Chin. Ann. Math.
38B(2), 2017, 591–600
DOI: 10.1007/s11401-017-1085-6

Chinese Annals of Mathematics, Series B

Singular Solutions to Conformal Hessian Equations

Nikolai NADIRASHVILI[1] Serge VLĂDUŢ[2]

(*Dedicated to Professor Haim Brezis on the occasion of his* 70*th birthday*)

Abstract The authors show that for any $\varepsilon \in]0,1[$, there exists an analytic outside zero solution to a uniformly elliptic conformal Hessian equation in a ball $B \subset \mathbb{R}^5$ which belongs to $C^{1,\varepsilon}(B) \setminus C^{1,\varepsilon+}(B)$.

Keywords Viscosity solutions, Conformal Hessian equation, Cartan's cubic
2000 MR Subject Classification 35J60, 53C38

1 Introduction

In this paper, we study a class of fully nonlinear second-order elliptic equations of the form

$$F(D^2u, Du, u) = 0 \tag{1.1}$$

defined in a domain of $\mathbb{R}^n$. Here D^2u denotes the Hessian of the function u, with Du being its gradient. We assume that F is a Lipschitz function defined on a domain in the space $\mathrm{Sym}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$, with $\mathrm{Sym}_2(\mathbb{R}^n)$ being the space of $n \times n$ symmetric matrices, and that F satisfies the uniform ellipticity condition, i.e., there exists a constant $C = C(F) \geq 1$ (called an ellipticity constant), such that

$$C^{-1}\|N\| \leq F(M+N) - F(M) \leq C\|N\|$$

for any non-negative definite symmetric matrix N. If $F \in C^1(\mathrm{Sym}_2(\mathbb{R}^n))$, then this condition is equivalent to

$$\frac{1}{C'}|\xi|^2 \leq F_{u_{ij}}\xi_i\xi_j \leq C'|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Here, u_{ij} denotes the partial derivative $\frac{\partial^2 u}{\partial x_i \partial x_j}$. A function u is called a classical solution to (1.1) if $u \in C^2(\Omega)$ and u satisfies (1.1). Actually, any classical solution to (1.1) is a smooth $C^{\alpha+1}$-solution, provided that F is a smooth C^α function of its arguments.

More precisely, we are interested in conformal Hessian equations (see, e.g., [9, pp. 5–6]), i.e., those of the form

$$F[u] := f(\lambda(A^u)) = \psi(u, x), \tag{1.2}$$

Manuscript received December 1, 2014. Revised August 11, 2015.
[1]Aix Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France.
E-mail: nicolas@cmi.univ-mrs.fr
[2]Aix Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France; IITP RAS, B. Karetnyi, 9, Moscow, Russia. E-mail: vladuts@univ-amu.fr

with f being a function on $\mathbb{R}^n$ invariant under permutations of the coordinates, and

$$\lambda(A^u) = (\lambda_1, \cdots, \lambda_n)$$

being the eigenvalues of the conformal Hessian in $\mathbb{R}^n$:

$$A^u := uD^2u - \frac{1}{2}|Du|^2 I_n, \tag{1.3}$$

where $n \geq 3, u > 0$. In fact, in our setting the function $\psi(u, x)$ is identically 0.

If F has this form, it is invariant under conformal mappings $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, i.e., transformations which preserve angles between curves. In contrast to the case $n = 2$, for $n \geq 3$, any conformal transformation of $\mathbb{R}^n$ is decomposed into a finitely many family of Möbius transformations, that is, mappings of the form

$$Tx = y + \frac{kA(x-z)}{|x-z|^a}$$

with $x, z \in \mathbb{R}^n$, $k \in \mathbb{R}$, $a \in \{0, 2\}$ and an orthogonal matrix A. In other words, each T is a composition of a translation, a homothety, a rotation and (may be) an inversion. If T is a conformal mapping and $v(x) = J_T^{-\frac{1}{n}} u(Tx)$, where J_T denotes the Jacobian determinant of T, then $F[v] = F[u]$.

