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Affinely Prime Dynamical Systems

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(*Dedicated to Professor Haim Brezis on the occasion of his* 70*th birthday*)

Abstract This paper deals with representations of groups by "affine" automorphisms of compact, convex spaces, with special focus on "irreducible" representations: equivalently "minimal" actions. When the group in question is $PSL(2,\mathbb{R})$, the authors exhibit a one-one correspondence between bounded harmonic functions on the upper half-plane and a certain class of irreducible representations. This analysis shows that, surprisingly, all these representations are equivalent. In fact, it is found that all irreducible affine representations of this group are equivalent. The key to this is a property called "linear Stone-Weierstrass" for group actions on compact spaces. If it holds for the "universal strongly proximal space" of the group (to be defined), then the induced action on the space of probability measures on this space is the unique irreducible affine representation of the group.

Keywords Irreducible affine dynamical systems, Affinely prime, Strong proximality, Möbius transformations, Harmonic functions
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1 Introduction

The classical theory of group representations deals with representing a group as automorphisms of vector spaces. In principle, one can take any category with its morphisms and study representing a group by automorphisms of objects in this category. In what follows, we shall do this for the category of compact convex spaces with morphisms preserving the affine structure. There is particular interest in the "irreducible" representations where no proper "subobject" is invariant under the action. A pleasant aspect of this theory is that for any group, there is a "universal" irreducible representation from which all others can be derived. Moreover, for many groups, the universal irreducible representation can be described explicitly. Following our preliminary discussion, we focus on the group $PSL(2,\mathbb{R})$, or equivalently, on the Möbius group of analytic maps preserving the unit disc of the complex plane. Denote the latter group by G. We show, following [3], that each bounded harmonic function on the disc leads to an irreducible representation of G on a compact, convex subset of $L^\infty(G)$. Since there is an abundance of bounded harmonic functions on the disc, we might expect to find a great variety of non-equivalent irreducible representations of $PSL(2,\mathbb{R})$. This was our initial guess and the motivation for the ensuing research. As it turns out, the universal irreducible representation

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of the Möbius group is given by the natural action on probability measures on the unit circle. Moreover, we show that this representation is "prime", meaning that no other irreducible representation can be derived from this one. This means, in particular, that all non-constant harmonic functions lead to equivalent irreducible representations.

In Section 2, we develop the rudiments of the theory of irreducible affine dynamical systems and introduce the notion of an affinely prime dynamical system. In Section 3, we consider the group G of Möbius transformations preserving the unit disk $D \subset \mathbb{C}$, which is topologically isomorphic to the group $PSL(2, \mathbb{R})$. As was shown in [3], the action of G on the boundary S^1 of D is minimal and strongly proximal, and moreover the system (S^1, G) is the universal minimal and strongly proximal G-action, denoted as $\Pi_s(G)$. This is the same as saying that the induced action of G on the space $M(S^1)$ of probability measures on S^1 is the universal irreducible affine action of G. We prove that in fact, up to affine isomorphism, the irreducible affine system $(M(\Pi_s(G)), G) = (M(S^1), G)$ is the unique irreducible affine G-system. In the last section we show, following [3], that there is a one-to-one correspondence between bounded harmonic functions h on the unit disk $D \cong G/K$ (where $K \subset G$ is the subgroup of rotations of D) and irreducible affine systems (Q_h, G) in $L^\infty(G)$, where each such irreducible system contains a unique K invariant function which is the lift of h from G/K to G. Moreover, as a consequence of the analysis of the previous section, all the affine systems Q_h are isomorphic to the universal irreducible affine system $(M(S^1), G) = (M(\Pi_s(G)), G)$.

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2 Affinely Prime Dynamical Systems

A dynamical system (X, G, ψ) is a triple consisting of a compact metric space X, a topological group G and a continuous homomorphism $\psi : G \to \text{Homeo(X)}$, the Polish group of homeomorphisms of X equipped with the compact open topology. As a rule, we will suppress the homomorphism ψ and, given $x \in X$ and $g \in G$, write gx for $\psi(g)(x)$. A dynamical system is nontrivial when it contains more than one point. Given two G dynamical systems, a homomorphism $\pi : (X, G) \to (Y, G)$ is a continuous map of X into Y which intertwines the G-actions. When π is surjective we say that it is a factor map and that Y is a factor of X. A dynamical system (X, G) is prime if every factor map $\pi : (X, G) \to (Y, G)$ with Y nontrivial is one-to-one.

