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Bolzano's Theorems for Holomorphic Mappings

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(*To Haïm Brezis, with friendship and admiration*)

Abstract The existence of a zero for a holomorphic functions on a ball or on a rectangle under some sign conditions on the boundary generalizing Bolzano's ones for real functions on an interval is deduced in a very simple way from Cauchy's theorem for holomorphic functions. A more complicated proof, using Cauchy's argument principle, provides uniqueness of the zero, when the sign conditions on the boundary are strict. Applications are given to corresponding Brouwer fixed point theorems for holomorphic functions. Extensions to holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ are obtained using Brouwer degree.

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1 Introduction

Bolzano's theorem (see [3]) states that any continuous function $f : [a, b] \to \mathbb{R}$ which takes opposite signs at a and b must vanish in $[a, b]$. Without loss of generality, we can assume that $f(a) \le 0 \le f(b)$.

If we consider $[a, b]$ as the ball in $\mathbb{R}$ of center $c = \frac{a+b}{2}$ and radius $r = \frac{b-a}{2}$, Bolzano's condition can be written equivalently

$$(x - c)f(x) \ge 0 \quad \text{for } |x - c| = r. \tag{1.1}$$

A possible n-dimensional generalization of Bolzano's theorem consists in considering a continuous mapping $f : \overline{B}(c, r) \to \mathbb{R}^n$, with $B(c, r)$ the open ball of center $c \in \mathbb{R}^n$ and radius $r > 0$, and generalize condition (1.1) in the form

$$\langle x - c, f(x)\rangle \ge 0 \quad \text{for } \|x - c\| = r, \tag{1.2}$$

where $\langle \cdot, \cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$, and $\|\cdot\|$ denotes the corresponding Euclidian norm. In his proof of Brouwer fixed point theorem given in 1910 in [8], Hadamard has shown, using his extension of Kronecker's integral to continuous mappings, that condition (1.2) implies the existence of a zero of f in $\overline{B}(c, r)$. This can be seen as an n-dimensional extension of Bolzano's theorem.

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Another possible extension is to consider an open parallellotope

$$P = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

in $\mathbb{R}^n$, a continuous mapping $f : \overline{P} \to \mathbb{R}^n$, with components $f_1, \cdots, f_n$, and to extend Bolzano's conditions by requesting that, for each $j = 1, \cdots, n$, one has $f_j \leq 0$ when $x \in P$ and $x_j = a_j$, and $f_j \geq 0$ when $x \in P$ and $x_j = b_j$. Geometrically, those sets are the couples of opposite faces of P. That f has at least one zero in P under those conditions was already stated, proved and used in celestial mechanics, in 1883, by Poincaré [13]. Because of its complicated history (see, e.g., [11]), the result is generally referred as Poincaré-Miranda's theorem, and recent proofs can be found in [9, 12].

A version of Bolzano theorem for a complex function f holomorphic on a suitable bounded open neighborhood $\Omega \subset \mathbb{C}$ of 0 and continuous on $\overline{\Omega}$, was proposed in 1982 by Shih, Mau-Hsiang [17]. He showed that f has a unique zero in Ω when $\Re[\overline{z} f(z)] > 0$ on $\partial\Omega$. His proof was based upon Rouché's theorem applied to the functions f and g, with $g(z) = \alpha z$ and $\alpha = \inf_{z \in \partial\Omega} \Re[\overline{z} f(z)] / \sup_{z \in \partial\Omega} |z|^2$. Noticing that

$$\Re[\overline{z} f(z)] = \Re z \cdot \Re f(z) + \Im z \cdot \Im f(z),$$

Shih's condition is just Hadamard's one with $c = 0$ and a strict inequality sign.

We first show in Section 2 that the existence of a zero of a holomorphic function when (non-strict) Hadamard-Shih's conditions hold on the boundary of a ball in $\mathbb{C}$ (see Theorem 2.1), or when (non-strict) Poincaré-Miranda's conditions hold on the boundary of a rectangle in $\mathbb{C}$ (see Theorem 2.2), follows in a very simple way from an immediate consequence of Cauchy's theorem for holomorphic functions (see Proposition 2.3). Direct applications are corresponding versions of Brouwer fixed point theorem for holomorphic functions on a closed ball or a closed rectangle (see Corollaries 2.2 and 2.5), and an intermediate value property for a holomorphic function on a rectangle (see Corollary 2.4).

Using a more sophisticated tool of function theory, namely the argument principle (see [1]), we show, in a simple way in Section 3, the existence of a unique zero of a holomorphic function f, when strict Hadamard-Shih's conditions hold on the boundary of a suitable open bounded set, and when strict Poincaré-Miranda's conditions hold on the boundary of an open rectangle. As consequences, corresponding Brouwer theorems with a unique fixed point are obtained when the holomorphic function maps the boundary of the corresponding set into its interior (see Corollaries 3.1 and 3.2).

Dealing with holomorphic mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$, the approach of Section 3 can be extended by replacing Cauchy's integral of f'/f by the Brouwer degree of the corresponding mapping between $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n}$. In this way, using results on Brouwer degree of holomorphic mappings proved in a particular simple way by Rabinowitz in [14] (see also [15]), we obtain in Section 4 the existence of a unique zero under corresponding strict Hadamard-Shih's inequalities (see Theorem 4.1) (generalizing some results of Shih [16]), or corresponding strict Poincaré-Miranda's inequalities (see Theorem 4.3), and existence only when the inequalities are not strict (see Theorems 4.2 and 4.5). Applications are given to some intermediate value properties (see Theorems 4.4 and 4.6) and to some Brouwer fixed point theorems (see Corollaries 4.1–4.2).

Of course, the results of Sections 2–3 are special cases, for $n = 1$, of the results of Section 4, but they are deduced there either from the very fundamental, or from classical properties of holomorphic functions.

2 Existence of Zeros of Holomorphic Functions

2.1 A simple condition for the existence of a zero

Recall that a domain in $\mathbb{C}$ is an open connected set. Let $\Omega \subset \mathbb{C}$ be a domain, and $f : \Omega \to \mathbb{C}$ be a holomorphic function, in the sense that, for each $a \in \Omega$,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and is denoted by $f'(a)$.

Let $k \geq 1$ be an integer.

