{H}^1(S[s]) - \mathcal{H}^1(A[s]) - \mathcal{H}^1(B[s]))\, \mathrm{d}s. \tag{4.4}$$

Recall that $K = K^*$, so that the canonical projection $\pi(K)$ onto $\mathbb{R}^{n-1}$ is a ball. We denote it $B_R := \pi(K)$, and note that $R \leq nr_n$, with $r_n = r_n(K)$ given by John's lemma at the beginning of this proof. Then, since $|S\Delta K| \leq C\delta^\alpha$, we have

$$C\delta^\alpha \geq |S \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^\infty \mathcal{H}^1(S[s])\, \mathrm{d}s,$$

so, by (4.3),

$$|A \setminus \pi^{-1}(B_R)| + |B \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^\infty (\mathcal{H}^1(A[s]) + \mathcal{H}^1(B[s]))\, \mathrm{d}s \leq C\delta^\alpha. \tag{4.5}$$

Hence, recalling that $|A|$ and $|B|$ are $\geq 1 - \delta$, we deduce that

$$\int_0^{\mathcal{H}^{n-1}(B_R)} \mathcal{H}^1(A[s])\, \mathrm{d}s \geq \frac{1}{2}, \quad \int_0^{\mathcal{H}^{n-1}(B_R)} \mathcal{H}^1(B[s])\, \mathrm{d}s \geq \frac{1}{2},$$

and since $R$ is universally bounded (being less than $nr_n$) and both functions

$$s \mapsto \mathcal{H}^1(A[s]), \quad s \mapsto \mathcal{H}^1(B[s])$$

are decreasing, there exists a small dimensional constant $c' > 0$, such that

$$\min\{\mathcal{H}^1(A[s]), \mathcal{H}^1(B[s])\} \geq c', \quad \forall\, s \in (0, c'). \tag{4.6}$$

Also, by (4.4),

$$\int_0^{c'} \left(2\mathcal{H}^1(S[s]) - \mathcal{H}^1(A[s]) - \mathcal{H}^1(B[s])\right) \mathrm{d}s \leq 4\delta, \tag{4.7}$$

and since $|S\Delta K| \leq C\delta^\alpha$ and $K \subset \{x_n \leq \tau\}$,

$$\int_0^{c'} \mathcal{H}^1(S[s] \setminus (-\infty, \tau])\,\mathrm{d}s \leq |S \setminus \{x_n \leq \tau\}| \leq C\delta^\alpha. \tag{4.8}$$

Hence, thanks to (4.6)–(4.8), we use Theorem 1.2 and Chebishev's inequality to find a value

$$\overline{s} \in [\delta^{\frac{\alpha}{2}}, 2\delta^{\frac{\alpha}{2}}], \tag{4.9}$$

such that

$$\mathcal{H}^1(\mathrm{co}(A[\overline{s}]) \setminus A[\overline{s}]) + \mathcal{H}^1(\mathrm{co}(B[\overline{s}]) \setminus B[\overline{s}]) \leq C\delta^{1-\frac{\alpha}{2}} \leq C\delta^{\frac{\alpha}{2}}$$

(notice that $\alpha \leq 1$) and

$$\mathcal{H}^1(S[\overline{s}] \setminus (-\infty, \tau]) \leq C\delta^{\frac{\alpha}{2}}.$$

Since $\frac{1}{2}(A[\overline{s}] + B[\overline{s}]) \subset S[\overline{s}]$, this implies

$$\frac{\mathrm{co}(A[\overline{s}]) + \mathrm{co}(B[\overline{s}])}{2} \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}].$$

Hence, after applying opposite translations along the $e_n$-axis to $A$ and $B$, i.e.,

$$A \mapsto A + \ell e_n, \quad B \mapsto B - \ell e_n$$

for some $\ell \in \mathbb{R}$, we can assume that

$$\mathrm{co}(A[\overline{s}]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}], \quad \mathrm{co}(B[\overline{s}]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}].$$

Since the sets $s \mapsto A[s],\ B[s]$ are decreasing, we deduce that

$$\mathrm{co}(A[s]), \mathrm{co}(B[s]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}], \quad \forall\, s \geq \overline{s}. \tag{4.10}$$