If (X, G) is a dynamical system and $Y \subset X$ is an invariant closed subset, we say that (Y, G), the restriction of the action of G to Y, is a subsystem. When (X, G) has no proper subsystems, we say that it is minimal. This is of course the case if and only if the orbit Gx of every point $x \in X$ is dense. We say that two points x, y in a system X are proximal, if there exists a point $z \in X$ and a sequence $g_n \in G$ such that $\lim g_n x = \lim g_n y = z$. The system (X, G) is proximal, if every pair of points in X is proximal.

Lemma 2.1 *A nontrivial prime dynamical system is either minimal, or it has a unique fixed point, and every other point has a dense orbit.*

Proof Let (X, G) be a nontrivial prime dynamical system. If X properly contains a closed G-invariant subset $Y \subsetneq X$, which contains more than one point, form the set

$$R = (Y \times Y) \cup \{(x, x) : x \in X\} \subsetneq X \times X.$$

This is an icer (i.e., an invariant closed equivalence relation) on X, and the corresponding homomorphism $\pi : X \to X/R$ is non-trivial, contradicting primality.

Thus every proper closed invariant subset of X is a singleton. It follows that if X is not minimal, then it has a unique fixed point, and every other point has a dense orbit, as claimed.

The space $M(X)$ of probability measures on X will be equipped with its natural weak* topology which is inherited from $C(X)^*$ where a measure is identified with the corresponding linear functional on $C(X)$, the Banach algebra of real valued continuous functions on X. The compact metric space $M(X)$ also supports an affine structure and the G-action on X induces a continuous affine action of G on $M(X)$. In general, if Q is a compact convex metrisable subset of a locally convex topological vector space, and G acts on Q as a group of continuous affine maps (i.e., each $g \in G$ preserves convex combinations), we say that (Q, G) is an affine dynamical system.

For more details on the notions and results introduced below, see, e.g., [6].

Definition 2.1 (1) *Let (X, G) be a dynamical system and (Q, G) be an affine dynamical system. We say that Q is an affine compactification of X, if there is a homomorphism $\phi : X \to Q$ such that $\overline{co}(\phi(X)) = Q$, where for $A \subset Q$, $\overline{co}(A)$ denotes the closed convex hull of the set A. When ϕ is one-to-one, we say that it is faithful (or that it is an affine embedding).*

(2) *An affine dynamical system (Q, G) is irreducible, if it does not contain properly any affine subsystem, i.e., if whenever $Q' \subset Q$ is a closed convex and G-invariant subset, then $Q' = Q$.*

(3) *An affine dynamical system (Q, G) is affinely prime, if it does not admit any proper factor affine system, i.e., if whenever $\pi : Q \to Q'$ is an affine surjective homomorphism with Q' nontrivial, then π is one-to-one.*

(4) *A dynamical system (X, G) is affinely prime, if with respect to the canonical faithful affine compactification $\phi : X \to M(X)$ given by $\phi(x) = \delta_x$, the associated affine system $(M(X), G)$ is affinely prime.*

(5) *A dynamical system (X, G) is strongly proximal, if for every $\mu \in M(X)$, there is a sequence of elements $g_n \in G$ and a point $x \in X$ such that $\lim g_n \mu = \delta_x$, the point mass at x.*

The next proposition follows easily from Choquet's theory (see, e.g., [8]).

Proposition 2.1 (1) *If Q is an affine dynamical system and $X = \overline{\mathrm{ext}\,(Q)}$ (where $\mathrm{ext}\,(Q)$ denotes the set of extreme points of Q), then Q is a faithful affine compactification of X.*

(2) *For a dynamical system (X, G), the canonical affine compactification defined on $(M(X), G)$ is universal, i.e., for any affine compactification $\phi : X \to Q$, there is a uniquely defined (barycenter) map $\beta : M(X) \to Q$ with $\beta(\delta_x) = \phi(x)$ for every $x \in X$.*

Lemma 2.2 *If (Q, G) is an irreducible affine system and $A \subset Q$ is any closed G-invariant subset, then A contains* $\mathrm{ext}\,(Q)$.

Proof The barycenter map takes $M(A)$ onto Q, so, in particular, each extremal is the barycenter of a measure on A which by extremality must be the corresponding point mass.

Lemma 2.3 *For a dynamical system (X, G), the affine compactification $M(X)$ is irreducible if and only if (X, G) is minimal and strongly proximal.*

Proof If $Y \subset X$ is a proper closed invariant subset, then $M(Y) \subsetneq M(X)$. Thus irreducibility of $M(X)$ implies minimality of X. Given any element $\mu \in M(X)$, let $Z_\mu = \overline{\{g\mu : g \in G\}}$ and $Q_\mu = \overline{co}(Z_\mu)$. The latter is an affine sub-system of $M(X)$. If $M(X)$ is irreducible, it follows that $Q_\mu = M(X)$. From Lemma 2.2, we have $Z_\mu \supset \mathrm{ext}\,(M(X)) = \{\delta_x : x \in X\}$, whence X is strongly proximal.

