Chin. Ann. Math.
38B(2), 2017, 473–488
DOI: 10.1007/s11401-017-1078-5

Chinese Annals of Mathematics, Series B

Exact Boundary Controllability for a Coupled System of Wave Equations with Neumann Boundary Controls*

Tatsien LI[1] Bopeng RAO[2]

(*Dedicated to Professor Haim Brezis on the occasion of his* 70*th birthday*)

Abstract This paper first shows the exact boundary controllability for a coupled system of wave equations with Neumann boundary controls. In order to establish the corresponding observability inequality, the authors introduce a compact perturbation method which does not depend on the Riesz basis property, but depends only on the continuity of projection with respect to a weaker norm, which is obviously true in many cases of application. Next, in the case of fewer Neumann boundary controls, the non-exact boundary controllability for the initial data with the same level of energy is shown.

Keywords Compactness-uniqueness perturbation, Boundary observability, Exact boundary controllability, Non-exact boundary controllability, Coupled system of wave equations, Neumann boundary condition
2010 MR Subject Classification 93B05, 93B07, 93C20, 35L53

1 Introduction

Since the pioneering work of Russell [25], and the celebrated paper of Lions [19] in which a systematic approach, the so-called Hilbert Uniqueness Method, was developed, the control of wave equations has undergone a significant progress. In the last decades, the control of systems has become a very challenging issue. The aim of this paper is to investigate the exact boundary controllability and the non-exact boundary controllability for the following coupled system of wave equations with Neumann boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu U = DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ U = U_0, \ U' = U_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$, ∂_ν denotes the outward normal derivative on the boundary, A is a matrix of order N with constant elements, D is a full column-rank matrix of order $N \times M$ $(M \le N)$ with constant

Manuscript received February 16, 2015. Revised December 3, 2015.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China; Shanghai Key Laboratory for Contemporary Applied Mathematics; Nonlinear Mathematical Modeling and Methods Laboratory. E-mail: dqli@fudan.edu.cn
[2]Institut de Recherche Mathématique Avancée, Université de Strasbourg, 67084 Strasbourg, France. E-mail: bopeng.rao@math.unistra.fr
*This work was supported by the National Basic Research Program of China (No. 2013CB834100) and the National Natural Science Foundation of China (No. 11121101).

elements, $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$ and $H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}$ denote the state variables and the boundary controls, respectively. Here, the introduction of the boundary control matrix D makes the discussion on the control problem more flexible.

Let us denote

$$\mathcal{H}_0 = (L^2(\Omega))^N, \quad \mathcal{H}_1 = (H^1_{\Gamma_0}(\Omega))^N, \tag{1.2}$$

where $H^1_{\Gamma_0}(\Omega)$ is the subspace of $H^1(\Omega)$ composed of functions with null trace on Γ_0. We will show the exact boundary controllability of (1.1) for any given initial data $(U_0, U_1) \in \mathcal{H}_1 \times \mathcal{H}_0$ via the HUM approach. To this end, we first establish the following theorem.

Theorem 1.1 *There exist positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality*

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_0 \times \mathcal{H}_{-1}} \leq C \int_0^{\mathrm{T}} \int_{\Gamma_1} |\Phi|^2 \mathrm{d}\Gamma \mathrm{d}t \tag{1.3}$$

holds for all solutions Φ to the corresponding adjoint problem:

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\mathrm{T}}\Phi = 0 & \textit{in } (0,T) \times \Omega, \\ \Phi = 0 & \textit{on } (0,T) \times \Gamma_0, \\ \partial_\nu \Phi = 0 & \textit{on } (0,T) \times \Gamma_1, \\ t = 0: \ \Phi = \Phi_0, \ \Phi' = \Phi_1 & \textit{in } \Omega, \end{cases} \tag{1.4}$$

where $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^{\mathrm{T}}$ denotes the adjoint variables, $\mathcal{H}_{-1}$ is the dual space of $\mathcal{H}_1$, and the initial data (Φ_0, Φ_1) is taken in a subspace $\mathcal{F} \subset \mathcal{H}_0 \times \mathcal{H}_{-1}$.

Recall that without any assumption on the coupling matrix A, the usual multiplier method can not be applied directly. The absorption of coupling lower terms is a delicate issue even for a single wave equation (see [9, 13]). In order to deal with the lower order terms, we propose a method based on the compactness-uniqueness argument that we formulate in the following lemma.

Lemma 1.1 *Let $\mathcal{F}$ be a Hilbert space endowed with the p-norm. Assume that*

$$\mathcal{F} = \mathcal{N} \oplus \mathcal{L}, \tag{1.5}$$

where $\oplus$ denotes the direct sum and $\mathcal{L}$ is a finite co-dimensional closed subspace in $\mathcal{F}$. Assume that q is another norm in $\mathcal{F}$ such that the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the q-norm. Assume furthermore that

$$q(y) \leq p(y), \quad \forall y \in \mathcal{L}. \tag{1.6}$$

Then there exists a positive constant $C > 0$ such that

$$q(z) \leq Cp(z), \quad \forall z \in \mathcal{F}. \tag{1.7}$$

Following the above lemma, we have to first show the observability inequality for the initial data with higher frequencies in $\mathcal{L}$. In order to extend it to the whole space $\mathcal{F}$, it is sufficient to verify the continuity of the projection from $\mathcal{F}$ into $\mathcal{N}$ for the q-norm. In many situations, it often occurs that the subspaces $\mathcal{N}$ and $\mathcal{L}$ are mutually orthogonal with respect to the q-inner product, and this is true in the present case. This new approach turns out to be particularly simple and efficient for getting the observability of some distributed systems with lower order terms.

As for the Dirichlet boundary problem (see [15–16]), we show the non-exact boundary controllability for the Neumann boundary problem (1.1) in the case of fewer boundary controls, i.e., $M < N$ (see Theorem 4.2). Roughly speaking, in the framework that all the components of initial data are in the same energy space, a coupled system of wave equations with Dirichlet or Neumann boundary controls is exactly controllable if and only if one applies the same number of boundary controls as the number of state variables or wave equations.

Let us comment the related literatures. The exact boundary controllability and the approximate boundary controllability for a coupled system of wave equations with Dirichlet boundary controls were established by Li and Rao in [14–16]. Moreover, in the case of fewer boundary controls, the authors also obtained various results on the exact boundary synchronization and the approximate boundary synchronization for the same system. Using Carleman estimates, Duyckaerts, Zhang and Zuazua studied in [7] the observability inequality for a coupled system of wave equations with Dirichlet boundary condition by means of internal or boundary observation. The optimality of the observability inequality was proved in even space dimensions. In a similar work [27], Zhang and Zuazua established the sharp observability inequality for Kirchhoff plate systems in any space dimensions. In a recent work [6], Dehman, Le Rousseau and Léautaud established the controllability of two coupled wave equations on a compact manifold with only one local distributed control. The optimal time of controllability and the controllable spaces were given in the cases that the waves propagate with the same speed or with different speeds. Using the Riemannian geometry method, Yao established in [26] the controllability of wave equation with variable coefficients for Dirichlet or Neumann boundary condition. We finally mention the work of Hu, Ji and Wang [8] for the exact boundary controllability of one space-dimensional quasilinear system of wave equations with various boundary controls.

