Chin. Ann. Math.
38B(2), 2017, 601–628
DOI: 10.1007/s11401-017-1086-5

Chinese Annals of Mathematics, Series B

Negative Index Materials and Their Applications: Recent Mathematics Progress

Hoai-Minh NGUYEN[1]

(*Dedicated to Haïm Brezis for his* 70*th birthday with esteem*)

Abstract Negative index materials are artificial structures whose refractive index has negative value over some frequency range. These materials were first investigated theoretically by Veselago in 1946 and were confirmed experimentally by Shelby, Smith, and Schultz in 2001. Mathematically, the study of negative index materials faces two difficulties. Firstly, the equations describing the phenomenon have sign changing coefficients, hence the ellipticity and the compactness are lost in general. Secondly, the localized resonance, i.e., the field explodes in some regions and remains bounded in some others as the loss goes to 0, might appear. In this survey, the author discusses recent mathematics progress in understanding properties of negative index materials and their applications. The topics are reflecting complementary media, superlensing and cloaking by using complementary media, cloaking a source via anomalous localized resonance, the limiting absorption principle and the well-posedness of the Helmholtz equation with sign changing coefficients.

Keywords Negative index materials, Localized resonance, Cloaking, Superlensing
2010 MR Subject Classification 35B34, 35B35, 35B40, 35J05, 78A25

1 Introduction

Negative index materials (NIMs for short) were first investigated theoretically by Veselago [49]. The existence of such materials was confirmed by Shelby, Smith and Schultz [48]. The study of NIMs attracted a lot attention in the scientific community thanks to their many possible applications such as superlensing and cloaking by using complementary media, and cloaking a source via anomalous localized resonance (ALR for short). Mathematically, the study of NIMs faces two difficulties. Firstly, the equations describing the phenomena have sign changing coefficients, hence the ellipticity and the compactness are lost in general. Secondly, the localized resonance, i.e., the field explodes in some regions and remains bounded in some others as the loss goes to 0, might appear.

In this survey, we present recent mathematics progress in understanding properties of NIMs and their applications. The following five topics are discussed: reflecting complementary media, superlensing by using complementary media, cloaking by using complementary media, cloaking a source via ALR, and the limiting absorption principle and the well-posedness of the Helmholtz equations with sign changing coefficients. The choice of these topics is related to the author's expertise for which he made contribution in [27–33] and in his joint work with Nguyen [34–35]. An interesting topic of NIMs, the construction of NIMs via various processes of homogeneization,

Manuscript received July 15, 2015. Revised October 8, 2015.
[1]EPFL SB MATHAA CAMA, Station 8, CH-1015 Lausanne, Switzerland.
E-mail: hoai-minh.nguyen@epfl.ch

is completely ignored; Concerning this aspect, the reader can consult [10, 12, 16, 20, 46] and references therein.

The outline of the paper is as follows. Reflecting complementary media is discussed in Section 2. To motivate this concept, we first illustrate peculiar properties of NIMs by presenting an interesting result due to Nicorovici, McPhedran and Milton [42]. We then discuss its non-trivial extension in [27] via the concept of reflecting complementary media using the reflecting technique introduced there. Superlensing using complementary media is presented in Section 3. Superlensing using complementary media was suggested by Veselago [49], Nicorovici, McPhedran and Milton [42], Pendry [44–45], and Ramarkrishna and Pendry [47]. Concerning this topic, we first present a class of superlensing schemes, which is a subclass of schemes given in [28], and is inspired on one hand by the suggestion of superlenses in [42, 45, 47] and on another hand by the study of reflecting complementary media in [27]. We then provide the proof of superlensing for this class. The proof presented here uses the results in Section 2, and is simpler than the first one given in [28]. Cloaking using complementary media is given in Section 4. This was suggested by Lai et al. [21]. Concerning this topic, we present a cloaking scheme, which is related to [21], and the first proof of cloaking by using complementary media from [31], where the removing localized singularity technique was introduced to handle the localized resonance. This technique was inspired by the influential work of Bethuel, Brezis and Hélein on the Ginzburg Landau equation in [4]. Cloaking a source via ALR is discussed in Section 5. This was discovered by Milton and Nicorovici [23] for a constant radial symmetric plasmonic structure in the two dimensional quasi-static regime. Section 5, which is based on [30], is on various properties on cloaking a source via ALR for doubly complementary media, a subclass of reflecting complementary media introduced there. As an application of these properties, one can construct a cloaking device to cloak a general source concentrate on a manifold of codimension 1 in an arbitrary medium. The limiting absorption principle and the well-posedness of the Helmholtz equations with sign changing coefficients are given in Section 6 and based on [32]. Concerning this topic, we discuss various conditions on the coefficients for which the limiting absorption principle holds and the equation is well-posed. The unique solution which might not be in L^2_{loc} is obtained from the limiting absorption principle. The results presented here extend largely known results by using the integral method, the pseudo differential operator theory, and the T-coercivity approach. From Section 2 to Section 5, we mainly concentrate on the quasi-static regime in a bounded domain, even thought the results in the finite frequency regime are also mentioned. In the last section, we consider the finite frequency regime in the whole space for which the uniqueness can be established without imposing further assumptions.

