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Extremum Problems of Laplacian Eigenvalues and Generalized Polya Conjecture*

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(*Dedicated to Professor Haim Brezis on the occasion of his 70th birthday*)

Abstract In this survey on extremum problems of Laplacian-Dirichlet eigenvalues of Euclidian domains, the author briefly presents some relevant classical results and recent progress. The main goal is to describe the well-known conjecture due to Polya, its connections to Weyl's asymptotic formula for eigenvalues and shape optimizations. Many related open problems and some preliminary results are also discussed.

Keywords Extremum problems, Laplacian eigenvalues, Weyl asymptotics, Polya's conjecture, Spliting equality, Regularity of minimizers
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1 Introduction

This write up covers the author's mini-course consisting of three lectures at the 6th-symposium on Analysis and PDEs at the Purdue University, June 1st–4th, 2015. A somewhat detailed expositions on the relevant works were given in a special graduate course of the author (which was first given at the Courant Institute in the spring of 2014 and then in the fall 2014 at the NYU/Shanghai). My goal here is to give a rather brief survey of some extremum problems for Laplacian eigenvalues on bounded domains in Euclidean spaces with the zero Dirichlet boundary condition. I also want to explain how the solvability (existence and regularity) of these extremum problems is related to (a stronger version) a generalized version of the Polya conjecture.

It is with the deep respect and admiration that I write this article dedicating to Professor Haim Brezis.

2 Some Classical Results

2.1 Weyl's asymptotic formula and Polya's conjecture

Let us start with the following simplest examples of eigenvalues and eigenfunctions.

Example 2.1 Given an ODE,

$$\begin{cases}-u_{xx}(x) = \lambda u(x), \quad x \in [0,1],\\ u(0) = u(L) = 0.\end{cases}$$

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One has a set of (properly normalized) eigenfunctions

$$u_k(x) = \sin\Big(\frac{kx}{L}\Big)$$

and eigenvalues

$$\lambda_k = \frac{\pi^2}{L^2}k^2, \quad k = 1, 2, \cdots .$$

If we denote

$$C_N = \frac{(2\pi)^2}{V_N^{\frac{2}{N}}}, \quad N = 1, 2, \cdots .$$

Here V_N is the volume of the unit ball $B_1^N(0)$ in $\mathbb{R}^N$, then $\lambda_k = \frac{\pi^2}{L^2}k^2$ can be written as, for the dimension $N = 1$, that $\lambda_k = \frac{(2\pi)^2}{V_N^{\frac{2}{N}}} \frac{k^{\frac{2}{N}}}{L^{\frac{2}{N}}}$.

Example 2.2 Let Q be the unit square in $\mathbb{R}^2$, and we consider

$$\begin{cases} -\Delta u \equiv -(u_{xx} + u_{yy}) = \lambda u(x, y), \quad (x, y) \in Q, \\ u = 0 \quad \text{on } \partial Q. \end{cases}$$

We again have a set of eigenfunctions: $u_k(x, y) = \sin(mx)\sin(ny)$, and corresponding eigenvalues $\lambda_k = \pi^2(m^2 + m^2)$. Here $k = N(\lambda)$ is the number of lattice points in $\{(m, n) \in \mathbb{N}^2\ \pi^2(m^2 + n^2) \le \lambda\} \approx \frac{\lambda}{4\pi}$. One notices that for $N = 2$, $C_2 = \frac{4\pi^2}{\pi} = 4\pi$, and $\big(\frac{k}{|Q|}\big)^{\frac{2}{N}} = k$. Thus one has $\lambda_k \approx 4\pi k = C_N\big(\frac{k}{|Q|}\big)^{\frac{2}{N}}$ when $N = 2$.

What we have described in the above examples are nothing but special cases of the well-known Weyl's asymptotic formula (see, e.g., [1, 5, 16, 19, 25–26] and references therein).

Let Ω be a bounded domain in $\mathbb{R}^N$. The eigenvalue problems

$$\begin{cases} -\Delta u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has a sequence of (normalized) eigenfunctions: $\{u_k(x)\}$ that it forms an orthonormal basis of $L^2(\Omega)$. In particular,

$$\int_\Omega u_k^2(x)\mathrm{d}x = 1$$

and

$$\begin{cases} -\Delta u_k = \lambda_k u_k \quad \text{in } \Omega, \\ u_k = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The corresponding sequence of eigenvalues $\{\lambda_k\}$ satisfies in addition that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Weyl's asymptotic formula then implies that

$$\lambda_k \approx C_N\Big(\frac{k}{|\Omega|}\Big)^{\frac{2}{N}} \quad \text{as } k \to +\infty. \tag{$\star$}$$

Here $|\Omega|$ is the volume of Ω and N is the dimension of Ω. Weyl's formula can be further improved (as conjectured by Weyl himself) (see [1, 16, 25] and references therein):

$$N(\lambda) = (2\pi)^{-N}\lambda^{\frac{N}{2}}|\Omega| \mp \frac{1}{4}(2\pi)^{1-N}\lambda^{\frac{N-1}{2}}|\partial\Omega| + \cdots .$$

Here $N(\lambda)$ denotes number of eigenvalues $\leq \lambda$ (the minus sign in the formula is for the Dirichlet eigenvalues, and the plus sign corresponds to the eigenvalues with zero Neumann boundary conditions or we simply call them the Neumann eigenvalues).

Thus for a fixed domain Ω of finite perimeter in $\mathbb{R}^N$, one has for the Dirichlet eigenvalues that

$$\lambda_k(\Omega) \geq C_N\Big(\frac{k}{|\Omega|}\Big)^{\frac{2}{N}} + D(N, |\partial\Omega|, |\Omega|)k^{\frac{1}{N}} \quad \text{as } k \to +\infty,$$

where $D(N, |\Omega|, |\partial\Omega|)$ is a positive constant depending on N, $|\Omega|$ and $|\partial\Omega|$. This leads also naturally to the well-known Polya's conjecture (for the Dirichlet eigenvalues on a bounded domain $\Omega \subseteq \mathbb{R}^N$).

Polya's conjecture (see [27]) is as follows:

$$\lambda_k(\Omega) \geq C_N\Big(\frac{k}{|\Omega|}\Big)^{\frac{2}{N}} \tag{$\star\star$}$$

for every bounded domain in $\mathbb{R}^N$ and every positive integer k.

We note that, using the further improved expansions of Weyl's formula, if k and $|\Omega|$ are fixed, then the optimal shape, that is, the right-hand side reaches the minimum value, of the domain Ω is a ball. Weak versions of Polya's conjectured lower bound for Dirichlet eigenvalues are due to Berezin [4] and Li and Yau [23], with developments later by Laptev [22] and others using Riesz means and "universal" inequalities (see for example a survey article by Ashbaush [2]). Polya [27] proved the conjecture ($\star\star$) for planar tiling domains (see also [22]).

On the other hand, if we fix a k (sufficient large) and if the Weyl asymptotic formula ($\star$) is valid (uniformly in Ω), which is obviously an unknown issue, then one would expect, as k becomes larger and larger, that the "optimal" domain Ω of a fixed volume, which realizes the least value for the tight hand side of ($\star$), would converge to a ball which solves the isoperimetric (inequality) problem. We shall come back to this interesting and difficult point later.