Conversely, if (X, G) is minimal and strongly proximal, then it is easy to see that every $Q_\mu = M(X)$, i.e., $M(X)$ is irreducible.

In the following lemma, we recall some basic facts about affine systems and also provide the short proofs.

Lemma 2.4 (1) *A proximal system contains exactly one minimal subsystem.*

(2) *A minimal proximal system admits no endomorphisms other than the identity automorphism.*

(3) *A system (X, T) is strongly proximal if and only if the system $(M(X), G)$ is proximal. In particular, a strongly proximal system is proximal.*

(4) *For an affine irreducible system (Q, G), let X denote the closure of the extreme points of Q. Then X is the unique minimal subsystem of Q and the system (X, G) is strongly proximal.*

(5) *If there is a homomorphism $\pi : Q \to P$, where (Q, G) and (P, G) are irreducible affine systems then it is unique. In particular, the only affine endomorphism of an irreducible affine system is the identity.*

Proof (1) By Zorn's lemma, every dynamical system contains at least one minimal subsystem. But if $x, y \in X$ belong to two distinct minimal subsystems, they can not be proximal.

(2) Suppose that (X, G) is minimal and proximal, and that $\phi : X \to X$ is an endomorphism. Since the pair $(x, \phi(x))$ is proximal, there is a sequence $g_n \in G$ with $\lim g_n(x, \phi(x)) = (z, z)$ for some $z \in X$, whence $z = \phi(z)$. Since X is minimal, this implies that $\phi = \mathrm{id}$.

(3) Clearly proximality of $M(X)$ implies strong proximality of X. Conversely, let (X, G) be a strongly proximal system. Given $x, y \in X$, form the measure $\mu = \frac{1}{2}(\delta_x + \delta_y)$. There exists a point $z \in X$ and a sequence $g_n \in G$ with $\lim g_n \mu = \delta_z$, and, as δ_z is an extreme point of $M(X)$, it is easy to see that this implies that $\lim g_n x = \lim g_n y = z$. Thus any two points in X are proximal, i.e., X is a proximal system. It is now easy to see that $M(X)$ is also proximal.

(4) By Proposition 2.1 there is an affine surjective homomorphism $\beta : M(X) \to Q$. Given $\mu \in M(X)$, let $Z_\mu = \overline{\{g\mu : g \in G\}}$ and $Q_\mu = \overline{co}(Z_\mu)$. Then, by the irreducibility of Q, we have $\beta(Q_\mu) = Q$. In particular, for every extreme point $w \in \mathrm{ext}\,(Q) \subset X$, there is $\nu \in Q_\mu$ with

$\beta(\nu) = w$. As w is an extreme point, this implies that $\nu = \delta_w \in X$. It follows that $X \subset Q_\mu$, whence $Q_\mu = M(X)$. Thus $M(X)$ is also irreducible and an application of Lemma 2.3 concludes the proof.

(5) Suppose that $\pi : Q \to P$ and $\sigma : Q \to P$ are two affine homomorphisms. Let $X = \overline{\mathrm{ext}\,(Q)}$ and $Y = \overline{\mathrm{ext}\,(P)}$. We know that both X and Y are proximal and minimal systems. For every $x \in X$, we consider the pair $(\pi(x), \sigma(x))$. This is a proximal pair in Y and thus for some sequence $g_n \in G$, we have $\lim(g_n\pi(x), g_n\sigma(x)) = (y, y)$ for some $y \in Y$. However, we can also assume that the limit $\lim g_n x = z \in X$ exists, and then $(y, y) = (\pi(z), \sigma(z))$, hence $\pi(z) = \sigma(z)$. X being minimal, this implies that $\pi(z') = \sigma(z')$ for every $z' \in X$, and finally, as π and σ are affine maps, this leads to the conclusion that $\pi = \sigma$.

For any topological group G there exists a universal minimal strongly proximal system which we denote by $\Pi_s(G)$. Recalling the fact that a group G is amenable if and only if every compact dynamical system (X, G) admits an invariant probability measure, we see that a group G is amenable if and only if the space $\Pi_s(G)$ is a trivial one point space. The following is a consequence of (4).

Corollary 2.1 *The affine dynamical system $(M(\Pi_s(G)), G)$ is irreducible and it is the universal affine system for irreducible affine G systems, i.e., for any irreducible affine G system Q, there is a unique surjective affine homomorphism $\Theta : M(\Pi_s(G)) \to Q$. In particular, if $\Pi_s(G)$ is affinely prime, then $M(\Pi_s(G))$ is the only nontrivial irreducible affine G-system.*

The next definition is reminiscent of the classical Stone-Weierstrass theorem.