We are interested in the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u) = 0, \quad u > 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

Consider the problem of existence and regularity of solutions to the Dirichlet problem (1.4) which has always a unique viscosity (weak) solution for fully nonlinear elliptic equations. The viscosity solutions satisfy the equation (1.1) in a weak sense, and the best known interior regularity (see [1–2, 8]) for them is $C^{1+\varepsilon}$ for some $\varepsilon > 0$. For more details, see [2–3]. Recall that in [4], the authors constructed a homogeneous singular viscosity solution in 5 dimensions for Hessian equations of order δ for any $\delta \in]1, 2]$, that is, of any order compatible with the mentioned interior regularity results. In fact, we proved in [4] the following result.

Theorem 1.1 *The function*

$$w_{5,\delta}(x) = \frac{P_5(x)}{|x|^\delta}, \quad \delta \in [1, 2[$$

is a viscosity solution to a uniformly elliptic Hessian equation $F(D^2w) = 0$ *with a smooth functional* F *in a unit ball* $B \subset \mathbb{R}^5$ *for the isoparametric Cartan cubic form*

$$P_5(x) = x_1^3 + \frac{3x_1}{2}(z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2) + \frac{3\sqrt{3}}{2}(x_2z_1^2 - x_2z_2^2 + 2z_1z_2z_3)$$

with $x = (x_1, x_2, z_1, z_2, z_3)$.

It proves the optimality of the interior $C^{1+\varepsilon}$-regularity of viscosity solutions to fully nonlinear equations in 5 and more dimensions.

In this paper, we show that the same singularity result remains true for conformal Hessian equations.

Theorem 1.2 *Let* $\delta = 1 + \varepsilon \in]1, 2[,\ \varepsilon \in]0, 1[$. *The function*

$$u(x) := c + w_{5,\delta}(x) = c + \frac{P_5(x)}{|x|^\delta} = c + \frac{P_5(x)}{|x|^{1+\varepsilon}}$$

is a viscosity solution to a uniformly elliptic conformal Hessian equation (1.1) *in a unit ball* $B \subset \mathbb{R}^5$ *for a sufficiently large positive constant* c ($c = 10^6$ *is sufficient for* $\delta = \frac{3}{2},\ \varepsilon = \frac{1}{2}$).

The idea behind this choice of $u(x)$ is that the conformal Hessian of u has the form cD^2w plus a term which does not depend on c, that is, the conformal Hessian is (relatively) very close to cD^2w for large enough $c > 0$ which permits to use a very precise information on the spectrum of cD^2u obtained in a previous paper (see [4]).

Notice also that the result does not hold for $\delta = 1$, and we do not know how to construct a non-classical $C^{1,1}$-solution to a uniformly elliptic conformal Hessian equation.

The rest of the paper is organized as follows. In Section 2, we recall some necessary preliminary results, and we prove our main result in Section 3. To simplify the notation, we suppose that $\delta = \frac{3}{2}$ in Section 3. For any δ, the proof is along the same line, but more cumbersome. However, we give also some indications for a general δ. In fact, all proofs but one (Lemma 3.4 which is more cumbersome) remain valid for any $\delta \in]1, 2[$. In our proofs of Sections 2–3, we used MAPLE to verify some algebraic identities. However, these calculations of derivatives and eigenvalues do not exceed human capacities and could be verified by a hardworking reader.

2 Preliminary Results

Notation 2.1 For a real symmetric matrix A, we denote by $|A|$ the maximum absolute value of its eigenvalues.

Let u be a strictly positive function on B_1, $u \in C^1(B_1) \cap C^2(B_1 \setminus \{0\})$. Define the map

$$\Lambda : B_1 \setminus \{0\} \longrightarrow \lambda(S) \in \mathbb{R}^n,$$

where $\lambda(S) = \{\lambda_i : \lambda_1 \geq \cdots \geq \lambda_n\} \in \mathbb{R}^n$ is the ordered set of eigenvalues of the conformal Hessian

$$A^u := uD^2u - \frac{1}{2}|Du|^2 I_n\,.$$

Denote Σ_n the permutation group of $\{1, \cdots, n\}$. For any $\sigma \in \Sigma_n$, we denote by T_σ the linear transformation of $\mathbb{R}^n$ given by $x_i \mapsto x_{\sigma(i)},\ i = 1, \cdots, n$.

Let $a, b \in B_1$ and let $\mu_1(a,b) \geq \cdots \geq \mu_n(a,b)$ be the eigenvalues of the difference $A^u(a) - A^u(b)$. The following ellipticity criterion can be proved similarly to Lemma 2.1 of [5]. However, note that in the present setting, one needs the positivity of u which we suppose everywhere below.

