Chin. Ann. Math.
38B(2), 2017, 513–538
DOI: 10.1007/s11401-017-1081-x

Chinese Annals of Mathematics, Series B

Dirac Concentrations in a Chemostat Model of Adaptive Evolution*

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(*In honor of the immense scientific influence of Haïm Brezis*)

Abstract This paper deals with a non-local parabolic equation of Lotka-Volterra type that describes the evolution of phenotypically structured populations. Nonlinearities appear in these systems to model interactions and competition phenomena leading to selection. In this paper, the equation on the structured population is coupled with a differential equation on the nutrient concentration that changes as the total population varies.

Different methods aimed at showing the convergence of the solutions to a moving Dirac mass are reviewed. Using either weak or strong regularity assumptions, the authors study the concentration of the solution. To this end, BV estimates in time on appropriate quantities are stated, and a constrained Hamilton-Jacobi equation to identify where the solutions concentrates as Dirac masses is derived.

Keywords Adaptive evolution, Asymptotic behaviour, Chemostat, Dirac concentrations, Hamilton-Jacobi equations, Lotka-Volterra equations, Viscosity solutions
2010 MR Subject Classification 35B25, 35K57, 47G20, 49L25, 92D15

1 Introduction

We survey several methods developed to study concentration effects in parabolic equations of Lotka-Volterra type. Furthermore, we extend the theory to a coupled system motivated by models of chemostat, where we observe very rare mutations for a long time. These equations were established with the aim of describing how speciation occurs in biological populations, taking into account competition for resources and mutations in the populations. There is a large literature on the subject where the mutation-competition principles are illustrated in various mathematical terms: for instance, in [23, 28, 35] for an approach based on the study of the stability of differential systems, in [29–30, 45] for the evolutionary games theory, in [14] for the study of stochastic individual based models, or in [6, 36, 42] for the study of integro-differential models. We choose here the formalism using parabolic partial differential equations, widely developed in [5, 7, 21, 41] to describe the competition dynamics in a chemostat.

The chemostat is a bioreactor to which fresh medium containing nutrients is continuously added, while culture liquid is continuously removed to keep the culture volume constant. This device is used as an experimental ecosystem in evolutionary biology to observe mutation and

Manuscript received June 22, 2015. Revised November 5, 2015.
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*This work was supported by ANR-13-BS01-0004 funded by the French Ministry of Research (ANR Kibord).

selection processes driven by competition for resources. From the mathematical point of view, the theoretical description of the population dynamics in a chemostat leads to highly nonlinear models and questions of long term behaviour and convergence to an evolutionary steady state naturally arise (see [1, 19, 24, 39, 44]).

Our aim is to study a generalization of the chemostat model introduced in [34] with a representation of mutations by a diffusion term. In this model, each individual in the population is characterized by a quantitative phenotypic trait $x \in \mathbb{R}^d$, and $n_\varepsilon(t,x)$ denotes the population density at time t with the trait x. We study the following equations:

$$\varepsilon \partial_t n_\varepsilon(x,t) = n_\varepsilon R(x, S_\varepsilon(t)) + \varepsilon^2 \Delta n_\varepsilon(x,t), \quad x \in \mathbb{R}^d,\ t \geq 0, \tag{1.1}$$

$$\varepsilon \beta \frac{\mathrm{d}}{\mathrm{d}t} S_\varepsilon(t) = Q(S_\varepsilon(t), \rho_\varepsilon(t)), \tag{1.2}$$

$$\rho_\varepsilon(t) := \int_{\mathbb{R}^d} n_\varepsilon(x,t)\mathrm{d}x,$$

where the function $R(x, S_\varepsilon)$ represents a trait-dependent birth-death rate and S_ε denotes the nutrient concentration which changes over time with rate Q. Here ε is a small parameter which allows to consider very rare mutations and large time of order ε^{-1}. The idea of an ε^{-1} rescaling in the space and time variables goes back to [31–32] to study propagation for systems of reaction-diffusion PDE. The parameter β, introduced first in [34], gives a time scale which, as $\beta \to 0$, leads to the equation $Q(\rho, S) = 0$. In this case, under suitable assumptions, we deduce the existence of a function f by implicit function theorem, such that $S = f(\rho)$ and the concentration results are known to hold (see [7, 33]).

Such models can be derived from stochastic individual based models in the limit of large populations (see [16–17]).

A possible way to express mathematically the emergence of the fittest traits among the population is to prove that n_ε concentrates as a Dirac mass centered on a point $\overline{x}$ (or a sum of Dirac masses) when ε vanishes. This means the phenotypic selection of a quantitative trait denoted by $\overline{x}$ in long time. The main results of the paper can be summarized as follows.

Theorem 1.1 *For well-prepared initial data and two classes of assumptions* (*monotonic in one dimension or concavity in multi-dimensions*)*, the solution $n_\varepsilon(t,x)$ concentrates, i.e.,*

$$n_\varepsilon(t,x) \underset{\varepsilon \to 0}{\longrightarrow} \overline{\rho}(t)\delta(x - \overline{x}(t)) \quad \textit{in the sense of measure},$$

where the pair $(\overline{x}(t), \overline{\rho}(t))$ can be determined thanks to a constrained Hamilton-Jacobi equation given later on.

In order to describe these concentration effects, following earlier works on similar issues [5, 7, 13], we will use the Hopf-Cole transformation defining $u_\varepsilon(t,x) = \varepsilon \ln n_\varepsilon(t,x)$, and derive a Hamilton-Jacobi equation. Then we obtain by passing to the limit $\varepsilon \to 0$ a constrained Hamilton-Jacobi equation, whose solutions have a maximum value of 0. The point is that the concentration locations in the limit $\varepsilon \to 0$ can be identified among the maximum points of these solutions. This method, introduced in [24] and used for instance in [42–43], is very general and was extended to various systems (see [18, 33, 41]).

Singular perturbation problems in PDEs is a classical subject that was studied from different viewpoints. For instance, a seminal paper on parabolic equations involving measures is [11].

Also the above rescaling in parabolic equations or systems was deeply studied in reaction-diffusion equations (see [4, 25]) leading to front propagation where a state invades another as in the Fishher-KPP equation where the stable state $n_\varepsilon = 1$ invades the unstable state $n_\varepsilon = 0$. This is also the case of Ginzburg-Landau equations (see [8]), where the quadratic observable $n_\varepsilon = |u_\varepsilon|^2$ takes asymptotically the value 1. This is different from our case, as one can see in the above theorem and since we essentially derive L^1 bounds from the presented model.

To prove the main convergence results of this paper, we adapt the method introduced in [5, 7, 34] to find BV estimates for the appropriate quantities as a first step. Then we use the theory of viscosity solutions to Hamilton-Jacobi equations (see [2–3, 20, 27] for general introduction to this theory) to obtain the Dirac locations. In the first part, we proceed with assumptions of weak regularity of the growth rate in a first instance, and then we resume the study under concavity assumptions.

This paper is organized as follows. We first state (see Section 2) the framework of the general weak theory and its main results. We start the study by establishing BV estimates on ρ_ε^2 and S_ε in Section 3. Section 4 is devoted to the analysis of the solutions to the constrained Hamilton-Jacobi equations. We first prove some regularity results for u_ε. Then we study the asymptotic behaviour of u_ε and deduce properties of the concentration points. In Section 5, we set the simple case of our results when the dimension d equals 1 and prove concentration effects. In Section 6, we review the d-dimensional framework where we assume uniform concavity of the growth rate and initial conditions. We establish again the BV estimates in this specific case and prove the uniform concavity of u_ε. The regularity obtained for u_ε allows us to derive the dynamics of the concentration points in the form of a canonical equation. We complete these results by numerics in Section 7.

2 The Weak Theory: Assumptions and Main Results

First of all, we give assumptions to set a framework for the general weak theory. We use the same assumptions as in [34].

For the Lipschitz continuous functions R and Q, we assume that there are constants $S_0 > 0$, $K_Q > 0$, $\underline{K}_1 > 0$ and $\overline{K}_1 > 0$, such that

$$Q(0,\rho) > 0, \quad \max_{\rho \geq 0} Q(S_0,\rho) = 0, \quad Q_S(S,\rho) \leq -K_Q, \quad Q_\rho(S,\rho) \leq -K_Q, \tag{2.1}$$

$$0 < \underline{K}_1 \leq R_S(x,S) \leq \overline{K}_1, \tag{2.2}$$

$$\sup_{0 \leq S \leq S_0} \|R(\cdot,S)\|_{W^{2,\infty}(\mathbb{R}^d)} \leq K_2. \tag{2.3}$$

We complete the system with the initial conditions S_ε^0, n_ε^0, such that

$$S_m < S_\varepsilon^0 < S_0, \quad n_\varepsilon^0(x) > 0, \quad \forall x \in \mathbb{R}^d, \quad 0 < \rho_m \leq \rho_\varepsilon^0 := \int_{\mathbb{R}^d} n_\varepsilon^0(x)\mathrm{d}x \leq \rho_M, \tag{2.4}$$

where $\rho_m,\ \rho_M$ and S_m are defined below.

We add to these assumptions a smallness condition on β which can be written as

$$\min_{\substack{0 \leq \rho \leq \rho_M \\ S_m \leq S \leq S_0}} \frac{|Q_S|}{|Q_\rho|} \geq \ 4\beta \max_{\substack{0 \leq \rho \leq \rho_M \\ S_m \leq S \leq S_0}} \frac{\overline{K}_1 \rho_M}{|Q_S|} \tag{2.5}$$

with the definition of ρ_M stated below.

Note that from (2.1), we directly obtain the bounds

$$n_\varepsilon(t,x) > 0, \quad 0 < S_\varepsilon(t) \le S_0. \tag{2.6}$$

First we recall the following lemma, whose proof is given in [34].

Lemma 2.1 *Under the assumptions* (2.1)–(2.4)*, there are constants* ρ_m*,* ρ_M *and* $S_m > 0$*, such that*

$$0 < \rho_m \le \rho_\varepsilon(t) \le \rho_M, \quad S_m \le S_\varepsilon(t) \le S_0,$$

where the value $S_m < S_0$ *is defined by* $Q(S_m, \rho_M) = 0$.

This result is required to prove the following theorem.

Theorem 2.1 *Assuming also* (2.5)*,* $\rho_\varepsilon(t)$ *and* $S_\varepsilon(t)$ *have locally bounded total variation uniformly in* ε*. Consequently, there are limit functions* $\rho_m \le \overline{\rho} \le \rho_M$*,* $S_m \le \overline{S} \le S_0$*, such that after extraction of a subsequence, we have*

$$S_{\varepsilon_k}(t) \underset{\varepsilon_k \to 0}{\longrightarrow} \overline{S}(t), \quad \rho_{\varepsilon_k}(t) \underset{\varepsilon_k \to 0}{\longrightarrow} \overline{\rho}(t) \quad a.e.$$

and

$$Q(\overline{\rho}, \overline{S}) = 0 \quad a.e.$$

The next section is devoted to the proof of Theorem 2.2. Contrary to what we could expect, the establishment of the BV estimates will be more complicated than in the previous works (see [7, 33]), where the nutrients are represented by an integral term as $\int \psi(x) n_\varepsilon(t,x)\mathrm{d}x$. Here the main challenge comes from the equation (1.2) that we also have to consider to obtain BV estimates on S_ε. Another difficulty comes from the parameter β. For β large enough, it seems that we cannot derive BV estimates with our approach, but anyway we do observe the convergence of the solutions in the numerics we performed. This is not the case for inhibitory integrate-and-fire models for instance (see [12]), where delays generate periodic solutions.

