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On the C^1 Regularity of Solutions to Divergence Form Elliptic Systems with Dini-Continuous Coefficients*

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(*Dedicated to Haim Brezis on his* 70*th birthday with friendship and admiration*)

Abstract The author proves C^1 regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients.

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1 Introduction

This note addresses a question raised to the author by Haim Brezis, in connection with his solution to a conjecture of Serrin concerning divergence form second order elliptic equations (see [1] and [2]). If the coefficients of the equations (or systems) are Hölder continuous, then H^1 solutions are known to have Hölder continuous first derivatives. The question is what minimal regularity assumption of the coefficients would guarantee C^1 regularity of all H^1 solutions. See [3] for answers to some other related questions of Haim.

Consider the elliptic system for vector-valued functions $u = (u^1, \cdots, u^N)$,

$$\partial_\alpha(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j) = 0 \quad \text{in } B_4,\ i = 1, 2, \cdots, N,$$

where B_4 is the ball in $\mathbb{R}^n$ of 4 centered at the origin. The coefficients $\{A_{ij}^{\alpha\beta}\}$ satisfy, for some positive constants Λ and λ,

$$|A_{ij}^{\alpha\beta}(x)| \le \Lambda, \quad x \in B_4, \tag{1.1}$$

$$\int_{B_4} A_{ij}^{\alpha\beta}(x)\partial_\alpha \eta^i \partial_\beta \eta^j \ge \lambda \int_{B_4} |\nabla \eta|^2, \quad \forall\, \eta \in H_0^1(B_4, \mathbb{R}^N) \tag{1.2}$$

and

$$\int_0^1 r^{-1}\overline{\varphi}(r)\mathrm{d}r < \infty, \tag{1.3}$$

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where

$$\overline{\varphi}(r) := \sup_{x\in B_3}\Big(\fint_{B_r(x)}|A-A(x)|^2\Big)^{\frac{1}{2}}. \tag{1.4}$$

Main Theorem *Suppose that $\{A_{ij}^{\alpha\beta}\}$ satisfy the above assumptions, and $u\in H^1(B_4,\mathbb{R}^N)$ is a solution of the elliptic system. Then u is C^1 in B_1.*

Remark 1.1 For elliptic equations with coefficients satisfying α-increasing Dini conditions, a proof of the C^1 regularity of u can be found (see, e.g., [6, Theorem 5.1] as pointed out in [1–2]).

Question 1.1 *If we replace $\overline{\varphi}$ in (1.3) by*

$$\widehat{\varphi}(r) := \sup_{x\in B_3}\Big(\fint_{B_r(x)}|A-\overline{A}_{B_r(x)}|^2\Big)^{\frac{1}{2}} \tag{1.5}$$

with $\overline{A}_{B_r(x)} := \fint_{B_r(x)} A$, does the conclusion of the main theorem still hold?

2 Main Results and Proofs

Let $B_r(x)\subset\mathbb{R}^n$ denote the ball of radius r and centered at x. We often write B_r for $B_r(0)$, and rB_1 for B_r. Consider elliptic systems

$$\partial_\alpha(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j) = h_i + \partial_\beta f_i^\beta \quad \text{in } B_4,\ i=1,\cdots,N, \tag{2.1}$$

where α,β are summed from 1 to n, while i,j are summed from 1 to N. The coefficients $\{A_{ij}^{\alpha\beta}\}$, often denoted by A, satisfy, for some positive constants Λ and λ, (1.1)–(1.3), with $\overline{\varphi}$ given by (1.4).

Theorem 2.1 *For $B_4\subset\mathbb{R}^n$, $n\geq 1$, let A, Λ, λ, φ be as above, $\{h_i\},\{f_i^\beta\}\in C^\alpha(B_4)$ for some $\alpha>0$, and let $u\in H^1(B_4,\mathbb{R}^N)$, $N\geq 1$, be a solution to (2.1). Then $u\in C^1(B_1)$. Moreover, the modulus of continuity of ∇u in B_1 can be controlled in terms of $\overline{\varphi}$, n, N, Λ, λ, α, $\|h\|_{C^\alpha(B_2)}$ and $[f]_{C^\alpha(B_2)}$.*

Remark 2.1 Assumption (1.3) is weaker than A being Dini-continuous.