Lemma 2.1 *Suppose that the family*

$$\{M(a,b,O) := A^u(a) - O^{-1} \cdot A^u(b) \cdot O\ : a, b \in B_1, O \in \mathrm{O}(n)\} \setminus \{0\}$$

is uniformly hyperbolic, i.e., if $\{\mu_1(a,b,O) \geq \cdots \geq \mu_n(a,b,O)\}$ *is the ordered spectrum of* $M(a,b,O) \neq 0$, *then*

$$C^{-1} \leq -\frac{\mu_1(a,b,O)}{\mu_n(a,b,O)} \leq C, \quad \forall a, b \in B_1,\ \forall O \in \mathrm{O}(n)$$

for some constant $C > 1$. Then w is a viscosity solution in B_1 to a uniformly elliptic conformal Hessian equation (1).

We recall then some properties of the function $w := w_{5,\delta}(x) = \frac{P_5(x)}{|x|^\delta}$, and of its Hessian $D^2 w$ proved in [4].

Lemma 2.2 *There exists a 3-dimensional Lie subgroup G_P of* SO(5)*, such that P is invarant under its natural action and the orbit $G_P\mathbb{S}_1^1$ of the circle*

$$\mathbb{S}_1^1 = \{(\cos(\chi), 0, 0, \sin(\chi), 0) : \chi \in \mathbb{R}\} \subset \mathbb{S}_1^4$$

under this action is the whole $\mathbb{S}_1^4$.

This result permits to parametrize the values of $w_{5,\delta}(x)$ and the spectrum $\mathrm{Spec}(D^2 w_{5,\delta}(x))$ by a single number $p \in [0,1]$, where x lies in the orbit of $(p, 0, 0, \sqrt{1-p^2}, 0) \in \mathbb{S}_1^1$.

Lemma 2.3. (i) *Let $x \in \mathbb{S}_1^4$, and let $x \in G_P(p, 0, 0, r, 0)$ with $p^2 + r^2 = 1$.*
Then $w_{5,\delta}(x) = \frac{P(x) = p(3-p^2)}{2}$ and

$$\mathrm{Spec}(D^2 w_{5,\delta}(x)) = \{\mu_{1,\delta}, \mu_{2,\delta}, \mu_{3,\delta}, \mu_{4,\delta}, \mu_{5,\delta}\}$$

for

$$\begin{aligned}
\mu_{1,\delta} &= \frac{p(p^2\delta + 6 - 3\delta)}{2}, \\
\mu_{2,\delta} &= \frac{p(p^2\delta - 3 - 3\delta) + 3\sqrt{12 - 3p^2}}{2}, \\
\mu_{3,\delta} &= \frac{p(p^2\delta - 3 - 3\delta) - 3\sqrt{12 - 3p^2}}{2}, \\
\mu_{4,\delta} &= -\frac{p\delta(6-\delta)(3-p^2) + \sqrt{D(p,\delta)}}{4}, \\
\mu_{5,\delta} &= -\frac{p\delta(6-\delta)(3-p^2) - \sqrt{D(p,\delta)}}{4},
\end{aligned}$$

where

$$D(p,\delta) := (6-\delta)(4-\delta)(2-\delta)\delta(p^2-3)^2 p^2 + 144(\delta-2)^2 \geq 144(\delta-2)^2 > 0.$$

(ii) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_5$ be the ordered eigenvalues of $D^2 w_{5,\delta}(x)$. Then*

$$\begin{aligned}
\lambda_1 &= \mu_{2,\delta}, \quad \lambda_5 = \mu_{3,\delta}, \\
\lambda_2 &= \begin{cases} \mu_{4,\delta} & \text{for } p \in [-1, p_0(\delta)], \\ \mu_{1,\delta} & \text{for } p \in [p_0(\delta), 1], \end{cases} \\
\lambda_3 &= \begin{cases} \mu_{5,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\ \mu_{1,\delta} & \text{for } p \in [-p_0(\delta), p_0(\delta)], \\ \mu_{4,\delta} & \text{for } p \in [p_0(\delta), 1], \end{cases} \\
\lambda_4 &= \begin{cases} \mu_{1,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\ \mu_{5,\delta} & \text{for } p \in [-p_0(\delta), 1], \end{cases}
\end{aligned}$$

where

$$p_0(\delta) := \frac{3^{\frac14}\sqrt{2-\delta}}{(3 + 2\delta - \delta^2)^{\frac14}} = \frac{3^{\frac14}\sqrt{1-\varepsilon}}{(4-\varepsilon^2)^{\frac14}} \in]0,1[.$$