The non-exact boundary controllability for a coupled system of wave equations depends on the level of energy and the property of controllability. In fact, if the components of initial data are allowed to have different levels of energy, then the exact boundary controllability for a system of two wave equations was established by means of only one boundary control in [1, 21, 24], or more recently, for a cascade system of N wave equations by means of only one boundary control in [2]. In contrast with the exact boundary controllability, the approximate boundary controllability for a coupled system of wave equations is more flexible with respect to the number of boundary controls. It was recently shown in [17] that for Dirichlet boundary controls, this property could be characterized by means of the famous Kalman's criterion on the rank of an enlarged matrix composed of the coupling matrix A and the boundary control matrix D.

Differently from the hyperbolic systems, the exact boundary null controllability of coupled systems of parabolic equations can be realized in the case of fewer controls for the initial data with the same level of energy. There are a number of works on this topic. We only quote [3] and the references therein for the null controllability of coupled systems of parabolic equations with a local distributed control or with a boundary control.

The paper is organized as follows. In Section 2, we give the proof of the basic lemma of compact perturbation. Section 3 is devoted to the proof of Theorem 1.1. In order to clarify the idea, we divide the proof into several propositions. In Section 4, we prove the exact boundary controllability (see Theorem 4.1) and the non-exact boundary controllability (see Theorem 4.2) for a system of wave equations with Neumann boundary condition.

2 Proof of Lemma 1.1

Assume that (1.7) fails, then there exists a sequence $z_n \in \mathcal{F}$ such that

$$q(z_n) = 1 \quad \text{and} \quad p(z_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.1}$$

Using (1.5), we write $z_n = x_n + y_n$ with $x_n \in \mathcal{N}$ and $y_n \in \mathcal{L}$. Since the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the q-norm, there exists a positive constant $c > 0$ such that

$$q(x_n) \le cq(z_n) \le c, \quad \forall n \ge 1. \tag{2.2}$$

Since $\mathcal{N}$ is of finite dimension, we may assume that there exists $x \in \mathcal{N}$ such that $x_n \to x$ in $\mathcal{N}$. Then, since the second relation of (2.1) means that $z_n \to 0$ in $\mathcal{F}$, we deduce that $y_n \to -x$ in $\mathcal{L}$ for the p-norm. Therefore, we get $x \in \mathcal{L} \cap \mathcal{N}$, which leads to $x = 0$. Then, we have

$$q(x_n) \to 0 \quad \text{and} \quad p(y_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.3}$$

Then using (1.6), we get a contradiction:

$$1 = q(z_n) \le q(x_n) + q(y_n) \le q(x_n) + p(y_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.4}$$

The proof is then complete.

Remark 2.1 Noting that $\mathcal{L}$ is not necessarily closed with respect to the weaker q-norm, so, a priori, the projection $z \to x$ is not continuous with respect to the q-norm (see [5]).

3 Proof of Theorem 1.1

Theorem 1.1 will be proved at the end of this section. We first start with some useful preliminary results.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary Γ. Let $\Gamma = \Gamma_1 \cup \Gamma_0$ be a partition of Γ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$. Throughout this paper, we assume that Ω satisfies the usual geometric control condition (see [4]). More precisely, assume that there exists $x_0 \in \mathbb{R}^n$, such that setting $m = x - x_0$, we have

$$(m, \nu) \le 0, \quad \forall x \in \Gamma_0; \quad (m, \nu) > 0, \quad \forall x \in \Gamma_1, \tag{3.1}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Let

$$\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^{\mathrm{T}}. \tag{3.2}$$

We consider the following homogenous adjoint problem:

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\mathrm{T}}\Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \Phi = \Phi_0, \ \Phi' = \Phi_1 & \text{in } \Omega. \end{cases} \tag{3.3}$$

Let

$$\mathcal{H}_0 = (L^2(\Omega))^N. \tag{3.4}$$

We define the linear unbounded operator $-\Delta$ in $\mathcal{H}_0$ by

$$D(-\Delta) = \{\Phi \in H^2(\Omega)^N : \ \Phi|_{\Gamma_0} = 0, \ \partial\Phi_\nu|_{\Gamma_1} = 0\}. \tag{3.5}$$

Clearly, $-\Delta$ is a densely defined self-adjoint and coercive operator with a compact resolvent in $\mathcal{H}_0$. Then we can define the power operator $(-\Delta)^{\frac{s}{2}}$ for any given $s \in \mathbb{R}$ (see [18]). Moreover, the domain $\mathcal{H}_s = D((-\Delta)^{\frac{s}{2}})$ endowed with the norm $\|\Phi\|_s = \|(-\Delta)^{\frac{s}{2}}\Phi\|_0$, $\|\cdot\|_0$ being the norm of $\mathcal{H}_0$, is a Hilbert space, and its dual space with respect to the pivot space $\mathcal{H}_0$ is $\mathcal{H}'_s = \mathcal{H}_{-s}$. In particular, we have

$$\mathcal{H}_1 = D(\sqrt{-\Delta}) = \{\Phi \in H^1(\Omega)^N : \ \Phi = 0 \text{ on } \Gamma_0\}. \tag{3.6}$$

Then we formulate (3.3) into an abstract evolution problem:

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\mathrm{T}}\Phi = 0, \\ t = 0: \ \Phi = \Phi_0, \ \Phi' = \Phi_1. \end{cases} \tag{3.7}$$

Clearly, the problem (3.7) generates a C^0-semigroup in the space $\mathcal{H}_s \times \mathcal{H}_{s-1}$. Moreover, we have the following result (see [18, Chapter III-8] and [23, Chapter III]).

Proposition 3.1 *For any given initial data* $(\Phi_0, \Phi_1) \in \mathcal{H}_s \times \mathcal{H}_{s-1}$ *with* $s \in \mathbb{R}$*, the problem* (3.7) *admits a unique weak solution in the sense of* C^0*-semigroups, such that*

$$\Phi \in C^0([0, +\infty); \mathcal{H}_s) \cap C^1([0, +\infty); \mathcal{H}_{s-1}). \tag{3.8}$$

Now let e_m be the normalized eigenfunction defined by

$$\begin{cases} -\Delta e_m = \mu_m^2 e_m & \text{in } \Omega, \\ e_m = 0 & \text{on } \Gamma_0, \\ \partial_\nu e_m = 0 & \text{on } \Gamma_1, \end{cases} \tag{3.9}$$

where the sequence of positive terms $\{\mu_m\}_{m\geq 1}$ is increasing so that $\mu_m \to +\infty$ as $m \to +\infty$. Clearly, $\{e_m\}_{m\geq 1}$ is a Hilbert basis in $L^2(\Omega)$.