2.2 Min-Max principle and nodal domains

Let us recall the classical min-max principle for Laplacian eigenvalues [14]:

(1)

$$\lambda_k = \min_{E_k \subset H_0^1(\Omega)} \max_{u \in E_k \setminus \{0\}} R(u).$$

(2)

$$\lambda_k = \max_{E_{k-1} \subset H_0^1(\Omega)} \min_{u \perp E_{k-1}, u \neq 0} R(u).$$

Here

$$R(u) = \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_\Omega u^2 \mathrm{d}x}, \quad 0 \neq u \in H_0^1(\Omega)$$

is the Rayleigh quotient, and E_j's are j-dimensional subspaces.

One of the uses of (1) and (2) is to obtain estimates for both upper bounds (via (1)) and lower bounds (via (2)).

In particular, $\lambda_1 = \inf\{R(u) : 0 \neq u \in H_0^1(\Omega)\}$ and $\lambda_k = \inf\{R(u) : 0 \neq u \perp \mathrm{Span}\{u_1, u_2, \cdots, u_{k-1}\}\}$ for $k = 2, 3, \cdots$.

Next, we have the nodal domains theorem of Courant.

Theorem 2.1 *Let u be a k-th eigenfunction, i.e., $\Delta u+\lambda_k u=0$ in Ω and $u\neq 0$ (identically) in Ω with $u\mid_{\mathcal{D}\Omega}=0$, then the number of nodal domains of $u\leq k$. Here an open connected subset D of Ω is called a nodal domain of u if u never vanishes on D and if $u\equiv 0$ on ∂D.*

When $N=1$, then the above nodal domain theorem is an easy consequence of the Sturm-Liouville theory. Moreover, in the latter case, the number of nodal domains (intervals) of a k-th eigenfunction is exactly k. A simple proof of the Courant's nodal domain theorem can be deduced from the min-max principle.

Proof of Courant's Nodal Domain Theorem Suppose that u is a k-th eigenfunction. Let us denote all eigenfunctions with eigenvalue λ_k by $E(\lambda_k)\setminus\{0\}$ (one has to eliminate the trivial zero function). If u has more than k nodal domains, then $\Omega=\coprod\limits_{j=1}^{m}\Omega_j$, $m>k$, here each Ω_j is a nodal domain (and the union means that they have no intersecting interior). Since $v_j=u\mid_{\Omega_j}$ does not change the signs, hence a first eigenfunction of the Laplacian on Ω_j with the zero Dirichlet boundary condition $v_j=0$ on $\partial\Omega_j$ (see [14]). We extend each $v_j, j=1,2,\cdots,k$, to be 0 outside Ω_j, and consider $v=\sum\limits_{j=1}^{k}c_jv_j\in H_0^1(\Omega)$, such that $v\perp \mathrm{Span}\{u_1,\cdots,u_{k-1}\}$, and that $\int_\Omega v^2\mathrm{d}x=\sum\limits_{j=1}^{k}c_j^2(\int_{\Omega_j}v_j^2\mathrm{d}x)=1$. It is obvious that one can find $(c_1,\cdots,c_k)\in\mathbb{R}^k\setminus\{0\}$, such that these conditions are valid. A direct computation yields that

$$R(v)=\frac{\int_\Omega|\nabla v|^2\mathrm{d}x}{\int_\Omega v^2\mathrm{d}x}=\lambda_k,$$

and hence, via min-max principle, one has $v\in E(\lambda_k)\setminus\{0\}$. Consequently, $\Delta v+\lambda_k v=0$ in Ω, $\int_\Omega v^2\mathrm{d}x=1$ and $\mathrm{spt}(v)\subseteq\bigcup\limits_{j=1}^{k}\Omega_j$. Therefore v vanishes on a nonempty open subset $\subseteq\Omega$. The latter is impossible by the analyticity of v in Ω (see [14]) or by the unique continuation theorem.

Corollary 2.1 *Suppose $u\in E(\lambda_2)\setminus\{0\}$. Then the number of nodal domains of u is exactly 2.*

Proof We have shown the number of nodal domains of a second eigenfunction $u_2\leq 2$. On the other hand, since u_1 has only one sign on Ω (an easy exercise) and $\int_\Omega u_1u_2\mathrm{d}x=0$, u_2 has to change signs on Ω.

2.3 Partitions of a domain

In general, we let $u\in E(\lambda_k)\setminus\{0\}$ be a k-th eigenfunction of the Laplacian on Ω with Dirichlet boundary condition. Let $\mu(u)$ be the number of nodal domains of u. We say there is an m-nodal partition of Ω if there is an eigenfunction $u\in E(\lambda_k)\setminus\{0\}$ (for some $k\geq m$) such that $m=\mu(u)$ (see also [17] for related discussion). A natural question is how large $\mu(u)$ can be, for a $u\in E(\lambda_k)\setminus\{0\}$. Courant's nodal domain theorem says that, for any $u\in E(\lambda_k)\setminus\{0\}$, $\mu(u)\leq k$. It turns out that $\mu_k=\max\{\mu(u):u\in E(\lambda_k)\setminus\{0\}\}$ is definitely smaller than k, for k large. The following elegant theorem was due to Pleijel.

Theorem 2.2 (Pleijel)

$$\limsup_{k\to\infty}\frac{\mu_k}{k}=\eta_0(N)<1$$

for any bounded domain Ω in $\mathbb{R}^N$. Here μ_k is defined above.

Proof Suppose that $u \in E(\lambda_k) \setminus \{0\}$ has μ_k nodal domains so that $\Omega = \coprod_{1 \le j \le \mu_k} \Omega_j$, each Ω_j is a nodal domain. Then there is at least one of the nodal domain, say Ω_j, satisfies $|\Omega_j| \le \frac{|\Omega|}{\mu_k}$. Note, by Courant's theorem, $\mu_k \le k$.

Next, we apply the Faber-Krahn inequality (see discussions below) to Ω_j to obtain (since Ω_j is a nodal domain of $u \in E(\lambda_k) \setminus \{0\}$) that $\lambda_k(\Omega) \equiv \lambda_1(\Omega_j) \ge \lambda_1(B_j)$, where B_j is a ball in $\mathbb{R}^N$ with $|B_j| \equiv |\Omega_j| \le \frac{|\Omega|}{\mu_k}$. To proceed, one examines, for example, the case $N = 2$. Then one has

$$\lambda_1(B_j) \ge \frac{\mu_k}{|\Omega|}\pi j_0^2 \ge (2.4)^2 \pi \frac{\mu_k}{|\Omega|}$$

(here $j_0 = 2.4$ is the first positive zero of the 0-th order Bessel function).

On the other hand, Weyl asymptotic formula implies that $\lambda_k(\Omega) \approx \frac{4\pi k}{|\Omega|}$ when $N = 2$. Hence $(2.4)^2\mu_k \le 4k$ or $\limsup\limits_{k\to\infty} \frac{\mu_k}{k} \le \eta_0(2) = \frac{4}{(2.4)^2} < 1$ when $N = 2$.

Finally, the cases $N > 2$ can be handled in the same way, and the conclusion of the theorem follows.