In the following proofs, C denotes a constant which may change from line to line.

3 BV Estimates on $\rho_\varepsilon^2(t)$ and $S_\varepsilon(t)$

3.1 Bounds for ρ_ε

We follow the lines of [34] to give the bounds ρ_m and ρ_M. By integrating the equation (1.1) and using the assumptions (2.2)–(2.3), we arrive to the inequalities

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon \le \rho_\varepsilon (K_2 + \overline{K_1} S_\varepsilon)$$

and

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{ln}\rho_\varepsilon \le K_2 + \overline{K_1} S_0.$$

Notice that $Q(S_\varepsilon, \rho_\varepsilon) \le -K_Q \rho_\varepsilon + Q(0,0)$ from the assumptions in (2.1). By adding the equation (1.2) to the inequation above, we arrive to

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{ln}\rho_\varepsilon + \beta S_\varepsilon) \le K_2 + \overline{K_1} S_0 + Q(0,0) - K_Q \rho_\varepsilon \tag{3.1}$$

$$\le K_2 + \overline{K_1} S_0 + Q(0,0) - \frac{K_Q}{\mathrm{e}^{\beta S_0}} \mathrm{e}^{\mathrm{ln}\rho_\varepsilon + \beta S_\varepsilon}. \tag{3.2}$$

It follows that, for C_2, the root in $\ln\rho_\varepsilon + \beta S_\varepsilon$ of the right-hand side,

$$\ln\rho_\varepsilon \le \ln\rho_\varepsilon + \beta S_\varepsilon \le \max(\ln\rho_M^0 + \beta S_0, C_2).$$

Hence we have the upper bound ρ_M for $\rho_\varepsilon(t)$.

Thanks to this upper bound, we obtain the lower bound S_m on $S_\varepsilon(t)$, since, by using the assumption (2.1) on Q, we remark that

$$\varepsilon\beta\frac{\mathrm{d}}{\mathrm{d}t}S_\varepsilon(t) = Q(S_\varepsilon(t), \rho_\varepsilon(t)) \ge Q(S_\varepsilon(t), \rho_M).$$

Then there is a unique value S_m such that $Q(S_m, \rho_M) = 0$, and from the initial conditions (2.4), we deduce that $S_m \le S_\varepsilon(t)$ for $t \ge 0$.

Next, let us look for the lower bound. It follows, from the integration of (1.1) as above, that we have

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\ln\rho_\varepsilon \ge -K_2 + \underline{K_1}S_m.$$

By subtracting (1.2) and still using (2.1), we obtain

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(\ln\rho_\varepsilon - \beta S_\varepsilon) &\ge -K_2 + \underline{K_1}S_m - Q(S_\varepsilon, \rho_\varepsilon)\\ &\ge -K_2 - Q(0,0) + K_Q\rho_\varepsilon\\ &\ge -K_2 - Q(0,0) + K_Q\mathrm{e}^{\ln\rho_\varepsilon - \beta S_\varepsilon}\mathrm{e}^{\beta S_m}. \end{aligned} \tag{3.3}$$

Taking C_3 the root in $\ln\rho_\varepsilon - \beta S_\varepsilon$ of the right-hand side in (3.3), we have the lower bound

$$\rho_\varepsilon(t) \ge \min(\rho_m^0, C_3),$$

which ends the proof of Lemma 2.1.

3.2 Local *BV* estimates

To find local BV bounds for ρ_ε and S_ε which are uniform in $\varepsilon > 0$, we apply the method described in [34] that we explain in detail in this section.

Let us first define $J_\varepsilon := \dot{S}_\varepsilon$ and $P_\varepsilon := \dot{\rho_\varepsilon}$. With these definitions, we have the equations

$$\varepsilon P_\varepsilon = \int n_\varepsilon R(x, S_\varepsilon(t))\,\mathrm{d}x, \quad \varepsilon\beta J_\varepsilon = Q(\rho_\varepsilon(t), S_\varepsilon(t)). \tag{3.4}$$

Defining α_ε and γ_ε as

$$\alpha_\varepsilon(t) := \int n_\varepsilon R_S(x, S_\varepsilon(t))\,\mathrm{d}x \quad \text{and} \quad \gamma_\varepsilon(t) := \int n_\varepsilon R^2\,\mathrm{d}x,$$

respectively, we differentiate both equations above, then we obtain the following equations on J_ε and P_ε:

$$\begin{aligned}\varepsilon\dot{P}_\varepsilon &= J_\varepsilon\int n_\varepsilon R_S(x, S_\varepsilon(t))\,\mathrm{d}x + \int \partial_t n_\varepsilon R(x, S_\varepsilon(t))\,\mathrm{d}x\\ &= \alpha_\varepsilon(t)J_\varepsilon + \varepsilon\int n_\varepsilon \Delta R\mathrm{d}x + \frac{1}{\varepsilon}\gamma_\varepsilon(t), \end{aligned} \tag{3.5}$$

$$\varepsilon\beta\dot{J}_\varepsilon = Q_S J_\varepsilon + Q_\rho P_\varepsilon. \tag{3.6}$$

However, at this stage, we cannot obtain directly the BV bounds on ρ_ε and S_ε which we expect. Thus we consider a linear combination of P_ε and J_ε. Let $\mu_\varepsilon(t)$ be a function which we will determine later. By combining the equalities above, we obtain the following equation on $P_\varepsilon + \mu_\varepsilon J_\varepsilon$:

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(P_\varepsilon+\beta\mu_\varepsilon J_\varepsilon) &= \alpha_\varepsilon J_\varepsilon + \varepsilon\int n_\varepsilon \Delta R \mathrm{d}x + \beta\dot{\mu}_\varepsilon J_\varepsilon + \mu_\varepsilon(Q_S J_\varepsilon + Q_\rho P_\varepsilon) + \frac{1}{\varepsilon}\gamma_\varepsilon \\ &= \mu_\varepsilon Q_\rho(P_\varepsilon+\beta\mu_\varepsilon J_\varepsilon) + (\varepsilon\beta\dot{\mu}_\varepsilon - \beta Q_\rho \mu_\varepsilon^2 + \mu_\varepsilon Q_S + \alpha_\varepsilon)J_\varepsilon \\ &\quad + \varepsilon\int n_\varepsilon \Delta R \mathrm{d}x + \frac{1}{\varepsilon}\gamma_\varepsilon. \end{aligned} \tag{3.7}$$

First we prove the following result.

Lemma 3.1 *Considering the solution μ_ε of the differential equation*

$$\varepsilon\beta\dot{\mu}_\varepsilon = -\beta|Q_\rho|\mu_\varepsilon^2 + \mu_\varepsilon|Q_S| - \alpha_\varepsilon,$$

there exist constants $0 < \mu_m < \mu_M$, such that, choosing initially $\mu_m < \mu_\varepsilon(0) < \mu_M$, we have

$$\mu_m \le \mu_\varepsilon(t) \le \mu_M, \quad \forall t \ge 0.$$

Furthermore, we have the following estimate concerning the negative part of the linear combination:

$$(P_\varepsilon(t) + \beta\mu(t)J_\varepsilon(t))_- \le (P_\varepsilon(0) + \beta\mu(0)J_\varepsilon(0))_- \mathrm{e}^{\frac{-K_Q\mu_m}{\varepsilon}t} + \varepsilon C(1 - \mathrm{e}^{\frac{-K_Q\mu_m}{\varepsilon}t}). \tag{3.8}$$

Proof Our goal is to choose a function $\mu_\varepsilon(t)$ which solves the differential equation

$$\varepsilon\beta\dot{\mu}_\varepsilon = -\beta|Q_\rho|\mu_\varepsilon^2 + \mu_\varepsilon|Q_S| - \alpha_\varepsilon. \tag{3.9}$$

We use the same argument as in [34]. Therefore, we concentrate on the main ideas.

Note that, because the solution might blow up to $-\infty$ in finite time, we need to prove that solutions of (3.9) remain strictly positive for all time. To do so, we first notice that the zeroes of $-\beta|Q_\rho|\mu_\varepsilon^2 + \mu_\varepsilon|Q_S| - \alpha_\varepsilon$ are

$$\mu_{\varepsilon,\pm}(t) := \frac{1}{2\beta|Q_\rho|}\Big(|Q_S| \pm \sqrt{Q_S^2 - 4\alpha_\varepsilon\beta|Q_\rho|}\Big),$$

and from the smallness condition (2.5), both zeros are positive.

We need to find two constants $0 < \mu_m < \mu_M$, such that, choosing initially $\mu_m < \mu_\varepsilon(0) < \mu_M$, then we have for all time,

$$0 < \mu_m \le \mu_\varepsilon(t) \le \mu_M. \tag{3.10}$$

This condition is satisfied with the following constants:

$$\mu_M := \frac{1}{\beta}\max_{\substack{\rho_m\le\rho\le\rho_M \\ S_m\le S\le S_0}} \frac{|Q_S|}{|Q_\rho|}, \tag{3.11}$$

and μ_m defined as

$$\max_t \mu_{\varepsilon,-}(t) \le \mu_m := \min_t \mu_{\varepsilon,+}(t), \tag{3.12}$$

which defines a positive constant because of the smallness condition for β (2.5).

Coming back to (3.7), we arrive to

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(P_\varepsilon+\beta\mu J_\varepsilon)\geq -\mu\,|Q_\rho|\,(P_\varepsilon+\beta\mu J_\varepsilon)+\varepsilon\int n_\varepsilon\Delta R\mathrm{d}x\geq -\mu|Q_\rho|(P_\varepsilon+\beta\mu J_\varepsilon)-\varepsilon C,$$

and we conclude that, for all $t\geq 0$,

$$(P_\varepsilon(t)+\beta\mu(t)J_\varepsilon(t))_-\leq (P_\varepsilon(0)+\beta\mu(0)J_\varepsilon(0))_-\mathrm{e}^{\frac{-K_Q\mu_m}{\varepsilon}t}+\varepsilon C(1-\mathrm{e}^{\frac{-K_Q\mu_m}{\varepsilon}t}), \tag{3.13}$$

which concludes the proof of Lemma 3.1.