Remark 2.2 The conclusion of Theorem 2.1 still holds (the dependence on α, $\|h\|_{C^\alpha(B_2)}$ and $[f]_{C^\alpha(B_2)}$ is changed accordingly), if $\{h_i\}\in L^p(B_4)$ for some $p>n$, and f satisfies

$$\int_0^1 r^{-1}\overline{\psi}(r)\mathrm{d}r<\infty, \quad \text{where } \overline{\psi}(r) := \sup_{x\in B_3}\Big(\fint_{B_r(x)}|f-f(x)|^2\Big)^{\frac{1}{2}}.$$

Remark 2.3 This note was written in 2008. It was intended to be published after having an answer to the question raised above.

Theorem 2.1 follows from the following two propositions.

Proposition 2.1 *For $B_4 \subset \mathbb{R}^n$, $n \geq 1$, let Λ, λ, N be as above, and let A satisfy* (1.1)–(1.2), *and*

$$\Big(\fint_{B_r} |A - A(0)|^2 \Big)^{\frac{1}{2}} \leq \varphi(r), \quad 0 < r < 1 \tag{2.2}$$

for some non-negative function φ on $(0,1)$ satisfying, for some $\mu > 1$,

$$\max_{r/2 \leq s \leq 2r} \varphi(s) \leq \mu \varphi(r), \quad \int_0^1 r^{-1} \varphi(r) \mathrm{d}r < \infty. \tag{2.3}$$

Assume that $h, f \in C^\alpha(B_4)$ for some $\alpha > 0$, and $u \in H^1(B_4, \mathbb{R}^N)$ is a solution to (2.1). *Then there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$, such that*

$$\fint_{B_r} |u(x) - [a + b \cdot x]| \mathrm{d}x \leq r\delta(r)[\|u\|_{L^2(B_2)} + \|h\|_{C^\alpha(B_2)} + [f]_{C^\alpha(B_2)}], \quad \forall\, 0 < r < 1, \tag{2.4}$$

where $\delta(r)$, depending only on φ, $n, \lambda, \Lambda, N, \mu, \alpha$, satisfies $\lim_{r \to 0} \delta(r) = 0$.

Proposition 2.2 *Let u be a Lebesgue integrable function on $B_1 \subset \mathbb{R}^n$, $n \geq 1$, and let $\delta(r)$ be a monotonically increasing positive function defined on $(0,1)$ satisfying $\lim_{r \to 0} \delta(r) = 0$. Assume that for every $\overline{x} \in B_{\frac{1}{4}}$, there exist $a(\overline{x}) \in \mathbb{R}$, $b(\overline{x}) \in \mathbb{R}^n$, such that*

$$\fint_{B_r(\overline{x})} |u(x) - [a(\overline{x}) + b(\overline{x}) \cdot (x - \overline{x})]| \mathrm{d}x \leq r\delta(r), \quad \forall\, 0 < r < \frac{1}{2}. \tag{2.5}$$

Then u, after changing its values on a zero Lebesgue measure set, belongs to $C^1(B_{\frac{1}{4}})$, with $u \equiv a$ and $\nabla u \equiv b$. Moreover, for some dimensional constant C,

$$|\nabla u(x) - \nabla u(y)| \leq C\delta(4|x - y|), \quad \forall\, x, y \in B_{\frac{1}{4}}. \tag{2.6}$$

Similar results hold for Dirichlet problem: Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a domain with smooth boundary, let Λ and λ be positive constants, and let A satisfy, for $N \geq 1$,

$$\begin{gathered} |A_{ij}^{\alpha\beta}(x)| \leq \Lambda, \quad x \in \Omega, \\ \int_\Omega A_{ij}^{\alpha\beta}(x) \partial_\alpha \eta^i \partial_\beta \eta^j \geq \lambda \int_\Omega |\nabla \eta|^2, \quad \forall\, \eta \in H_0^1(\Omega, \mathbb{R}^N), \\ \int_0^1 r^{-1} \overline{\psi}(r) \mathrm{d}r < \infty, \end{gathered}$$

where

$$\overline{\psi}(r) := \sup_{x \in \Omega} \Big(\fint_{B_r(x) \cap \Omega} |A - A(x)|^2 \Big)^{\frac{1}{2}}.$$