A closely related concept to the nodal partition of a domain Ω is the following notion of the spectral equal partition. A partition of $\Omega = \coprod_{j=1}^{m} \Omega_j$ is called a spectral equal partition, if $\lambda_1(\Omega_j)$'s for $j = 1, 2, \cdots, m$, are all equal. Thus a nodal partition is surely a spectral equal partition. But the converse is not true in general. For example, for a disc D in $\mathbb{R}^2$ centered at the origin, one can divide D into three equal (up to rotations) circular sectors (each of 120° opening) $\Omega_j, j = 1, 2, 3$, then $\lambda_1(\Omega_j)$'s are all equal. However, there is no eigenfunction u of the Laplacian on D with nodal set $N(u) = \{x \in D : u(x) = 0\}$ consisting of $E = \{(r, \theta) : 0 \le r \le r_0, \theta = \theta_i = \frac{2\pi i}{3}$ for $i = 1, 2, 3\}$ (it is an easy exercise). E cannot be a zero set of an analytic function on D (an analytic variety). On the other hand, it is somewhat no trivial that $D = \coprod_{j=1}^{3} \Omega_j$ is in fact a minimal partition (see discussions below).

As in [17], one can introduce the so-called l^p-minimal partitions: A partition of $\Omega = \coprod_{j=1}^{m} \Omega_j$ is called an l^p-minimal partition, if for "any partition" of Ω into m-subsets, $\Omega = \coprod_{j=1}^{m} \Omega'_j$, then

$$\Big(\sum_{j=1}^{m} \lambda_1^p(\Omega_j)\Big)^{\frac{1}{p}} \le \Big(\sum_{j=1}^{m} \lambda_1^p(\Omega'_j)\Big)^{\frac{1}{p}}.$$

We remark that

(a) Any l^∞-minimal partition is a spectral equal partition (this may be viewed as an interesting exercise).

(b) The existence, regularity and regularity of free interfaces of minimal partition have been studied by many authors (see, for example, the survey article [17], and also [11–12]).

In [11], we also conjectured, in the case $N = 2$, that

(i) $\lim\limits_{m\to\infty} L_m^1(\Omega) = c_0$ exists, and it is independent of Ω.

(ii) $\lim\limits_{m\to\infty} L_m^1(\Omega) = \lambda_1$ (Hexa).

Here $L_m^1(\Omega) = \frac{\sum\limits_{j=1}^{m} \lambda_1(\Omega_j)}{m^2}$, $\Omega = \coprod\limits_{j=1}^{m} \Omega_j \subseteq \mathbb{R}^2$ is a bounded domain with $|\Omega| = 1$ and where λ_1 (Hexa) is the first Dirichlet eigenvalue of a regular Hexagon in $\mathbb{R}^2$ with area 1. We also note that the first item in the above conjecture has been established by the author a few years ago (to be published). Similar conjectures for l^∞-minimal partitions were also proposed in [17].

3 Extremum Problems

3.1 Special cases

One of the main purposes of this article is to address the following extremum problem:

$$\inf\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ with } |\Omega| = 1\} \tag{EP}$$

for a positive integer k and $N \geq 2$. Here $\lambda_k(\Omega)$ is the k-th Dirichlet eigenvalue of the Laplacian on $\Omega \subseteq \mathbb{R}^N$. It is already unclear, at the first look, that what type of measurable subsets Ω in $\mathbb{R}^N$ with volume 1, so that $\lambda_k(\Omega)$ would be well-defined.

When $k = 1$, we have the following well-known Faber-Krahn inequality:

$$\lambda_1(B) \leq \inf\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ with } |\Omega| = 1\},$$

where B is a ball in $\mathbb{R}^N$ with $|B| = 1$. Here infimum is taking among any measurable sets Ω in $\mathbb{R}^N$ with $|\Omega| = 1$, so that $\lambda_1(\Omega)$ is defined. For example, Ω being a bounded, open subset of $\mathbb{R}^N$ will work. This inequality may be proved by many arguments including the classical symmetrization method (see [3, 13, 15, 18, 28]).

The case $k = 2$ is already drastically different from the case $k = 1$. In fact, there is no connected open set Ω that could solve the extremum problem (EP) when $k = 2$. Indeed, let us assume that $\Omega^* \subseteq \mathbb{R}^N$ solves (EP) with $k = 2$, and let u_2 be the corresponding second eigenfunction:

$$\begin{cases} -\Delta u_2 = \lambda_2^* u_2 & \text{in } \Omega^*, \\ u_2 = 0 & \text{on } \partial\Omega^* \end{cases}$$

with $\int_\Omega u_2^2 \mathrm{d}x = 1$, $|\Omega^*| = 1$, and λ_2^* be the infimum value for (EP) when $k = 2$. By the min-max principle, u_2 has exactly two nodal domains: Ω_1, Ω_2 so that $\Omega = \Omega_1 \coprod \Omega_2$. Hence $|\Omega| = |\Omega_1| + |\Omega_2|$, and u_2 restricted to both Ω_1 and Ω_2 are the Dirichlet first eigenfunctions with eigenvalue λ_2^*. We let balls $B_1, B_2 \subseteq \mathbb{R}^N$ be such that $|B_1| = |\Omega_1|$, $|B_2| = |\Omega_2|$, then Rayleigh-Faber-Krahn inequality implies that $\lambda_1(B_1) \leq \lambda_2^*$, $\lambda_1(B_2) \leq \lambda_2^*$ (with equality if and only if the domains Ω_1 and Ω_2 are also balls). Now it is clear, then one must have $|B_1| = |B_2| = \frac{1}{2}|\Omega|$. For otherwise, say $|B_1| < |B_2|$, by a simple scaling and by the monotonicity of the first eigenvalues depending on domain inclusions, one has $\lambda_1(B_2) < \lambda_2^*$. Thus one can scale up B_1 by a factor > 1 and B_2 by a factor < 1, so that the resulting balls $\widetilde{B_1}, \widetilde{B_2}$ would satisfy $|\widetilde{B_1}| + |\widetilde{B_2}| = |\Omega| = 1$ and that $\lambda_1(\widetilde{B_2}) < \lambda_2^*$ remains to be true while $\lambda_1(\widetilde{B_1}) < \lambda_1(B_1) \leq \lambda_2^*$. Therefore, the new domain $\widetilde{\Omega} = \widetilde{B_1} \coprod \widetilde{B_2}$ would satisfy the properties that $|\widetilde{\Omega}| = 1$ and $\lambda_2(\widetilde{\Omega}) < \lambda_2^*$. We obtain a contradiction.

The above argument has showed also that the solution to (EP) when $k = 2$ is given by a disjoint union of two balls of the equal volume $\frac{1}{2}$.

These preliminary observations lead to a couple rather basis questions:

(Q1) What type sets Ω would be admissible for the extremum problem (EP)?

(Q2) How one can handle the usual "concentration compactness" problems when a minimizing sequence $\{\Omega_j\}$ of (EP) in $\mathbb{R}^N$ may be stretched and splitted to infinitely?

(Q3) Is it possible to find minimizers Ω of (EP) such that Ω would be open and connected?

There were numerous works addressing the first two questions. Much of discussion of these may be find in the excellent monograph by Henrot [18] and references therein. Here we shall be concentrated mainly on the third question and to discuss some recent progress on it. Before we do so, let us discuss a couple more specific cases.

The first special class of subsets $\Omega \subseteq \mathbb{R}^N$ one would consider is the class of convex domains. We have the following.