We now want to bound $\sup_{s>0} \mathcal{H}^1(A[s])$. (Recall that we cannot assume that $A$ and $B$ are $M$-normalized, since we already made an affine transformation to ensure that $B_{r_n} \subset K \subset B_{nr_n}$.) Since $A = A^*$, we have $\sup_{s>0} \mathcal{H}^1(A[s]) = \sup_{y\in\mathbb{R}^{n-1}} \mathcal{H}^1(A_y)$, so, by Lemma 2.2,

$$\sup_{s>0} \mathcal{H}^1(A[s]) \leq \frac{M}{\mathcal{H}^{n-1}(\pi(B))}, \quad \frac{\mathcal{H}^{n-1}(\pi(A))}{\mathcal{H}^{n-1}(\pi(B))} \in (M^{-1}, M). \tag{4.11}$$

In addition, since $\pi(A)$ and $\pi(B)$ are $(n-1)$-dimensional disks centered on the $e_n$-axis, $|S\Delta K| \leq C\delta^\alpha$ and $B_{r_n} \subset K \subset B_{nr_n}$, we easily deduce that

$$\frac{\pi(A) + \pi(B)}{2} = \pi(S) \supset B_{\frac{r_n}{2}}, \tag{4.12}$$

provided that $\delta$ is small enough. Hence, combining (4.11) and (4.12), we deduce that $\mathcal{H}^{n-1}(\pi(B))$ is bounded from away from zero by a dimensional constant, thus

$$\sup_{s>0} \mathcal{H}^1(A[s]) \leq C. \tag{4.13}$$

Hence, by (4.5), (4.10), (4.13) and (4.9),

$$\begin{aligned} |A \setminus \{x_n \leq \tau\}| &\leq |A \setminus \pi^{-1}(B_R)| + |\pi^{-1}(B_R) \cap \{\tau \leq x_n \leq \tau + C\delta^{\frac{\alpha}{2}}\}| + \int_0^{\overline{s}} \mathcal{H}^1(A[s])\, \mathrm{d}s \\ &\leq C\delta^\alpha + C\delta^{\frac{\alpha}{2}} + C\overline{s} \leq C\delta^{\frac{\alpha}{2}}, \end{aligned} \tag{4.14}$$

and, analogously,

$$|B \setminus \{x_n \leq \tau\}| \leq C\delta^{\frac{\alpha}{2}}. \tag{4.15}$$

Now, given $r \geq 0$, let us define the sets

$$A'_r := A \cap \{x_n \leq \tau - r\}, \quad B'_r := B \cap \{x_n \leq \tau - r\}, \quad S'_r := S \cap \{x_n \leq \tau - r\}.$$

By (4.14)–(4.15), we know that

$$|A'_0|, |B'_0| \geq 1 - C\delta^{\frac{\alpha}{2}},$$

and it is immediate to check that

$$\frac{A'_0 + B'_r}{2} \subset S'_{\frac{r}{2}}, \quad \frac{A'_r + B'_0}{2} \subset S'_{\frac{r}{2}}.$$

Also, since $K$ is a convex set satisfying $B_{r_n} \subset K \subset B_{nr_n}$, there exists a dimensional constant $c_n > 0$ such that

$$\left| K \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \geq c_n \min\{r^n, 1\}.$$

Hence

$$\begin{aligned} |S'_{r/2}| &\leq |S| - \left| S \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \\ &\leq |S| + |S \Delta K| - \left| K \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \\ &\leq 1 + C\delta^\alpha - c_n \min\{r^n, 1\}, \end{aligned}$$

and by (1.2) applied to $A'_r$ and $B'_0$, we get

$$\begin{aligned} 1 - C\delta^{\frac{\alpha}{2}} - C|A \cap \{\tau - r \leq x_n \leq \tau\}| &\leq \frac{|A'_r|^{\frac{1}{n}} + |B'_0|^{\frac{1}{n}}}{2} \leq |S'_{\frac{r}{2}}|^{\frac{1}{n}} \\ &\leq 1 + C\delta^\alpha - c_n \min\{r^n, 1\}, \end{aligned}$$

which gives

$$C|A \cap \{\tau - r \leq x_n \leq \tau\}| \geq c_n \min\{r^n, 1\} - C\delta^{\frac{\alpha}{2}} \tag{4.16}$$

(and analogously for $B$).