Definition 2.1 *A C^k-cycle in Ω is a mapping $\gamma \in C^k([a,b],\Omega)$ such that*

$$\gamma(a) = \gamma(b). \tag{2.1}$$

Definition 2.2 *If γ is a C^k-cycle in Ω, and $f : \Omega \to \mathbb{C}$ is holomorphic on Ω, the integral of f along γ is defined by*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f[\gamma(t)]\gamma'(t)\,\mathrm{d}t.$$

Definition 2.3 *A C^k-family of C^k-cycles in Ω is a mapping $\Phi \in C^k([a,b] \times [0,1], \Omega)$ such that*

$$\Phi(b,\cdot) = \Phi(a,\cdot). \tag{2.2}$$

So, for each $s \in [0,1]$, $\Phi(\cdot, s)$ is a C^k-cycle in Ω, and condition (2.2) immediately implies that

$$\partial_s\Phi(b,\cdot) = \partial_s\Phi(a,\cdot). \tag{2.3}$$

We first state and prove in an elementary way the needed simple versions of Cauchy theorem. The proof is reminiscent of Cauchy's one in 1825 (see [4]) for the independence of the path of $\int_{z_0}^z f(z)\,\mathrm{d}z$, based upon the variation of the path (see [4]), and reworked in a more rigorous way by Falk in 1883, in a letter to Hermite [7]. The setting here is integration over cycles.

Proposition 2.1 *If $f : \Omega \to \mathbb{C}$ is holomorphic and Φ is a C^2 family of C^2-cycles in Ω, the mapping*

$$s \mapsto \int_{\Phi(\cdot,s)} f(z)\,\mathrm{d}z$$

is constant on $[0,1]$.

Proof We have (with differentiation under the integral sign easily justified)

$$\begin{aligned}\partial_s \int_{\Phi(\cdot,s)} f(z)\,\mathrm{d}z &= \partial_s \int_a^b f[\Phi(t,s)]\partial_t\Phi(t,s)\,\mathrm{d}t \\ &= \int_a^b \{f'[\Phi(t,s)]\partial_s\Phi(t,s)\partial_t\Phi(t,s) + f[\Phi(t,s)]\partial_s\partial_t\Phi(t,s)\}\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b [\partial_t\{f[\Phi(t,s)]\}\partial_s\Phi(t,s) + f[\Phi(t,s)]\partial_t\partial_s\Phi(t,s)]\,\mathrm{d}t \\
&= \int_a^b \partial_t\{f[\Phi(t,s)]\partial_s\Phi(t,s)\}\,\mathrm{d}t \\
&= f[\Phi(b,s)]\partial_s\Phi(b,s) - f[\Phi(a,s)]\partial_s\Phi(a,s) = 0,
\end{aligned}
$$

by using (2.2)–(2.3).

Definition 2.4 *A piecewise C^k-cycle in Ω is a mapping $\gamma \in C([a,b],\Omega)$, such that there exists*

$$a = a_0 < a_1 < a_2 < \cdots < a_q - 1 < a_q = b$$

with the property that $\gamma \in C^k([a_{j-1},a_j],\Omega)$ $(j = 1,\cdots,q)$ (with one-sided derivatives at a_{j-1} and a_j) $(j = 1,\cdots,q)$, and condition (2.1) *holds.*

Definition 2.5 *A C^k family of piecewise C^k-cycles in Ω is a mapping $\Phi \in C([a,b] \times [0,1],\mathbb{C})$, such that there exists*

$$a = a_0 < a_1 < a_2 < \cdots < a_q - 1 < a_q = b$$

with the property that $\Phi \in C^k([a_{j-1},a_j] \times [0,1],\Omega)$ $(j = 1,\cdots,q)$ (with one-sided derivatives at a_{j-1} and a_j) $(j = 1,\cdots,q)$, and condition (2.2) *holds.*

Proceeding exactly like for Proposition 2.1, one obtains the following extension to a C^2 family of piecewise C^2-cycles.

Proposition 2.2 *If $f : \Omega \to \mathbb{C}$ is holomorphic and Φ is a C^2 family of piecewise C^2-cycles in Ω, the mapping*

$$s \mapsto \int_{\Phi(\cdot,s)} f(z)\,\mathrm{d}z$$

is constant on $[0,1]$.

Definition 2.6 *A piecewise C^k-cycle reducible to a constant in Ω is a mapping $\gamma : [a,b] \to \Omega$, such that $\gamma = \Phi(\cdot,0)$ and $\Phi(\cdot,1) \equiv c \in \Omega$ for some C^k family Φ of piecewise C^k-cycles in Ω.*

A version of Cauchy formula is a direct consequence of Proposition 2.2.

Corollary 2.1 *If $\Omega \subset \mathbb{C}$ is a domain, $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a piecewise C^2-cycle in Ω reducible to a constant in Ω, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof Let $\Phi : [a,b] \times [0,1] \to \Omega$ be the C^2 family of piecewise C^2-cycles given by Definition 2.6. By Proposition 2.1, $s \mapsto \int_{\Phi(\cdot,s)} f(z)\,\mathrm{d}z$ is constant, and, furthermore, $\partial_t\Phi(\cdot,1) = 0$, so that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\Phi(\cdot,0)} f(z)\,\mathrm{d}z = \int_{\Phi(\cdot,1)} f(z)\,\mathrm{d}z = 0.$$

Proposition 2.3 *Let $\Omega \subset \mathbb{C}$ be a domain, $g : \Omega \to \mathbb{C}$ be holomorphic on Ω, $\gamma : [a,b] \to \Omega$ be a piecewise C^2-cycle reducible to a constant in Ω, whose image $\gamma([a,b])$ is the boundary of an open set Δ with $\overline{\Delta} \subset \Omega$. If g is different from zero on $\gamma([a,b])$, and if*

$$\int_\gamma \frac{\mathrm{d}z}{g(z)} \neq 0, \tag{2.4}$$

then g has at least one zero in Δ.

Proof If g has no zero in Δ, then $z \mapsto \frac{1}{g(z)}$ is holomorphic on some open neighborhood of $\overline{\Delta}$ contained in Ω. Consequently, by Corollary 2.1,

$$\int_\gamma \frac{\mathrm{d}z}{g(z)} = 0,$$

a contradiction to (2.4).

2.2 Hadamard-Shih's conditions on a circle

Let $B(c,r) \subset \mathbb{C}$ denote the open disk of center c and radius $r > 0$, $\partial B(c,r)$ denote its boundary, and $\overline{B}(c,r)$ denote its closure.

Theorem 2.1 *If $f : \Omega \to \mathbb{C}$ is holomorphic on some domain $\Omega \supset \overline{B}(c,r)$, and if*

$$\Re[(\overline{z} - \overline{c}) f(z)] \geq 0 \quad \text{for all } z \in \partial B(c,r), \tag{2.5}$$

then f has at least one zero in $\overline{B}(c,r)$.