Theorem 3.1 *There is a convex domain that solves the* (EP)*:*

$$\inf\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \textit{ is convex and } |\Omega| = 1\}$$

for every positive integer k.

Proof Let $\{\Omega_n\}$ be a minimizing sequence. Since Ω_n's are convex and $|\Omega_n| = 1$, one has that either Ω_n converges in the Hausdorff distance to a bounded convex domain Ω^* with $|\Omega^*| = 1$, or there is a subsequence of $\{\Omega_{n'}\}$ such that $\Omega_{n'}$ would be contained in strips (after suitable rotations and translations) of form $[0, \delta_n] \times [-L_n, L_n]^{N-1}$, such that $\delta_n \to 0^+$ while $L_n \to +\infty$. If the latter is the case, then an easy calculation of the first eigenvalues of the regions of the form $Q_n = [0, \delta_n] \times [-L_n, L_n]^{N-1}$ yields that $\lambda_1(Q_n) \geq \frac{\pi^2}{\delta_n^2}$ (no matter what are the sized of L_n's). In particular,

$$\lambda_k(\Omega_{n'}) \geq \lambda_1(\Omega_{n'}) \geq \lambda_1(Q_n) \geq \frac{\pi^2}{\delta_n^2} \to +\infty$$

as n tends to infinite. This would contradict to the fact that $\lambda_k(\Omega_{n'})$ converge to the value

$$\inf\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ is convex and } |\Omega| = 1\},$$

and hence the latter is not possible. For the former case, an easy fact in the convex geometry implies that if Ω_n's converges to Ω^* (convex) with $|\Omega^*| = 1$ in the Hausdorff distance, then Ω_n's are uniformly Lipschitz domains. Moreover, $\lambda_k(\Omega_n)$ converges to $\lambda_k(\Omega^*)$ as $n \to \infty$ can be easily established. Hence the conclusion of the theorem follows.

The second class of subsets $\Omega \subseteq \mathbb{R}^N$ we would discuss here are bounded sets. Here is one of the basis existence result (see [7–9, 18] and references therein).

Theorem 3.2 *There is a quasi-open set $\Omega_* \subseteq B$ that solves the following constrained extremum eigenvalue problem:*

$$\inf\{\lambda_k(\Omega) : \Omega \textit{ is a quasi-open subset of } B \textit{ with } |\Omega| = 1\}. \tag{EPc}$$

Here B is a large ball (or any bounded, Lipschitz domain with $|B| > 1$).

The proof of the above theorem is contained in the references [7–9, 18], and it takes some pages to describe it. Here we shall discuss the relevant notion of quasi-open sets and some properties of such sets in the next section as these would be important to other parts of discussions in the paper.

3.2 Quasi-open sets

Let $f(x)$ be a real valued continuous function on $\mathbb{R}^N$. Then for any $c \in \mathbb{R}$, $O = \{x \in \mathbb{R}^N : f(x) > c\}$ is an open set. The converse is also true, that is, if O is open in $\mathbb{R}^N$, then there is a (smooth) continuous function on $\mathbb{R}^N$ such that $O = \{x \in \mathbb{R}^N : f(x) > 0\}$. To define quasi-open subsets of $\mathbb{R}^N$, we introduce the notion of quasi-continuous functions. A real valued function $f(x)$ is called quasi-continuous if and only if, $\forall \varepsilon > 0$, there is a subset $\Omega_\varepsilon \subset \mathbb{R}^N$, such that f is continuous on Ω_ε and that the classical capacity of $\mathbb{R}^N \setminus \Omega_\varepsilon$ is less than ε. A subset Ω of $\mathbb{R}^N$ is called quasi-open, if there is a quasi-continuous function f such that $\Omega = \{x \in \mathbb{R}^N : f(x) > 0\}$ (which is defined upto zero capacity sets). One can check that a set Ω is quasi-open, if, $\forall \varepsilon > 0$, $\exists$ an open set O_ε such that $\mathrm{cap}(\Omega \Delta O_\varepsilon) < \varepsilon$. Equivalently, a set Ω is quasi-open, if there is a sequence of open sets $\{\Omega_n\}$, such that, $\Omega_n \supseteq \Omega_{n+1} \supseteq \cdots \supseteq \Omega$ and that $\mathrm{cap}(\Omega_n \setminus \Omega) \to 0^+$.

A theorem of Federer-Ziemer says that if f is an $H^1(\mathbb{R}^N)$ function, then f is quasi-continuous. It is then not hard to show that a subset $\Omega \subseteq \mathbb{R}^N$ is quasi-open if and only if there is a non-negative $H^1(\mathbb{R}^N)$ function f such that $\Omega = \{x \in \mathbb{R}^N : f(x) > 0\}$.

Next, it is necessary to discuss also a few natural topologies on the space of quasi-open subsets in $\mathbb{R}^N$ in order to solve (EP). For conveniences, let us assume that these quasi-open sets are contained in a fixed bounded domain.

Definition 3.1 *Let $\{\Omega_n\}$ be a sequence of quasi-open sets in a bounded domain B. We say that Ω_n is γ-convergent to Ω if and only if the associated potential functions of the domains are convergent, that is, $u_{\Omega_n} \to u_\Omega$ in $L^2(B)$ as $n \to \infty$. Here*

$$\begin{cases} -\Delta u_{\Omega_n} = \chi_{\Omega_n}, \\ u_{\Omega_n} = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_n, \end{cases}$$
$$\begin{cases} -\Delta u_{\Omega} = \chi_{\Omega}, \\ u_{\Omega} = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Suppose that u_{Ω_n} and u_Ω are defined as above, and let $d_\gamma(\Omega_n, \Omega) = \int_B |u_{\Omega_n} - u_\Omega| \mathrm{d}x$. Then it is an exercise to check that $\Omega_n \Rightarrow \Omega$ in the sense of γ-convergence if and only if $d_\gamma(\Omega_n, \Omega) \to 0$ as n converges to infinite. The following theorem is due to Sverak.

Theorem 3.3 (see [29]) *Let $\{\Omega_n\}$ and Ω be quasi-open sets in B. Then Ω_n is γ-convergent to Ω if and only if, $\forall f \in L^2(B)$, the solution of*

$$\begin{cases} -\Delta v_n = f \quad \text{in } \Omega_n, \\ v_n = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_n \end{cases}$$

converges in $L^2(B)$ to v, the solution of

$$\begin{cases} -\Delta v = f \quad \text{in } \Omega, \\ v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We note that $v,\ v_n \in H_0^1(B)$. Let us sketch a proof of the above theorem when Ω_n and Ω are open and smooth domains.