2 Reflecting Complementary Media

Let $0 < r_1 < r_2 < R$, and $f \in L^2(B_R)$ [1]. Set $r_3 = \frac{r_2^2}{r_1}$. Assume that $R > r_3$ and $\operatorname{supp} f \cap B_{r_3} = \emptyset$. Let $u_\delta \in H^1_0(B_R)$ $(\delta > 0)$ be the unique solution to the equation

$$\operatorname{div}(\varepsilon_\delta \nabla u_\delta) = f \quad \text{in } B_R, \tag{2.1}$$

where, for $\delta \geq 0$, [2]

$$\varepsilon_\delta(x) = \begin{cases} -1 - i\delta, & \text{if } r_1 < |x| < r_2, \\ 1, & \text{otherwise.} \end{cases} \tag{2.2}$$

[1] Here and in what follows, for $r > 0$, B_r denotes the ball centered at the origin and of radius r.

[2] In [42], ε_δ is given by $-1 + i\delta$ instead of $-1 - i\delta$ for $r_1 < |x| < r_2$; nevertheless, this point is not essential.

Physically, the imaginary part of ε_δ is the loss of the medium. In [42], Nicorovici, McPhedran and Milton obtained, by separation of variables, the following interesting result:

$$u_\delta \to \widehat{u} \quad \text{for } |x| > r_3, \tag{2.3}$$

where $\widehat{u} \in H^1_0(B_R)$ is the unique solution to the equation

$$\Delta \widehat{u} = f \quad \text{in } B_R.$$

The surprising fact of this result is that (2.3) holds for any f with $\operatorname{supp} f \cap B_{r_3} = \emptyset$. From (2.3), one might say that the region $\{r_2 < |x| < r_3\}$ is canceled by the one in $\{r_1 < |x| < r_2\}$ and the total system is effectively equivalent to the free space; invisibility appeared. From (2.3), one might as well say that the shell $B_{r_2} \setminus B_{r_1}$ magnifies the core B_{r_1} $\frac{r_3}{r_1} = \frac{r_2^2}{r_1^2}$ times to make it like B_{r_3}, i.e., superlensing is revealed.

Under the condition (2.2), the property (2.3) does not hold in three dimensions, and its natural extension for the finite frequency regime is not valid in two dimensions. In [27], we extended the above results for quite general settings in two and three dimensions both in the quasistatic and the finite frequency regimes by using a completely different approach. This is given in the next two subsections. In the first one, we present an heuristic argument for (2.3), and in the second one, we discuss new results inspired from the heuristic argument.

2.1 An heuristic argument for Nicorovici, McPhedran and Milton's result

In this subsection, we assume that $u_\delta \to u_0 \in H^1(B_R)$ as $\delta \to 0$, and present an heuristic argument to obtain (2.3) from [27]. From the assumption, $u_0 \in H^1_0(B_R)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 \nabla u_0) = f \quad \text{in } B_R.$$

Let $F : B_{r_2} \setminus \{0\} \mapsto \mathbb{R}^2 \setminus \overline{B}_{r_2}$ be the Kelvin transform with respect to ∂B_{r_2}, i.e., $F(x) = \frac{r_2^2 x}{|x|^2}$. Define

$$u_{1,0}(x) = u_0 \circ F^{-1}(x) \quad \text{in } \mathbb{R}^2 \setminus B_{r_2}.$$

From the transmission conditions on ∂B_{r_2}, we have

$$u_{1,0} = u_0 \quad \text{and} \quad \partial_r u_{1,0}\big|_{r \to r_2+} = \partial_r u_0\big|_{r \to r_2+} \quad \text{on } \partial B_{r_2}. \tag{2.4}$$