Proof For each integer $k \geq 1$, define $f_k : \overline{B}(c,r) \to \mathbb{C}$ by

$$f_k(z) = k^{-1}(z - c) + f(z).$$

Each f_k has the same regularity properties than f and, for any $z \in \partial B(c,r)$,

$$(\overline{z} - \overline{c}) f_k(z) = k^{-1} r^2 + (\overline{z} - \overline{c}) f(z) > 0, \tag{2.6}$$

using assumption (2.5), so that $f_k(z) \neq 0$ for all $z \in \partial B(c,r)$. Let

$$\gamma_{c,r} : [0, 2\pi] \to \Omega,\ t \mapsto c + r \exp(\mathrm{i}t) \tag{2.7}$$

be a C^∞-cycle with image $\partial B(c,r)$. $\gamma_{c,r}$ is deformable into c in Ω. Furthermore,

$$\begin{aligned}
\Im\Big[\int_{\gamma_{c,r}} \frac{\mathrm{d}z}{f_k(z)}\Big] &= \Im\Big[\int_{\gamma_{c,r}} \frac{(z-c)(\overline{z}-\overline{c})}{(\overline{z}-\overline{c}) f_k(z)} \frac{\mathrm{d}z}{z-c}\Big] \\
&= \Im\Big[\int_{\gamma_{c,r}} \frac{|z-c|^2 \{\Re[(\overline{z}-\overline{c}) f_k(z)] - \mathrm{i}\Im[(\overline{z}-\overline{c}) f_k(z)]\}}{|(\overline{z}-\overline{c}) f_k(z)|^2} \frac{\mathrm{d}z}{z-c}\Big] \\
&= \Im\Big[\int_0^{2\pi} \frac{\mathrm{i}\Re[r\mathrm{e}^{-\mathrm{i}t} f_k(c + r\mathrm{e}^{\mathrm{i}t})] + \Im[r\mathrm{e}^{-\mathrm{i}t} f(c + r\mathrm{e}^{\mathrm{i}t})]}{|f_k(c + r\mathrm{e}^{\mathrm{i}t})|^2} \mathrm{d}t\Big] \\
&= \int_0^{2\pi} \frac{\Re[r\mathrm{e}^{-\mathrm{i}t} f_k(c + r\mathrm{e}^{\mathrm{i}t})]}{|f_k(c + r\mathrm{e}^{\mathrm{i}t})|^2} \mathrm{d}t > 0,
\end{aligned}$$

because of (2.6). By Proposition 2.3, for each integer $k \geq 1$, f_k has at least one zero z_k in $B(c,r)$, and Bolzano-Weierstrass theorem implies the existence of a subsequence $(z_{k_n})_{n\geq 1}$ of $(z_k)_{k\geq 1}$ converging to some $z^* \in \overline{B}(c,r)$. Letting $n \to \infty$ in

$$0 = f_{k_n}(z_{k_n}) = k_n^{-1}(z_{k_n} - c) + f(z_{k_n}), \quad n = 1, 2, \cdots,$$

we deduce that $0 = f(z^*)$.

The following version of Brouwer fixed point theorem on a ball, first introduced for continuous mappings in $\mathbb{R}^n$ by Birkhoff and Kellog [2] in 1922, is a direct consequence of Theorem 2.1.

Corollary 2.2 *If $\Omega \supset \overline{B}(c,r)$, any function $h : \Omega \to \mathbb{C}$ is holomorphic on Ω, such that $h(\partial B(c,r)) \subset \overline{B}(c,r)$ has at least one fixed point in $\overline{B}(c,r)$.*

Proof It is essentially the one given in [8] to prove Brouwer fixed point theorem for continuous mappings in dimension n. For each $z \in \partial B(c,r)$, one has

$$\begin{aligned}\Re\{(\overline{z} - \overline{c})[z - h(z)]\} &= |z - c|^2 - \Re[(\overline{z} - \overline{c})(c - h(z))] \\ &\geq |z - c|^2 - |z - c||h(z) - c| \\ &= r^2 - |z - c||h(z) - c| \\ &\geq r^2 - r^2 = 0.\end{aligned}$$

The result follows from Theorem 2.1 applied to $f(z) = z - h(z)$.

Example 2.1 For any integer $m \geq 1$, the mapping h defined by $h(z) = \frac{z}{2}(z^m + 1)$ has at least one fixed point in $\overline{B}(0,1)$. Indeed, if $|z| = 1$,

$$|h(z)| \leq \frac{|z|}{2}(|z|^m + 1) \leq 1.$$

There is no uniqueness as h has the two fixed points 0 and 1 in $\overline{B}(0,1)$.

2.3 Poincaré-Miranda's conditions on a rectangle

Let $a < b,\ c < d$,

$$P = \{z \in \mathbb{C} : \Re z \in (a,b) \text{ and } \Im z \in (c,d)\}$$

be an open rectangle in $\mathbb{C}$, and let

$$P_a = \{a + \mathrm{i}y : y \in [c,d]\}, \quad P_b = \{b + \mathrm{i}y : y \in [c,d]\}$$

and

$$P^c = \{x + \mathrm{i}c : x \in [a,b]\}, \quad P^d = \{x + \mathrm{i}d : x \in [a,b]\}$$

be the opposite vertical and horizontal sides of $\overline{P}$, respectively.

Theorem 2.2 *If $f : \Omega \to \mathbb{C}$ is holomorphic on some domain $\Omega \supset \overline{P}$ and if*
(i) $\Re f(z) \leq 0$ *for all* $z \in P_a$, $\Re f(z) \geq 0$ *for all* $z \in P_b$,
(ii) $\Im f(z) \leq 0$ *for all* $z \in P^c$, $\Im f(z) \geq 0$ *for all* $z \in P^d$,
then f has at least one zero in $\overline{P}$.

Proof Let w be the center of P defined by

$$w = \frac{1}{2}[(a+b) + \mathrm{i}(c+d)]. \tag{2.8}$$

For each positive integer k, it is easy to check that the function f_k defined by

$$f_k(z) = k^{-1}(z-w) + f(z), \quad z \in \overline{P}$$

is such that $\Re f_k(z) < 0$ for $z \in P_a$, $\Re f_k(z) > 0$ for $z \in P_b$, $\Im f_k(z) < 0$ for $z \in P^c$, and $\Im f_k(z) < 0$ for $z \in P^d$, so that $f_k(z) \neq 0$ for each $z \in \partial P$. Let $\rho : [0,4] \to \Omega$ be the piecewise C^2 cycle defined by

$$\rho(t) = \begin{cases} a + t(b-a) + \mathrm{i}c, & \text{if } t \in [0,1], \\ b + \mathrm{i}[c + (t-1)(d-c)], & \text{if } t \in [1,2], \\ b + (t-2)(a-b) + \mathrm{i}d, & \text{if } t \in [2,3], \\ a + \mathrm{i}[d + (t-3)(c-d)], & \text{if } t \in [3,4], \end{cases} \tag{2.9}$$