Proof It suffices to verify that if Ω_n is γ-convergent to Ω, then $v_n \to v$ in $L^2(B)$. For this purpose, we consider first that $0 \leq f \leq M$, for a large constant M. Then the maximum principle implies that $0 \le v \le Mu_\Omega$ and $0 \le v_n \le Mu_{\Omega_n}$. A simple calculation yields

$$\begin{aligned}0 &= \int_B \Delta(|v_n - v|)^2 \mathrm{d}x \\ &= 2\int_B |\nabla(v_n - v)|^2 \mathrm{d}x - \int_{(\partial\Omega_n)\cap\Omega} v\Big|\frac{\mathcal{D}v_n}{\mathcal{D}\nu_n}\Big| - \int_{(\partial\Omega)\cap\Omega_n} v_n\Big|\frac{\mathcal{D}v}{\mathcal{D}\nu}\Big| \\ &\quad - \int_{\Omega_n\setminus\Omega} 2fv_n\ \mathrm{d}x - \int_{\Omega\setminus\Omega_n} 2fv\ \mathrm{d}x.\end{aligned}$$

One notices that the same formula is valid also for u_Ω and u_{Ω_n}. Since Ω_n is γ-convergent to Ω, hence $u_{\Omega_n} \to u_\Omega$ in $L^2(B)$. Consequently, $\int_B |u_{\Omega_n} - u_\Omega| \mathrm{d}x \to 0$ as $n \to \infty$. Therefore, the followings are true:

(i) $\int_{\Omega_n\setminus\Omega} u_{\Omega_n}\mathrm{d}x = o_n(1)$, $\int_{\Omega\setminus\Omega_n} u_\Omega \mathrm{d}x = o_n(1)$;

(ii) $\int_B |\nabla u_{\Omega_n}|^2\mathrm{d}x - \int_B |\nabla u_\Omega|^2 \mathrm{d}x = o_n(1)$;

(iii)

$$\int_{(\partial\Omega_n)\cap\Omega} u_\Omega\Big|\frac{\mathcal{D}u_{\Omega_n}}{\mathcal{D}\nu_n}\Big| + \int_{(\partial\Omega)\cap\Omega_n} u_{\Omega_n}\Big|\frac{\mathcal{D}u_\Omega}{\mathcal{D}\nu}\Big| = o_n(1).$$

Note the second item above, which follows from (i) and an integration, implies that $u_{\Omega_n} \to u_\Omega$ in $H_0^1(B)$. Now (i) and the maximum principle imply also that

$$\begin{aligned}0 &\le \int_{\Omega_n\setminus\Omega} v_n f \mathrm{d}x \le M^2 \int_{\Omega_n\setminus\Omega} u_{\Omega_n}\mathrm{d}x \to 0, \\ 0 &\le \int_{\Omega\setminus\Omega_n} vf\mathrm{d}x \le M^2\int_{\Omega\setminus\Omega_n} u_\Omega \mathrm{d}x \to 0\end{aligned}$$

and that

$$\int_{(\partial\Omega_n)\cap\Omega} v\Big|\frac{\mathcal{D}v_n}{\mathcal{D}\nu_n}\Big| + \int_{(\partial\Omega)\cap\Omega_n} v_n\Big|\frac{\mathcal{D}v}{\mathcal{D}\nu}\Big| \le M^2 \times \text{ (iii) } \to 0.$$

Thus $v_n \to v$ in $H_0^1(B)$ in this case.

Now for any $f \in L^2(B)$ (or $H^{-1}(B)$), one can find $f_M \in L^2(B)$ with $|f_M| \le M$. Let the corresponding solutions be v_n^M and v^M, then the above arguments yields $\|v_n^M - v^M\|_{L^2(B)} \to 0$ as $n \to \infty$. On the other hand, standard elliptic estimates imply that $\|v_n^M - v_n\|_{L^2(B)} + \|v - v^M\|_{L^2(B)} \le c\|f - f_M\|_{H^{-1}(B)} \to 0$ as $M \to +\infty$. We conclude $v_n \to v$ in $L^2(B)$.

Let us also introduce a convergence of Hilbert spaces, $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$, in the sense defined by Mosco (see [7]), here Ω_n, $\Omega \subseteq B$. A sequence $H_0^1(\Omega_n)$ is called to be convergent to $H_0^1(\Omega)$ in the sense of Mosco, if the following two statements are held:

(a) $\forall v \in H_0^1(\Omega)$, there is a sequence $\{v_n\} \subset \{H_0^1(\Omega_n)\}$ such that $v_n \to v$ in $H_0^1(B)$.

(b) If $\{v_n\} \subset \{H_0^1(\Omega_n)\}$ such that there is a subsequence $\{v_{n_k}\}$ with $v_{n_k} \rightharpoonup v$ in $H_0^1(B)$, then $v \in H_0^1(\Omega)$.

It is not hard to show, via min-max principle, that the statement (a) above implies that

$$\limsup_n \lambda_k(\Omega_n) \le \lambda_k(\Omega).$$

On the other hand, the statement (b) would imply that

$$\liminf_n \lambda_k(\Omega_n) \ge \lambda_k(\Omega).$$

Though the γ-convergence (and convergence in the sense of Masco defined above) would imply the convergence of Laplacian eigenvalues with the zero Dirichlet boundary condition, they are in a way strong convergences. Consequently, it is not easy to work with for our extremum problems. The following weak-convergence of domains would be more suitable to solve the variational problems in shape optimizations.

Definition 3.2 *A sequence of quasi-open domains $\Omega_n \subseteq B$ is said to converge to Ω weakly with $\Omega \subset B$, if $u_{\Omega_n} \to w$ in $L^2(B)$, and $\Omega = \{x \in B : w(x) > 0\}$.*

We note that w in general is not equal to u_Ω. One also notices that potential functions are continuous from the De Girogi elliptic regularity theory. The following proposition is trivial.

Proposition 3.1 (Compactness) *Let $\{\Omega_n\}$ be a sequence of quasi-open subdomains in B. Then there is a subsequence $\{\Omega_{n_k}\}$ that converges weakly to Ω, a quasi-open subdomain in B.*

With this easy compactness result, one can show the constrained extremum-problem (EPc for short) has a solution (see [8–9]). For example, when $k = 1$, if $\{\Omega_n\}$ converges weakly to Ω^* in B, then it is easy to show that $\lambda_1(\Omega^*) \le \liminf_n \lambda_1(\Omega_n)$. This latter fact can be easily deduced by considering the Rayleigh quotients that characterizing eigenvalues, as the lower semicontinuity of the Dirichlet integral along with the fact that the L^2 norms eigenfunctions are conserved under the weak convergence in $H_0^1(B)$. Moreover, $|\Omega^*| \ge 1$ (indeed, if $|\Omega^*| < 1$, then it would contradict to $\lambda_1(\Omega^*) \le \inf\{\lambda_1(\Omega) : \Omega \subset B$ quasi-open with $|\Omega| = 1\}$). On the other hand, if $|\Omega^*| > 1$, then for small $\sigma > 0$ such that $|\Omega^*| \approx |\{u_* > \sigma\}|$, u_* is the positive first Dirichlet eigenfunctions on Ω^*. Let u_n be the first Dirichlet eigenfunction on Ω_n, then $u_n \to u_*$ in $L^2(B)$ (by taking subsequence if needed). Thus

$$|\{u_* > \sigma\}| \le \left|\left\{u_n > \frac{\sigma}{2}\right\}\right| \le |\Omega_n| = 1,$$

and we would also obtain a contradiction. For general k-th, $k > 1$, eigenvalues, it could be also verified in the same way using the min-max principle and an induction on k.