Definition 2.2 *We say that a dynamical system (X, G) has the linear Stone-Weierstrass property (LSW), if for every non-constant function $f \in C(X)$ the uniformly closed linear span V_f of the set $\{f^g : g \in G\} \cup \{\mathbf{1}\}$ is all of $C(X)$ (here $f^g(x) = f(gx)$).*

Proposition 2.2 *A dynamical system has LSW if and only if it is affinely prime.*

Proof For a function $f \in C(X)$, we denote by $\widehat{f} \in Aff(M(X))$ the map

$$\mu \mapsto \int f \,\mathrm{d}\mu, \quad \mu \in M(X).$$

Suppose first that X has the LSW property, and let $\pi : M(X) \to Q$ be an affine homomorphism with nontrivial Q. Let $Aff(Q)$ denote the collection of continuous affine real valued functions on Q, and let

$$\mathcal{A}(Q) = \{f \in C(X) : \widehat{f} = F \circ \pi, \text{ for some } F \in Aff(Q)\}.$$

The LSW property implies that $\mathcal{A}(Q) = C(X)$. Suppose now that $\pi(\mu) = \pi(\nu)$ and $\mu \neq \nu$. Then there is $f \in C(X)$ with $\widehat{f}(\mu) \neq \widehat{f}(\nu)$, and, as $\widehat{f} = F \circ \pi$, for some $F \in Aff(Q)$, we have $\widehat{f}(\mu) = F \circ \pi(\mu) = F \circ \pi(\nu) = \widehat{f}(\nu)$, a contradiction. Thus π is indeed one-to-one.

Conversely, suppose that (X, G) is affinely prime, and let f be a non-constant function in $C(X)$. Let V_f be as in Definition 2.2. If V_f is a proper subspace of $C(X)$, then the restriction map $\mu \to \mu \restriction V_f,\ M(X) \to Q$, where the latter is

$$Q = S(V_f) = \{\xi \in V_f^*,\ \xi \geq 0 \text{ on non-negative functions, and } \ \xi(\mathbf{1}) = 1\},$$

the state space of V_f, yields a non-injective affine homomorphism of $M(X)$.

Proposition 2.3 *If (X,G) is affinely prime, then it is prime, whence it is either minimal or it has a unique fixed point and every other point has a dense orbit.*

Proof Observe that if $\pi : (X,G) \to (Y,G)$ is a surjective but non-injective factor map, then the induced map $\pi_* : M(X) \to M(Y)$ is a surjective but non-injective affine homomorphism. Thus an affinely prime system is prime. The rest follows from Lemma 2.1.

Definition 2.3 *We say that a dynamical system (X,G) is completely uniquely ergodic, if it admits a unique G-invariant probability measure, say η, and $\{\eta\}$ is the only irreducible affine subsystem of $M(X)$.*

Proposition 2.4 *If (X,G) is affinely prime, then the dynamical system (X,G) satisfies*

(1) *It is prime;*

(2) *It is either minimal, or it has a unique fixed point, and every other point has a dense orbit;*

(3) *It is either completely uniquely ergodic, or it is strongly proximal;*

(4) *For a minimal affinely prime system which is not completely uniquely ergodic, $M(X)$ is irreducible.*

Proof (1) Observe that if $\pi : (X,G) \to (Y,G)$ is a surjective but non-injective factor map, then the induced map $\pi_* : M(X) \to M(Y)$ is a surjective but non-injective affine homomorphism. Thus an affinely prime system is prime.

(2) This now follows from Lemma 2.1.

(3) Assume that X is not strongly proximal. Then there is a probability measure $\xi \in M(X)$ whose orbit closure $Z_\xi = \overline{(G\xi)}$ does not meet X. It follows that $Q_\xi = \overline{\mathrm{co}}(Z_\xi) \subsetneq M(X)$ is a nonempty closed convex and G-invariant proper subset of $M(X)$.

Now given any nonempty closed convex and G-invariant proper subset Q of $M(X)$, set $L = w^*\text{-}\overline{\mathrm{span}(Q)} \subset C(X)^*$.

Suppose first that $L = C(X)^*$. Then in particular, every point mass δ_x is in L, and there is a sequence $\phi_n \in \mathrm{span}(Q - Q)$ such that $\phi_n \overset{w^*}{\to} \delta_x$. Let $\phi_n = a_n\mu_n - b_n\nu_n$ with $\mu_n, \nu_n \in Q$ and $a_n, b_n \geq 0$. It follows that $b_n\nu_n \to 0$ and $\mu_n \to \delta_x$. We conclude that $Q = M(X)$. Thus in this case X is minimal and strongly proximal.