whose image $\rho([0,4]) = \partial P$. Then,

$$\begin{aligned}
\Im\Big[\int_\rho \frac{\mathrm{d}z}{f_k(z)}\Big] &= \Im\Big\{\int_\rho |f_k(z)|^{-2}[\Re f_k(z) - i\Im f_k(z)]\,\mathrm{d}z\Big\} \\
&= \int_\rho |f_k(z)|^{-2}[-\Im f_k(z)\,\mathrm{d}x + \Re f_k(z)\,\mathrm{d}y] \\
&= -\int_0^1 |f_k(\rho(t))|^{-2}\Im f_k[\rho(t)](b-a)\,\mathrm{d}t + \int_1^2 |f_k(\rho(t))|^{-2}\Re f_k[\rho(t)](d-c)\,\mathrm{d}t \\
&\quad - \int_2^3 |f_k(\rho(t))|^{-2}\Im f_k[\rho(t)](a-b)\,\mathrm{d}t + \int_3^4 |f_k(\rho(t))|^{-2}\Re f_k[\rho(t)](c-d)\,\mathrm{d}t \\
&= -\int_a^b |f_k(s+\mathrm{i}c)|^{-2}\Im f_k(s+\mathrm{i}c) + \int_c^d |f_k(b+\mathrm{i}t)|^{-2}\Re f_k(b+\mathrm{i}t)\,\mathrm{d}s \\
&\quad + \int_a^b |f_k(s+\mathrm{i}d)|^{-2}\Im f_k(s+\mathrm{i}d)\,\mathrm{d}t - \int_c^d |f_k(a+\mathrm{i}s)|^{-2}\Re f_k(a+\mathrm{i}s)\,\mathrm{d}s \\
&= \int_a^b [-|f_k(s+\mathrm{i}c)|^{-2}\Im f_k(s+\mathrm{i}c) + |f_k(s+\mathrm{i}d)|^2\Im f_k(s+\mathrm{i}d)]\,\mathrm{d}s \\
&\quad + \int_c^d [|f_k(b+\mathrm{i}s)|^{-2}\Re f_k(b+\mathrm{i}s) - |f_k(a+\mathrm{i}s)|^{-2}\Re f_k(a+\mathrm{i}s)]\,\mathrm{d}s > 0,
\end{aligned}$$

using the assumptions (i)–(ii). For each integer $k \geq 1$, Proposition 2.3 implies the existence of some $z_k \in P$ such that

$$k^{-1}(z_k - w) + f(z_k) = 0.$$

Using Bolzano-Weierstrass theorem, a subsequence $(z_{k_n})_{n\geq 1}$ of the sequence $(z_k)_{k\geq 1}$ converges to some $z^* \in \overline{P}$. Hence

$$0 = \lim_{n\to\infty} f(z_{k_n}) = \lim_{n\to\infty} [k_n^{-1}(z_{k_n} - w) + f(z_{k_n})] = f(z^*).$$

Example 2.2 Let the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = z^3 + 4z + 1 + \mathrm{i}, \quad z \in \mathbb{C},$$

so that

$$\Re f(x+\mathrm{i}y) = x^3 - 3xy^2 + 4x + 1, \quad \Im f(x+\mathrm{i}y) = 3x^2y - y^3 + 4y + 1.$$

Taking $P = \{z \in \mathbb{C} : \Re z \in (-1,1) \text{ and } \Im z \in (-1,1)\}$, one has

$$\begin{aligned} z \in P_{-1} &\Rightarrow \Re f(z) = -4 + 3y^2 < 0, \quad z \in P_1 \Rightarrow \Re f(z) = 6 - 3y^2 > 0, \\ z \in P^{-1} &\Rightarrow \Im f(z) = -3x^2 - 2 < 0, \quad z \in P^1 \Rightarrow \Im f(z) = 3x^2 + 4 > 0, \end{aligned}$$

and f satisfies the assumptions (i)–(ii) of Theorem 2.2. Thus f has a zero in $[-1,1] \times [-1,1]$.

Corollary 2.3 *If f is holomorphic on some domain $\Omega \supset \overline{P}$, and if*

$$\begin{aligned} &\Re(z-w) \cdot \Re f(z) \geq 0 \quad \text{for all } z \in P_a \cup P_b, \\ &\Im(z-w) \cdot \Im f(z) \geq 0 \quad \text{for all } z \in P^c \cup P^d, \end{aligned} \tag{2.10}$$

then f has at least one zero in $\overline{P}$.

Proof For all $z \in P_a$, $\Re(z-w) < 0$ and for all $z \in P_b$, $\Re(z-w) > 0$, so that the first condition in (2.10) implies the assumption (i) of Theorem 2.2. Similarly, the second condition in (2.10) implies the assumption (ii) of Theorem 2.2.

Remark 2.1 In the case of P, Hadamard-Shih's condition can be written

$$\Re(z-w) \cdot \Re f(z) + \Im(z-w) \cdot \Im f(z) > 0 \quad \text{for all } z \in \partial P, \tag{2.11}$$

which shows that the Hadamard-Shih's and Poincaré-Miranda's conditions are independent.

For a real function of a real variable, Bolzano's theorem implies the intermediate value property. A similar result holds in the complex case. If $f : \Omega \to \mathbb{C}$ satisfies the assumptions of Theorem 2.2, define

$$R := [\max_{P_a} \Re f, \min_{P_b} \Re f] \times [\max_{P^c} \Im f, \min_{P^d} \Im f] \subset \mathbb{C}. \tag{2.12}$$

Because of the assumptions (i)–(ii) of Theorem 2.2, $0 \in R$.

Corollary 2.4 *If $f : \Omega \to \mathbb{C}$ satisfies the assumptions of Theorem* 2.2*, then $f(P) \supset R$, with R defined in* (2.12).

Proof For $v \in R$ given, let $g : \Omega \to \mathbb{C}$ be defined by $g(z) = f(z) - v$ $(z \in \Omega)$. Then, for $z \in P_a$, we have

$$\Re g(z) = \Re f(z) - \Re v \leq \Re f(z) - \max_{P_a} \Re f \leq 0.$$

Similarly, $\Re g$ and $\Im g$ satisfy the other inequalities in the assumptions (i)–(ii) of Theorem 2.2, and the existence of at least one zero of g in P follows.

Example 2.3 Corollary 2.4 and the computations of Example 2.2 imply that, for $f(z) = z^3 + 4z + 1 + \mathrm{i}$, we have, with $P = \{z \in \mathbb{C} : \Re z \in (-1,1) \text{ and } \Im z \in (-1,1)\}$ and $R = \{z \in \mathbb{C} : \Re z \in (-1,3) \text{ and } \Im z \in (-2,4)\}$, $f(P) \supset R$.

The following version of Brouwer fixed point theorem on a rectangle is a direct consequence of Theorem 2.2.