4 Connected Minimizers

The existence of minimizers of (EP) without boundedness constraint has been established recently in the work of Mazzoleni and Pratelli [24] and Bucur [6]. In fact, in [24] a more general class of extremum problems for Laplacian-Dirichlet eigenvalues was considered, and existence of bounded minimizers was proven. As a consequence of their proofs, they also showed that for any quasi-open set $A \subseteq \mathbb{R}^N$, one has $\lambda_k(A) \le M(k,N)\lambda_1(A)$. In [6], Bucur proved that minimizers of (EP) exists. Moreover, he showed every minimizer is bounded and has a finite perimeter. The last result will be discussed in the final section of this paper. The aim of this section is to study when such minimizers are connected domains. We should also point out that in another recent work, by Bucur-Mazzoleni-Pratelli-Velichkov [10], it was shown that minimizers are open sets.

4.1 Splitting (in)equality

Theorem 4.1 *Assume that there is a multiply connected domain Ω_k^* that solves, for given k, the following problem:*

$$\inf\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \textit{ is quasi-open with } |\Omega| = 1\}. \tag{EP}$$

Denote the infimum value of above extremum problem by $\Lambda(k,N)$. Then, for some $1 \le m \le k$,

$$\Lambda(k,N)^{\frac{N}{2}} = \sum_{j=1}^{m} \Lambda(k_j,N)^{\frac{N}{2}}.$$

Here Ω_k^* can be decomposed into mutually disjoint subdomains $\Omega_{k_1}, \cdots, \Omega_{k_m}$, such that $\Omega_k^* = \coprod_{j=1}^{m} \Omega_{k_j}$, and $\sum_{j=1}^{m} |\Omega_{k_j}| = 1$, and that the positive integers $k_1, k_2, \cdots, k_m$ satisfy $k_1+k_2+\cdots+k_m = k$. And each Ω_{k_j} can be scaled properly (so that its volume becomes 1) to solve (EP) with $k = k_j$.

This result may be viewed as an extension of a theorem due to Keller-Wolf [20], and we shall see that it is an easy consequence of the min-max principle. On the other hand, using arguments from the proofs of concentration-compactness (see [8]), one may derive a similar statement for minimizing sequences.

Theorem 4.2 *For any positive integer k, there is a minimizing sequence $\{\Omega_n\}$ of the problem (EP) and an integer m with $1 \le m \le k$ such that $\Omega_n = \coprod_{j=1}^{m} \Omega_{n,k_j}$, $\sum_{j=1}^{m} k_j = k$, for some positive integers k_j's, $j = 1, 2, \cdots, m$. Moreover, each sequence $\{\Omega_{n,k_j}\}$ after suitable scalings would itself form a minimizing sequence for (EP) when $k = k_j$. In particular,*

$$\Lambda(k,N)^{\frac{N}{2}} \ge \sum_{j=1}^{m} \Lambda(k_j,N)^{\frac{N}{2}}.$$

Let us sketch a proof of Theorem 4.1.

Proof of Theorem 4.1 Assume that Ω_k^* is not connected, and we write $\Omega_k^* = \Omega_1 \coprod \Omega_2$, a union of two subdomains (which are not necessarily connected) such that $|\Omega_i| > 0$ $(i = 1, 2)$ and $|\Omega_1| + |\Omega_2| = 1$. Let u_k^* be an eigenfunction of the Dirichlet-Laplacian on Ω_k^*. Then $u_k^* |_{\Omega_i}$ $(i = 1, 2)$ are eigenfunctions of the Dirichlet-Laplacian on Ω_i also. Assume that u_k^* is the j_1-th eigenvalue on Ω_1 (with j_1 is the maximum so that $\lambda_{j_1}(\Omega_1) = \lambda_k(\Omega_k^*) = \Lambda(k,N)$). We claim first that $j_1 < k$. Indeed, otherwise, we may choose u_k^* to be zero on Ω_2 and u_k^* will be a k-th Dirichlet eigenfunction on Ω_1. But the latter is not possible as $|\Omega_1| < |\Omega_k^*| = 1$, so that one can properly scaled up Ω_1 to obtain a new domain of volume 1 but strictly less k-th eigenvalue, and hence it contradicts to the fact that $\lambda_k(\Omega_k^*) = \Lambda(k,N)$ solves (EP).

Next, we claim that there are at least $(k - j_1)$ eigenvalues of Ω_2 which are smaller than $\Lambda(k,N)$. Otherwise, for $\lambda_{k-j_1}(\Omega_2) > \Lambda(k,N)$ (while $\lambda_{j_1}(\Omega_1) = \Lambda(k,N)$), the min-max principle (here we may choose the $k-1$ dimensional subspace of $H_0^1(\Omega_k^*)$ to be spanned by j_1 eigenfunctions on Ω_1 and the first $k-1-j_1$ eigenfunctions on Ω_2) is as follows:

$$\lambda_k(\Omega_k^*) = \max_{E_{k-1} \subset H_0^1(\Omega_k^*)} \min_{u \perp E_{k-1} \setminus \{0\}} R(u) \ge \min(\lambda_{k-j_1}(\Omega_2), \lambda_{j_1+1}(\Omega_1)) > \Lambda(k,N).$$

The latter is not possible. Thus $\lambda_{k-j_1}(\Omega_2) \le \Lambda(k,N)$. On the other hand, if $\lambda_{k-j_1}(\Omega_2) < \Lambda(k,N)$, then the other min-max principle

$$\lambda_k(\Omega_k^*) = \min_{E_k \subset H_0^1(\Omega_k^*)} \max_{u \in E_k \setminus \{0\}} R(u)$$

would imply $\lambda_k(\Omega_k^*) < \Lambda(k,N)$, again impossible.

Finally, if we replace Ω_1 by $|\Omega_1|^{\frac{1}{N}} \Omega_{j_1}^* = \widetilde{\Omega}_1$ and Ω_2 by $|\Omega_2|^{\frac{1}{N}} \Omega_{k-j_1}^* = \widetilde{\Omega}_2$. Then we have $|\widetilde{\Omega}_1| + |\widetilde{\Omega}_2| = 1$ (note that $|\Omega_1| = |\widetilde{\Omega}_1|$, $|\Omega_2| = |\widetilde{\Omega}_2|$) and $\lambda_{j_1}(\widetilde{\Omega}_1) = |\Omega_1|^{-\frac{2}{N}} \Lambda(j_1,N)$,

$\lambda_{k-j_1}(\widetilde{\Omega}_2) = |\Omega_2|^{-\frac{2}{N}}\Lambda(k-j_1, N)$. By the minimality of $\Lambda(k,N)$, we thus conclude that the last two inequalities above are equalities. That is,

$$\Lambda(k,N)^{\frac{N}{2}} = \Lambda(j_1,N)^{\frac{N}{2}} + \Lambda(k-j_1,N)^{\frac{N}{2}}.$$

An easy induction leads to the conclusion of Theorem 4.1.

As a byproduct of the above proof, we have the following statement which again follow from the min-max principle.

Proposition 4.1 *Let Ω be a bounded open set in $\mathbb{R}^N$, and assume that Ω has m connected components $\Omega_1, \Omega_2, \cdots, \Omega_m$. Suppose that u is the k-th eigenfunction of the Laplacian on Ω with the Dirichlet boundary condition. Then $u\,|_{\Omega_j}$ are eigenfunctions of Dirichlet-Laplacian on Ω_j's, $j = 1, 2, \cdots, m$ (unless that u vanished identically on some of Ω_j's). Let k_j be positive integers (or zero if $u\,|_{\Omega_j} \equiv 0$), such that $u\,|_{\Omega_j}$ is a k_j-th eigenfunction. Then $\sum_{j=1}^{m} k_j = k$.*

There are a few simple consequence of the above proposition.