Suppose next that L is a proper subspace of $C(X)^*$. Fix some $\phi \in C(X)^* \setminus L$. By the Hahn-Banach separation theorem (see, e.g., [2, Corollary 11, p. 418]), there is a function $f \in C(X)$ such that $\phi(f) = 1$ and $\psi(f) \geq 0$ for all $\psi \in L$. Since L is a subspace, it follows that $\psi(f) = 0$ for all $\psi \in L$.

Thus f is an element of the norm closed G-invariant subspace $L_\perp \subset C(X)$ defined by

$$L_\perp = \{h \in C(X) : \psi(h) = 0,\ \forall \psi \in L\}.$$

Next define $V = L_\perp \oplus \mathbb{R}\mathbf{1}$, where the latter stands for the space of constant functions. If V is a proper subspace of $C(X)$, this contradicts the assumption that X has the LSW property. So we now assume that $V = C(X)$.

Case 1 There exists Q as above which contains more than one element.

Let ν_1, ν_2 be two distinct elements of Q, and let $F \in C(X)$ be such that $\nu_1(F) \neq \nu_2(F)$. We write $F = h + c\mathbf{1}$ with $h \in L_\perp$ and $c \in \mathbb{R}$, and then get

$$\nu_1(F) = \nu_1(h + c\mathbf{1}) = \nu_1(h) + \nu_1(c\mathbf{1}) = \nu_1(c\mathbf{1}) = c,$$
$$\nu_2(F) = \nu_2(h + c\mathbf{1}) = \nu_2(h) + \nu_2(c\mathbf{1}) = \nu_2(c\mathbf{1}) = c,$$

a contradiction.

Case 2 Every closed G-invariant convex proper subset of $M(X)$ is a singleton.

In this case, the collection K of G-invariant probability measures is a closed convex G-invariant subspace of $M(X)$. Now, as we assume that (X, G) is not trivial, the case where K is not a singleton can be ruled out, as in Case 1 above, and we are left with the case, where $K = Q = \{\eta\}$ is the only closed convex G-invariant subset of $M(X)$, which is, by definition, the case of complete unique ergodicity.

(4) This follows from part (3) and Lemma 2.3.

The following diagram sums up the various possible situations described in Proposition 2.4.

Table 1 Affinely prime systems

X \ $M(X)$	irreducible	reducible
minimal	minimal strongly proximal	completely uniquely ergodic
x_0 fixed, $\overline{Gx} = X\ \forall x \neq x_0$	vacuous	completely uniquely ergodic

Remark 2.1 The converse of Proposition 2.4(4), of course, does not hold. There are many minimal strongly proximal systems (so with $M(X)$ irreducible) which are not even prime (see, e.g., Examples 3.1 and 3.2 below).

Example 2.1 (1) For every prime p, the map $Tx = x+1 \pmod p$ generates a prime system $(\mathbb{Z}_p, T)$. It is affinely prime (over $\mathbb{R}$) only for $p = 2, 3$.

(2) Let X be the Cantor set and $G = \text{Homeo(X)}$, the group of self-homeomorphisms of X. The system (X, G) has LSW.

(3) Let $X = S^2$, the two dimensional sphere in $\mathbb{R}^3$, and $G = \text{Homeo(X)}$, the group of self-homeomorphisms of X. The system (X, G) has LSW.

(4) Take $X = S^2$ again, but now consider the action of $H < G = \text{Homeo(X)}$, the subgroup consisting of those homeomorphisms which fix the north pole. (X, H) is affinely prime again, this time strongly proximal with a unique fixed point.

(5) Let $X = \mathbb{Z} \cup \{\infty\}$ be the one point compactification of the integers, and T be the translation $Tx = x + 1$ on $\mathbb{Z}$ which fixes the point at infinity. It is easy to check that X is prime and strongly proximal. However, it does not have the LSW property.

Proof (1) It is clear when one considers the associated Koopman representation on $C(\mathbb{Z}_p) \cong \mathbb{R}^p$.

(2) Let f be a non-constant function in $C(X)$. Rescaling we can assume that $0 \le f(x) \le 1$ for every $x \in X$, and that the values 0 and 1 are attained, say $f(x_0) = 0$ and $f(x_1) = 1$.

Suppose

$$V_f = \overline{\mathrm{span}(\{f^g : g \in G\} \cup \{\mathbf{1}\})} \subsetneq C(X).$$

Then there exists a functional $0 \ne \mu \in C(X)^*$ such that $\mu(h) = 0$ for every $h \in V_f$. We think of μ as a signed measure and write $\mu = \mu_0 - \mu_1$, where μ_0 and μ_1 are non-negative measures, such that for some Borel set $B \subset X$, $\mu_0(B) = \mu_0(X)$ and $\mu_1(X \setminus B) = \mu_1(X)$. Since $\mathbf{1} \in V_f$, we have $\mu(X) = \mu_0(X) - \mu_1(X) = 0$, whence $\mu_0(X) = \mu_1(X) = a > 0$. Again without loss of generality, we assume that $\mu_0(X) = \mu_1(X) = a = 1$.