Remark 2.1 Notice the oddness property of the spectrum:

$$\lambda_{1,\delta}(-p) = -\lambda_{5,\delta}(p), \quad \lambda_{2,\delta}(-p) = -\lambda_{4,\delta}(p), \quad \lambda_{3,\delta}(-p) = -\lambda_{3,\delta}(p).$$

Proposition 2.1 *Let $N_\delta(x) = D^2 w_\delta(x),\ 1 \le \delta < 2$. Suppose that $a \ne b \in B_1 \setminus \{0\}$, and let $O \in O(5)$ be an orthogonal matrix s.t.*

$$N_\delta(a, b, O) := N_\delta(a) - {}^t O \cdot N_\delta(b) \cdot O \ne 0.$$

Denote $\Lambda_1 \ge \Lambda_2 \ge \cdots \ge \Lambda_5$ the eigenvalues of the matrix $N_\delta(a, b, O)$. Then

$$\frac{1}{C} \le -\frac{\Lambda_1}{\Lambda_5} \le C$$

for $C := C(\delta) := \frac{1000\delta(4-\delta)}{3(2-\delta)^2}$. For $\delta \in [1, \frac{3}{2}]$, one can choose $C = 1000$.

As an immediate consequence we get the following result.

Corollary 2.1 *In the notation of Proposition* 2.1 *we have*

$$\Lambda_1 \ge \frac{|N_\delta(a, b, O)|}{C(\delta)}, \quad |\Lambda_5| \ge \frac{|N_\delta(a, b, O)|}{C(\delta)}.$$

We need also the following classical Hermann Weyl's result.

Lemma 2.4 *Let $A \ne B$ be two real symmetric $n \times n$ matrices with the eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\lambda'_1 \ge \lambda'_2 \ge \cdots \ge \lambda'_n$, respectively. Then for the eigenvalues $\Lambda_1 \ge \Lambda_2 \ge \cdots \ge \Lambda_n$ of the matrix $A - B$, we have*

$$\Lambda_1 \ge \max_{i=1,\cdots,n} (\lambda_i - \lambda'_i), \quad \Lambda_n \le \min_{i=1,\cdots,n} (\lambda_i - \lambda'_i).$$

3 Proofs

Let $n = 5, u(x) = c + w_{5,\delta}(x)$. We begin with $\delta = 1$ and show that the result is false in this case. Indeed, let $a = (1, 0, 0, 0, 0),\ b = \left(\frac{1}{2}, 0, 0, 0, 0\right),\ O = I_5$. Then

$$\begin{aligned} &w(a) = 1, \quad w(b) = \frac{1}{4}, \quad |Du(a)|^2 = |Dw(a)|^2 = 9, \quad |Du(b)|^2 = |Dw(b)|^2 = \frac{9}{4}, \\ &D^2 u(a) = D^2 w(a) = D^2 u(b) = D^2 w(b), \end{aligned}$$

and

$$A^u(a) - A^u(b) = \frac{1}{2} D^2 w(a) - \frac{27}{4} I_5$$

which is negative since the spectrum of $D^2 w(a)$ is $(2, 2, 2, -7, -7)$. The reason is clearly that $D^2 w(a)$ for $\delta = 1$ is homogeneous order 0 and does not depend on $|a|$.

Remark 3.1 More generally, the same argument applied to the points

$$a = (1, 0, 0, 0, 0), \quad b' = (\lambda, 0, 0, 0, 0)$$

for small enough $\lambda > 0$ shows that a solution of the form $c + v$ for a constant c and an order 2 homogeneous function v is impossible for a conformal Hessian equation.

Suppose now that $\delta \in]1,2[$. We formulate below the results which we need to prove the main theorem for any $\delta \in]1,2[$, but give detailed proofs only for $\delta = \frac{3}{2}$ (and $c = 10^6$). However, we point out how to modify the arguments for a general $\delta \in]1,2[$. First we spell out Lemma 2.3 for $\delta = \frac{3}{2}$.