Consider

$$\begin{gathered} \partial_\alpha (A_{ij}^{\alpha\beta}(x) \partial_\beta u^j) = h_i + \partial_\beta f_i^\beta \quad \text{in } \Omega, \ i = 1, \cdots, N, \\ u = g \quad \text{on } \partial\Omega. \end{gathered}$$

Theorem 2.2 (see [4]) *Assume the above, and let $h, f \in C^\alpha(\overline{\Omega})$ and $g \in C^{1,\alpha}(\partial\Omega)$ for some $\alpha > 0$. Then an $H^1(\Omega, \mathbb{R}^N)$ solution u to the above Dirichlet problem is in $C^1(\overline{\Omega})$.*

Our proof of Proposition 2.1, based on the general perturbation in Lemma 3.1 in [5], is similar to that of Proposition 4.1 in [5].

Proof of Proposition 2.2 For any $\overline{x} \in B_1$, we see from (2.5) that as $r \to 0$,

$$\fint_{B_r(\overline{x})} |u(x) - a(\overline{x})| \mathrm{d}x \le \fint_{B_r(\overline{x})} |u(x) - [a(\overline{x}) + b(\overline{x}) \cdot (x - \overline{x})]| \mathrm{d}x + \fint_{B_r(\overline{x})} |b(\overline{x}) \cdot (x - \overline{x})| \mathrm{d}x \to 0.$$

Thus, by a theorem of Lebesgue, $a = u$ a.e. in B_1. We now take $u \equiv a$, after changing the values of u on a zero measure set. Let $\overline{x}, \overline{y} \in B_{\frac{1}{4}}$ satisfy, for some positive integer k, $2^{-(k+1)} \le |\overline{x} - \overline{y}| \le 2^{-k}$. By (2.5), we have, for some dimensional constant C,

$$\begin{aligned} & |u(\overline{x}) - [u(\overline{y}) + b(\overline{y}) \cdot (\overline{x} - \overline{y})]| \\ = & \Big| \fint_{B_{2^{-k}}(\overline{x})} \{[u(\overline{x}) + b(\overline{x}) \cdot (x - \overline{x})] - [u(\overline{y}) + b(\overline{y}) \cdot (x - \overline{y})]\} \mathrm{d}x \Big| \\ \le & \fint_{B_{2^{-k}}(\overline{x})} |u(x) - [u(\overline{x}) + b(\overline{x}) \cdot (x - \overline{x})]| \mathrm{d}x \\ & + \fint_{B_{2^{-k}}(\overline{x})} |u(x) - [u(\overline{y}) + b(\overline{y}) \cdot (x - \overline{y})]| \mathrm{d}x \\ \le & \fint_{B_{2^{-k}}(\overline{x})} |u(x) - [u(\overline{x}) + b(\overline{x}) \cdot (x - \overline{x})]| \mathrm{d}x \\ & + 2^n \fint_{B_{2^{-(k-1)}}(\overline{y})} |u(x) - [u(\overline{y}) + b(\overline{y}) \cdot (x - \overline{y})]| \mathrm{d}x \\ \le & C 2^{-k} \delta(2^{-(k-1)}) \le C |\overline{x} - \overline{y}| \delta(4 |\overline{x} - \overline{y}|). \end{aligned}$$

Switching the roles of $\overline{x}$ and $\overline{y}$ leads to

$$|u(\overline{y}) - [u(\overline{x}) + b(\overline{x}) \cdot (\overline{y} - \overline{x})]| \le C |\overline{y} - \overline{x}| \delta(4 |\overline{y} - \overline{x}|). \tag{2.7}$$

Thus, by the above two inequalities and the triangle inequality,

$$|b(\overline{x}) - b(\overline{y})| \le 2C \delta(4 |\overline{x} - \overline{y}|). \tag{2.8}$$

The conclusion of Proposition 2.2 follows from (2.7)–(2.8).