Given $0 < \epsilon < \frac{1}{8}$, we can find closed subsets $K_0 \subset B$ and $K_1 \subset X \setminus B$, such that $\mu_i(K_i) > (1-\epsilon),\ i = 0, 1$.

Next choose a sequence $g_n \in G$, such that $g_n(K_i) \to x_i,\ i = 0, 1$, in the sense that for every two open neighbourhoods U_i of x_i, there is n_0 with $g_n K_i \subset V_i$ for all $n \ge n_0$.

We also assume, as we may, that the limits $g_n\mu_i \to \nu_i$ $(i = 0, 1)$ exist, and that $\nu_i = c_i\delta_{x_i} + (1 - c_i)\nu_i'$, where $(1 - \epsilon) < c_i \le 1$ and the measures ν_i' are probability measures.

Now

$$\int f \, \mathrm{d}g_n\mu_0 \to c_0 f(x_0) + (1 - c_0)\nu_0'(f) = (1 - c_0)\nu_0'(f) \le \epsilon,$$
$$\int f \, \mathrm{d}g_n\mu_1 \to c_1 f(x_1) + (1 - c_1)\nu_1'(f) = c_1 + (1 - c_1)\nu_1'(f) \ge 1 - \epsilon.$$

But, as $f \in V_f$, these two limits are equal, and we arrive at the absurd inequality $\frac{7}{8} < 1 - \epsilon \le \epsilon < \frac{1}{8}$.

(3) As in the previous proof, given f a non-constant function is $C(X)$, we rescale f, form the space V_f and proceed as above. When we choose the closed disjoint sets K_0, K_1, we can assume that they are Cantor sets. We claim that there is a smooth closed simple Jordan curve with $A \subset \mathrm{ins}(\gamma)$ and $B \subset \mathrm{out}(\gamma)$. In fact, this follows easily e.g. from [1, Proposition 1.8, p. 4]. Now we again proceed as in part (2) above, and choose the homeomorphisms g_n, so that their restriction to a sufficiently small neighborhood of γ is the identity. The rest of the proof goes verbatim as in part (2).

(4) A similar argument.

(5) In order to see this observe first that $C(X) \cong c(\mathbb{Z})$, the Banach space of converging sequences in $\mathbb{R}^{\mathbb{Z}}$. It is now sufficient to show that the closed Banach subspace $c_0(\mathbb{Z})$ (consisting of those sequences whose limit is zero) contains a closed T-invariant proper subspace. However, such (even symmetric, i.e., $S_\infty(\mathbb{Z})$-invariant) subspaces exist in abundance (see, e.g., [4–5]).

Remark 2.2 (1) For the case where $X = S^1$ and $G = \mathrm{Homeo}(\mathrm{S}^1)$, see Corollary 3.2 below.

(2) With some more work, one can show that, with $X = S^n,\ n = 3, 4, \cdots$, or $X = Q$, the Hilbert cube, the systems $(X, \mathrm{Homeo(X)})$ are affinely prime.

Problem 2.1 *Is there a non-trivial, minimal, weakly mixing, uniquely ergodic cascade* (X, T) *which is affinely prime*?

Remark 2.3 We note that if a cascade (X, T) as in Problem 2.1 exists and μ is its unique invariant measure, then the ergodic measure preserving system (X, μ, T) has necessarily simple spectrum.

3 The Group of Möbius Transformations Preserving the Unit Disc

Let G be the group of Möbius transformations preserving the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ (see, e.g., [7, p. 72]),

$$G = \Big\{ z \mapsto \frac{az + \overline{c}}{cz + \overline{a}} : \ a\overline{a} - c\overline{c} = 1 \Big\}.$$

G also acts on the circle $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. As was shown in [3], the system (S^1, G) is the universal minimal strongly proximal G-system, $\Pi_s(G)$. Another representation of this system is as the group $PSL(2, \mathbb{R})$ acting on the projective line $\mathbb{P}^1$ comprising the lines through the origin in $\mathbb{R}^2$.

Theorem 3.1 *The system $(\mathbb{P}^1, PSL(2, \mathbb{R}))$ is affinely prime. Equivalently, the group G of Möbius transformations preserving the unit disk acting on the circle S^1 has the LSW property.*

Proof We will work with the version, where G is the Möbius group acting on $X = S^1$.