Lemma 3.1 (i) *Let $x \in \mathbb{S}^4_1$, and let $x \in G_P(p,0,0,r,0)$ with $p^2 + r^2 = 1$. Then*

$$\mathrm{Spec}(D^2u(x)) = \mathrm{Spec}(D^2w(x)) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$$

for

$$\begin{aligned}
\mu_1 &= \frac{3p(p^2+1)}{4},\\
\mu_2 &= \frac{3p(p^2-5) + 6\sqrt{12-3p^2}}{4},\\
\mu_3 &= \frac{3p(p^2-5) - 6\sqrt{12-3p^2}}{4},\\
\mu_4 &= \frac{27p(p^2-3) + 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16},\\
\mu_5 &= \frac{27p(p^2-3) - 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16}.
\end{aligned}$$

(ii) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_5$ be the ordered eigenvalues of* $\mathrm{Spec}(D^2u(x)) = \mathrm{Spec}(D^2w(x))$. *Then*

$$\begin{aligned}
&\lambda_1 = \mu_2, \quad \lambda_5 = \mu_3,\\
&\lambda_2 = \begin{cases} \mu_4 & \text{for } p \in [-1, p_0],\\ \mu_1 & \text{for } p \in [p_0, 1], \end{cases}\\
&\lambda_3 = \begin{cases} \mu_5 & \text{for } p \in [-1, -p_0],\\ \mu_1 & \text{for } p \in [-p_0, p_0],\\ \mu_4 & \text{for } p \in [p_0, 1], \end{cases}\\
&\lambda_4 = \begin{cases} \mu_1 & \text{for } p \in [-1, -p_0],\\ \mu_5 & \text{for } p \in [-p_0, 1], \end{cases}
\end{aligned}$$

where

$$p_0 = 5^{-\frac{1}{4}} \simeq 0.6687403050.$$

We will need also the derivatives of the eigenvalues.

Lemma 3.2 *Let $d_i(p) := \frac{d(\mu_i)}{dp}$. Then*

$$\begin{aligned}
d_1(p) &= \frac{3(3p^2+1)}{4},\\
d_2(p) &= -\frac{3(5-3p^2)}{4} + \frac{9p}{2\sqrt{12-3p^2}},\\
d_3(p) &= -\frac{3(5-3p^2)}{4} - \frac{9p}{2\sqrt{12-3p^2}},
\end{aligned}$$

$$d_4(p) = \frac{81(1-p^2)}{16}\Big(\frac{35p(3-p^2)}{3\sqrt{105p^6-630p^4+945p^2+64}} - 1\Big),$$
$$d_5(p) = -\frac{81(1-p^2)}{16}\Big(\frac{35p(3-p^2)}{3\sqrt{105p^6-630p^4+945p^2+64}} + 1\Big).$$

For the general $\delta \in]1,2[$, we give only the two most complicated derivatives:

$$d_4(p,\delta) := \frac{d\mu_{4,\delta}}{dp} = \frac{3\delta(p^2-1)(6-\delta)}{4} + \frac{A(p,\delta)}{4\sqrt{D(p,\delta)}},$$
$$d_5(p,\delta) := \frac{d\mu_{5,\delta}}{dp} = \frac{3\delta(p^2-1)(6-\delta)}{4} - \frac{A(p,\delta)}{4\sqrt{D(p,\delta)}},$$

where

$$A(p,\delta) := 3\delta p(\delta-6)(\delta+2)(\delta-4)(p-1)(p+1)(p^2-3),$$

and $D(p,\delta)$ is defined in Lemma 2.3.

Simple calculus gives the following result.

Corollary 3.1 *Define*

$$D_\delta := \max\{|d_i(p,\delta)| : p \in [-1,1],\ i = 1,\cdots,5\}.$$

Then

$$D_\delta < \frac{d}{2-\delta}$$

for an absolute constant $d > 0$ (*one can take* $d = 100$). *For* $\delta = \frac{3}{2}$, *one has* $D = D_{\frac{3}{2}} < 10$.

Indeed, the only problem to bound the derivatives $d_i(p,\delta)$ is the expression $\sqrt{D(p,\delta)} \geq 12(2-\delta)$ in the denominator of $d_4(p,\delta)$ and $d_5(p,\delta)$, while the other derivatives are easily bounded by an absolute constant (say, by 100).