Proof of Proposition 2.1 For simplicity, we prove it for $h = 0, f = 0$ (the general case only requires minor changes). We may assume without loss of generality that $\varphi(1) \le \epsilon_0, \int_0^1 r^{-1} \varphi(r) \mathrm{d}r \le \epsilon_0$ for some small universal constant $\epsilon_0 > 0$. This can be achieved by working with $u(\delta_0 x)$ for some δ_0 satisfying $\varphi(\delta_0) \le \epsilon_0$ and $\int_0^{\delta_0} r^{-1} \varphi(r) \mathrm{d}r < \epsilon_0$. The smallness of ϵ_0 will be either obvious or specified in the proof. In the proof, a universal constant means that it depends only on $\varphi, n, \lambda, \Lambda, N, \mu$. We assume that u is normalized to satisfy $\|u\|_{L^2(B_2)} = \varphi(4^{-1})$. We often write $\partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j)$ as $\partial(A \partial u)$. For $k \ge 0$, let

$$A_{k+1}(x) = A(4^{-(k+1)} x), \quad \overline{A} = A(0).$$

We will find $w_k \in H^1(\frac{3}{4^{k+1}}B_1, \mathbb{R}^N)$, such that for all $k \geq 0$,

$$\partial(\overline{A}\partial w_k) = 0 \quad \text{in } \frac{3}{4^{k+1}}B_1, \tag{2.9}$$

$$\|w_k\|_{L^2(\frac{2}{4^{k+1}}B_1)} \leq C'4^{-\frac{k(n+2)}{2}}\varphi(4^{-k}), \quad \|\nabla w_k\|_{L^\infty(\frac{1}{4^{k+1}}B_1)} \leq C'\varphi(4^{-k}), \tag{2.10}$$

$$\|\nabla^2 w_k\|_{L^\infty(\frac{1}{4^{k+1}}B_1)} \leq C'4^k\varphi(4^{-k}), \tag{2.11}$$

$$\Big\|u - \sum_{j=0}^{k} w_j\Big\|_{L^2((\frac{1}{4})^{k+1}B_1)} \leq 4^{-\frac{(k+1)(n+2)}{2}}\varphi(4^{-(k+1)}). \tag{2.12}$$

An easy consequence of (2.10) is

$$\|w_k\|_{L^\infty(4^{-(k+1)}B_1)} \leq C'4^{-k}\varphi(4^{-k}). \tag{2.13}$$

Here and in the following, C, C' and ϵ_0 denote various universal constants. In particular, they are independent of k. C will be chosen first and will be large, then C' (much larger than C), and finally $\epsilon_0 \in (0,1)$ (much smaller than $\frac{1}{C'}$).

By Lemma 3.1 in [5], we can find $w_0 \in H^1(\frac{3}{4}B_1, \mathbb{R}^N)$, such that

$$\partial(\overline{A}\partial w_0) = 0 \quad \text{in } \frac{3}{4}B_1,$$
$$\|u - w_0\|_{L^2(\frac{1}{2}B_1)} \leq C\epsilon_0^\gamma\|u\|_{L^2(B_1)} \leq 4^{-\frac{n+2}{2}}\varphi(4^{-1}).$$

So

$$\|w_0\|_{L^2(\frac{1}{2}B_1)},\ \|\nabla w_0\|_{L^\infty(\frac{1}{4}B_1)},\ \|\nabla^2 w_0\|_{L^\infty(\frac{1}{4}B_1)} \leq C\varphi(1) \leq C'\varphi(1).$$

We have verified (2.9)–(2.12) for $k = 0$. Suppose that (2.9)–(2.12) hold up to k ($k \geq 0$). We will prove them for $k+1$. Let

$$W(x) = \Big[u - \sum_{j=0}^{k} w_j\Big](4^{-(k+1)}x),$$

$$g_{k+1}(x) = 4^{-(k+1)}\Big\{[\overline{A} - A_{k+1}](x)\sum_{j=0}^{k}(\partial w_j)(4^{-(k+1)}x)\Big\}.$$

Then W satisfies

$$\partial(A_{k+1}\partial W) = \partial(g_{k+1}) \quad \text{in } B_1.$$

A simple calculation yields, using (2.3),

$$\|A_{k+1} - \overline{A}\|_{L^2(B_1)} = \sqrt{|B_1|}\varphi(4^{-(k+1)}) \leq C(n,\mu)\varphi(4^{-(k+2)}).$$