Corollary 2.5 *If $\Omega \supset \overline{P}$, any function $h : \Omega \to \mathbb{C}$ is holomorphic on Ω, such that $h(\partial P) \subset \overline{P}$ has at least one fixed point in P.*

Proof Define $f : \Omega \to \mathbb{C}$ by $f(z) = z - h(z)$ for all $z \in \Omega$. The assumption $h(\partial P) \subset \overline{P}$ is equivalent to

$$a \leq \Re h(z) \leq b, \quad c \leq \Im h(z) \leq d \quad \text{for all } z \in \partial P,$$

and hence, if $z \in P_a$, $\Re f(z) = a - \Re h(z) \leq 0$, if $z \in P_b$, $\Re f(z) = b - \Re h(z) \geq 0$, if $z \in P^c$, $\Im f(z) = c - \Im h(z) \leq 0$, and if $z \in P^d$, $\Im f(z) = d - \Im h(z) \geq 0$. Thus, the assumptions (a) and (b) of Theorem 2.2 are satisfied, f has a zero in $\overline{P}$, and h has a fixed point in $\overline{P}$.

3 Uniquenesss of Zeros of Holomorphic Functions

3.1 Zeros of holomorphic functions

This section will rely upon more sophisticated properties of holomorphic functions than the ones used in Section 2. They can be found for example in [1]. Let $\Omega \subset \mathbb{C}$ be a domain, and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω.

Definition 3.1 *A point $a \in \mathbb{C}$ such that*

$$0 = f(a) = f'(a) = \cdots = f^{(m-1)}(a), \quad f^{(m)}(a) \neq 0$$

for some integer $m \geq 1$, is called a zero of multiplicity m of f.

It follows easily from Taylor's expansion of f at a that if f is not identically zero on Ω, then its zeros are isolated.

Definition 3.2 *A piecewise C^1-cycle $\gamma : [a, b] \to \Omega$ is said to bound a domain $\Delta \subset \Omega$, if $\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z-c}$ is defined and equal to one for all $c \in \Delta$ and either undefined or equal to zero for all $c \notin \Delta$.*

The following result, which goes back to Cauchy in one of his last notes to the Comptes rendus (see [5]), is called the argument principle.

Proposition 3.1 *Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to \mathbb{C}$ be holomorphic, Δ be a bounded domain such that $\overline{\Delta} \subset \Omega$, and γ be a piecewise C^1 cycle which bounds Δ. Then if $f(z) \neq 0$ on $\gamma([a, b])$, f has at most a finite number of zeros $a_1, \cdots, a_p$ in Δ and*

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{j=1}^{p} m_j, \tag{3.1}$$

where m_j denotes the multiplicity of a_j $(j = 1, \cdots, p)$.

Finally, let $F : \Omega \times [0, 1] \to \mathbb{C}$ be a continuous function, such that $F(z, \cdot)$ is of class C^1 on $[0, 1]$ for each $z \in \Omega$, $F(\cdot, \lambda)$ is analytic on Ω for each $\lambda \in [0, 1]$, and let $\gamma : [a, b] \to \Omega$ be a piecewise C^1-cycle.

Proposition 3.2 *If $F(z, \lambda) \neq 0$ for each $(z, \lambda) \in \gamma([a, b]) \times [0, 1]$, the mapping*

$$\lambda \mapsto \int_\gamma \frac{\partial_z F(z, \lambda)}{F(z, \lambda)} \,\mathrm{d}z$$

is constant on $[0, 1]$.

Proof We have, for each $(z,\lambda) \in \gamma([a,b]) \times [0,1]$,

$$\begin{aligned}\partial_\lambda\Big[\frac{\partial_z F(z,\lambda)}{F(z,\lambda)}\Big] &= \frac{\partial_\lambda\partial_z F(z,\lambda)F(z,\lambda) - \partial_z F(z,\lambda)\partial_\lambda F(z,\lambda)}{F^2(z,\lambda)}\\ &= \frac{\partial_z\partial_\lambda F(z,\lambda)F(z,\lambda) - \partial_\lambda F(z,\lambda)\partial_z F(z,\lambda)}{F^2(z,\lambda)}\\ &= \partial_z\Big[\frac{\partial_\lambda F(z,\lambda)}{F(z,\lambda)}\Big].\end{aligned}$$

Hence, with differentiation under the integral sign easily justified,

$$\begin{aligned}\partial_\lambda \int_\gamma \frac{\partial_z F(z,\lambda)}{F(z,\lambda)}\,\mathrm{d}z &= \int_\gamma \partial_\lambda\Big[\frac{\partial_z F(z,\lambda)}{F(z,\lambda)}\Big]\,\mathrm{d}z = \int_\gamma \partial_z\Big[\frac{\partial_\lambda F(z,\lambda)}{F(z,\lambda)}\Big]\,\mathrm{d}z\\ &= \int_a^b \partial_z\Big[\frac{\partial_\lambda F(\gamma(t),\lambda)}{F(\gamma(t),\lambda)}\Big]\gamma'(t)\,\mathrm{d}t\\ &= \int_a^b \partial_t\Big[\frac{\partial_\lambda F(\gamma(t),\lambda)}{F(\gamma(t),\lambda)}\Big]\,\mathrm{d}t\\ &= \frac{\partial_\lambda F(\gamma(b),\lambda)}{F(\gamma(b),\lambda)} - \frac{\partial_\lambda F(\gamma(a),\lambda)}{F(\gamma(a),\lambda)} = 0.\end{aligned}$$

3.2 Strict Hadamard-Shih's condition on the boundary of a bounded domain

Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to \mathbb{C}$ be holomorphic, Δ be a bounded domain such that $\overline{\Delta} \subset \Omega$, and γ be a piecewise C^1-cycle which bounds Δ.

Lemma 3.1 *If for some $c \in \Delta$, $\Re[(\overline{z} - \overline{c})f(z)] > 0$ for each $z \in \partial\Delta$, then*

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 1.$$

Proof Define $F : \Omega \times [0,1] \to \mathbb{C}$ by

$$F(z,\lambda) = (1-\lambda)(z-c) + \lambda f(z).$$

Then F satisfies the regularity conditions of Proposition 3.2 and, for each $(z,\lambda) \in \partial\Delta \times [0,1]$,

$$\begin{aligned}\Re[(\overline{z}-\overline{c})F(z,\lambda)] &= \Re[(1-\lambda)|z-c|^2 + \lambda(\overline{z}-\overline{c})f(z)]\\ &= (1-\lambda)|z-c|^2 + \lambda\Re[(\overline{z}-\overline{c})f(z)] > 0,\end{aligned}$$

so that $F(z,\lambda) \neq 0$. By Propositions 3.2 and 2.1,

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\partial_z F(z,1)}{F(z,1)}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\partial_z F(z,0)}{F(z,0)}\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}z}{z-c} = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_{c,r}} \frac{\mathrm{d}z}{z-c} = 1,\end{aligned}$$

where $r > 0$ is sufficiently small, so that $\overline{B}(c,r) \subset \Delta$ and $\gamma_{c,r}$ is defined in (2.7).