Below we denote $D_i(p) := \frac{d(\lambda_i)}{dp}$; the relation of $D_i(p)$ and $d_i(p)$ is clear from Lemma 3.1 (ii); for example, $D_1(p) = d_2(p)$, $D_5(p) = d_3(p)$.

The proof of Theorem 1.2 is based on some auxiliary lemmas which use the following notation. Let us take two points

$$a, b \in B_1 \setminus \{0\} \quad \text{with } |a| = s \leq 1,\ |b| = t \leq 1, \quad \text{and a matrix } O \in \mathrm{O}(5),$$

and let $a' := \frac{a}{s} \in G_P(p,0,0,r,0) \in \mathbb{S}_1^4$, $b' := \frac{b}{t} \in G_P(q,0,0,r',0) \in \mathbb{S}_1^4$. Below we use the following quantity K depending on the pair (a,b):

$$K := K(a,b) = K(p,q,s,t) = |s-t| + |p-q| > 0,$$

and work with the following matrices depending on (a,b) and on an orthogonal matrix O (and also on δ):

$$M_1 := M_1(a,b,O) := D^2u(a) - O^{-1}D^2u(b)\cdot O = N_\delta(a,b,O),$$
$$M_2 := M_2(a,b,O) := w(a)D^2u(a) - O^{-1}w(b)D^2u(b)O = w(a)D^2w(a) - O^{-1}w(b)D^2w(b)O.$$

Lemma 3.3 *There holds*

$$||Du(a)|^2 - |Du(b)|^2| \leq C_1(\delta)K$$

with $C_1(\delta) = c_1(2-\delta)$ *for an absolute constant* c_1. *For* $\delta = \frac{3}{2}$, *we can take* $C_1 := C_1(\frac{3}{2}) = 21$.

Proof Notice first that $|Du(a)|^2 = |Dw(a)|^2, |Du(b)|^2 = |Dw(b)|^2$. Since $P = P_5(x)$ can be represented as the generic traceless norm in the Jordan algebra $\mathrm{Sym}_3(\mathbb{R})$, it verifies the eiconal equation $|DP(x)|^2 = 9|x|^4$ (see, e.g., [7]). Then let $\delta = \frac{3}{2}$. An easy calculation gives

$$|Du(a)|^2 = \frac{9s(16-3p^2(p^2-3)^2)}{16}, \quad |Du(b)|^2 = \frac{9t(16-3q^2(q^2-3)^2)}{16},$$

since $P(a) = \frac{s^3p(3-p^2)}{2}$, $P(b) = \frac{t^3q(3-q^2)}{2}$. Thus

$$\begin{aligned}
||Du(a)|^2 - |Du(b)|^2| &\le \Big|\frac{9s(16-3p^2(p^2-3)^2)}{16} - \frac{9t(16-3p^2(p^2-3)^2)}{16}\Big| \\
&\quad + \Big|\frac{9t(16-3p^2(p^2-3)^2)}{16} - \frac{9t(16-3q^2(q^2-3)^2)}{16}\Big| \\
&= \Big|\frac{9(s-t)(16-3p^2(p^2-3)^2)}{16}\Big| \\
&\quad + \Big|\frac{27t(p-q)(p+q)((q^2-3)^2-(p^2-3)^2)}{16}\Big| \\
&\le |9(s-t)| + \Big|\frac{81(p-q)}{4}\Big| \le 21K.
\end{aligned}$$

Same calculation gives for the general δ,

$$\begin{aligned}
|Du(a)|^2 &= s^{4-2\delta}\Big(9 + \delta(\delta-6)\frac{p^2(p^2-3)^2}{4}\Big), \\
|Du(b)|^2 &= t^{4-2\delta}\Big(9 + \delta(\delta-6)\frac{q^2(q^2-3)^2}{4}\Big).
\end{aligned}$$

Repeating the argument, we obtain the conclusion.

Lemma 3.4 *Let* $M := |M_1| = |D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O|$. *Then*

$$M \ge C_2(\delta)K$$

for a positive constant $C_2(\delta)$ *depending only on* δ. *For* $\delta = \frac{3}{2}$, *we can take* $C_2 := C_2(\frac{3}{2}) = \frac{1}{8}$.