For each $m \geq 1$, we define the subspace Z_m by

$$Z_m = \{\alpha e_m : \ \alpha \in \mathbb{R}^N\}. \tag{3.10}$$

It is clear that the subspaces Z_m $(m \geq 1)$ are invariant with respect to A^{T}. Moreover, for any given integers $m \neq n$ and any given vectors $\alpha, \beta \in \mathbb{R}^N$, we have

$$\begin{aligned} (\alpha e_m, \beta e_n)_{\mathcal{H}_s} &= (\alpha, \beta)((-\Delta)^{\frac{s}{2}} e_m, (-\Delta)^{\frac{s}{2}} e_n)_{L^2(\Omega)} \\ &= (\alpha, \beta)\mu_m^s \mu_n^s (e_m, e_n)_{L^2(\Omega)} \\ &= (\alpha, \beta)\mu_m^s \mu_n^s \delta_{mn}. \end{aligned} \tag{3.11}$$

Then the subspaces Z_m $(m \geq 1)$ are mutually orthogonal in the Hilbert space $\mathcal{H}_s$ with any given $s \in \mathbb{R}$ and in particular, we have

$$\|\Phi\|_{\mathcal{H}_s} = \frac{1}{\mu_m}\|\Phi\|_{\mathcal{H}_{s+1}}, \quad \forall \Phi \in Z_m. \tag{3.12}$$

Let $m_0 \geq 1$ be an integer. We denote by $\bigoplus_{m \geq m_0} (Z_m \times Z_m)$ the linear hull of the subspaces $Z_m \times Z_m$ for $m \geq m_0$. In other words, $\bigoplus_{m \geq m_0} (Z_m \times Z_m)$ is composed of all finite linear combinations of elements of $Z_m \times Z_m$ for $m \geq m_0$.

Proposition 3.2 *Let Φ be the solution to the problem* (3.7) *with the initial data* $(\Phi_0, \Phi_1) \in \mathcal{H}_1 \times \mathcal{H}_0$*, which satisfies an additional condition:*

$$\Phi \equiv 0 \quad \text{on } [0, T] \times \Gamma_1 \tag{3.13}$$

for $T > 0$ large enough. Then, we have $\Phi_0 \equiv \Phi_1 \equiv 0$, i.e., $\Phi \equiv 0$.

Proof By Schur's theorem, we may assume that A is an upper triangular matrix so that the problem (3.7) with the additional condition (3.13) can be rewritten as

$$\begin{cases} (\phi^{(k)})'' - \Delta\phi^{(k)} + \sum\limits_{p=1}^{k} a_{pk}\phi^{(p)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi^{(k)} = 0 & \text{on } (0, +\infty) \times \Gamma, \\ \partial_\nu \phi^{(k)} = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \phi^{(k)} = \phi_0^{(k)}, \ (\phi^{(k)})' = \phi_1^{(k)} & \text{in } \Omega \end{cases} \tag{3.14}$$

for $k = 1, \cdots, N$. Then using Holmgren's uniqueness theorem (see [20, Chapter I-8]), there exists a positive constant $T > 0$ large enough and independent of the initial data $(\phi_0^{(1)}, \phi_1^{(1)})$, such that $\phi^{(1)} \equiv 0$. Then, we get successively $\phi^{(k)} \equiv 0$ for $k = 1, \cdots, N$. The proof is then complete.

Proposition 3.3 *Let $m_0 \geq 1$ be an integer and Φ be the solution to the problem* (3.7) *with the initial data* $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$*. Define the energy by*

$$E(t) = \frac{1}{2} \int_\Omega (|\Phi'|^2 + |\nabla\Phi|^2) \mathrm{d}x. \tag{3.15}$$

Let σ denote the Euclidian norm of the matrix A. Then we have the following energy estimates:

$$\mathrm{e}^{\frac{-\sigma t}{\mu_{m_0}}} E(0) \leq E(t) \leq \mathrm{e}^{\frac{\sigma t}{\mu_{m_0}}} E(0), \quad t \geq 0 \tag{3.16}$$

for all $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

Proof First, a straightforward computation yields

$$E'(t) = -\int_\Omega (A\Phi', \Phi) \mathrm{d}x. \tag{3.17}$$

Then, using (3.12), we get

$$\Big| \int_\Omega (A\Phi', \Phi) \mathrm{d}x \Big| \leq \sigma \|\Phi'\|_{\mathcal{H}_0} \|\Phi\|_{\mathcal{H}_0} \leq \frac{\sigma}{\mu_{m_0}} \|\Phi'\|_{\mathcal{H}_0} \|\Phi\|_{\mathcal{H}_1} \leq \frac{\sigma}{\mu_{m_0}} E(t). \tag{3.18}$$

It follows that

$$-\frac{\sigma}{\mu_{m_0}} E(t) \leq E'(t) \leq \frac{\sigma}{\mu_{m_0}} E(t). \tag{3.19}$$

Therefore, the function $E(t)\mathrm{e}^{\frac{\sigma t}{\mu_{m_0}}}$ is increasing, while, the function $E(t)\mathrm{e}^{-\frac{\sigma t}{\mu_{m_0}}}$ is decreasing, which implies (3.16). The proof is then complete.

Proposition 3.4 *There exist an integer $m_0 \geq 1$ and positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality:*

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_1 \times \mathcal{H}_0} \leq C \int_0^T \int_{\Gamma_1} |\Phi'|^2 \mathrm{d}\Gamma \mathrm{d}t \tag{3.20}$$

holds for all solutions Φ of the adjoint problem (3.7) *with initial data* $(\Phi_0, \Phi_1) \in \bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$.

Proof First we write (3.7) as

$$\begin{cases} (\phi^{(k)})'' - \Delta \phi^{(k)} + \sum\limits_{p=1}^N a_{pk} \phi^{(p)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi^{(k)} = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi^{(k)} = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \phi^{(k)} = \phi_0^{(k)}, \ (\phi^{(k)})' = \phi_1^{(k)} & \text{in } \Omega \end{cases} \tag{3.21}$$

for $= 1, 2, \cdots, N$. Then, multiplying the k-th equation of (3.21) by

$$M^{(k)} := 2m \cdot \nabla \phi^{(k)} + (N-1)\phi^{(k)} \quad \text{with } m = x - x_0 \tag{3.22}$$

and integrating by parts over the domain $[0, T] \times \Omega$, we get easily the following identities (see [9, Chapter III], [20, Chapter III]):

$$\begin{aligned} &\int_0^T \int_\Gamma (\partial_\nu \phi^{(k)} M^{(k)} + (m, \nu)(|(\phi^{(k)})'|^2 - |\nabla \phi^{(k)}|^2) \mathrm{d}\Gamma \mathrm{d}t \\ = &\Big[\int_\Omega (\phi^{(k)})' M^{(k)} \mathrm{d}x \Big]_0^T + \int_0^T \int_\Omega (|(\phi^{(k)})'|^2 + |\nabla \phi^{(k)}|^2) \mathrm{d}x \mathrm{d}t \\ &+ \sum_{p=1}^N \int_0^T \int_\Omega a_{kp} \phi^{(p)} M^{(k)} \mathrm{d}x \mathrm{d}t, \quad k = 1, \cdots, N. \end{aligned} \tag{3.23}$$