Theorem 3.1 *If there is some $c \in \Delta$ such that $\Re[(\overline{z}-\overline{c})f(z)] > 0$ for each $z \in \partial\Delta$, then f has a unique zero in Δ and the zero is simple.*

Proof By Proposition 3.1 and Lemma 3.1,

$$\sum_{j=1}^{p} m_j = 1,$$

so that $p = 1$ and $m_1 = 1$.

Like in the case of non-strict conditions, one deduces from Theorem 3.1 the following version of Brouwer fixed point theorem.

Corollary 3.1 *If $h : \Omega \to \mathbb{C}$ is holomorphic and $h(\partial\Delta) \subset \Delta$, then h has a unique fixed point in Δ.*

3.3 Strict Poincaré-Miranda's condition on the boundary of a rectangle

Let P, P_a, P_b, P^c, P^d be defined in the beginning of Subsection 2.3, w be the center of P defined in (2.8), and ρ be the cycle defined in (2.9).

Lemma 3.2 *If f is holomorphic on $\Omega \supset \overline{P}$ and if*
(i) $\Re f(z) < 0$ *for all* $z \in P_a$, $\Re f(z) > 0$ *for all* $z \in P_b$,
(ii) $\Im f(z) < 0$ *for all* $z \in P^c$, $\Im f(z) > 0$ *for all* $z \in P^d$,
then

$$\frac{1}{2\pi \mathrm{i}} \int_\rho \frac{f'(z)}{f(z)} \, \mathrm{d}z = 1.$$

Proof Define $F : \Omega \times [0,1] \to \mathbb{C}$ by

$$F(z,\lambda) = (1-\lambda)(z-w) + \lambda f(z).$$

Then F satisfies the regularity conditions of Proposition 3.2. Now, for all $\lambda \in [0,1]$, we have

$$\begin{aligned}
\Re F(z,\lambda) &= (1-\lambda)\frac{a-b}{2} + \lambda \Re f(z) < 0, \quad \forall\, z \in P_a, \\
\Re F(z,\lambda) &= (1-\lambda)\frac{b-a}{2} + \lambda \Re f(z) > 0, \quad \forall\, z \in P_b, \\
\Im F(z,\lambda) &= (1-\lambda)\frac{c-d}{2} + \lambda \Im f(z) < 0, \quad \forall\, z \in P^c, \\
\Im F(z,\lambda) &= (1-\lambda)\frac{d-c}{2} + \lambda \Im f(z) > 0, \quad \forall\, z \in P^d.
\end{aligned}$$

Hence, $F(z,\lambda) \neq 0$ for all $(z,\lambda) \in \partial P \times [0,1]$, and, by Proposition 3.2,

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_\rho \frac{f'(z)}{f(z)} \, \mathrm{d}z &= \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{\partial_z F(z,1)}{F(z,1)} \, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{\partial_z F(z,0)}{F(z,0)} \, \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{\mathrm{d}z}{z-w} \, \mathrm{d}z.
\end{aligned} \tag{3.2}$$

Using Proposition 2.1, if $r > [(b-a)^2 + (d-c)^2]^{\frac{1}{2}}$, we have, with $\gamma_{w,r}$ defined in (2.7),

$$\frac{1}{2\pi \mathrm{i}} \int_\rho \frac{\mathrm{d}z}{z-w} = \frac{1}{2\pi \mathrm{i}} \int_{\partial \gamma_{w,r}} \frac{\mathrm{d}z}{z-w} = 1, \tag{3.3}$$

and the result follows from (3.2)–(3.3).

Theorem 3.2 *If*
(i) $\Re f(z) < 0$ *for all* $z \in P_a$, $\Re f(z) > 0$ *for all* $z \in P_b$,
(ii) $\Im f(z) < 0$ *for all* $z \in P^c$, $\Im f(z) > 0$ *for all* $z \in P^d$,
then f *has a unique zero in* P, *and this zero is simple.*

Proof It is similar to the proof of Theorem 3.1 with Lemma 3.1 replaced by Lemma 3.2.

Example 3.1 If $P = \{z \in \mathbb{C} : \Re z \in (0,1) \text{ and } \Im z \in (-1,+1)\}$ and $h(z) = \mathrm{e}^{-z}$ $(z \in \mathbb{C})$, then

$$\Re h(z) = \mathrm{e}^{-x}\cos y, \quad \Im h(z) = -\mathrm{e}^{-x}\sin y,$$

and

$$\begin{aligned}
z \in P_0 &\Rightarrow \cos 1 \le \Re h(z) \le 1, \quad -\sin 1 \le \Im h(z) \le \sin 1,\\
z \in P_1 &\Rightarrow \mathrm{e}^{-1}\cos 1 \le \Re h(z) \le \mathrm{e}^{-1}, \quad -\mathrm{e}^{-1}\sin 1 \le \Im h(z) \le \mathrm{e}^{-1}\sin 1,\\
z \in P^{-1} &\Rightarrow \mathrm{e}^{-1}\cos 1 \le \Re h(z) \le \cos 1, \quad \mathrm{e}^{-1}\sin 1 \le \Im h(z) \le \sin 1,\\
z \in P^{+1} &\Rightarrow \mathrm{e}^{-1}\cos 1 \le \Re h(z) \le \cos 1, \quad -\sin 1 \le \Im h(z) \le -\mathrm{e}^{-1}\sin 1.
\end{aligned}$$

Hence $h(\partial P) \subset P$ and there exists a unique $z \in P$ such that $z = \mathrm{e}^{-z}$.

Like in the case of non-strict conditions, one deduces from Theorem 3.2 a corresponding Brouwer fixed point theorem.

Corollary 3.2 *If* $h : \Omega \to \mathbb{C}$ *is holomorphic on* $\Omega \supset \overline{P}$ *such that* $h(\partial P) \subset P$, *then* h *has a unique fixed point in* P.

Remark 3.1 Because one has $h(\partial P) \in P$ in Example 3.1, the fixed point is unique in P.