We begin by analyzing the case of complex valued functions. Let V be a closed linear subspace of $C(S^1, \mathbb{C})$ invariant under G that contains a non-constant function f. For all $0 \neq n \in \mathbb{Z}$, the convolution of f with $\mathrm{e}^{in\theta}$

$$\int \mathrm{e}^{\mathrm{i}n\phi} f(\theta - \phi)\, \mathrm{d}\phi = \int \mathrm{e}^{\mathrm{i}n(\theta-\psi)} f(\psi)\, \mathrm{d}\psi = \mathrm{e}^{\mathrm{i}n\theta} \widehat{f}(n)$$

is also contained in V. Therefore, if $\widehat{f}(n) \neq 0$, it follows that the function $\mathrm{e}^{\mathrm{i}n\theta}$ is in V. As f is not a constant, there is some $n \neq 0$ for which $\widehat{f}(n) \neq 0$. We fix such an n, and, applying the transformation $\frac{\mathrm{e}^{\mathrm{i}\theta}+t}{1+t\mathrm{e}^{\mathrm{i}\theta}}$ to $\mathrm{e}^{in\theta}$, we see that for all t, the function $\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+t}{1+t\mathrm{e}^{\mathrm{i}\theta}}\right)^n$ belongs to V. Upon differentiating with respect to t at $t = 0$, we see that the function $n(\mathrm{e}^{\mathrm{i}(n-1)\theta} - \mathrm{e}^{\mathrm{i}(n+1)\theta})$, and hence also the functions $\mathrm{e}^{\mathrm{i}(n-1)\theta}$ and $\mathrm{e}^{\mathrm{i}(n+1)\theta}$, are all in V.

This procedure can be iterated, and we conclude that V contains either

$$\{\mathrm{e}^{\mathrm{i}n\theta} : n \geq 0\} \quad \text{or} \quad \{\mathrm{e}^{-\mathrm{i}n\theta} : n \geq 0\}, \quad \text{or both.}$$

Of course in the latter case, we have $V = C(S^1, \mathbb{C})$.

The first alternative happens when V consists of the boundary values of analytic functions in D which are continuous on $\overline{D}$; the second happens, when V consists of the boundary values of anti-analytic functions in D which are continuous on $\overline{D}$.

Now, for real valued functions, these first two cases do not apply since a non-constant analytic function cannot map the boundary to the real line. Thus starting with a G-invariant closed subspace $U \subset C(S^1, \mathbb{R})$ which contains a non-constant function and considering its complexification $V = \mathbb{C} \otimes U$, we conclude that $U = C(S^1, \mathbb{R})$ as claimed.

From Corollary 2.1, we now get the following result.

Corollary 3.1 *For $G = PSL(2,\mathbb{R})$, the affine system $M(\Pi_s(G)) = M(\mathbb{P}^1)$ is the only nontrivial irreducible affine G-system.*

Another immediate consequence of Theorem 3.1 is the following.

Corollary 3.2 *The dynamical system $(S^1, \mathrm{Homeo}(S^1))$ is affinely prime.*

Example 3.1 As was shown in [3], $\Pi_s(G)$, the universal minimal strongly proximal dynamical system for the group $G = PSL(3,\mathbb{R})$ is the flag manifold,

$$\mathcal{F} = \{(\ell, V) : \ell \subset V \subset \mathbb{R}^3, \text{ where } \ell \text{ is a line and } V \text{ a plane in } \mathbb{R}^3\}.$$

The dynamical system $(\mathcal{F}, G)$ however is not affinely prime, since it admits (up to conjugacy) two (isomorphic) proper factors, namely the actions of G on the Grassman varieties $Gr(3,1)$ and $Gr(3,2)$ consisting of the lines and planes through the origin in $\mathbb{R}^3$, respectively (both are copies of the projective plane $\mathbb{P}^2$). More generally, the corresponding flag manifold is the universal minimal strongly proximal dynamical system for all the groups $G = PSL(d+1,\mathbb{R})$, $d \geq 2$ and a similar situation occurs. See Remark 3.1 below.

Remark 3.1 Let $G = PSL(d+1,\mathbb{R})$, $d \geq 2$ and $X = \mathbb{P}^d$ be the projective space. With the natural action of G on X, the system (X, G) is minimal, strongly proximal and prime. In fact, we can show that these actions as well are affinely prime. We plan to return to this in a future work.

Example 3.2 Let X denote the one-sided reduced sequences on the symbols $\{a, a^{-1}, b, b^{-1}\}$, and let $G = F_2$, the free group on the symbols a and b, act on X by concatenation and cancelation. The dynamical system (X, G) is minimal and strongly proximal (see, e.g., [6, pp. 26, 41]). However, it is not prime and a fortiori, not affinely prime. To see this, let $x = a^\infty = aaa\cdots$ and $y = a^{-\infty} = a^{-1}a^{-1}a^{-1}\cdots$, and consider the set

$$R = \{(gx, gy), (gy, gx) : g \in G\} \cup \Delta_X.$$

It is easy to see that this is a closed G-invariant equivalence relation on X, and consequently the induced map $\pi : X \to X/R$ yields a proper factor of X.