Noting the geometrical control condition (3.1), we have

$$\begin{cases} \partial_\nu \phi^{(k)} M^{(k)} + (m, \nu)(|(\phi^{(k)})'|^2 - |\nabla \phi^{(k)}|^2) \\ \quad = (m, \nu) |\partial_\nu \phi^{(k)}|^2 \leq 0 & \text{on } (0, T) \times \Gamma_0, \\ \partial_\nu \phi^{(k)} M^{(k)} + (m, \nu)(|(\phi^{(k)})'|^2 - |\nabla \phi^{(k)}|^2) \\ \quad = (m, \nu)(|(\phi^{(k)})'|^2 - |\nabla \phi^{(k)}|^2) \leq (m, \nu) |\phi'^{(k)}|^2 & \text{on } (0, T) \times \Gamma_1, \end{cases} \tag{3.24}$$

then, it follows from (3.23) that

$$\begin{aligned} \int_0^T \int_\Omega (|(\phi^{(k)})'|^2 + |\nabla \phi^{(k)}|^2) \mathrm{d}x \mathrm{d}t \leq &\int_0^T \int_{\Gamma_1} (m, \nu) |(\phi^{(k)})'|^2 \mathrm{d}\Gamma \mathrm{d}t - \Big[\int_\Omega (\phi^{(k)})' M^{(k)} \mathrm{d}x \Big]_0^T \\ &- \sum_{p=1}^N \int_0^T \int_\Omega a_{kp} \phi^{(p)} M^{(k)} \mathrm{d}x \mathrm{d}t, \quad 1 \leq k \leq N. \end{aligned} \tag{3.25}$$

Taking the summation of (3.25) with respect to $k = 1, \cdots, N$, we get

$$2 \int_0^T E(t) \mathrm{d}t \leq \int_0^T \int_{\Gamma_1} (m, \nu) |\Phi'|^2 \mathrm{d}\Gamma \mathrm{d}t - \Big[\int_\Omega (\Phi', M) \mathrm{d}x \Big]_0^T - \int_0^T \int_\Omega (\Phi, AM) \mathrm{d}x \mathrm{d}t, \tag{3.26}$$

where M is the vector composed of $M^{(k)}$ $(k=1,\cdots,N)$.

Next, we estimate the last two terms on the right-hand side of (3.26). First, it follows from (3.22) that

$$\|M\|_{\mathcal{H}_0} \le 2R\|\nabla\Phi\|_{\mathcal{H}_0} + (N-1)\|\Phi\|_{\mathcal{H}_0} \le \gamma\|\Phi\|_{\mathcal{H}_1}, \tag{3.27}$$

where $R=\|m\|_\infty$ is the diameter of Ω and

$$\gamma = \sqrt{4R^2+(N-1)^2}. \tag{3.28}$$

On the other hand, since Z_m is invariant with respect to A^{T}, then for all $(\Phi_0,\Phi_1) \in \bigoplus\limits_{m\ge m_0}(Z_m\times Z_m)$, the corresponding solution $\Phi \in \bigoplus\limits_{m\ge m_0} Z_m$. Then, using (3.12) and (3.27), we have

$$\Big|\int_\Omega(\Phi, AM)\mathrm{d}x\Big| \le \sigma\|\Phi\|_{\mathcal{H}_0}\|M\|_{\mathcal{H}_0} \le \gamma\sigma\|\Phi\|_{\mathcal{H}_0}\|\Phi\|_{\mathcal{H}_1} \le \frac{\gamma\sigma}{\mu_{m_0}}\|\Phi\|^2_{\mathcal{H}_1} \le \frac{2\gamma\sigma}{\mu_{m_0}}E(t). \tag{3.29}$$

Similarly, we have

$$\Big|\int_\Omega(\Phi', M)\mathrm{d}x\Big| \le \|\Phi'\|_{\mathcal{H}_0}\|M\|_{\mathcal{H}_0} \le \gamma\|\Phi'\|_{\mathcal{H}_0}\|\Phi\|_{\mathcal{H}_1} \le \gamma E(t). \tag{3.30}$$

Thus, setting

$$T=\frac{\mu_{m_0}}{\sigma} \tag{3.31}$$

and noting (3.16), we get

$$\Big|\Big[\int_\Omega \Phi' M\mathrm{d}x\Big]_0^{\mathrm{T}}\Big| \le \gamma(E(T)+E(0)) \le \gamma(1+\mathrm{e})E(0). \tag{3.32}$$

Inserting (3.29) and (3.32) into (3.26) gives

$$2\int_0^{\mathrm{T}} E(t)\mathrm{d}t \le \int_0^{\mathrm{T}}\int_{\Gamma_1}(m,\nu)|\Phi'|^2\mathrm{d}\Gamma\mathrm{d}t + \gamma(1+\mathrm{e})E(0) + \frac{2\sigma\gamma}{\mu_{m_0}}\int_0^{\mathrm{T}} E(t)\mathrm{d}t. \tag{3.33}$$

Thus, we have

$$\int_0^{\mathrm{T}} E(t)\mathrm{d}t \le R\int_0^{\mathrm{T}}\int_{\Gamma_1}|\Phi'|^2\mathrm{d}\Gamma\mathrm{d}t + \gamma(1+\mathrm{e})E(0), \tag{3.34}$$

provided that m_0 is so large that

$$\mu_{m_0} \ge 2\sigma\gamma. \tag{3.35}$$

Now, integrating the inequality on the left-hand side of (3.16) over $[0,T]$, we get

$$\frac{\mu_{m_0}}{\sigma}\big(1-\mathrm{e}^{-\frac{\sigma T}{\mu_{m_0}}}\big)E(0) \le \int_0^{\mathrm{T}} E(t)\mathrm{d}t, \tag{3.36}$$

then, noting (3.31), we get

$$T(1-\mathrm{e}^{-1})E(0) \le \int_0^{\mathrm{T}} E(t)\mathrm{d}t. \tag{3.37}$$

Thus, it follows from (3.34) and (3.37) that

$$E(0) \le \frac{R}{T(1-\mathrm{e}^{-1})-\gamma(1+\mathrm{e})}\int_0^{\mathrm{T}}\int_{\Gamma_1}|\Phi'|^2\mathrm{d}\Gamma\mathrm{d}t \tag{3.38}$$

holds for all $(\Phi_0, \Phi_1) \in \bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$, provided that

$$T > \frac{\gamma \mathrm{e}(1+\mathrm{e})}{\mathrm{e}-1}, \tag{3.39}$$

which is guaranteed by the following choice (see (3.31), (3.35) and (3.39)):

$$\mu_{m_0} > \frac{2\sigma\gamma \mathrm{e}(1+\mathrm{e})}{\mathrm{e}-1}. \tag{3.40}$$

The proof is then complete.

Proposition 3.5 *There exist an integer $m_0 \geq 1$ and positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality:*

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_0 \times \mathcal{H}_{-1}} \leq C \int_0^{\mathrm{T}} \int_{\Gamma_1} |\Phi|^2 \mathrm{d}\Gamma \mathrm{d}t \tag{3.41}$$

holds for all solutions Φ of the adjoint problem (3.7) *with initial data* $(\Phi_0, \Phi_1) \in \bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$.