By the induction hypothesis (see (2.10)–(2.12)),

$$\sum_{j=0}^{k}|(\nabla w_j)(4^{-(k+1)}x)| \leq C'\sum_{j=0}^{k}\varphi(4^{-j}) \leq C(n)C'\int_0^1 r^{-1}\varphi(r)\mathrm{d}r \leq C(n)C'\epsilon_0, \quad x \in B_1,$$

$$\sum_{j=0}^{k}|(\nabla^2 w_j)(4^{-(k+1)}x)| \leq C'\sum_{j=0}^{k}4^j\varphi(4^{-j}), \quad x \in B_1,$$

$$\|W\|_{L^2(B_1)} \leq 4^{-(k+1)}\varphi(4^{-(k+1)}) \leq C(\mu)4^{-(k+2)}\varphi(4^{-(k+2)}),$$

$$\|g_{k+1}\|_{L^2(B_1)} \leq C(n,\mu)C'\epsilon_0 4^{-(k+2)}\varphi(4^{-(k+2)}).$$

By Lemma 3.1 in [5], there exists $v_{k+1} \in H^1(\frac{3}{4}B_1, \mathbb{R}^N)$, such that

$$\partial(\overline{A}\partial v_{k+1}) = 0 \quad \text{in } \frac{3}{4}B_1,$$

and, for some universal constant $\gamma > 0$,

$$\begin{aligned}\|W - v_{k+1}\|_{L^2(\frac{1}{2}B_1)} &\le C(\|g_{k+1}\|_{L^2(B_1)} + \epsilon_0^\gamma \|W\|_{L^2(B_1)}) \\ &\le C(C'\epsilon_0 + \epsilon_0^\gamma)4^{-(k+2)}\varphi(4^{-(k+2)}).\end{aligned} \tag{2.14}$$

Let

$$w_{k+1}(x) = v_{k+1}(4^{k+1}x), \quad x \in \frac{3}{4^{k+2}}B_1.$$

A change of variables in (2.14) and in the equation of v_{k+1} yields (2.9) and (2.12) for $k+1$. It follows from the above that

$$\|\nabla^2 v_{k+1}\|_{L^\infty(\frac{1}{4}B_1)} + \|\nabla v_{k+1}\|_{L^\infty(\frac{1}{4}B_1)} \le C\|v_{k+1}\|_{L^2(\frac{1}{2}B_1)} \le C4^{-(k+1)}\varphi(4^{-(k+1)}).$$

Estimates (2.10) for $k+1$ follow from the above estimates for v_{k+1}. We have, thus, established (2.9)–(2.12) for all k.

For $x \in 4^{-(k+1)}B_1$, using (2.10)–(2.11), (2.13), (2.3) and Taylor expansion,

$$\begin{aligned}&\Big|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} \nabla w_j(0)\cdot x\Big| \\ &\le \sum_{j=k+1}^{\infty} (|w_j(0)| + |\nabla w_j(0)||x|) + \sum_{j=0}^{k} \|\nabla^2 w_j\|_{L^\infty(4^{-(k+1)}B_1)}|x|^2 \\ &\le C\sum_{j=k+1}^{\infty} (4^{-j}\varphi(4^{-j}) + \varphi(4^{-j})|x|) + C\sum_{j=0}^{k} 4^j\varphi(4^{-j})|x|^2 \\ &\le C4^{-(k+1)}\int_0^{4^{-k}} r^{-1}\varphi(r)\mathrm{d}r + C|x|^2\int_{\frac{|x|}{2}}^{1} r^{-2}\varphi(r)\mathrm{d}r.\end{aligned} \tag{2.15}$$

It is easy to see that $\lim_{|x|\to 0} |x|\int_{\frac{|x|}{2}}^1 r^{-2}\varphi(r)\mathrm{d}r = 0$, since (2.3) implies $\lim_{r\to 0^+}\varphi(r) = 0$.