Since the point $\overline{x}_0 = (\tau+\rho)e_n$ belongs to $S = \frac{A+B}{2}$, there as to be a point $\overline{x} \in A \cup B$ such that $\overline{x} \cdot e_n \geq (\tau+\rho)$. Without loss of generality, assume that $\overline{x} \in B$. Then, by (4.16) applied with $r = \rho$, we get

$$S \cap \{x_n \geq \tau\} \supset \frac{\overline{x} + (A \cap \{\tau - \rho \leq x_n \leq \tau\})}{2},$$

so

$$C\delta^\alpha \geq |S \cap \{x_n \geq \tau\}| \geq \frac{|A \cap \{\tau - \rho \leq x_n \leq \tau\}|}{2^n} \geq \frac{c_n}{C} \min\{\rho^n, 1\} - C\delta^{\frac{\alpha}{2}},$$

which implies $\rho \leq C\delta^{\frac{\alpha}{2n}}$, proving (4.1).

Hence $\mathrm{co}(S) \subset (1 + C\delta^{\frac{\alpha}{2n}})K$, from which the result follows immediately.

### 4.2 Proof of Proposition 2.2

Since

$$\frac{\mathrm{co}(A) + \mathrm{co}(B)}{2} = \mathrm{co}(S),$$

by (1.2), (2.4)–(1.5), we have

$$\begin{aligned} |\,\mathrm{co}(A)|^{\frac{1}{n}} + |\,\mathrm{co}(B)|^{\frac{1}{n}} &\leq |\,\mathrm{co}(A) + \mathrm{co}(B)|^{\frac{1}{n}} \\ &= 2|\,\mathrm{co}(S)|^{\frac{1}{n}} \leq 2|S|^{\frac{1}{n}} + C\delta^\beta \\ &\leq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} + C\delta^\beta \\ &\leq |\,\mathrm{co}(A)|^{\frac{1}{n}} + |\,\mathrm{co}(B)|^{\frac{1}{n}} + C\delta^\beta, \end{aligned}$$

from which we deduce that

$$|\,\mathrm{co}(A) \setminus A| + |\,\mathrm{co}(B) \setminus B| \leq C\delta^\beta. \tag{4.17}$$

Also, by Theorem 1.3 and the fact that $||\,\mathrm{co}(A)| - |\,\mathrm{co}(B)|| \leq C\delta^{\beta\alpha_n}$ (see (4.17)), we obtain that, up to a translation,

$$|\,\mathrm{co}(A)\Delta\,\mathrm{co}(B)| \leq C(\delta^{\frac{\beta}{2}} + \delta^\beta) \leq C\delta^{\frac{\beta}{2}}. \tag{4.18}$$

This estimate combined with (4.17) implies that

$$|A\Delta B| \leq C\delta^{\frac{\beta}{2}}.$$

In addition, if we define $\mathcal{K} := \mathrm{co}(A \cup B)$, then we will conclude our argument by showing that

$$|\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq C\delta^{\frac{\beta}{2n}}. \tag{4.19}$$

Indeed, by John's lemma (see [16]), after a volume preserving affine transformation, we can assume that $B_r \subset \mathrm{co}(A) \subset B_{nr}$ for some radius $r$ bounded above and below by positive dimensional constants. By (4.18) and a simple geometric argument, we easily deduce that

$$\mathrm{co}(B) \subset (1 + C\delta^{\frac{\beta}{2n}})\,\mathrm{co}(A).$$

Thus

$$\mathrm{co}(A) \cup \mathrm{co}(B) \subset \mathcal{K} \subset (1 + C\delta^{\frac{\beta}{2n}})\,\mathrm{co}(A),$$

and (4.19) follows by (4.17)–(4.18).

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