Proof Noting that $\mathrm{Ker}(-\Delta + A^{\mathrm{T}})$ is of finite dimension, there exists an integer $m_0 \geq 1$ so large that

$$\mathrm{Ker}(-\Delta + A^{\mathrm{T}}) \bigcap \bigoplus_{m \geq m_0} Z_m = \{0\}. \tag{3.42}$$

Let

$$\mathcal{W} = \overline{\Big\{ \bigoplus_{m \geq m_0} Z_m \Big\}}^{\mathcal{H}_0} \subseteq \mathcal{H}_0. \tag{3.43}$$

Since $\bigoplus\limits_{m \geq m_0} Z_m$ is an invariant subspace of $(-\Delta + A^{\mathrm{T}})$, by Fredholm's alternative, $(-\Delta + A^{\mathrm{T}})^{-1}$ is an isomorphism from $\mathcal{W}$ onto $\mathcal{W}'$. Moreover, we have

$$\|(-\Delta + A^{\mathrm{T}})^{-1}\Psi\|^2_{\mathcal{H}_0} \sim \|\Psi\|^2_{\mathcal{H}_{-1}}, \quad \forall \Psi \in \mathcal{W}. \tag{3.44}$$

For any given $(\Phi_0, \Phi_1) \in \bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$, let

$$\Psi_0 = (\Delta - A^{\mathrm{T}})^{-1}\Phi_1, \quad \Psi_1 = \Phi_0. \tag{3.45}$$

We have

$$\|\Psi_0\|^2_{\mathcal{H}_1} + \|\Psi_1\|^2_{\mathcal{H}_0} \sim \|\Phi_1\|^2_{\mathcal{H}_{-1}} + \|\Phi_0\|^2_{\mathcal{H}_0}. \tag{3.46}$$

Next let Ψ be the solution of the problem (3.7) with the initial data (Ψ_0, Ψ_1) given by (3.45). We have

$$t = 0: \quad \Psi' = \Phi_0, \quad \Psi'' = (\Delta - A^{\mathrm{T}})\Psi_0 = \Phi_1. \tag{3.47}$$

By the well-posedness, we get

$$\Psi' = \Phi. \tag{3.48}$$

On the other hand, since the subspace $\bigoplus\limits_{m \geq m_0} Z_m$ is invariant for $(-\Delta + A^{\mathrm{T}})$, we have

$$(\Psi_0, \Psi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m). \tag{3.49}$$

Then, applying (3.20) to Ψ, we get

$$\|(\Psi_0,\Psi_1)\|^2_{\mathcal{H}_1\times\mathcal{H}_0}\le C\int_0^{\mathrm{T}}\int_{\Gamma_1}|\Psi'|^2\mathrm{d}\Gamma\mathrm{d}t. \tag{3.50}$$

Thus, using (3.46) and (3.48), we get immediately (3.41). The proof is finished.

Finally, we give the proof of Theorem 1.1.

For any given $(\Phi_0,\Phi_1)\in\bigoplus\limits_{m\ge1}(Z_m\times Z_m)$, define

$$p(\Phi_0,\Phi_1)=\sqrt{\int_0^{\mathrm{T}}\int_{\Gamma_1}|\Phi|^2\mathrm{d}\Gamma\mathrm{d}t}, \tag{3.51}$$

where Φ is the solution to the corresponding adjoint problem (3.7). By Proposition 3.2, for $T>0$ large enough, $p(\cdot)$ defines well a norm in $\bigoplus\limits_{m\ge1}(Z_m\times Z_m)$. Then, we denote by $\mathcal{F}$ the completion of $\bigoplus\limits_{m\ge1}(Z_m\times Z_m)$ with respect to the p-norm. Clearly, $\mathcal{F}$ is a Hilbert space. Moreover, we have $\mathcal{F}\subset\mathcal{H}_0\times\mathcal{H}_{-1}$.

We next define

$$\mathcal{F}=\mathcal{N}\oplus\mathcal{L} \tag{3.52}$$

with

$$\mathcal{N}=\bigoplus_{1\le m<m_0}(Z_m\times Z_m),\quad \mathcal{L}=\overline{\Big\{\bigoplus_{m\ge m_0}(Z_m\times Z_m)\Big\}}^p. \tag{3.53}$$

Clearly, $\mathcal{N}$ is a finite-dimensional subspace and $\mathcal{L}$ is a closed subspace in $\mathcal{F}$. In particular, the observability inequality (3.41) can be extended to all initial data (Φ_0,Φ_1) in the whole subspace $\mathcal{L}$.

Now we introduce the q-norm by

$$q(\Phi_0,\Phi_1)=\|(\Phi_0,\Phi_1)\|_{\mathcal{H}_0\times\mathcal{H}_{-1}},\quad \forall(\Phi_0,\Phi_1)\in\mathcal{F}. \tag{3.54}$$

By (3.11), the subspaces $(Z_m\times Z_m)$ are mutually orthogonal in $\mathcal{H}_0\times\mathcal{H}_{-1}$ for all $m\ge1$, then the subspace $\mathcal{N}$ is an orthogonal complement of $\mathcal{L}$ in $\mathcal{H}_0\times\mathcal{H}_{-1}$. In particular, the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the q-norm. On the other hand, since the observability inequality (3.41) holds for all initial data (Φ_0,Φ_1) in the subspace $\mathcal{L}$, the condition (1.6) is verified. Then, applying Lemma 1.1, we get the inequality (1.7), which precisely means that (3.41) can be extended to all $(\Phi_0,\Phi_1)\in\mathcal{F}$, we then get (1.3). The proof of Theorem 1.1 is now completed.

Remark 3.1 Without any additional assumptions on the coupling matrix A, the adjoint system (3.7) is not conservative and the usual multiplier method can not be applied directly. However, since each subspace Z_m is invariant with respect to the matrix A^{T}, for any given initial data $(\Phi_0,\Phi_1)\in Z_m\times Z_m$, the corresponding solution Φ of (3.7) still lies in the subspace Z_m. Then because of the identity (3.12), the coupling term $\|A^{\mathrm{T}}\Phi\|_{\mathcal{H}_0}$ is negligible comparing with $\frac{1}{\mu_m}\|\Phi\|_{\mathcal{H}_1}$. Therefore, we first expect the observability inequality (3.15) only for the initial data (Φ_0,Φ_1) with higher frequencies lying in the sub-linear hull $\bigoplus\limits_{m\ge m_0}(Z_m\times Z_m)$ with an integer $m_0\ge1$ large enough. We next extend it to the closure of the whole linear hull $\bigoplus\limits_{m\ge1}(Z_m\times Z_m)$ by an argument of compact perturbation as shown in Lemma 1.1.

Remark 3.2 The compactness-uniqueness arguments are frequently used in the study of the observability of distributed parameter systems. It turns out that this method is particularly simple and efficient for dealing with some systems with lower order terms. A natural formulation is to consider the problem as a compact perturbation of a skew-adjoint operator (see [10, 22]). This approach requests that the eigen-system of the underlying system forms a Riesz basis in the energy space. Since the Riesz basis is not stable even for the compact perturbation, this may cause serious problems in the application. By contrast, the method proposed here does not require any spectral conditions on the underlying system. In particular, instead of Riesz basis property, we assume only that the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the q-norm. Moreover, it often occurs that the subspaces $\mathcal{N}$ and $\mathcal{L}$ are mutually orthogonal with respect to the q-inner product, hence, the continuity of the projection from $\mathcal{F}$ into $\mathcal{N}$ is much easier to be checked than the Riesz basis property.