From the estimate of the Lemma 3.1, we can deduce the local BV bounds uniform in ε. We start with P_ε. Adding $\alpha_\varepsilon\frac{P_\varepsilon}{\beta\mu_\varepsilon}$ to (3.5) and using (2.3) and Lemma 2.1, we find

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}P_\varepsilon+\alpha_\varepsilon\frac{P_\varepsilon}{\beta\mu_\varepsilon}=\alpha_\varepsilon\Big(J_\varepsilon+\frac{P_\varepsilon}{\beta\mu_\varepsilon}\Big)+\varepsilon\int n_\varepsilon\Delta R\mathrm{d}x+\frac{1}{\varepsilon}\gamma_\varepsilon\geq -\alpha_\varepsilon\Big(J_\varepsilon+\frac{P_\varepsilon}{\beta\mu_\varepsilon}\Big)_- -C\varepsilon.$$

Notice that $0<\underline{K}_1\rho_\varepsilon(t)\leq\alpha_\varepsilon(t)\leq\overline{K}_1\rho_M$. By considering the negative parts of P_ε and using (2.2) and (3.8), we arrive to the inequality

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(P_\varepsilon)_-+\alpha_\varepsilon\frac{(P_\varepsilon)_-}{\beta\mu_\varepsilon}&\leq\alpha_\varepsilon\Big(J_\varepsilon+\frac{P_\varepsilon}{\beta\mu_\varepsilon}\Big)_-+C\varepsilon\\ &\leq\alpha_\varepsilon(P_\varepsilon(0)+\beta\mu_\varepsilon(0)J_\varepsilon(0))_-\frac{\mathrm{e}^{\frac{-\mu_mK_Q}{\varepsilon}t}}{\beta\mu_m}+\varepsilon\alpha_\varepsilon C(1-\mathrm{e}^{\frac{-\mu_mK_Q}{\varepsilon}t})+C\varepsilon\\ &\leq\overline{K}_1\rho_M(P_\varepsilon(0)+\beta\mu_\varepsilon(0)J_\varepsilon(0))_-\frac{\mathrm{e}^{\frac{-\mu_mK_Q}{\varepsilon}t}}{\beta\mu_m}+C\varepsilon.\end{aligned} \tag{3.14}$$

With this inequality, the BV bounds follow. Since $\varepsilon P_\varepsilon$ is bounded, by integrating the inequality above, we have

$$\int_0^T\alpha_\varepsilon(t)(P_\varepsilon(t))_-\mathrm{d}t\leq C_1(T)+\varepsilon C_2(T),\quad\forall T\geq 0.$$

Consequently, we obtain

$$\underline{K}_1\int_0^T\rho_\varepsilon\Big(\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon\Big)_-\mathrm{d}x=\frac{\underline{K}_1}{2}\int_0^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon^2\Big)_-\mathrm{d}x\leq\frac{C_1(T)+\varepsilon C_2(T)}{2},\quad\forall T\geq 0.$$

Since $\rho_\varepsilon(t)$ is bounded, we have finally that ρ_ε^2 has local bounded variations. Therefore, up to an extraction, there exists a function $\overline{\rho}$ on $(0,\infty)$ satisfying

$$\rho_\varepsilon\longrightarrow\overline{\rho}\quad\text{in }L^1_{\mathrm{loc}}(0,\infty).$$

Since we have the lower bound $\rho_\varepsilon\geq\rho_m$ by Lemma 1.1, we obtain the bound for the negative part of the derivative of ρ_ε

$$\int_0^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon\Big)_-\mathrm{d}x\leq\frac{C_1+C_2\varepsilon}{2\underline{K}_1\rho_m}.$$

Finally, it remains to study S_ε. To do so, we rewrite (3.6) as

$$\varepsilon\beta\frac{\mathrm{d}}{\mathrm{d}t}J_\varepsilon=Q_SJ_\varepsilon+Q_\rho P_\varepsilon=Q_SJ_\varepsilon+Q_\rho\frac{\dot{(\rho_\varepsilon^2)}}{2\rho_\varepsilon}. \tag{3.15}$$

With our assumptions (2.1) on the Lipschitz function Q, we have

$$\varepsilon\beta\frac{\mathrm{d}}{\mathrm{d}t}(-J_\varepsilon) = Q_S(-J_\varepsilon) - Q_\rho\frac{(\dot{\rho_\varepsilon^2})}{2\rho} \le Q_S(-J_\varepsilon) + L_Q\frac{|(\dot{\rho_\varepsilon^2})|}{2\rho_m} \tag{3.16}$$

and

$$\varepsilon\beta\frac{\mathrm{d}}{\mathrm{d}t}(J_\varepsilon)_- \le -K_Q(J_\varepsilon)_- + L_Q\frac{|(\dot{\rho_\varepsilon^2})|}{2\rho_m}. \tag{3.17}$$

The term $\varepsilon J_\varepsilon$ is bounded because of our assumptions on Q. So, integrating this equation, we have, for $T>0$,

$$\int_0^T (J_\varepsilon)_-\mathrm{d}x \le C + \frac{L_Q}{2\rho_m K_Q}\int_0^T |(\dot{\rho_\varepsilon^2})|\mathrm{d}x, \tag{3.18}$$

and we deduce that $\int_0^T (J_\varepsilon)_-\mathrm{d}x$ is uniformly bounded from our previous results on ρ_ε^2.

And then, since S_ε is uniformly bounded, we conclude that there exists a function $\overline{S}(t)$, such that, after extraction of a subsequence,

$$S_\varepsilon \longrightarrow \overline{S} \quad \text{in } L^1_{\mathrm{loc}}(0,\infty) \quad \text{and} \quad Q(S_\varepsilon,\rho_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} Q(\overline{S},\overline{\rho}) \quad \text{a.e.}$$

To conclude, it follows that $\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}S_\varepsilon$ converges in measure to 0 as ε vanishes, and thus $Q(\overline{S},\overline{\rho})=0$.

4 Concentration and Constrained Hamilton-Jacobi Equation

In order to prove the concentration of n_ε in a sum of Dirac masses as ε vanishes, we perform the change of unknown $n_\varepsilon(t,x) = \mathrm{e}^{\frac{u_\varepsilon(t,x)}{\varepsilon}}$, and study the regularity properties of $u_\varepsilon(t,x)$. With the definition of u_ε, we obtain the following equation which is equivalent to (1.1):

$$\begin{cases} \partial_t u_\varepsilon(t,x) = |\nabla u_\varepsilon|^2 + R(x, S_\varepsilon(t)) + \varepsilon\Delta u_\varepsilon, \\ u_\varepsilon(t=0,x) = u_\varepsilon^0(x) := \varepsilon \ln n_\varepsilon^0. \end{cases} \tag{4.1}$$

We complete assumption (2.4) on the initial data with

$$u_\varepsilon^0(x) \le A - K_2\sqrt{1+|x|^2}, \quad \|\nabla u_\varepsilon^0\| \le B, \quad \forall x\in\mathbb{R}^d \tag{4.2}$$

with $A,B>0$.

We prove in this section the following result.

Theorem 4.1 *Under the assumptions* (2.1)–(2.5) *and* (4.2), *then after extraction of a subsequence* $(u_\varepsilon)_\varepsilon$ *converges locally uniformly to a Lipschitz continuous viscosity solution* u *to the constrained Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u(t,x) = |\nabla u|^2 + R(x,\overline{S}(t)), \\ \max\limits_{x\in\mathbb{R}^d} u(t,x) = 0, \quad \forall t\ge 0. \end{cases} \tag{4.3}$$

In the simple case, when dimension d is equal to 1 and when $R(x,S)$ is monotonic in x for all S, n concentrates in one single point.

We first prove that u_ε is equi-bounded, then the equi-continuity, and finally we explain how to pass to the limit in (4.1).

4.1 Local bounds and equi-continuity in space

We first set the upper bound for u_ε. Let $T > 0$ be given. Defining $\overline{u}(t,x) = A + Ct - K_2\sqrt{1+|x|^2}$ with $C = K_2(1+K_2)$, we have

$$\partial_t \overline{u} - \varepsilon\Delta\overline{u} - |\nabla\overline{u}|^2 - R(x, S_\varepsilon(t)) \geq C + \varepsilon K_2 \frac{d-1}{\sqrt{1+|x|^2}} - K_2^2 - K_2 \geq 0.$$

Since $\overline{u}(0,x) \geq u_\varepsilon^0(x)$ from initial data (4.2), we conclude that $\overline{u}$ is a super-solution and $u_\varepsilon(t,x) \leq A + CT - K_2\sqrt{1+|x|^2}$ for all $t \in [0,T]$.

Next we prove that u_ε is uniformly Lipschitz continuous in space on $[0,T]\times\mathbb{R}^d$. We define for h small enough, $w_\varepsilon(t,x) = u_\varepsilon(t,x+h) - u_\varepsilon(t,x)$. Since the initial condition u_ε^0 are uniformly continuous, given $\delta > 0$, for h small enough, we have $|w_\varepsilon(0,x)| < \frac{\delta}{2}$. From (4.1), we arrive to

$$\begin{aligned} &\partial_t w_\varepsilon(t,x) - \varepsilon\Delta w_\varepsilon(t,x) - (\nabla u_\varepsilon(t,x+h) + \nabla u_\varepsilon(t,x))\cdot\nabla w_\varepsilon(t,x) \\ &= R(x+h, S_\varepsilon(t)) - R(x, S_\varepsilon(t)) \leq K_2 h. \end{aligned} \tag{4.4}$$

Thus by the maximum principle, we deduce that

$$|w_\varepsilon(t,x)| \leq \left|\max_{\mathbb{R}^d} w_\varepsilon(0,x)\right| + K_2|h|t \leq (\|\nabla u_\varepsilon^0\|_{L^\infty(\mathbb{R}^d)} + K_2 t)|h|.$$

We conclude that u_ε is uniformly Lipschitz in space on $[0,T]\times\mathbb{R}^d$, and set

$$L(t) = \sup_{\substack{\varepsilon\leq\varepsilon_0 \\ 0\leq s\leq t \\ x\in\mathbb{R}^d}} \|\nabla u_\varepsilon(s,x)\|_{L^\infty}. \tag{4.5}$$

To conclude, we show that u_ε is also uniformly bounded from below on compact subsets of $[0,\infty)\times\mathbb{R}^d$. Let $0 < T$ and $r > 0$. For all $t \in [0,T]$ and $x \in B(0,r)$, we recall that $u_\varepsilon(t,x) \leq A + CT - K_2\sqrt{1+|x|^2}$, and thus

$$\int_{|x|>r} e^{\frac{u_\varepsilon}{\varepsilon}} dx < \int_{|x|>r} e^{\frac{A+CT-K_2|x|}{\varepsilon}} dx < \frac{\rho_m}{2}$$

for $0 < \varepsilon < \varepsilon_0$, ε_0 small enough and r large enough. We also have from Lemma 3.1 that $\rho_\varepsilon \geq \rho_m$, then for $0 < \varepsilon < \varepsilon_0$ and r large enough, we obtain

$$\frac{\rho_m}{2} < \int_{|x|\leq r} e^{\frac{u_\varepsilon}{\varepsilon}} dx \leq B_r e^{\max_{B_r}\frac{u_\varepsilon}{\varepsilon}}.$$

This implies

$$\max_{B_r} u_\varepsilon \geq \varepsilon \ln\frac{\rho_m}{2|B_r|}.$$

Using the Lipschitz bound (4.5), we obtain

$$u_\varepsilon(t,x) > \varepsilon\ln\frac{\rho_m}{2|B_r|} - 2L(t)r, \quad \forall x \in B(0,r).$$

Hence we have the local lower bound on u_ε.