4 Holomorphic Mappings in $\mathbb{C}^n$

4.1 Brouwer degree

Let us now consider holomorphic mappings $f : \Omega \to \mathbb{C}^n$, with components $f_1, \cdots, f_n$, where $\Omega \subset \mathbb{C}^n$ is open. By the definition in [6], they are such that, for each $a \in \Omega$, one can find a $\mathbb{C}$-linear mapping $L : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\lim_{z \to a} \frac{\|f(z) - f(a) - L(z-a)\|}{\|z-a\|} = 0.$$

This can be shown to be equivalent for f to be differentiable on Ω such that

$$\begin{aligned}
&z = (z_1, \cdots, z_n) = (x_1 + \mathrm{i}y_1, \cdots, x_n + \mathrm{i}y_n),\\
&\partial_{x_j} f_k = \frac{1}{\mathrm{i}} \partial_{y_j} f_k (:= \partial_{z_j} f_k), \quad j,k = 1,2,\cdots,n.
\end{aligned}$$

When $n \ge 1$, it seems difficult to use the n-dimensional extensions of Cauchy theorem like in Section 2, and hence the approach of Section 3 is extended. To this effect, we use Brouwer degree for holomorphic mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ considered as mappings from $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ (see, e.g., [10] for a definition). The following result is proved in an elegant way in [14].

Proposition 4.1 *If* $\Omega \subset \mathbb{C}$ *is open,* $f : \Omega \to \mathbb{C}^n$ *is holomorphic on* Ω, D *is open, bounded, with* $\overline{D} \subset \Omega$, *and* $0 \notin f(\partial D)$, *the Brouwer degree* $d_B[f, D, 0]$ *is a nonnegative integer such that* $d_B[f, D, 0] > 0$ *if and only if* $0 \in f(D)$.

Those properties are consequences of the definition of Brouwer degree and the fact that if $\widehat{f} : \Omega \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes the map associated to f, $J_f(z)$ denotes its Jacobian at $z \in \Omega$ (in terms of partial derivatives $\partial_{z_j} f_k(z)$), $J_{\widehat{f}}(x)$ denotes its Jacobian at the corresponding $x = (x_1, y_1, \cdots, x_n, y_n)$ (in terms of partial derivatives $\partial_{x_j}\Re f_k(x)$, $\partial_{x_j}\Im f_k(x)$, $\partial_{y_j}\Re f_k(x)$, $\partial_{y_j}\Im f_k(x)$), one has

$$J_{\widehat{f}}(x) = |J_f(z)|^2 \geq 0,$$

and $J_{f+\varepsilon I}(z) \neq 0$ for all sufficiently small $\varepsilon \neq 0$.

Notice that, for $n = 1$ and if a piecewise C^1-cycle γ bounds D,

$$d_B[f, D, 0] = \frac{1}{2\pi} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z,$$

and Proposition 3.1 states that f has in D a finite number of isolated zeros $a_1, \cdots, a_p$ and

$$d_B[f, D, 0] = \sum_{j=1}^{p} m_j \geq p, \tag{4.1}$$

where m_j denotes the multiplicity of a_j $(j = 1, \cdots, p)$. The following extension of this result to holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ is given in [14].

Proposition 4.2 *Under the conditions above for $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ and D, $d_B[f, D, 0]$ is greater or equal to the number of isolated zeros of f in D.*

The following consequence is also proved in [14].

Proposition 4.3 *Under the conditions above for $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ and D, $d_B[f, D, 0] = 1$ if and only if f has a unique zero ζ in D and $J_f(\zeta) \neq 0$.*

4.2 Hadamard-Shih's conditions for holomorphic mappings

The results of the previous section provide a simpler proof of an extension of Bolzano's theorem to holomorphic functions from $\mathbb{C}^n$ into $\mathbb{C}^n$ given by Shih in [16].

Theorem 4.1 *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}^n$ be holomorphic on Ω, D be open and bounded with $\overline{D} \subset \Omega$, and assume that, for some $c \in D$,*

$$\sum_{j=1}^{n} \Re[(\overline{z_j} - \overline{c_j}) f_j(z)] > 0, \quad \forall\, z \in \partial D. \tag{4.2}$$

Then f has a unique zero in D, and this zero is non-degenerate.

Proof Consider the homotopy $H : \overline{D} \times [0, 1] \to \mathbb{C}^n$ defined by

$$H(z, \lambda) = (1 - \lambda)(z - c) + \lambda f(z).$$

By the assumption (4.2), we have

$$H(z, 0) = z - c \neq 0, \quad H(z, 1) = f(z) \neq 0, \quad \forall\, z \in \partial D.$$

Furthermore, $\forall\, (z, \lambda) \in \partial D \times (0, 1)$,

$$\sum_{j=1}^{n} \Re[(\overline{z_j} - \overline{c_j}) H_j(z, \lambda)] = \sum_{j=1}^{n} \{(1 - \lambda)|z_j - c_j|^2 + \lambda \Re[(\overline{z_j} - \overline{c_j}) f_j(z)]\} > 0,$$

and hence $H(z,\lambda)\neq 0$. The homotopy invariance property of Brouwer degree implies that

$$d_B[f,D,0]=d_B[I,D,0]=1,$$

and the result follows from Proposition 4.3.

We still have existence conclusion in Theorem 4.1 under weak inequalities in the assumptions.

Theorem 4.2 *Let $\Omega\subset\mathbb{C}^n$ be open, $f:\Omega\to\mathbb{C}^n$ be holomorphic, D be open, bounded, such that $\overline{D}\subset\Omega$, and assume that , for some $c\in D$,*

$$\sum_{j=1}^{n}\Re[(\overline{z_j}-\overline{c_j})f_j(z)]\geq 0,\quad \forall\, z\in\partial D.$$

Then f has at least one zero in $\overline{D}$.

Proof For each positive integer k, the mapping $f_k:\Omega\to\mathbb{C}^n$ defined by

$$f^k(z)=k^{-1}(z-c)+f(z)$$

satisfies the assumptions of Theorem 4.1. Thus, for each $k\geq 1$, f^k has a unique zero $z^k\in D$. By Bolzano-Weierstrass theorem, there exists a subsequence $(z_{k_n})_{n\in\mathbb{N}}$ converging to some $z^*\in\overline{D}$. Letting $n\to\infty$ in

$${k_n}^{-1}(z_{k_n}-c)+f(z_{k_n})=0,\quad n\in\mathbb{N}$$

gives $f(z^*)=0$.

In the same way as for $n=1$, we can also deduce from Theorems 4.1–4.2, the corresponding versions of Brouwer fixed point theorem.