Remark 3.3 The present method can be generalized to a larger class of problems, for example, to the coupled system of wave equations with different speeds of propagation, which will be considered in a forthcoming work.

Remark 3.4 As to the optimality of controllability time T, we refer to [6–7, 27] for some related discussions.

4 Exact Boundary Controllability with Neumann Boundary Controls

Let D be a boundary control matrix of order $N \times M$ $(M \leq N)$ and denote

$$U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}, \quad H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}. \tag{4.1}$$

We consider the following inhomogeneous problem:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu U = DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ U = U_0, \ U' = U_1 & \text{in } \Omega. \end{cases} \tag{4.2}$$

We will first show the exact boundary controllability of (4.2) by a standard application of the HUM method of Lions [20]. We next show the non-exact boundary controllability in the case of fewer boundary controls ($M < N$) by an argument of compactness, which requires more regularity of the weak solution as indicated in (4.7) below.

Obviously, we have

$$\mathcal{H}_s \subset (H^s(\Omega))^N, \quad s \geq 0. \tag{4.3}$$

On the other hand, by (1.3) and the trace embedding $H^s(\Omega) \to L^2(\Gamma_1)$ for all $s > \frac{1}{2}$, we get the following continuous embedding:

$$\mathcal{H}_s \times \mathcal{H}_{s-1} \subset \mathcal{F} \subset \mathcal{H}_0 \times \mathcal{H}_{-1}, \quad s > \frac{1}{2}. \tag{4.4}$$

Multiplying the equation in (4.2) by a solution Φ of the adjoint problem (3.7) and integrating by parts, we get

$$\begin{aligned} &(U'(t), \Phi(t))_{\mathcal{H}_0} - (U(t), \Phi'(t))_{\mathcal{H}_0} \\ =\ &(U_1(t), \Phi_0)_{\mathcal{H}_0} - (U_0, \Phi_1)_{\mathcal{H}_0} + \int_0^t \int_{\Gamma_1} (DH(\tau), \Phi(\tau)) \mathrm{d}\Gamma \mathrm{d}\tau. \end{aligned} \tag{4.5}$$

Taking $\mathcal{H}_0$ as the pivot space and noting (4.4), (4.5) can be written as

$$\begin{aligned}&\langle (U'(t), -U(t)), (\Phi(t), \Phi'(t))\rangle_{\mathcal{H}'_s\times\mathcal{H}'_{s-1};\mathcal{H}_s\times\mathcal{H}_{s-1}}\\ =&\langle (U_1, -U_0), (\Phi_0, \Phi_1)\rangle_{\mathcal{H}'_s\times\mathcal{H}'_{s-1};\mathcal{H}_s\times\mathcal{H}_{s-1}} + \int_0^t\int_{\Gamma_1}(DH(\tau), \Phi(\tau))\mathrm{d}\Gamma\mathrm{d}\tau.\end{aligned}\tag{4.6}$$

Definition 4.1 *U is a weak solution to the problem* (4.2)*, if*

$$(U', U) \in C^0([0, T]; \mathcal{H}'_s \times \mathcal{H}'_{s-1}),\tag{4.7}$$

such that the variational equation (4.6) *holds for any given $(\Phi_0, \Phi_1) \in \mathcal{H}_s \times \mathcal{H}_{s-1}$ with $s > \frac{1}{2}$.*

Proposition 4.1 *For any given $H \in L^2(0, T; L^2(\Gamma_1))^M$ and any given $(U_1, U_0) \in \mathcal{H}'_s\times\mathcal{H}'_{s-1}$ with $s > \frac{1}{2}$, the problem* (4.2) *admits a unique weak solution U. Moreover, the linear map*

$$(U_1, U_0, V) \to (U', U)\tag{4.8}$$

is continuous with respect to the corresponding topologies.

Proof Define the linear form

$$\begin{aligned}L_t(\Phi_0, \Phi_1) = &\langle (U_1, -U_0), (\Phi_0, \Phi_1)\rangle_{\mathcal{H}'_s\times\mathcal{H}'_{s-1};\mathcal{H}_s\times\mathcal{H}_{s-1}}\\ &+ \int_0^t\int_{\Gamma_1}(DH(\tau), \Phi(\tau))\mathrm{d}\Gamma\mathrm{d}\tau.\end{aligned}\tag{4.9}$$

By the definition (3.51) of the p-norm and the continuous embedding (4.4), the linear form L_t is bounded in $\mathcal{H}_s \times \mathcal{H}_{s-1}$ for any given $t \geq 0$. Let S_t be the semi-group associated to the homogeneous problem (3.7) on the Hilbert space $\mathcal{H}_s \times \mathcal{H}_{s-1}$, which is an isomorphism on $\mathcal{H}_s \times \mathcal{H}_{s-1}$. The composed linear form $L_t \circ S_t^{-1}$ is bounded in $\mathcal{H}_s \times \mathcal{H}_{s-1}$. Then, by Riesz-Frêchet's representation theorem, there exists a unique element $(U'(t), -U(t)) \in \mathcal{H}'_s \times \mathcal{H}'_{s-1}$ such that

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = \langle (U'(t), -U(t)), (\Phi(t), \Phi'(t))\rangle_{\mathcal{H}'_s\times\mathcal{H}'_{s-1};\mathcal{H}_s\times\mathcal{H}_{s-1}}$$

for any given $(\Phi_0, \Phi_1) \in \mathcal{H}_s \times \mathcal{H}_{s-1}$. Since

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = L_t(\Phi_0, \Phi_1),$$

we get (4.6) for any given $(\Phi_0, \Phi_1) \in \mathcal{H}_s \times \mathcal{H}_{s-1}$. Moreover, we have

$$\begin{aligned}\|(U'(t), -U(t))\|_{\mathcal{H}'_s\times\mathcal{H}'_{s-1}} &= \|L_t \circ S_t^{-1}\|_{\mathcal{L}(\mathcal{H}_s\times\mathcal{H}_{s-1})}\\ &\leq C_T(\|(U_1, U_0)\|_{\mathcal{H}'_s\times\mathcal{H}'_{s-1}} + \|H\|_{L^2(0,T;L^2(\Gamma_1))^M}).\end{aligned}\tag{4.10}$$

Then, by a classic argument of density, we get the regularity (4.7). The proof is thus complete.