4.2 The equi-continuity in time

For given T, η and $r > 0$, we fix $(s, x) \in [0, T[\times B(0, \frac{r}{2})$, and define

$$\xi_\varepsilon(t, y) = u_\varepsilon(s, x) + \eta + E|y - x|^2 + D(t - s) \quad \text{for } (t, y) \in [s, T] \times B(0, r),$$

where E and D are constants to be determined. We prove in this section the uniform continuity in time. The idea of the proof is to find constants E and D large enough such that, for any $x \in R(0, \frac{r}{2})$, and for all $\varepsilon < \varepsilon_0$,

$$u_\varepsilon(t, y) \leq \xi_\varepsilon(t, y) = u_\varepsilon(s, x) + \eta + E|y - x|^2 + D(t - s), \quad \forall (t, y) \in [0, T] \times B(0, r) \tag{4.6}$$

and

$$u_\varepsilon(t, y) \geq \phi_\varepsilon(t, y) := u_\varepsilon(s, x) - \eta - E|y - x|^2 - D(t - s), \quad \forall (t, y) \in [0, T] \times B(0, r). \tag{4.7}$$

Then by taking $y = x$, we have the uniform continuity in time on compact subsets of $[0, \infty) \times \mathbb{R}^d$. We prove here the inequality (4.6), and the proof of (4.7) is analogous.

First we prove that $\xi_\varepsilon(t, y) > u_\varepsilon(t, y)$ on $[s, T] \times \partial B(0, r)$, for all η, D and $x \in B(0, \frac{r}{2})$. Since u_ε are locally uniformly bounded according to Sections 4.1 and 4.3, by taking E large enough, such that

$$E \geq \frac{8\|u_\varepsilon\|_{L^\infty([0,T]\times B(0,r))}}{r^2},$$

we obtain

$$\begin{aligned}
\xi_\varepsilon(t, y) &\geq u_\varepsilon(t, x) + \eta + 2\|u_\varepsilon\|_{L^\infty([0,T]\times B(0,r))} + D(t - s) \\
&\geq \|u_\varepsilon\|_{L^\infty([0,T]\times B(0,r))} \\
&\geq u_\varepsilon(t, y).
\end{aligned}$$

Next we prove that, for E large enough, $\xi_\varepsilon(s, y) \geq u_\varepsilon(s, y)$ for all $y \in B(0, r)$. We argue by contradiction. Assume that there exists $\eta > 0$, such that for all constants $E > 0$, there exists $y_E \in B(0, r)$, such that

$$u_\varepsilon(s, y_E) - u_\varepsilon(s, x) > \eta + E|y_E - x|^2.$$

This implies

$$|y_E - x| \leq \sqrt{\frac{2M}{E}},$$

where M is a uniform upper bound for $\|u_\varepsilon\|_{L^\infty([0,T]\times B(0,r))}$. For $E \to \infty$, we have $|y_E - x| \to 0$. Since u_ε are uniformly continuous in space, this is a contradiction.

Finally, from assumption (2.3), if D is large enough, ξ_ε is a super-solution to (4.3) in $[s, T] \times B(0, r)$,

$$u_\varepsilon(t, y) \leq u_\varepsilon(s, x) + \eta + E|y - x|^2 + D(t - s), \quad \forall (t, y) \in [0, T] \times B(0, r).$$

With the proof of (4.7) which is similar, we deduce that the sequence u_ε is uniformly continuous in time on compact subsets of $[0, \infty) \times \mathbb{R}^d$.

4.3 Passing to the limit

We proceed as in [5] to prove the convergence of (4.1) to (4.3) as ε goes to 0. Considering the regularity results above, the point at this step is to pass to the limit in the term $R(x, S_\varepsilon)$. To avoid the complications of the discontinuity, we define

$$\phi_\varepsilon(t,x) := u_\varepsilon(t,x) - \int_0^t R(x, S_\varepsilon(s))\mathrm{d}s,$$

and it follows that ϕ_ε satisfies the equation

$$\begin{aligned}&\partial_t\phi_\varepsilon(t,x) - \varepsilon\Delta\phi_\varepsilon(t,x) - |\nabla\phi_\varepsilon(t,x)|^2 - 2\nabla\phi_\varepsilon(t,x)\cdot\int_0^t \nabla R(x, S_\varepsilon(s))\mathrm{d}s\\ &= \varepsilon\int_0^t \Delta R(x, S_\varepsilon(s))\mathrm{d}s + \Big|\int_0^t \nabla R(x, S_\varepsilon(s))\mathrm{d}s\Big|^2. \qquad (4.8)\end{aligned}$$

As $S_\varepsilon(t)$ converges to $\overline{S}(t)$ for all $t \geq 0$ and $R(x, I)$ is a Lipschitz continuous function, we have

$$\lim_{\varepsilon\to 0}\int_0^t R(x, S_\varepsilon(s))\mathrm{d}s = \int_0^t R(x, \overline{S}(s))\mathrm{d}s,$$

$$\lim_{\varepsilon\to 0}\int_0^t \nabla R(x, S_\varepsilon(s))\mathrm{d}s = \int_0^t \nabla R(x, \overline{S}(s))\mathrm{d}s,$$

$$\lim_{\varepsilon\to 0}\int_0^t \Delta R(x, S_\varepsilon(s))\mathrm{d}s = \int_0^t \Delta R(x, \overline{S}(s))\mathrm{d}s$$

for all $t \geq 0$. Furthermore, the limit functions

$$\int_0^t R(x, \overline{S}(s))\mathrm{d}s, \quad \int_0^t \nabla R(x, \overline{S}(s))\mathrm{d}s, \quad \int_0^t \Delta R(x, \overline{S}(s))\mathrm{d}s$$

are locally uniformly continuous.

After extraction of a subsequence by the Arzela-Ascoli theorem, $u_\varepsilon(t,x)$ converges locally uniformly to the continuous function $u(t,x)$ as ε vanishes. Consequently, $\phi_\varepsilon(t,x)$ converges locally uniformly to the continuous function $\phi(t,x) = u(t,x) - \int_0^t R(x, \overline{S}(s))\mathrm{d}s$ and ϕ is a viscosity solution to the equation

$$\partial_t\phi(t,x) - |\nabla\phi(t,x)|^2 - 2\nabla\phi(t,x)\cdot\int_0^t \nabla R(x, \overline{S}(s))\mathrm{d}s = \Big|\int_0^t \nabla R(x, \overline{S}(s))\mathrm{d}s\Big|^2. \qquad (4.9)$$

Then u is a solution to the following equation in the viscosity sense:

$$\partial_t u(t,x) = |\nabla u|^2 + R(x, \overline{S}(t)).$$

It remains to prove that $\max\limits_{x\in\mathbb{R}^d} u(t,x) = 0$ for all $t \geq 0$. We argue by contradiction. Assume that there exists $a > 0$, such that for some $t > 0$ and $x \in \mathbb{R}^d$, we have $0 < a \leq u(t,x)$. It follows that, from the continuity of u, $u(t,y) \geq \frac{a}{2}$ on $B(x,r)$ for some $r > 0$, and then $n_\varepsilon(t,y) \to \infty$ as ε goes to 0, which is a contradiction to the statements of Lemma 2.1. Thus we have $\max\limits_{x\in\mathbb{R}^d} u(t,x) \leq 0$ for all $t \geq 0$.

From Subsection 4.1, we have for $0 < \varepsilon < \varepsilon_0$ and for some $r > 0$ large enough,

$$\lim_{\varepsilon \to 0} \int_{|x| \le r} n_\varepsilon(t,x) \mathrm{d}x > \frac{\rho_m}{2}, \quad t \ge 0. \tag{4.10}$$

Furthermore, recall that we have

$$u_\varepsilon(t,x) \le A + Ct - K_2\sqrt{1+|x|^2} \le A + Ct - K_2|x|, \quad \forall t \ge 0, \ x \in \mathbb{R}^d.$$

Then it follows that, for r large enough,

$$\lim_{\varepsilon \to 0} \int_{|x| \ge r} n_\varepsilon(t,x) \mathrm{d}x \le \lim_{\varepsilon \to 0} \int_{|x| \ge r} \mathrm{e}^{\frac{A+Ct-K_2|x|}{\varepsilon}} \mathrm{d}x = 0.$$

We argue by contradiction again. Assume that $u(t,x) < 0$ for all $t \ge 0$ and $|x| < r$. It implies that $\lim\limits_{\varepsilon \to 0} n_\varepsilon(t,x) = 0$ and thus $\lim\limits_{\varepsilon \to 0} \int_{|x|<r} n_\varepsilon(t,x)\mathrm{d}x = 0$. This is a contradiction with (4.10) and it follows that $\max\limits_{x \in \mathbb{R}^d} u(t,x) = 0$ for all $t \ge 0$.

It is an open problem to know if the full sequence u_ε converges, and it is equivalent to the question of uniqueness of the solution to the Hamilton-Jacobi equation. We consider in Section 5 a special case where uniqueness holds.

In the next section, we derive some properties of the concentration points that also hold in the concavity framework (Section 6), and is useful in what follows.

4.4 Properties of the concentration points

We prove in the rest of this section the following theorem.

Theorem 4.2 *Let* (2.3) *hold. For any $u^0 \in W^{1,\infty}(\mathbb{R}^d)$, the solution of* (4.3) *is semi-convex in x for any $t > 0$, i.e., there exists a $C(t)$ such that, for any unit vector $\xi \in \mathbb{R}^d$, we have the following inequality:*

$$\frac{\partial^2}{\partial \xi^2} u \ge -C(t).$$

Consequently, $u(t,\cdot)$ is differentiable in x at maximum points, and we have

$$\nabla u(t, \overline{x}(t)) = 0,$$

where $\overline{x}(t)$ is a maximum point of $u(t,\cdot)$.