We then derive from (2.12) and the above, using Hölder inequality, that, for some $\delta(r) = \circ(1)$ (as $r \to 0$), depending only on φ, $n, \lambda, \Lambda, N, \mu$,

$$\begin{aligned}&\int_{4^{-(k+1)}B_1} \Big|u(x) - \Big(\sum_{j=0}^{\infty} w_j(0) + \sum_{j=0}^{\infty}\nabla w_j(0)\cdot x\Big)\Big|\mathrm{d}x \\ &\le \Big\|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty}(w_j(0) - \nabla w_j(0)\cdot x)\Big\|_{L^1(4^{-(k+1)}B_1)} + \Big\|u - \sum_{j=0}^{k} w_j(x)\Big\|_{L^1(4^{-(k+1)}B_1)} \\ &= 4^{-(k+1)(n+1)}\delta(4^{-(k+1)}).\end{aligned}$$

Proposition 2.1 follows from the above with $a = \sum_{j=0}^{\infty} w_j(0)$ and $b = \sum_{j=0}^{\infty} \nabla w_j(0)\cdot x)$.

Proof of Theorem 2.1 Fix a $\rho \in C_c^\infty(B_4)$, $\rho \equiv 1$ on B_3, and let

$$\varphi(r) := \sup_{x \in B_3} \Big(\fint_{B_r(x)} |(\rho A) - (\rho A)(x)|^2 \Big)^{\frac{1}{2}}.$$

It is easy to see that for some $\mu > 1$, φ satisfies (2.3). Indeed, since it is easily seen that

$$\varphi(r) \le C(\overline{\varphi}(r) + r),$$

the second inequality follows. For the first inequality, we only need to show that $\varphi(2r) \le C(n)\varphi(r)$, since the rest is obvious. For any $\overline{x}$, let $x_1 = \overline{x}, x_2, \cdots, x_m$, $m = m(n)$, satisfy $B_{2r}(\overline{x}) \subset \bigcup_{i=1}^m B_{\frac{r}{9}}(x_i)$, and $|x_i - x_{i+1}| \le \frac{r}{9}$. Then

$$\begin{aligned}
&\Big(\fint_{B_{2r}(\overline{x})} |(\rho A) - (\rho A)(\overline{x})|^2 \Big)^{\frac{1}{2}} \\
&\le C(n) \sum_{i=1}^m \Big(\fint_{B_{r/9}(x_i)} |(\rho A) - (\rho A)(\overline{x})|^2 \Big)^{\frac{1}{2}} \\
&\le C(n) \sum_{i=1}^m \Big\{ \Big(\fint_{B_{r/9}(x_i)} |(\rho A) - (\rho A)(x_i)|^2 \Big)^{\frac{1}{2}} + |(\rho A)(\overline{x}) - (\rho A)(x_i)| \Big\} \\
&\le C(n)\varphi(r) + C(n) \sum_{i=1}^{m-1} |(\rho A)(x_i) - (\rho A)(x_{i+1})|.
\end{aligned}$$

Since

$$\begin{aligned}
&|(\rho A)(x_i) - (\rho A)(x_{i+1})| \\
&= \Big| \fint_{B_{r/9}(x_i)} [(\rho A) - (\rho A)(x_i)] - \fint_{B_{r/9}(x_i)} [(\rho A) - (\rho A)(x_{i+1})] \Big| \\
&\le C(n) \Big(\fint_{B_r(x_i)} |(\rho A) - (\rho A)(x_i)| + \fint_{B_r(x_{i+1})} |(\rho A) - (\rho A)(x_{i+1})| \Big) \\
&\le C(n)\varphi(r),
\end{aligned}$$

we have

$$\Big(\fint_{B_{2r}(\overline{x})} |(\rho A) - (\rho A)(\overline{x})|^2 \Big)^{\frac{1}{2}} \le C(n)\varphi(r).$$

Thus $\varphi(2r) \le C(n)\varphi(r)$.

For any $\overline{x} \in B_2$,

$$\Big(\fint_{B_r(\overline{x})} |A - A(\overline{x})|^2 \Big)^{\frac{1}{2}} \le \varphi(r), \quad 0 < r < \frac{1}{4}.$$

Thus Theorem 2.1 follows from Propositions 2.1–2.2.

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