Definition 4.2 *The problem* (4.2) *is exactly null controllable at the time T in the space $\mathcal{H}_0\times\mathcal{H}_1$, if for any given $(U_1, U_0) \in \mathcal{H}_0\times\mathcal{H}_1$, there exists a boundary control $H \in L^2(0, T; L^2(\Gamma_1))^M$ such that the problem* (4.2) *admits a unique weak solution U satisfying the final condition*

$$t = T: \quad U = U' = 0\tag{4.11}$$

and the continuous dependance

$$\|H\|_{L^2(0,T;L^2(\Gamma_1))^M} \le C\|(U_1,U_0)\|_{\mathcal{H}_0\times\mathcal{H}_1}. \tag{4.12}$$

Theorem 4.1 *Assume that $M = N$. Then there exists a positive constant $T > 0$ such that the problem (4.2) is exactly null controllable at the time T for any given initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$.*

Proof Let Φ be the solution to the adjoint problem (3.7) in $\mathcal{H}_s \times \mathcal{H}_{s-1}$ with $s > \frac{1}{2}$. Let

$$H = D^{-1}\Phi|_{\Gamma_1}. \tag{4.13}$$

Because of the first inclusion in (4.4), we have $H \in L^2(0,T;L^2(\Gamma_1))^N$. Then by Proposition 4.1, the corresponding backward problem

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,T)\times\Omega,\\ U = 0 & \text{on } (0,T)\times\Gamma_0,\\ \partial_\nu U = \Phi & \text{on } (0,T)\times\Gamma_1,\\ t = T:\ U = 0,\ U' = 0 & \text{in } \Omega \end{cases} \tag{4.14}$$

admits a unique weak solution U with (4.7). Accordingly, we define the linear map

$$\Lambda(\Phi_0,\Phi_1) = (-U'(0), U(0)). \tag{4.15}$$

Clearly, Λ is a continuous map from $\mathcal{H}_s \times \mathcal{H}_{s-1}$ into $\mathcal{H}_s' \times \mathcal{H}_{s-1}'$.

Next, using (4.6), it follows that

$$\langle \Lambda(\Phi_0,\Phi_1), (\Psi_0,\Psi_1)\rangle_{\mathcal{H}_s'\times\mathcal{H}_{s-1}';\mathcal{H}_s\times\mathcal{H}_{s-1}} = \int_0^{\mathrm{T}}\int_{\Gamma_1} \Phi(\tau)\Psi(\tau)\mathrm{d}\Gamma\mathrm{d}\tau,$$

where Ψ is the solution to the problem (3.7) with the initial data (Ψ_0,Ψ_1). It follows that

$$\langle \Lambda(\Phi_0,\Phi_1), (\Psi_0,\Psi_1)\rangle_{\mathcal{H}_s'\times\mathcal{H}_{s-1}';\mathcal{H}_s\times\mathcal{H}_{s-1}} \le \|(\Phi_0,\Phi_1)\|_{\mathcal{F}}\|(\Psi_0,\Psi_1)\|_{\mathcal{F}}.$$

By definition, $\mathcal{H}_s \times \mathcal{H}_{s-1}$ is dense in $\mathcal{F}$, then the linear form

$$(\Psi_0,\Psi_1) \to \langle \Lambda(\Phi_0,\Phi_1), (\Psi_0,\Psi_1)\rangle_{\mathcal{H}_s'\times\mathcal{H}_{s-1}';\mathcal{H}_s\times\mathcal{H}_{s-1}}$$

can be continuously extended to $\mathcal{F}$, so that $\Lambda(\Phi_0,\Phi_1) \in \mathcal{F}'$. Moreover, we have

$$\|\Lambda(\Phi_0,\Phi_1)\|_{\mathcal{F}'} \le \|(\Phi_0,\Phi_1)\|_{\mathcal{F}}.$$

Once again, by the density of $\mathcal{H}_s \times \mathcal{H}_{s-1}$ in $\mathcal{F}$, the linear map Λ can be continuously extended to $\mathcal{F}$, so that Λ becomes a continuous linear map from $\mathcal{F}$ to $\mathcal{F}'$. Therefore, the symmetric bilinear form $\langle \Lambda(\Phi_0,\Phi_1), (\Psi_0,\Psi_1)\rangle_{\mathcal{H}_s'\times\mathcal{H}_{s-1}';\mathcal{H}_s\times\mathcal{H}_{s-1}}$ is continuous and coercive in the product space $\mathcal{F}\times\mathcal{F}$. By Lax and Milgram's lemma, Λ is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. Then for any given $(-U_1, U_0) \in \mathcal{F}'$, there exists an element $(\Phi_0,\Phi_1) \in \mathcal{F}$, such that

$$\Lambda(\Phi_0,\Phi_1) = (-U_1, U_0). \tag{4.16}$$

This is precisely the exact boundary null controllability of the problem (4.2) for any given initial data $(U_1, -U_0) \in \mathcal{F}'$, in particular, for any given initial data $(U_1, -U_0) \in \mathcal{H}_0 \times \mathcal{H}_1 \subset \mathcal{F}'$, because of the second inclusion in (4.4).

Finally, from (4.13) and (4.16), we deduce the continuous dependance

$$\|H\|_{L^2(0,T;L^2(\Gamma_1))^N} \le C\|(\Phi_0,\Phi_1)\|_{\mathcal{F}} \le C\|\Lambda^{-1}\|_{\mathcal{L}(\mathcal{F}',\mathcal{F})}\|(U_1,U_0)\|_{\mathcal{H}_0\times\mathcal{H}_1}.$$

The proof is thus complete.

In the case of fewer boundary controls, we have the following negative result.

Theorem 4.2 *Assume that $M < N$. Then the problem* (4.2) *is not exactly null controllable for all initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$ at any time $T > 0$.*

Proof Since $M < N$, there exists a vector $e \in \mathbb{R}^N$ such that $D^{\mathrm{T}} e = 0$. We choose a special initial data as

$$U_0 = \theta e, \quad U_1 = 0, \tag{4.17}$$

where $\theta \in \mathcal{D}(\Omega)$ is arbitrarily given. If the problem (4.2) is exactly null controllable, we can find a boundary control H with the least norm such that

$$\|H\|_{L^2(]0,T[;L^2(\Gamma_1))^M} \le C\|\theta\|_{H^1(\Omega)}. \tag{4.18}$$

Then, by Proposition 4.1, we have

$$\|U\|_{L^2(0,T;\mathcal{H}_{1-s}(\Omega))} \le C\|\theta\|_{H^1(\Omega)}, \quad \forall s > \frac{1}{2}. \tag{4.19}$$

Now, taking the inner product of e with (4.2) and noting $\phi = (e, U)$, we get

$$\begin{cases} \phi'' - \Delta\phi = -(e, AU) & \text{in } (0,T)\times\Omega, \\ \phi = 0 & \text{on } (0,T)\times\Gamma_0, \\ \partial_\nu \phi = 0 & \text{on } (0,T)\times\Gamma_1, \\ t = 0:\ \phi = \theta,\ \phi' = 0 & \text{in } \Omega, \\ t = T:\ \phi = 0,\ \phi' = 0 & \text{in } \Omega. \end{cases} \tag{4.20}$$

Noting (4.3), by the well-posedness, we get

$$\|\theta\|_{H^{2-s}(\Omega)} \le C\|U\|_{L^2(0,T;H_{1-s}(\Omega))} \le C'\|\theta\|_{H^1(\Omega)} \tag{4.21}$$

for any given $\theta \in \mathcal{D}(\Omega)$. Choosing $1 > s > \frac{1}{2}$, we have $2 - s > 1$. This gives a contradiction. The proof is then complete.