Furthermore, for all Lebesgue points of $\overline{S}$, we have

$$R(\overline{x}(t), \overline{S}(t)) = 0.$$

Step 1 The semi-convexity To increase readability, we use the notation $u_\xi := \frac{\partial u_\varepsilon}{\partial \xi}$, $u_{\xi\xi} := \frac{\partial^2 u_\varepsilon}{\partial \xi^2}$ for a unit vector ξ. We obtain from (4.1),

$$\frac{\partial}{\partial t} u_\xi = 2\nabla u_\varepsilon \cdot \nabla u_\xi + R_\xi(x, S_\varepsilon(t)) + \varepsilon \Delta u_\xi \tag{4.11}$$

and

$$\frac{\partial}{\partial t} u_{\xi\xi} = 2\nabla u_\varepsilon \cdot \nabla u_{\xi\xi} + 2|\nabla u_\xi|^2 + R_{\xi\xi}(x, S_\varepsilon(t)) + \varepsilon \Delta u_{\xi\xi}. \tag{4.12}$$

Notice that $|\nabla u_\xi| \geq |u_{\xi\xi}|$ because $u_{\xi\xi} = \nabla u_\xi \cdot \xi$. Therefore, the function $w := u_{\xi\xi}$ satisfies

$$\frac{\partial}{\partial t} w \geq 2\nabla u_\varepsilon \cdot \nabla w + 2w^2 - K_2 + \varepsilon \Delta w,$$

from the assumption (2.3). The semi-convexity follows from the comparison principle with the subsolution given by the solution to the ODE $\dot{y} = 2y^2 - K_2,\ y(0) = -\infty$.

Step 2 $\boldsymbol{\nabla u(t, \overline{x}(t)) = 0}$ The semi-convexity implies that u is differentiable at its maximum points. Therefore, we have for $t > 0$,

$$\nabla u(t, \overline{x}(t)) = 0.$$

Moreover, we also have the property that, for any sequence (t_k, x_k) of x-differentiability point of u which converges to $(t, \overline{x}(t))$, we have

$$\nabla u(t_k, x_k) \to 0 \quad \text{as } k \to \infty.$$

In fact, we deduce that, for $h, r > 0,\ h, r \to 0$,

$$\frac{1}{rh} \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} |\nabla u(s, y)|^2 \mathrm{d}s\mathrm{d}y \to 0$$

and

$$\frac{1}{rh} \int_{t-h}^{t} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} |\nabla u(s, y)|^2 \mathrm{d}s\mathrm{d}y \to 0.$$

We obtain these convergence results by applying Lebesgue's dominated convergence theorem to the integral

$$\int_0^1 \int_{-1}^1 |\nabla u(t + h\tau, x(t) + r\sigma)|^2 \mathrm{d}\tau\mathrm{d}\sigma$$

given by a change of variable, combined with the local Lipschitz continuity of u.

Step 3 Proof of $\boldsymbol{R(\overline{x}(t), \overline{S}(t)) = 0}$ We first integrate the equation on rectangles $(t, t + h) \times (\overline{x}(t) - r, \overline{x}(t) + r)$. We obtain

$$\int_{\overline{x}(t)-r}^{\overline{x}(t)+r} [u(t+h, y) - u(t, y)]\mathrm{d}y = \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} R(y, \overline{S}(s))\mathrm{d}s\mathrm{d}y + \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} |\nabla u(s, y)|^2 \mathrm{d}s\mathrm{d}y.$$

By the semi-convexity, we have

$$0 \geq u(t, y) \geq u(t, \overline{x}(t)) - C(t)|y - \overline{x}(t)|^2 = O(r^2),$$

and also $u(t + h, y) \leq 0$. We deduce

$$\frac{1}{rh} \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} R(y, \overline{S}(s))\mathrm{d}s\mathrm{d}y + \frac{1}{rh} \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} |\nabla u(s, y)|^2 \mathrm{d}s\mathrm{d}y \leq \frac{1}{rh} O(r^2).$$

Therefore, we obtain

$$\frac{1}{rh} \int_t^{t+h} \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} R(y, \overline{S}(s))\mathrm{d}s\mathrm{d}y \leq \frac{1}{rh} O(r^2).$$

We conclude that at any Lebesgue point of $\overline{S}$, we have

$$R(\overline{x}(t), \overline{S}(t)) \leq 0.$$

Next, we prove the opposite inequality. By integrating on the rectangle $(t-h, t) \times (\overline{x}(t) - r, \overline{x}(t) + r)$, we have

$$\int_{\overline{x}(t)-r}^{\overline{x}(t)+r} (u(t,y) - u(t-h,y))\mathrm{d}y \geq \int_{\overline{x}(t)-r}^{\overline{x}(t)+r} u(t,y)\mathrm{d}y$$

and

$$\frac{1}{rh}\int_{t-h}^{t}\int_{\overline{x}(t)-r}^{\overline{x}(t)+r} R(y, \overline{S}(s))\mathrm{d}s\mathrm{d}y + \frac{1}{rh}\int_{t-h}^{t}\int_{\overline{x}(t)-r}^{\overline{x}(t)+r} |\nabla u(s,y)|^2 \mathrm{d}s\mathrm{d}y \geq \frac{O(r)}{h}.$$

Hence, we have that, at any Lebesgue point of $\overline{S}$,

$$R(\overline{x}(t), \overline{S}(t)) \geq 0.$$

Hence the statement of Theorem 4.2 holds.

5 The Monomorphic Case in Dimension $d = 1$

In the case when dimension d equals 1 and $R(x, S)$ is monotonic in x for each S, we have the expected convergence toward a single Dirac mass under the additional assumption (which holds for instance when R is monotonic in x)

$$\forall S_m < S < S_0, \text{ there is a unique } X(S) \in \mathbb{R} \text{ such that } R(X(S), S) = 0. \tag{5.1}$$

Theorem 5.1 *Assume* (2.1)–(2.5) *and* (5.1) *hold, and that u_ε^0 are uniformly continuous in $\mathbb{R}^d$. Then, the solution n_ε of* (1.1)*, still after extraction of a subsequence, converges in the weak sense of measures*

$$n_{\varepsilon_k}(t,x) \longrightarrow n(t,x) := \overline{\rho}(t)\delta(x - \overline{x}(t)), \tag{5.2}$$

and we also obtain the relations

$$\overline{x}(t) = X(\overline{S}(t)), \quad R(\overline{x}(t), \overline{S}(t)) = 0 \quad a.e.$$

Moreover, the full sequence n_ε converges when R has one of the following form, for some functions $b > 0, \ d > 0, \ F > 0$,

$$R(x, S) = b(x) - d(x)F(S) \quad \textit{with } F'(S) < 0 \tag{5.3}$$

or

$$R(x, S) = b(x)F(S) - d(x) \quad \textit{with } F'(S) > 0. \tag{5.4}$$

We do not prove this result in detail. It is a consequence of the following observation. As the measure n defined in (5.2) satisfies the condition supp $n(t, \cdot) \subset \{u(t, \cdot)\}$ from the properties obtained in the previous section (see details in [5, 7]), n is monomorphic. Indeed, from the condition (5.1), the set $\{u(t, \cdot)\}$ is reduced to an isolated point for all $t \geq 0$. The uniqueness of the solution when R is written as (5.3) or (5.4) is entirely explained in [7]. The idea of the proof is to consider for instance the function

$$\phi(t,x) = u(t,x) - b(x) \int_0^t F(S(\sigma))\mathrm{d}\sigma,$$

and, by noticing that ϕ satisfies the equation

$$\partial_t \phi(t,x) = -d(x) + \left| \nabla(\phi(t,x) + b(x)) \int_0^t F(S(\sigma))\mathrm{d}\sigma \right|,$$

to derive an estimate on the derivative of the difference between two different solutions ϕ_1 and ϕ_2 with the same initial data. By considering the different quantities at the maximum points of $u(t,\cdot)$, we see that there exists a constant $C > 0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_1 - \phi_2\|_\infty \le C\|\phi_1 - \phi_2\|_\infty,$$

and the uniqueness follows.

6 The Concavity Framework in $\mathbb{R}^d$

In this section, we are going to assume more regularity in order to prove the convergence of n_ε to a Dirac mass in the sense of measure. The specific feature of this framework is that uniform concavity of the growth rate and initial data induce uniform concavity of the solutions u_ε to the Hamilton-Jacobi equations, which implies that u_ε has only one maximum point. The main technical difficulty is that uniform bounds are not possible because of the quadratic growth at infinity. Therefore, following the work [33], we start with assumptions on $R \in C^2$ as follows:

$$\max_{x\in\mathbb{R}^d} R(x, S_m) = 0 = R(0, S_m), \tag{6.1}$$

$$-\underline{K}_2|x|^2 \le R(x,S) \le \overline{K}_0 - \overline{K}_2|x|^2, \tag{6.2}$$

$$0 < \underline{K}_1 \le R_S(x,S) \le \overline{K}_1, \tag{6.3}$$

$$-2\underline{K}_2 \le D^2R(x,S) \le -2\overline{K}_2. \tag{6.4}$$

We also need the uniform concavity of the initial data

$$n_\varepsilon^0 = \mathrm{e}^{\frac{u_\varepsilon^0}{\varepsilon}}, \tag{6.5}$$

$$-\underline{L}_0 - \underline{L}_1|x|^2 \le u_\varepsilon^0 \le \overline{L}_0 - \overline{L}_1|x|^2, \tag{6.6}$$

$$-2\underline{L}_1 \le D^2 u_\varepsilon^0 \le -2\overline{L}_1, \tag{6.7}$$

and we add some compatibility conditions

$$4\overline{L}_1^2 \le \overline{K}_2 \le \underline{K}_2 \le 4\underline{L}_1^2. \tag{6.8}$$

For this section, we need

$$D^3R(\cdot,S) \in L^\infty(\mathbb{R}^d), \tag{6.9}$$

$$D^3u_\varepsilon^0 \in L^\infty(\mathbb{R}^d) \quad \text{uniformly in } \varepsilon, \tag{6.10}$$

$$n_\varepsilon^0(x) \longrightarrow \overline{\rho}^0\delta(x - \overline{x}^0) \quad \text{weakly in the sense of measures.} \tag{6.11}$$

We keep the same assumptions on Q and S_ε as in the previous section. Next we are going to prove the following result.

Theorem 6.1 *Under assumptions* (6.2)–(6.8) *and the assumptions on* Q, ρ_ε *and* S_ε *have locally bounded total variations uniformly in* ε. *Therefore, there exist functions* $\overline{\rho}$ *and* $\overline{S}$, *such that, after extraction of a subsequence, we have*

$$S_{\varepsilon_k}(t) \underset{\varepsilon_k \to 0}{\longrightarrow} \overline{S}(t), \quad \rho_{\varepsilon_k}(t) \underset{\varepsilon_k \to 0}{\longrightarrow} \overline{\rho}(t), \quad a.e.$$

Furthermore, we have weakly in the sense of measures for a subsequence n_ε,

$$n_\varepsilon(t,x) \underset{\varepsilon \to 0}{\longrightarrow} \overline{\rho}(t)\delta(x - \overline{x}(t)), \tag{6.12}$$

and the pair $(\overline{x}(t), \overline{S}(t))$ *also satisfies*

$$R(\overline{x}(t), \overline{S}(t)) = 0, \quad a.e. \tag{6.13}$$

As a first step, we give estimates on u_ε. Next, we adapt the proof of Section 3 to give BV estimates on ρ_ε and S_ε, and then pass to the limit as ε goes to 0. Finally, we prove the following theorem.