4 Harmonic Functions and Irreducible Affine Dynamical Systems

Let G be the group of Möbius transformations which preserve the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, as in Theorem 3.1. We let K denote the subgroup of rotations in G. The disk D can be identified with the quotient G/K by the map $g \mapsto g(0) \in D$. G is a locally compact, unimodular group with Haar measure dg, and we can associate G with the Banach spaces $L^1(G)$ and its dual $L^\infty(G)$. With respect to the weak* topology, B_R, the ball of radius R centered at the origin in $L^\infty(G)$, is compact and metrizable. The group G operates on B_R by $f \mapsto {}^{g'}\!f$, where ${}^{g'}\!f(g) = f(gg')$, $g, g' \in G$.

Recall that a real valued function h on D is harmonic, if it satisfies the mean value property:

$$h(z) = \int_{S^1} h(z + r\zeta)\, \mathrm{d}\zeta \quad \text{for every sufficiently small } r.$$

We will show that a harmonic function $f(z),\ z \in D,\ |f(z)| \le R$ induces an irreducible affine dynamical system (Q_f, G) with $Q_f \subset B_R$. Moreover, we will see that any irreducible affine subsystem $Q \subset B_R$ contains a unique function arising from a bounded harmonic function on D. For more background and details on the topic of this section, see [3].

Given f bounded harmonic on D, define $\widetilde{f} \in L^\infty(G)$ by $\widetilde{f}(g) = f(g(0))$. That is, $\widetilde{f}$ is the function on G obtained by lifting f from G/K to G. The mean value property of harmonic functions implies that for $z' \in D$,

$$f(0) = \int_K f(kz')\,\mathrm{d}k.$$

Setting $z' = g'(0)$, we have

$$f(0) = \int_K f(kg'(0))\,\mathrm{d}k,$$

and since for any $g \in G$, $f \circ g$ is again harmonic

$$f(g(0)) = \int_K f(gkg'(0))\,\mathrm{d}k,$$

or

$$\widetilde{f}(g) = \int_K \widetilde{f}(gkg')\,\mathrm{d}k \quad \text{for any } g, g' \in G. \tag{4.1}$$

Now let Q_f denote the closed, convex span of $\{{}^g\widetilde{f} : g \in G\}$ in $L^\infty(G)$. Equation (4.1) implies that for any $F \in Q_f$,

$$\widetilde{f}(g) = \int_K F(gk)\,\mathrm{d}k.$$

Thus $\widetilde{f}$ belongs to the closed convex span of $\{{}^kF : k \in K\}$ for any $F \in Q_f$. This shows that (Q_f, G) is an irreducible affine system.

Now let $Q \subset L^\infty(G)$ be any invariant closed convex subset, such that (Q, G) is irreducible. The universal minimal strongly proximal space, $\Pi_s(G)$ is the unit circle S^1 and so, by Corollary 2.1, $(M(S^1), G)$ is the universal irreducible affine system for G. In $M(S^1)$, there is a unique K-invariant measure, and it follows that in Q as well, there is a unique K-invariant point. As Q is a space of functions on G, its unique K fixed point is a function $H(g)$ satisfying $H(gk) = H(g)$ for $g \in G,\ k \in K$. Thus H depends on gK and is the pullback of a function h on D. For any fixed $g' \in G$, consider the function

$$H'(g) = \int_K H(gkg')\,\mathrm{d}k.$$

We have $H' \in Q$ and for $k \in K$, $H'(gk) = H'(g)$. So H' is K-invariant. But this function is unique. So $H' = H$. We have $H(g) = \int_K H(gkg')\,\mathrm{d}k$ or

$$h(g) = \int_K h(gkz')\,\mathrm{d}k \tag{4.2}$$

for any $z' \in D$. But, in fact, equation (4.2) characterises harmonic functions.

This discussion, combined with Theorem 3.1 proves the following result.

Theorem 4.1 *There is a one-to-one correspondence between bounded (non-constant) harmonic functions h on D and irreducible affine subsystems (Q,G) of $L^\infty(G)$. Namely,*

$$h \longleftrightarrow Q = Q_h,$$

where $\widetilde{h}$, the lift of h to G, is the unique K-invariant function in Q. Moreover, all the affine systems Q_h are isomorphic to the universal irreducible affine system $(M(S^1),G) = (M(\Pi_s(G)),G)$.

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