Corollary 4.1 *The minimization problem* (EP) *has a solution Ω_k^* which, in general, would have at most m connected components with $m \le c_0 k$ for some $c_0 < 1$.*

Proof Suppose that Ω_k^* has m connected components. Then one of the component, say Ω_{k_1}, must have its volume $\le \frac{1}{m}$. If u_k^* is a k-th eigenfunction on Ω_k^* with the Dirichlet boundary condition, then u_k^* is a k_1-th eigenfunction on Ω_{k_1}. Thus $\Lambda(k,N) = \lambda_{k_1}(\Omega_{k_1}) \ge \lambda_1(\Omega_{k_1}) \ge \lambda_1(\widetilde{B}) \ge (1+\delta_N)C_N m^{\frac{2}{N}}$ (by Faber-Krahn inequality), where $\widetilde{B}$ is a ball of volume $\frac{1}{m}$. On the other hand, $\Lambda(k,N) \le \lambda_k(B)$, for ball B of volume 1. Weyl's asymptotic formula implies that $\lambda_k(B) = C_N k^{\frac{2}{N}}$. We thus conclude that

$$m \le \frac{1}{(1+\delta_N)^{\frac{2}{N}}} k \le c_0 k.$$

We note that the above proof is very similar in the spirit to the proof of Pleijel's theorem.

Corollary 4.2 *For every $N \ge 2$,* (EP) *has a solution for some $k \ge 3$, which is a connected open set.*

Proof The existence of a bounded (depending on N and k) minimizer which is also an open set in $\mathbb{R}^N$ for the (EP) was known (see [6, 10, 24]). To show, for some k, it is connected, we assume, to the contrary, that Ω_k^* for all $k \ge 3$ are disconnected. Then by discussions in this section, one would conclude that Ω_k^* must consist of exactly k connected components. The latter is not possible by Corollary 4.1.

For $k = 3$ and $N = 2$ or 3, Keller-Wolf [20] observed earlier that solutions of (EP) are connected. Indeed, for $N = 2$, $k = 3$, if Ω_3^* is disconnected, then one would conclude that $\Lambda(3,2) = 3\Lambda(1,2)$, i.e., Ω_3^* is consisting of 3 disjoint equal balls of volume $\frac{1}{3}$ each. As $\Lambda(1,2) = \pi j_0^2 \ge 18$, and $\Lambda(3,2) \le \lambda_3(D) = 46$ (here D is a disc in $\mathbb{R}^2$ of area 1), one sees that it is not possible.

For $N = 3$, $k = 3$, one calculates $\Lambda(1,3) \le 26$, and $\lambda_3(B)^{\frac{3}{2}} = 380$ (here B is a ball in $\mathbb{R}^3$ of volume 1). We thus have $\Lambda(3,3)^{\frac{3}{2}} = 3\Lambda(1,3)^{\frac{3}{2}} \ge 389$, and again it is not possible.

4.2 Generalized Polya's conjecture

The classical Polya's conjecture can be stated as follows.

Conjecture 4.1

$$\Lambda(k,N) \geq C_N k^{\frac{2}{N}}.$$

Polya proved for any planar tilling domain Ω of area 1, $\lambda_k(\Omega) \geq C_2 k \equiv 4\pi k$ for $k = 1, 2, \cdots$. In fact, one can see, for any finite $k \in \mathbb{N}$, the strict inequality is true from Polya's proof. We believe the following conjecture may be also valid.

Conjecture 4.2 (Generalized Polya's Conjecture)

$$\Lambda(k,N) \geq (1+\delta_{k,N})^{\frac{2}{N}} C_N k^{\frac{2}{N}}$$

for all $N > 2$ and $k > 1$. Here $\delta_{k,N}$ are positive numbers depending only on N and k.

Proposition 4.2 *Assume that the generalized Polya's conjecture is true, then there are infinitely many k's, such that the extremum problem has a solution Ω_k^* which is a connected open set.*

Proof Suppose that the conclusion of the above proposition is not true. Then for any $k \geq k_0$, one has

$$\Lambda(k,N)^{\frac{N}{2}} = \sum_{j=1}^{m} \Lambda(k_j, N)^{\frac{N}{2}},$$

such that $1 \leq k_j \leq k_0$ and that $\sum\limits_{j=1}^{m} k_j = k$. Note that this implies in particular that $m \geq c_1 k$, $c_1 > 0$ for all k large. It is clear that

$$\limsup_{k\to\infty} \frac{\Lambda(k,N)^{\frac{N}{2}}}{k} \leq C_N^{\frac{N}{2}}.$$

On the other hand,

$$\begin{aligned}
\Lambda(k,N)^{\frac{N}{2}} &= \sum_{j=1}^{k_0} m_j \ \Lambda(k_j,N)^{\frac{N}{2}} \\
&\geq \sum_{j=1}^{k_0} k_j m_j \ (1+\delta_{k_j,N}) C_N^{\frac{N}{2}} \\
&\geq \Big(\sum_{j=1}^{k_0} k_j m_j\Big)(1+\delta_0) C_N^{\frac{N}{2}} \\
&= k(1+\delta_0) C_N^{\frac{N}{2}}.
\end{aligned}$$

Here m_j is the number of times of $\Lambda(k_j, N)$ appeared in the summation of the splitting equality, where $\delta_0 = \min\{\delta_{k,N} : 1 \leq k \leq k_0\} > 0$. We therefore obtain an contradiction, when k is sufficiently large.

4.3 Regularity of minimizers

The regularity of minimizers, Ω_k^*, of the extremum problem (EP) is a challenging problem. We are going to describe a work (in progress) of the author with Dennis Kriventsov. Before we do so, let us describe a recent interesting work of Bucur [6] in which he proved that Ω_k^*'s are bounded and of finite perimeter.

We let Ω_k^* be a minimizer of the problem (EP). For quasi-open set $\Omega \subset \mathbb{R}^N$, we define u_Ω be the potential function of Ω:

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega, \\ u_\Omega = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

For A_1, A_2 quasi-open and bounded sets, we define

$$d_\gamma(A_1, A_2) = \int_{\mathbb{R}^N} |u_{A_1} - u_{A_2}| \, \mathrm{d}x.$$

It is easy to see that a sequence $\{\Omega_n\}$ of quasi-open sets (contained in a fixed ball), such that Ω_n is γ convergent to Ω, here Ω is a quasi-open subset (of the same ball) if and only if $d_\gamma(\Omega_n, \Omega) \to 0$ as $n \to \infty$.

Definition 4.1 *A quasi-open set A of finite Lebesgue measure is called a local shape subsolution for $E(A)$, if there is an $\eta > 0$ and $\Lambda > 0$, such that, for all $\widetilde{A} \subseteq A$ and $\widetilde{A}$ quasi-open with $d_\gamma(A, \widetilde{A}) \le \eta$, one has*

$$E(A) + \Lambda |A| \le E(\widetilde{A}) + \Lambda |\widetilde{A}|,$$

where

$$\begin{aligned} E(A) &= \min_{u \in H_0^1(A)} \left\{ \frac{1}{2} \int_A |\nabla u|^2 \, \mathrm{d}x - \int_A u \, \mathrm{d}x \right\} \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_A|^2 - \int_{\mathbb{R}^N} u_A \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} u_A \mathrm{d}x \equiv -\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_A|^2 \mathrm{d}x. \end{aligned}$$

Note $d_\gamma(A, \widetilde{A}) = \int_{\mathbb{R}^N} (u_A - u_{\widetilde{A}}) \mathrm{d}x$ in this case.