Theorem 6.2 *Assuming that* (6.1)–(6.11) *hold,* $\overline{x}(t)$ *is a* $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^d)$*-function, and its dynamics is described by the equation*

$$\dot{\overline{x}}(t) = (-D^2 u(t, \overline{x}(t)))^{-1} \cdot \nabla_x R(\overline{x}(t), \overline{S}(t)), \quad \overline{x}(0) = \overline{x}^0 \tag{6.14}$$

with $u(t,x)$ *given below in* (6.29) *and* (6.11). *Furthermore,* $\overline{S}(t)$ *is a* $W^{1,\infty}(\mathbb{R}_+)$*-function. From this equation, it follows that* $\overline{S}(t)$ *is a decreasing function and*

$$\overline{S}(t) \underset{t \to \infty}{\longrightarrow} S_m, \quad \overline{x}(t) \underset{t \to \infty}{\longrightarrow} 0. \tag{6.15}$$

6.1 Uniform concavity of u_ε

Again we use the Hopf-Cole transformation defining $u_\varepsilon = \varepsilon \ln n_\varepsilon$, and we obtain the same equation as in Section 4

$$\begin{cases} \partial_t u_\varepsilon(t,x) = |\nabla u_\varepsilon|^2 + R(x, S_\varepsilon(t)) + \varepsilon \Delta u_\varepsilon, \\ \\ u_\varepsilon(t=0,x) = u_\varepsilon^0(x) := \varepsilon \ln n_\varepsilon^0. \end{cases} \tag{6.16}$$

We focus now on the study of the properties of the sequence u_ε.

We first prove the following lemma.

Lemma 6.1 *Under assumptions* (6.2) *and* (6.8), *we have for* $t \geq 0$ *and for* $x \in \mathbb{R}^d$,

$$-\underline{L}_0 - \underline{L}_1|x|^2 - \varepsilon(2d\underline{L}_1)t \leq u_\varepsilon(t,x) \leq \overline{L}_0 - \overline{L}_1|x|^2 + (\overline{K}_0 + 2\mathrm{d}\varepsilon\overline{L}_1)t. \tag{6.17}$$

Proof First we achieve an upper bound for u_ε. By defining $\overline{u}_\varepsilon(t,x) := \overline{L}_0 - \overline{L}_1|x|^2 + C_0(\varepsilon)t$ with $C_0(\varepsilon) := \overline{K}_0 + 2\mathrm{d}\varepsilon\overline{L}_1$, we obtain from assumptions (6.2), (6.6) and (6.8) that $\overline{u}_\varepsilon(t=0) \geq u_\varepsilon^0$ and

$$\partial_t \overline{u}_\varepsilon - |\nabla \overline{u}_\varepsilon|^2 - R(x, I_\varepsilon) - \varepsilon \Delta \overline{u}_\varepsilon \geq C_0(\varepsilon) - 4\overline{L}_1^2|x|^2 - \overline{K}_0 + \overline{K}_2|x|^2 - 2\mathrm{d}\varepsilon\overline{L}_1 \geq 0.$$

Then by a comparison principle, we conclude that $u_\varepsilon(t,x) \le \overline{L}_0 - \overline{L}_1|x|^2 + (\overline{K}_0 + 2\mathrm{d}\varepsilon\overline{L}_1)t$ for all $t \ge 0$ and $x \in \mathbb{R}^d$.

Next for the lower bound, we define $\underline{u}_\varepsilon(t,x) := -\underline{L}_0 - \underline{L}_1|x|^2 - \varepsilon C_1 t$ with $C_1 := 2d\underline{L}_1$. Thus, we have $\underline{u}_\varepsilon(t=0) \le u_\varepsilon^0$ and

$$\partial_t \underline{u}_\varepsilon - |\nabla \underline{u}_\varepsilon|^2 - R(x, I_\varepsilon) - \varepsilon\Delta\underline{u}_\varepsilon \le -\varepsilon C_1 - 4\underline{L}_1^2|x|^2 + \underline{K}_2|x|^2 + \varepsilon 2d\underline{L}_1 \le 0.$$

Consequently, we obtain that $u_\varepsilon(t,x) \ge -\underline{L}_0 - \underline{L}_1|x|^2 - \varepsilon(2d\underline{L}_1)t$ for all $t \ge 0$ and $x \in \mathbb{R}^d$. Hence we have the estimates on $u_\varepsilon(t,x)$.

The next point is to show that the semi-convexity and the concavity of the initial data are preserved by (1.1). In other words, we are going to show the following lemma.

Lemma 6.2 *Under assumptions* (6.2)–(6.8)*, we have for $t \ge 0$ and $x \in \mathbb{R}^d$,*

$$-2\underline{L}_1 \le D^2 u_\varepsilon(t,x) \le -2\overline{L}_1. \tag{6.18}$$

Proof For a unit vector ξ, we use the notation $u_\xi := \nabla_\xi u_\varepsilon$ and $u_{\xi\xi} := \nabla_{\xi\xi}^2 u_\varepsilon$ to obtain

$$\begin{aligned} u_{\xi t} &= R_\xi(x, I) + 2\nabla u \cdot \nabla u_\xi + \varepsilon\Delta u_\xi, \\ u_{\xi\xi t} &= R_{\xi\xi}(x, I) + 2\nabla u_\xi \cdot \nabla u_\xi + 2\nabla u \cdot \nabla u_{\xi\xi} + \varepsilon\Delta u_{\xi\xi}. \end{aligned}$$

By using $|\nabla u_\xi| \ge |u_{\xi\xi}|$ and the definition $\underline{w}(t,x) := \min\limits_\xi u_{\xi\xi}(t,x)$, we arrive at the inequality

$$\partial_t \underline{w} \ge -2\underline{K}_2 + 2\underline{w}^2 + 2\nabla u \cdot \nabla\underline{w} + \varepsilon\Delta\underline{w}.$$

Finally by a comparison principle and assumptions (6.7)–(6.8), we obtain

$$\underline{w} \ge -2\underline{L}_1. \tag{6.19}$$

Hence the uniform semi-convexity of u_ε is proved.

To prove the uniform concavity, we first recall that, at every point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, we can choose an orthonormal basis, such that $D^2 u_\varepsilon(t,x)$ is diagonal. Thus, we can estimate the mixed second derivatives in terms of $u_{\xi\xi}$, and consequently, we have

$$|\nabla u_\xi| = |u_{\xi\xi}|. \tag{6.20}$$

By defining $\overline{w}(t,x) := \max\limits_\xi u_{\xi\xi}(t,x)$ and using assumptions (6.4) and (6.20), we obtain the following inequality:

$$\partial_t \overline{w} \le -2\overline{K}_2 + 2\overline{w}^2 + 2\nabla u \cdot \nabla\overline{w} + \varepsilon\Delta\overline{w}.$$

By a comparison principle and assumption, we obtain the estimate

$$\overline{w} \le -2\overline{L}_1, \tag{6.21}$$

which ends the proof of Lemma 6.2.

6.2 BV estimates on $\rho_\varepsilon^2, S_\varepsilon$ and their limits

We use exactly the same proof as in Section 3 to obtain BV estimates on ρ_ε^2 and S_ε. To obtain these estimates, an important point is the bounds on $\varepsilon P_\varepsilon$. We need to confirm that $\varepsilon P_\varepsilon$ is bounded, which is clear in Section 3 thanks to the bounds on the growth rate. Here the growth rate has a quadratic decrease at infinity, which does not give an immediate lower bound on $\varepsilon P_\varepsilon$. Furthermore, we do not have a lower bound on ρ_ε either because of the same argument and we cannot obtain directly a BV estimate on S_ε as in Subsection 3.2. However, we derive a lower bound for $\varepsilon P_\varepsilon$, and we use the uniform concavity of u_ε for that purpose.

By the definition of P_ε, it follows from (6.2) and (6.17) that

$$\begin{aligned}\varepsilon P_\varepsilon = \int_{\mathbb{R}^d} n_\varepsilon R(x, S_\varepsilon(t))\mathrm{d}x &\geq \int_{\mathbb{R}^d} \mathrm{e}^{\frac{1}{\varepsilon}(-\underline{L}_0 - \underline{L}_1|x|^2 - \varepsilon C_1 t)}(-\underline{K}_2|x|^2)\mathrm{d}x \\ &\geq -\underline{K}_2 \mathrm{e}^{\frac{1}{\varepsilon}(-\underline{L}_0 - \varepsilon C_1 t)} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{\varepsilon}\underline{L}_1|x|^2}|x|^2\mathrm{d}x \\ &= -\underline{K}_2 \mathrm{e}^{\frac{1}{\varepsilon}(-\underline{L}_0 - \varepsilon C_1 t)} \frac{\mathrm{d}\varepsilon}{2\underline{L}_1}\Big(\sqrt{\frac{\pi\varepsilon}{\underline{L}_1}}\Big)^{d-1}. \end{aligned} \tag{6.22}$$

And we have a bound for $(\varepsilon P_\varepsilon)_-$.

We recall that (3.14) also holds true in this framework

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(P_\varepsilon)_- + \alpha_\varepsilon \frac{(P_\varepsilon)_-}{\beta\mu_\varepsilon} \leq \overline{K}_1 \rho_M (P_\varepsilon(0) + \beta\mu_\varepsilon(0)J_\varepsilon(0))_- \frac{\mathrm{e}^{\frac{-\mu_m K_Q}{\varepsilon}t}}{\beta\mu_m} + C\varepsilon.$$

Then, we integrate this inequality over $[0, T]$ for $T > 0$ and by the same arguments used in Subsection 3.1, it follows that ρ_ε^2 has local BV bounds. Therefore there exists a function $\overline{\rho}$, such that after extraction of a subsequence,

$$\rho_\varepsilon \longrightarrow \overline{\rho} \quad \text{in } L^1_{\mathrm{loc}}(0, \infty).$$

The next aim is to show that S_ε has local BV bounds. We go back to (3.6), and we recall

$$\varepsilon\beta \frac{\mathrm{d}}{\mathrm{d}t} J_\varepsilon = Q_S J_e + Q_\rho P_\varepsilon.$$

Then we have the following inequality:

$$\varepsilon\beta \frac{\mathrm{d}}{\mathrm{d}t}(-J_\varepsilon) \leq Q_S(-J_\varepsilon) + L_Q|P_\varepsilon| \tag{6.23}$$

and

$$\varepsilon\beta \frac{\mathrm{d}}{\mathrm{d}t}(J_\varepsilon)_- \leq Q_S(J_\varepsilon)_- + L_Q((P_\varepsilon)_+ + (P_\varepsilon)_-). \tag{6.24}$$

By integrating this inequality over $[0, T]$ for $T > 0$, using

$$\int_0^T L_Q|P_\varepsilon|\mathrm{d}x \leq L_Q\Big(\int_0^T (\dot{\rho}_\varepsilon)_+\mathrm{d}x + \int_0^T (P_\varepsilon)_-\mathrm{d}x\Big), \tag{6.25}$$

and since ρ_ε is bounded above, we deduce from (3.14) that

$$\int_0^T (J_\varepsilon)_-\mathrm{d}x \leq C_1 T + \underset{\varepsilon\to 0}{o}(1). \tag{6.26}$$

To conclude, we can extract a subsequence from S_ε which locally converges in $L^1_{\mathrm{loc}}(0, \infty)$ to a limit function $\overline{S}$.