Remark 4.1 As shown in the proof of Theorem 4.1, a weaker regularity such as $(U', U) \in C^0([0,T];\mathcal{H}_{-1}\times\mathcal{H}_0)$ is sufficient to make sense to the value $(U'(0), U(0))$, therefore, sufficient for proving the exact boundary controllability. At this stage, it is not necessary to pay much attention to the regularity of the weak solution with respect to the space variable. However, in order to establish the non-exact boundary controllability in Theorem 4.2, this regularity becomes indispensable for the argument of compact perturbation. In the case of Dirichlet boundary controls, the weak solution has the same smoothness as the controllable initial data. This regularity yields the non-exact boundary controllability in the case of fewer boundary controls (see [14–16]). But for Neumann boundary controls, the direct inequality is much weaker than the inverse inequality. For example, in Proposition 4.1, we can get only $(U', U) \in C^0([0,T];\mathcal{H}_{-s}\times H_{1-s})$ for any $s > \frac{1}{2}$, while, the controllable initial data (U_1, U_0) lies in the space $\mathcal{H}_0\times\mathcal{H}_1$. Even though this regularity is not sharp in general (see [11, Theorem 1.1] and

[12, Main Theorems 1.2–1.3]), it is already sufficient for the proof of the non-exact boundary controllability of the problem (4.2).

Remark 4.2 As for the case of Dirichlet boundary controls discussed in [14–16], we have shown in Theorems 4.1–4.2 that for Neumann boundary controls the problem (4.2) is exactly null controllable if and only if the boundary controls have the same number as the state variables or the wave equations. Of course, the non-exact boundary controllability is valid only in the framework that all the components of the initial data are in the same energy space. In fact, if the components of the initial data are allowed to have different levels of finite energy, then we can realize the exact boundary controllability by means of only one boundary control for a system of two wave equations (see [1, 21]), or more generally, for a cascade system of N wave equations (see [2]). On the other hand, in contrast with the exact boundary controllability, the approximate boundary controllability is more flexible with respect to the number of boundary controls, and is closely related to the so-called Kalman's criterion on the rank of an enlarged matrix composed of the coupling matrix A and the boundary control matrix D (see [15, 17]).

Acknowledgements Part of the work was done during the visit of the second author at the Laboratoire International Associé Sino-Français de Mathématiques Appliquées (LIASFMA) and the School of Mathematical Sciences of Fudan University during June–August 2014. He would like to thank their hospitality and support.

The authors are very grateful to Professor Xu Zhang for bringing their attention to the references [11, 12] and for the valuable discussions on several occasions, and would like also to thank the referees for their very valuable comments and remarks, which were greatly appreciated to improve the presentation of the paper.

References

[1] Alabau-Boussouira, F., A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems, *SIAM J. Control Optim.*, **42**, 2003, 871–904.

[2] Alabau-Boussouira, F., A hierarchic multi-level energy method for the control of bidiagonal and mixed n-coupled cascade systems of PDEs by a reduced number of controls, *Adv. Diff. Equ.*, **18**, 2013, 1005–1072.

[3] Ammar Khodja, F., Benabdallah, A., González-Burgos, M. and de Teresa, L., Recent results on the controllability of linear coupled parabolic problems: A survey, *Math. Control Relat. Fields*, **1**, 2011, 267–306.

[4] Bardos, C., Lebeau, G. and Rauch, J., Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30**, 1992, 1024–1064.

[5] Brezis, H., Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer-Verlag, New York, 2011.

[6] Dehman, B., Le Rousseau, J. and Léautaud, M., Controllability of two coupled wave equations on a compact manifold, *Arch. Ration. Mech. Anal.*, **211**, 2014, 113–187.

[7] Duyckaerts, T., Zhang, X. and Zuazua, E., On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **25**, 2008, 1–41.

[8] Hu, L., Ji, F. Q. and Wang, K., Exact boundary controllability and exact boundary observability for a coupled system of quasilinear wave equations, *Chin. Ann. Math., Ser. B*, **34**(4), 2013, 479–490.

[9] Komornik, V., Exact Controllability and Stabilization, The Multiplier Method, Masson, Paris, 1994.

[10] Komornik, V. and Loreti, P., Observability of compactly perturbed systems, *J. Math. Anal. Appl.*, **243**, 2000, 409–428.

[11] Lasiecka, I. and Triggiani, R., Trace regularity of the solutions of the wave equation with homogeneous Neumann boundary conditions and data supported away from the boundary, *J. Math. Anal. Appl.*, **141**, 1989, 49–71.

[12] Lasiecka, I. and Triggiani, R., Sharp regularity for mixed second-order hyperbolic equations of Neumann type, I. L_2 nonhomogeneous data, *Ann. Mat. Pura Appl.*, **157**, 1990, 285–367.

[13] Lasiecka, I., Triggiani, R. and Zhang, X., Nonconservative wave equations with unobserved Neumann B. C.: Global uniqueness and observability in one shot, Differential geometric methods in the control of partial differential equations, Contemp. Math., **268**, A. M. S., Providence, RI, 2000, 227–325.

[14] Li, T.-T. and Rao, B. P., Exact synchronization for a coupled system of wave equations with Dirichlet boundary controls, *Chin. Ann. Math., Ser. B*, **34**(1), 2013, 139–160.

[15] Li, T.-T. and Rao, B. P., Asymptotic controllability and asymptotic synchronization for a coupled system of wave equations with Dirichlet boundary controls, *Asym. Anal.*, **86**, 2014, 199–226.

[16] Li, T.-T. and Rao, B. P., A note on the exact synchronization by groups for a coupled system of wave equations, *Math. Methods Appl. Sci.*, **38**, 2015, 241–246.

[17] Li, T.-T. and Rao, B. P., Criteria of Kalman's type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls, *C. R. Acad. Sci. Paris, Ser. I*, **353**, 2015, 63–68.

[18] Lions, J.-L. and Magenes, E., Problèmes aux Limites Non Homogènes et Applications, Vol. 1, Dunod, Paris, 1968.

[19] Lions, J.-L., Exact controllability, stabilization and perturbations for distributed systems, *SIAM Rev.*, **30**, 1988, 1–68.

[20] Lions, J.-L., Contrôlabilité Exacte, Perturbations et Stabilisation de Systm̀es Distribués, Vol. 1, Masson, Paris, 1988.

[21] Liu, Z. and Rao, B. P., A spectral approach to the indirect boundary control of a system of weakly coupled wave equations, *Discrete Contin. Dyn. Syst.*, **23**, 2009, 399–414.

[22] Mehrenberger, M., Observability of coupled systems, *Acta Math. Hungar.*, **103**, 2004, 321–348.

[23] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[24] Rosier, L. and de Teresa, L., Exact controllability of a cascade system of conservative equations, *C. R. Math. Acad. Sci., Paris*, **349**, 2011, 291–296.

[25] Russell, D. L., Controllability and stabilization theory for linear partial differential equations: Recent progress and open questions, *SIAM Rev.*, **20**, 1978, 639–739.

[26] Yao, P. F., On the observability inequalities for exact controllability of wave equations with variable coefficients, *SIAM J. Control Optim.*, **37**, 1999, 1568–1599.

[27] Zhang, X. and Zuazua, E., A sharp observability inequality for Kirchhoff plate systems with potentials, *Comput. Appl. Math.*, **25**, 2006, 353–373.