Remark 4.1 If Ω_k^* is a minimizer of (EP), then $t\Omega_k^*$ is a minimizer of $\lambda_k(A) + |A|$, where A quasi-open in $\mathbb{R}^N$, for some dilation constant $t > 0$. The converse is also true. Indeed, for any quasi-open set A with $0 < |A| < \infty$, one has $\lambda_k(A) + |A| = |A|^{-\frac{2}{N}} \lambda_k(\widetilde{A}) + |A| \cdot |\widetilde{A}|$, where $\widetilde{A}$ is a homothety of A such that $|\widetilde{A}| = 1$. Hence if we let $a = |A| \in \mathbb{R}^+$, then $\lambda_k(A) + |A|$ $= a + a^{-\frac{2}{N}} \lambda_k(\widetilde{A})$.

If Ω_k^* is a minimizer of (EP), then $t\Omega_k^*$ is a minimizer of $\lambda_k(A) + |A|$, where t is the unique positive critical point of $f(a) = a + a^{-\frac{2}{N}} \Lambda(k, N)$. Conversely, if A quasi-open, $0 < |A| < \infty$ is a minimizer of $\lambda_k(A) + |A|$, then $\widetilde{A}$ is a minimizer of (EP). Here $\widetilde{A}$ is the homothety of A with $|\widetilde{A}| = 1$.

Theorem 4.3 (Bucur) *If A is a quasi-open set that minimizes $\{\lambda_k(B) + |B| : B$ quasi-open in $\mathbb{R}^N$ with $0 < |B| < \infty\}$, then A is a local shape subsolution of $E(\cdot)$.*

The proof of this statement is based on the fact that

$$\left| \frac{1}{\lambda_k(A)} - \frac{1}{\lambda_k(\widetilde{A})} \right| \le \|R_A - R_{\widetilde{A}}\| \le c_{k,N}(A) d_\gamma(A, \widetilde{A}) = c_{k,N}(A)(E(\widetilde{A}) - E(A)),$$

where R_A, $R_{\widetilde{A}}$ are resolvent operators of Laplacian on A and $\widetilde{A}$, respectively, with the zero Dirichlet boundary condition, and where $\widetilde{c}_{k,N}$ is a constant depending only on k and N. Thus

$\lambda_k(\widetilde{A}) - \lambda_k(A) \le \widetilde{c}_{k,N}(E(\widetilde{A}) - E(A))$. On the other hand, $|A \setminus \widetilde{A}| \le \lambda_k(\widetilde{A}) - \lambda_k(A)$ (see the remark above), thus one has proven that Ω_k^* is a local shape subsolution of the energy $E(\cdot)$.

Theorem 4.4 (Bucur) *If A is a local shape subsolution of the energy $E(\cdot)$, then A is bounded and $\chi_A \in BV$. That is, A is a set of finite perimeter.*

Proof Let $u = u_A$, $u_\varepsilon = (u-\varepsilon)_+$ and $\widetilde{A} = \{u_\varepsilon > 0\}$.

$$\begin{aligned} E(A) + \Lambda|A| &= \frac{1}{2}\int |\nabla u|^2 \mathrm{d}x - \int u \ \mathrm{d}x + \Lambda\{u > 0\} \\ &\le \frac{1}{2}\int |\nabla u_\varepsilon|^2 \mathrm{d}x - \int u_\varepsilon \ \mathrm{d}x + \Lambda|\{u_\varepsilon > 0\}. \end{aligned}$$

Note that as $\varepsilon \to 0$, $d_\gamma(\widetilde{A}, A) \to 0$. Thus we obtain

$$\frac{1}{2}\int_{0\le u\le\varepsilon} |\nabla u|^2 \mathrm{d}x + \Lambda|\{0 < u \le \varepsilon\}| \le \int (u - u_\varepsilon)\mathrm{d}x = \int_{0\le u\le\varepsilon} u \ \mathrm{d}x + \varepsilon|\{u > \varepsilon\}| \le \varepsilon|A|.$$

Consequently,

$$\int_{0\le u\le\varepsilon} |\nabla u|\mathrm{d}x \le \Big(\int_{0\le u\le\varepsilon} |\nabla u|^2\mathrm{d}x + |\{0 < u \le \varepsilon\}|\Big)^{\frac{1}{2}} \le \sqrt{\frac{2}{\Lambda}\varepsilon}|A|.$$

Co-area formula implies that there is a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, such that $\mathcal{H}^{N-1}(\mathcal{D}^*\{u > \varepsilon_n\}) \le \sqrt{\frac{2}{\Lambda}}|A|$, and $\mathcal{H}^{N-1}(\mathcal{D}^*\{u > 0\}) \le \sqrt{\frac{2}{\Lambda}}|A|$ follows. Finally, a direct construction and comparison, using the property of "local shape subsolution" of A, yields that for $\theta \in (0,1)$, there is an $r_0 > 0$, $c_0 > 0$ such that for all $x_0 \in \mathbb{R}^N$, $0 < r < r_0$, if $\sup\limits_{x\in B_{\theta r}(x_0)} \le c_0 r$, then $u \equiv 0$ in $B_{\theta^2 r}(x_0)$. Here $u = u_A$, $A = \Omega_k^*$.

We claim the latter implies the boundedness of A. Indeed, if there is a sequence $\{y_n\} \subset A$ such that $|y_n| \to +\infty$. Since $y_n \in A$, one has $\sup\limits_{B_{\theta_0 r_0}(y_0)} u_A \ge c_0 r_0 > 0$. Since $u_A(x) + \frac{|x-y_n|^2}{2N} = v$ is subharmonic in $\mathbb{R}^N$,

$$c_0 r_0 \le C(N) \fint_{B_{r_0}(y_n)} v \ \mathrm{d}x.$$

For r_0 suitably small, one has $\int_{B_{r_0}(y_n)} u_A \ \mathrm{d}x \ge c_0 r_0^{N+1}$ and $|y_n| \to +\infty$. It contradicts to the fact that $\int u_A \ \mathrm{d}x = \int |\nabla u_A|^2 \mathrm{d}x < \infty$.

Bucur's results described here provide a starting point for the regularity of $\partial^*\Omega_k^*$. The following is a statement that would be discussed in the forthcoming work (see [21]): If Ω_k^* is a non-degenerate minimizer of (EP), then $\partial\Omega_k^*$ is almost everywhere analytic. More precisely, away from an $\mathcal{H}^{N-1}$ measure zero set, it is real analytic. A key point of the proof of this last result is to reduce it to the case of the study of certain extremum domains that are associated with their first Dirichlet eigenvalues for the Laplacian. One may ask that if in the $2D$ case, the boundary of Ω_k^* consists of at most $c(k)$ analytic arcs. In general, one obviously has to understand much better the property of Ω_k^* in order to study the asymptotics of these minimizers as k becomes very large. In particular, it may be closely related to both the generalized Polya conjecture and the optimal partition problems.

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