6.3 The limit of the Hamilton-Jacobi equation

From the estimates obtained above on u_ε and D^2u_ε, we can deduce that ∇u_ε is locally uniformly bounded and thus from (4.1) for $\varepsilon < \varepsilon_0$ that $\partial_t u_\varepsilon$ is also locally uniformly bounded. Therefore there exists a function u such that, after extraction of a subsequence (see [10, 26] for compactness properties), we have for $T > 0$,

$$u_\varepsilon(t,x) \underset{\varepsilon\to 0}{\longrightarrow} u(t,x) \quad \text{strongly in } L^\infty(0,T;W^{1,\infty}_{\text{loc}}(\mathbb{R}^d)),$$
$$u_\varepsilon(t,x) \underset{\varepsilon\to 0}{\longrightarrow} u(t,x) \quad \text{weakly-}* \text{ in } L^\infty(0,T;W^{2,\infty}_{\text{loc}}(\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^\infty_{\text{loc}}(\mathbb{R}^d))$$

and

$$-\underline{L}_0 - \underline{L}_1|x|^2 \le u(t,x) \le \overline{L}_0 - \overline{L}_1|x|^2 + \overline{K}_0 t, \quad -2\underline{L}_1 \le D^2u(t,x) \le -2\overline{L}_1 \quad \text{a.e.,} \tag{6.27}$$
$$u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{6.28}$$

Then, passing to the limit as $\varepsilon \to 0$ in (4.1), we deduce that u satisfies in the viscosity sense the equation

$$\begin{cases} \dfrac{\partial}{\partial t}u = R(x,\overline{S}(t)) + |\nabla u|^2, \\ \max\limits_{\mathbb{R}^d} u(t,x) = 0. \end{cases} \tag{6.29}$$

In particular, u is strictly concave, therefore it has exactly one maximum. This proves that n stays monomorphic and characterizes the Dirac location by

$$\max_{\mathbb{R}^d} u(t,x) = 0 = u(t,\overline{x}(t)). \tag{6.30}$$

This completes the proof of Theorem 6.1.

6.4 The canonical equation

In this section, we establish from the regularity properties proved in the previous sections a form of the so-called canonical equation in the language of adaptive dynamics (see [15, 22]) as follows:

$$\dot{\overline{x}}(t) = (-D^2u(t,\overline{x}(t)))^{-1} \cdot \nabla_x R(\overline{x}(t),\overline{S}(t)).$$

This equation was formally introduced in [24] and holds true in our framework. The point of this differential equation is to describe the long time behaviour of the concentration point $\overline{x}(t)$.

Step 1 Bounds on third derivatives of u_ε For the unit vectors ξ and η, we use the notation $u_\xi := \nabla_\xi u_\varepsilon,\ u_{\xi\eta} := \nabla^2_{\xi\eta}u_\varepsilon$ and $u_{\xi\xi\eta} := \nabla^3_{\xi\xi\eta}u_\varepsilon$ to derive

$$\partial_t u_{\xi\xi\eta} = 4\nabla u_{\xi\eta}\cdot\nabla u_\xi + 2\nabla u_\eta \cdot \nabla u_{\xi\xi} + 2\nabla u\cdot\nabla u_{\xi\xi\eta} + R_{\xi\xi\eta} + \varepsilon\Delta u_{\xi\xi\eta}.$$

Let us define

$$M_1(t) := \max_{x,\xi,\eta} u_{\xi\xi\eta}(t,x).$$

Again, at every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we can choose an orthogonal basis, such that $D^2(\nabla_\eta u_\varepsilon(t,x))$ is diagonal. And since $-u_{\xi\xi\eta}(t,x) = \nabla_{-\eta}u_{\xi\xi}(t,x)$, we have $M_1(t) = \max\limits_{x,\xi,\eta}|u_{\xi\xi\eta}(t,x)|$. Then we obtain the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1 \le 4dM_1\|D^2u_\varepsilon\|_\infty + 2dM_1\|D^2u_\varepsilon\|_\infty + R_{\xi\xi\eta}.$$

As (6.10) gives a bound on $M_1(t=0)$, by using the Gronwall lemma, we obtain an L^∞-bound on the third derivative uniform in ε.

Step 2 Maximum point of $\boldsymbol{u_\varepsilon}$ We denote the maximum point of $u_\varepsilon(t,\cdot)$ by $\overline{x}_\varepsilon(t)$. Since we have $\nabla u_\varepsilon(t,\overline{x}_\varepsilon(t))=0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla u_\varepsilon(t,\overline{x}_\varepsilon(t))=0.$$

Then the chain rule gives

$$\frac{\partial}{\partial t}\nabla u_\varepsilon(t,\overline{x}_\varepsilon(t))+D_x^2 u_\varepsilon(t,\overline{x}_\varepsilon(t))\dot{\overline{x}}_\varepsilon(t)=0.$$

By using (6.16), it follows that, for all $t\geq 0$, we have

$$D_x^2 u_\varepsilon(t,\overline{x}_\varepsilon(t))\dot{\overline{x}}_\varepsilon(t)=-\frac{\partial}{\partial t}\nabla u_\varepsilon(t,\overline{x}_\varepsilon(t))=-\nabla_x R(\overline{x}_\varepsilon(t),S_\varepsilon(t))-\varepsilon\Delta\nabla_x u_\varepsilon.$$

Thanks to the uniform bound on $D^3 u_\varepsilon$ and the regularity on R, we pass to the limit

$$\dot{\overline{x}}(t)=(-D^2u(t,\overline{x}(t)))^{-1}\cdot\nabla_x R(\overline{x}(t),\overline{S}(t)),\quad \text{a.e.}$$

As we have $R(\overline{x}(t),\overline{S}(t))=0$ and (6.2), $\overline{x}(t)$ is bounded in $L^\infty(\mathbb{R}_+)$. Then it implies from the canonical equation that $\overline{x}(t)$ is bounded in $W^{1,\infty}(\mathbb{R}_+)$, and $\overline{S}(t)$ is also bounded in $W^{1,\infty}(\mathbb{R}^d)$, since $S\mapsto R(\cdot,S)$ is invertible by the implicit function theorem. We differentiate (6.13) and obtain the following differential equation:

$$\dot{\overline{x}}(t)\cdot\nabla_x R+\dot{\overline{S}}(t)\nabla_S R=0.$$

Step 3 Long time behaviour Using the canonical equation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R(\overline{x}(t),\overline{S}(t))&=\nabla R(\overline{x}(t),\overline{S}(t))\frac{\mathrm{d}}{\mathrm{d}t}\overline{x}(t)+\partial_S R(\overline{x}(t),\overline{S}(t))\frac{\mathrm{d}}{\mathrm{d}t}\overline{S}(t)\\&=\nabla R(\overline{x}(t),\overline{S}(t))(-D^2u)^{-1}\nabla R(\overline{x}(t),\overline{S}(t))+\partial_S R(\overline{x}(t),\overline{S}(t))\frac{\mathrm{d}}{\mathrm{d}t}\overline{S}(t).\end{aligned}$$

Since the left-hand side equals 0 from (6.13), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{S}(t)=\frac{-1}{\partial_S R(\overline{x}(t),\overline{S}(t))}\nabla R(\overline{x}(t),\overline{S}(t))(-D^2u)^{-1}\nabla R(\overline{x}(t),\overline{S}(t))\leq 0.$$

We deduce that $\overline{S}(t)$ decreases. Consequently, $\overline{S}(t)$ converges and subsequences of $\overline{x}(t)$ also converge, since $\overline{x}(t)$ is bounded. However, the possible limits $\overline{x}_\infty$ and $\overline{S}_\infty$ have to satisfy $\nabla R(\overline{x}_\infty,\overline{S}_\infty)=0$. Then from (6.1), (6.3) and (6.13), we conclude that

$$\overline{S}(t)\underset{t\to\infty}{\longrightarrow}S_m,\quad \overline{x}(t)\underset{t\to\infty}{\longrightarrow}\overline{x}_\infty=0,$$

which ends the proof of Theorem 6.2.

7 Numerical Results

We illustrate in this section the evolution of $n_\varepsilon, \rho_\varepsilon$ and S_ε in time with different values of β. We choose the following initial data:

$$n^0 = C_{\text{mass}} \exp\Big(-\frac{(x-0.8)^2}{\varepsilon} \Big), \tag{7.1}$$

and growth rate R and Q as follows:

$$R(x,S) = 0.2(-0.6 + 0.3S - (x-0.5)^2), \tag{7.2}$$

$$Q(\rho, S) = 10 - (1.5+\rho)S. \tag{7.3}$$

The numerics are performed in Matlab with parameters as follows. We consider the solution on interval $[0,1]$. We use a uniform grid with 1000 points on the segment and denote by n_i^k and S^k the numerical solutions at grid point $x_i = i\Delta x$ and at time $t_k = k\Delta t$. We choose as initial value of the nutrient concentration $S_\varepsilon(t=0) = 5$. We also choose β to be $2 \cdot 10^3$, the time step $\Delta t = 10^{-4}$ and C_{mass} such as the initial mass of the population in the computational domain is equal to 1. The equation is solved by an implicit-explicit finite-difference method with the following scheme:

$$n_i^{k+1} = n_i^k + \frac{\Delta t}{\varepsilon}((R_i^k)_+ n_i^k + (R_i^k)_- n_i^{k+1}) + \varepsilon \frac{\Delta t}{\Delta x^2}(n_{i+1}^{k+1} - 2n_i^{k+1} + n_{i-1}^{k+1}), \tag{7.4}$$

$$S^{k+1} = S^k + \frac{\Delta t}{\varepsilon\beta}(10 - (1.5+\rho^k)S^{k+1}). \tag{7.5}$$

We use Neumann boundary conditions $n_0^{k+1} = n_1^{k+1}$ and $n_{N-1}^{k+1} = n_N^{k+1}$. We use an implicit-explicit scheme for the growth term in order to maintain the positivity of the numerical solution.

Figure 1 shows the dynamics for $\varepsilon = 1 \cdot 10^{-3}$ and Figure 2 for $\varepsilon = 5 \cdot 10^{-4}$. We observe that, since ε is smaller in Figure 2, the concentration location of the population moves to the maximum point of fitness more quickly than in Figure 1, which illustrates the dynamics given by the canonical equation, and then the concentration point and the population density become stable.