Corollary 4.1 *Let $\Omega\subset\mathbb{C}^n$ be open, $h:\Omega\to\mathbb{C}^n$ be holomorphic on Ω, and D be open, bounded, convex such that $\overline{D}\subset\Omega$.*

(1) *If $h(\partial D)\subset D$, h has a unique fixed point in D.*

(2) *If $h(\partial D)\subset\overline{D}$, h has at least one fixed point in $\overline{D}$.*

4.3 Poincaré-Miranda's conditions for holomorphic mappings

We now consider the generalization of Theorem 3.2. Let $a_j<b_j,\ c_j<d_j$ $(1\leq j\leq n)$, and define the open set $P\subset\mathbb{C}^n$ by

$$\{z\in\mathbb{C}^n:\Re z_j\in(a_j,b_j),\ \Im z_j\in(c_j,d_j),\quad j=1,\cdots,n\}. \tag{4.3}$$

Theorem 4.3 *Let $\Omega\supset\overline{P}$ be an open subset of $\mathbb{C}^n$ and $f:\Omega\to\mathbb{C}^n$ be holomorphic on Ω, and such that, for $j=1,\cdots,n$,*

$$\begin{aligned}&\Re f_j(z)<0,\quad \forall\, z\in\overline{P},\ \Re z_j=a_j,\quad \Re f_j(z)>0,\quad \forall\, z\in\overline{P},\ \Re z_j=b_j,\\ &\Im f_j(z)<0,\quad \forall\, z\in\overline{P},\ \Im z_j=c_j,\quad \Im f_j(z)>0,\quad \forall\, z\in\overline{P},\ \Im z_j=d_j.\end{aligned} \tag{4.4}$$

Then f has a unique zero in P, and this zero is not degenerate.

Proof Let

$$w=\frac{1}{2}(a_1+b_1+\mathrm{i}(c_1+d_1),\cdots,a_n+b_n+\mathrm{i}(c_n+d_n))$$

be the center of P, and consider the homotopy $H : \overline{P} \times [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$H(z,\lambda) = (1-\lambda)(z-w) + \lambda f(z), \quad z \in \overline{P},\ \lambda \in [0,1].$$

By the assumption (4.4) and construction of w, for each $j = 1, \cdots, n$,

$$\begin{aligned}
&\Re H_j(z,0) < 0, \quad \forall\, z \in \overline{P},\ \Re z_j = a_j, \quad \Re H_j(z,0) > 0, \quad \forall\, z \in \overline{P},\ \Re z_j = b_j,\\
&\Im H_j(z,0) < 0, \quad \forall\, z \in \overline{P},\ \Im z_j = c_j, \quad \Im H_j(z,0) > 0, \quad \forall\, z \in \overline{P},\ \Im z_j = d_j,\\
&\Re H_j(z,1) < 0, \quad \forall\, z \in \overline{P},\ \Re z_j = a_j, \quad \Re H_j(z,1) > 0, \quad \forall\, z \in \overline{P},\ \Re z_j = b_j,\\
&\Im H_j(z,1) < 0, \quad \forall\, z \in \overline{P},\ \Im z_j = c_j, \quad \Im H_j(z,1) > 0, \quad \forall\, z \in \overline{P},\ \Im z_j = d_j,
\end{aligned}$$

and, for all $\lambda \in (0,1)$,

$$\begin{aligned}
&\Re H_j(z,\lambda) < 0, \quad \forall\, z \in \overline{P},\ \Re z_j = a_j, \quad \Re H_j(z,\lambda) > 0, \quad \forall\, z \in \overline{P},\ \Re z_j = b_j,\\
&\Im H_j(z,\lambda) < 0, \quad \forall\, z \in \overline{P},\ \Im z_j = c_j, \quad \Im H_j(z,\lambda) > 0, \quad \forall\, z \in \overline{P},\ \Im z_j = d_j.
\end{aligned}$$

Consequently, $H(z,\lambda) \neq 0,\ \forall\,(z,t) \in \partial P \times [0,1]$. The homotopy invariance of Brouwer degree implies that

$$d_B[f,P,0] = d_B[I-w,P,0] = 1,$$

and the result follows from Proposition 4.3.

We can deduce from Theorem 4.3 an "intermediate value property" for f on P. Define, for $j = 1, \cdots, n$,

$$\begin{aligned}
A_j &= \max_{z\in P, \Re z_j = a_j} \Re f_j, \quad B_j = \min_{z\in P, \Re z_j = b_j} \Re f_j,\\
C_j &= \max_{z\in P, \Im z_j = c_j} \Im f_j, \quad D_j = \min_{z\in P, \Im z_j = d_j} \Im f_j,
\end{aligned}$$

and let

$$Q = \{w \in \mathbb{C}^n : \Re w_j \in (A_j, B_j),\ \Im w_j \in (C_j, D_j), \quad j = 1, \cdots, n\}. \tag{4.5}$$

Theorem 4.4 *Under the assumptions of Theorem* 4.3*, for each $w \in Q$, equation $f(z) = w$ has a unique solution in P. Furthermore, $f : f^{-1}(Q) \to Q$ is a bi-holomorphic homeomorphism.*

Proof The open set Q is well defined because, by the assumption 4.4, $A_j < 0 < B_j$, $C_j < 0 < D_j$ for each $j = 1, \cdots, n$. The existence of a unique solution ζ in P of equation $f(z) = w$ follows easily from Theorem 4.3 applied to the holomorphic mapping $g(z) = f(z) - w$. Furthermore, its proof implies that $d_B[g,P,0] = 1$, which implies, using [14, Theorem 3], that the Jacobian $J_g(\zeta) \neq 0$. Hence, the result follows from the implicit function theorem for holomorphic mappings.

Proceeding like in the proof of Theorem 4.2 with c replaced by w, we still have existence results in Theorem 4.3 under weak inequalities in the assumptions.

Theorem 4.5 *Let $\Omega \supset \overline{P}$ be an open subset of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}^n$ be holomorphic such that, for $j = 1, \cdots, n$,*

$$\begin{aligned}
&\Re f_j(z) \le 0, \quad \forall\, z \in \overline{P},\ \Re z_j = a_j, \quad \Re f_j(z) \ge 0, \quad \forall\, z \in \overline{P},\ \Re z_j = b_j,\\
&\Im f_j(z) \le 0, \quad \forall\, z \in \overline{P},\ \Im z_j = c_j, \quad \Im f_j(z) \ge 0, \quad \forall\, z \in \overline{P},\ \Im z_j = d_j.
\end{aligned}$$

Then f has at least one zero in $\overline{P}$.

Theorem 4.5 implies, in a similar way as for Theorem 4.4, an intermediate value property.

Theorem 4.6 *Under the assumptions of Theorem* 4.5*, for each* $w \in \overline{Q}$*, with* Q *defined in* (4.5)*, equation* $f(z) = w$ *has at least one solution in* $\overline{P}$*.*

In the same way as for $n = 1$, we can also deduce from Theorems 4.3 and 4.5 the corresponding versions of Brouwer fixed point theorem.

Corollary 4.2 *Let* P *be given by* (4.3)*,* $\Omega \supset \overline{P}$ *be an open subset of* $\mathbb{C}^n$*, and* $h : \Omega \to \mathbb{C}^n$ *be a holomorphic mapping.*

(1) *If* $h(\partial P) \subset P$*,* h *has a unique fixed point in* P*.*

(2) *If* $h(\partial P) \subset \overline{P}$*,* h *has at least one fixed point in* $\overline{P}$*.*

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