Proof If one replaces a by $a' = \frac{a}{s}$ and b by $b'' = \frac{b}{s}$ the quantity M gets bigger and K gets smaller. Therefore, we can suppose that $|a| = s = 1$. Then we have

$$D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O = D^2u(a) - \frac{O^{-1} \cdot D^2u(b') \cdot O}{\sqrt{t}}.$$

By Lemma 2.4, we have

$$\begin{aligned}
M &\ge \max\{\lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \cdots, 5\}, \\
M &\ge \Big|\min\Big\{\lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \cdots, 5\Big\}\Big|.
\end{aligned}$$

Let then $\delta = \frac{3}{2}$. Suppose first $p \ge q$. If $q \ge -\frac{24}{25} = -0.96$, then

$$D_1(p') < -\frac{1}{4} = -0.25, \quad \lambda_1(p) > \frac{3}{2}, \quad \forall p' \in [q,p]$$

(by a simple calculation using the explicit formulas for D_1, λ_1). Therefore,

$$\lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} = \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \le -\frac{p-q}{4} - \frac{3}{2}\Big(\frac{1}{\sqrt{t}} - 1\Big) < -\frac{K}{4}.$$

If $q < -0.96$ but $p \ge -\frac{23}{25} = -0.92$, then

$$\begin{aligned}\lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} &= \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \\ &\le \lambda_1(p) - \lambda_1\Big(\frac{24}{25}\Big) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} - \frac{p+0.96}{4} - \frac{3}{2}\Big(\frac{1}{\sqrt{t}} - 1\Big) \\ &< -\frac{p-q}{8} - \frac{3}{2}\Big(\frac{1}{\sqrt{t}} - 1\Big) < -\frac{K}{8}.\end{aligned}$$

Then suppose that $q < -0.96$, $p < -0.92$. In this case, we have

$$d_2(p') > \frac{5}{2}, \quad \lambda_2(p') < -\frac{3}{2}, \quad \forall p' \in [q, p],$$

and thus

$$\lambda_2(p) - \frac{\lambda_2(q)}{\sqrt{t}} = \lambda_2(p) - \lambda_2(q) + \lambda_2(q) - \frac{\lambda_2(q)}{\sqrt{t}} \ge \frac{5(p-q)}{2} + \frac{3}{2}\Big(\frac{1}{\sqrt{t}} - 1\Big) \ge \frac{3K}{4},$$

which finishes the proof for $p \ge q$. The case $q \ge p$ is treated similarly (replace λ_1 by λ_5 and λ_2 by λ_4).

For the general $\delta \in]1, 2[$, the argument is similar, but more cumbersome. It shows that we can take $C_2(\delta) = c_2(2-\delta)^2$ for an absolute constant $c_2 > 0$ (say, $c_2 = 0.001$).

Remark 3.2 Notice that Lemma 3.4 is false for $\delta = 1$.

Lemma 3.5 *Let*

$$M' := |M_2| = |w(a)D^2u(a) - O^{-1}w(b)D^2u(b) \cdot O|.$$

Then

$$M' \le C_3(\delta)K$$

for a positive constant $C_3(\delta)$ depending only on δ. For $\delta = \frac{3}{2}$, we can take $C_3 := C_3(\frac{3}{2}) = 10$.

Proof Indeed for $\delta = \frac{3}{2}$, let $a' := \frac{a}{s}$, $b' := \frac{b}{s}$. Then by homogeneity,

$$\begin{aligned}|w(a)D^2w(a) - O^{-1}w(b)D^2w(b) \cdot O| &= |sD^2w(a') - O^{-1} \cdot tD^2w(b') \cdot O| \\ &\le s|D^2w(a') - O^{-1} \cdot D^2w(b') \cdot O| \\ &\quad + |s-t| \cdot |O^{-1} \cdot D^2w(b')| \\ &\le \max_{p,i}\{|D_i(p)|\}|p-q| + 7|s-t| \\ &= \max_{p,i}\{|d_i(p)|\}|p-q| + 7|s-t| \le 10K.\end{aligned}$$

For the general δ, the argument remains valid and permits to take

$$C_3(\delta) = D_\delta = \frac{100}{2-\delta}.$$

End of Proof of Theorem 1.2 We can now prove the uniform hyperbolicity of $M(a,b,O)$, and thus the theorem. In fact, for $\delta = \frac{3}{2}$, one can take $C = 2000$ in Lemma 2.1 for $c = 10^6$.