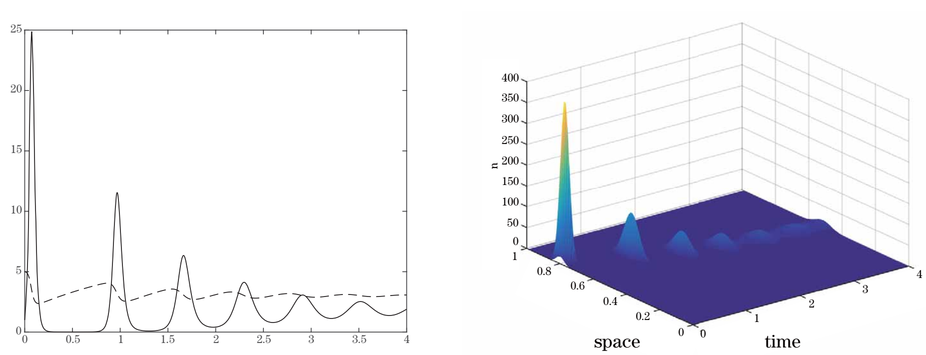

Figure 1 Dynamics of ρ_ε (————) and S_ε (– – – –) (left) and dynamics of the density n_ε for $\beta = 2 \cdot 10^3$ and $\varepsilon = 10^{-3}$.

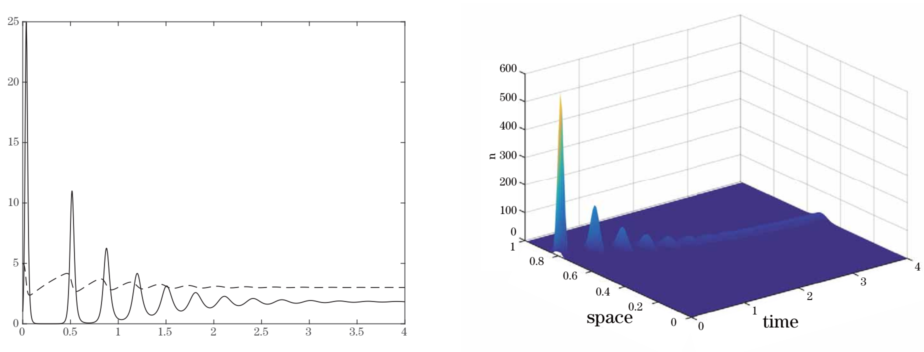

Figure 2 Dynamics of ρ_ε (————) and S_ε (– – – –) (left) and dynamics of the density n_ε for $\beta = 2 \cdot 10^3$ and $\varepsilon = 5 \cdot 10^{-4}$.

In Figure 3, we show the numerical results corresponding to the same data as in Figure 1, except that we choose $\beta = 2 \cdot 10^2$. We can observe oscillations of ρ_ε and S_ε in the first case ($\beta = 2 \cdot 10^3$), whereas there are very few variations of these quantities when β is smaller. Indeed the parameter β can be considered as a measure of the ecological dynamics: As β goes to 0, we approach the case of the quasi-stationary state of the resource level, and we then observe mostly the dynamics of the concentration location. However, as explained in the next section, the convergence to the quasi-stationary solutions as β goes to 0 cannot be proved with our approach and remains an open problem.

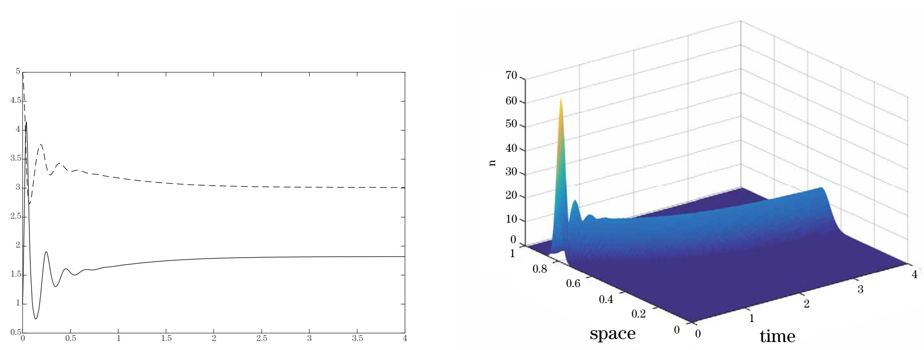

Figure 3 Dynamics of ρ_ε (————) and S_ε (– – – –) (left) and dynamics of the density n_ε for $\beta = 2 \cdot 10^2$ and $\varepsilon = 10^{-3}$.

In Figure 4, we show the numerical results for the 2-dimensional model with $\beta = 2 \cdot 10^2$, $\varepsilon = 1 \cdot 10^{-2}$ and $S_\varepsilon(t = 0) = 5$. We choose the time step Δt to be $5 \cdot 10^{-3}$ and Δx to be $1 \cdot 10^{-2}$. We also choose the initial condition

$$n^0(x, y) = \exp\Big(- \frac{(x - 0.8)^2}{\varepsilon} - \frac{(y - 0.2)^2}{\varepsilon} \Big), \tag{7.6}$$

and the functions

$$R(x, y, S) = 0.2(-0.6 + 0.3S - (1 + (x - 0.3)^2)(1 + (y - 0.6)^2)), \tag{7.7}$$

$$Q(\rho, S) = 10 - (1.5 + \rho)S. \tag{7.8}$$

As confirmed by the analysis we conducted, the population density concentrates at the maximum point of the growth rate.

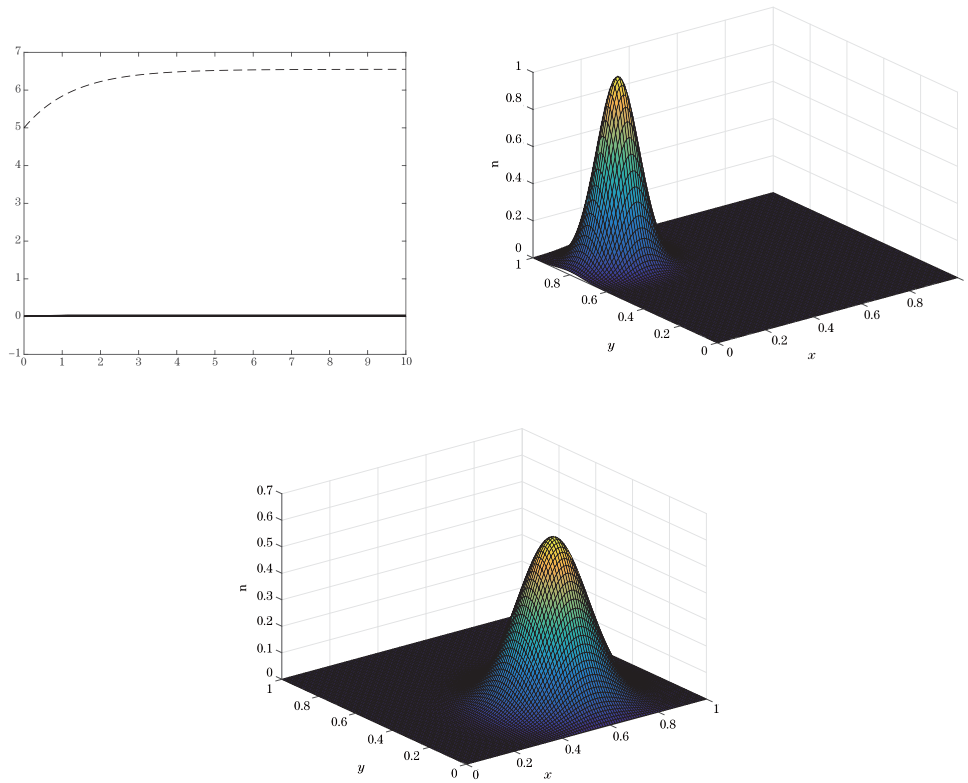

Figure 4 Dynamics in dimension 2 of ρ_ε (———) and S_ε (− − −−) (at top left), the initial condition for n_ε (at top right) and the stationary state n_ε converges to (at bottom), with $\beta = 2 \cdot 10^2$ and $\varepsilon = 10^{-2}$.

8 Discussion

The weak assumptions provide a generic framework to study the asymptotic behaviour of u_ε, but do not enable us to derive a canonical equation describing the dynamics of the concentration points, and we can observe jump phenomena of the concentration location. Indeed, the lack of regularity can produce a discontinuity of $\rho(t)$ and local maxima of $u(t, x)$ can become global maxima and jumps from a given concentration location to another one can occur, which means the extinction of a population quickly invaded by another growing one (see [7] for further details). The concavity assumptions are suitable to avoid the jump cases because these assumptions preserve regularity, and they ensure that the global maximum of u is the only maximum. The canonical equation derived in this framework describes the evolution of the selected trait in an evolutionary time scale.

Many models are studied to illustrate the diversity of evolutionary problems. For instance, the problem of coevolution was tackled in [7, 18, 33]. The branching phenomenon where a monomorphic population at some point becomes dimorphic is described in [24, 41]. In the chemostat model, the spatial component is neglected here with the hypothesis that the content of the chemostat is well-mixed, it was taken into account in [9, 37–38].

The inclusion of mutations in structured population models is necessary to generate phenotypic variability in a given population, which is a fundamental ingredient of the selection process. It implies the separation of the ecological time scale and the evolutionary one. In the presented model, the mutation term has little phenotypic effects due to the parameter ε. Especially in the canonical equation form, we observe that the pressure of mutants on the dynamics of $\overline{x}$ is small and, as ε goes to 0, it does not change the convergence of $\overline{x}_\varepsilon$ to the maximum point of fitness $\overline{x}$. It means that only the mutations with positive effects on the phenotypic trait can influence the dynamics: Mutants emerging with a better fitness than the residents can invade, while the other mutants go to extinction.

However, some open questions arise from the present study. First it seems that the method developed in this work does not give TV bounds for the full range $[0, \beta_0]$ for some small β_0, since the estimates providing the uniform BV estimates on ρ^2 in Subsection 3.2 are local in time and then it is not possible to prove uniform convergence of $S(t)$ as $\beta \to 0$ on $[0, \infty)$ at this stage. Thus we cannot obtain the asymptotic behaviour of the limit functions as β goes to 0, while the convergence of ε to 0 describes the dynamics of the presented system on a larger time scale, therefore local estimates are enough.

As mentioned in Section 4, the uniqueness of the solution of the Hamilton-Jacobi equation (6.29) has up to now been an open problem, apart from very particular cases (see for instance [6]), and the issue of the convergence of the full sequence u_ε has remained unsolved. However a recent work of Mirrahimi and Roquejoffre [40] has shown uniqueness of the constrained Hamilton-Jacobi equation related to the following selection-mutation model in the concavity framework:

$$\begin{aligned}
&\varepsilon\partial_t n_\varepsilon(t,x) = n_\varepsilon(t,x)R(x, I_\varepsilon(t)) + \varepsilon^2 \Delta n_\varepsilon(t,x),\\
&I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x)\mathrm{d}x,
\end{aligned}$$

and generalizes a result on a selection model with spatial structure (see [38]), where the proof relies on the uniqueness of the solution of the corresponding constrained Hamilton-Jacobi equation. The proof of the uniqueness property in our chemostat model is a forthcoming work.

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