Indeed, we have

$$M(a,b,O) = A^u(a) - O^{-1} \cdot A^u(b) \cdot O = cM_1 + M_2 - (|Du(a)|^2 - |Du(b)|^2)I_5.$$

Therefore,

$$|\mu_5(a,b,O)| \geq c|\Lambda_5(M_1)| - C_1(\delta)K - C_3(\delta)K \geq \frac{cM}{C(\delta)} - (C_1(\delta) + C_3(\delta))K,$$

$$|\mu_1(a,b,O)| \geq c|\Lambda_5(M_1)| - C_1(\delta)K - C_3(\delta)K \geq \frac{cM}{C(\delta)} - (C_1(\delta) + C_3(\delta))K$$

and

$$|M(a,b,O)| \leq cM + M' + ||Du(a)|^2 - |Du(b)|^2| \leq cM + (C_1(\delta) + C_3(\delta))K.$$

Thus

$$\max\left\{\frac{|\mu_1(a,b,O)|}{|\mu_5(a,b,O)|}, \frac{|\mu_5(a,b,O)|}{|\mu_1(a,b,O)|}\right\} \leq C(\delta) \cdot \frac{cM + (C_1(\delta) + C_3(\delta))K}{cM - C(\delta)(C_1(\delta) + C_3(\delta))K},$$

since $|M(a,b,O)| = \max\{|\mu_1(a,b,O)|, |\mu_5(a,b,O)|\}$. Therefore, for a sufficiently large c, we get

$$\max\left\{\frac{|\mu_1(a,b,O)|}{|\mu_5(a,b,O)|}, \frac{|\mu_5(a,b,O)|}{|\mu_1(a,b,O)|}\right\} \leq \frac{C(\delta)\left(1 + \dfrac{C_1(\delta) + C_3(\delta)}{cC_2(\delta)}\right)}{1 - \dfrac{C(\delta)(C_1(\delta) + C_3(\delta))}{cC_2(\delta)}} \leq 2C(\delta) =: C,$$

since $M \geq C_2(\delta)K$ which finishes the proof. Taking for $\delta = \frac{3}{2}$ the values

$$C_1\left(\frac{3}{2}\right) = 21, \quad C_2\left(\frac{3}{2}\right) = \frac{1}{8}, \quad C_3\left(\frac{3}{2}\right) = 10$$

from Lemmas 3.2–3.5, choosing $c := 10^6$, and making an elementary calculation, we see that $C = 2C\left(\frac{3}{2}\right) = 2000$ is admissible in this case.

Acknowledgements The authors are deeply grateful to the anonimous referee whose advise permitted to ameliorate significantly our exposition.

References

[1] Caffarelli, L., Interior a priory estimates for solutions of fully nonlinear equations, *Ann. Math.*, **130**, 1989, 189–213.

[2] Caffarelli, L. and Cabre, X., Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, **43**, Amer. Math. Soc., Providence, RI, 1995.

[3] Crandall, M. G., Ishii, H. and Lions, P.-L., User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* (*N.S.*), **27**, 1992, 1–67.

[4] Nadirashvili, N. and Vlăduţ, S., Singular solutions of Hessian elliptic equations in five dimensions, *J. Math. Pures Appl.*, **100**(9), 2013, 769–784.

[5] Nadirashvili, N. and Vlăduţ, S., Singular solutions of Hessian fully nonlinear elliptic equations, *Adv. Math.*, **228**, 2011, 1718–1741.

[6] Nadirashvili, N., Tkachev, V. G. and Vlăduţ, S., Nonlinear elliptic equations and nonassociative algebras, Math. Surv. and Monogr., **200**, Amer. Math. Soc., Providence, RI, 2014.

[7] Tkachev, V. G., A Jordan algebra approach to the eiconal, *J. of Algebra*, **419**, 2014, 34–51.

[8] Trudinger, N., Hölder gradient estimates for fully nonlinear elliptic equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **108**, 1988, 57–65.

[9] Trudinger, N., Fully nonlinear elliptic equations in geometry. CBMS Lectures, October 2004 draft. $\text{http://maths} - \text{people.anu.edu.au/}\tilde{\text{n}}\text{eilt/RecentPapers/notes1.pdf}$