Since F is a Kelvin transform and $\operatorname{supp} f \cap B_{r_3} = \emptyset$, it follows that

$$\operatorname{div}(\breve{\varepsilon}_0 \nabla u_{1,0}) = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{r_2},$$

where

$$\breve{\varepsilon}_0(x) = \begin{cases} 1 & \text{in } B_{r_3} \setminus B_{r_2}, \\ -1 & \text{in } \mathbb{R}^2 \setminus B_{r_3}. \end{cases}$$

Note that F maps ∂B_{r_1} into ∂B_{r_3}. By the unique continuation principle, we have

$$u_{1,0} = u_0 \quad \text{in } B_{r_3} \setminus B_{r_2}. \tag{2.5}$$

Let $G : \mathbb{R}^2 \setminus \overline{B}_{r_3} \mapsto B_{r_3}$ be the Kelvin transform with respect to ∂B_{r_3}, i.e., $G(x) = \frac{r_3^2 x}{|x|^2}$. Define

$$u_{2,0}(x) = u_{1,0} \circ G^{-1}(x) \quad \text{ in } B_{r_3}.$$

Similar to (2.4), we have

$$u_{2,0} = u_{1,0} \quad \text{ and } \quad \partial_r u_{2,0}\big|_{r \to r_{3-}} = \partial_r u_{1,0}\big|_{r \to r_{3-}} \quad \text{ on } \partial B_{r_3}.$$

It follows from (2.5) that

$$u_{2,0} = u_0 \quad \text{ and } \quad \partial_r u_{2,0}\Big|_{r \to r_{3-}} = \partial_r u_0 \quad \text{ on } \partial B_{r_3}. \tag{2.6}$$

We also have

$$\Delta u_{2,0} = 0 \quad \text{in } B_{r_3} \tag{2.7}$$

by the property of the Kelvin transforms and the definition of $u_{2,0}$. Define

$$\widehat{u}(x) = \begin{cases} u_0(x) & \text{if } B_R \setminus B_{r_3}, \\ u_{2,0}(x) & \text{in } B_{r_3}. \end{cases}$$

Since $\Delta u_0 = f$ in $B_R \setminus \overline{B}_{r_3}$, it follows from (2.6)–(2.7) that

$$\Delta \widehat{u} = f \quad \text{in } B_R.$$

Therefore, we obtain (2.3).

Remark 2.1 If $\operatorname{supp} f \cap (B_{r_3} \setminus B_{r_2}) \neq \emptyset$ and $\operatorname{supp} f \cap B_{r_2} = \emptyset$, then, instead of (2.5), one has

$$\Delta w_1 = f \quad \text{in } B_{r_3} \setminus B_{r_2}, \quad w_1 = \partial_r w_1 = 0 \quad \text{on } \partial B_{r_2},$$

where $w_1 = u_{1,0} - u_0$ in $B_{r_3} \setminus B_{r_2}$. In general, this Cauchy problem does not have a solution in $H^1(B_{r_3} \setminus \overline{B}_{r_2})$ or even in $L^2(B_{r_3} \setminus B_{r_2})$. In fact, one can easily see from the heuristic argument that there are two Cauchy problems in this context, another one is related to (2.7). The non-existence mentioned here is the origin of the concept of compatibility in Definition 2.2.

2.2 Reflecting complementary media

Motivated by the heuristic argument in Subsection 2.1 and the change of variables for the Helmholtz equations (see Lemma 2.1 below), in [27], we introduced the concept of reflecting complementary media and extended (2.3) to this class. Let $k \geq 0$ and $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ be smooth connected bounded open subsets of $\mathbb{R}^d$ ($d = 2, 3$). Let A be a measurable matrix-valued function and Σ be a measurable real function defined in Ω. Here and in what follows, we always assume that

$$\frac{1}{\Lambda}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall\, \xi \in \mathbb{R}^d \tag{2.8}$$

for a.e. $x \in \Omega$ and for some $0 < \Lambda < +\infty$, and

$$0 < \operatorname*{ess\,inf}_{\Omega} \Sigma \leq \operatorname*{ess\,sup}_{\Omega} \Sigma < +\infty. \tag{2.9}$$

Set

$$s_\delta(x) = \begin{cases} -1 - i\delta, & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ 1, & \text{otherwise.} \end{cases} \tag{2.10}$$

In [27], we were interested in the behavior of the unique solution $u_\delta \in H^1_0(\Omega)$ to the equation

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta = f \quad \text{in } \Omega \tag{2.11}$$

as $\delta \to 0$, under the condition that (A, Σ) in $\Omega_3 \setminus \Omega_2$ is reflecting complementary to $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. To motivate the definition of reflecting complementary media, let us recall the change of variables for the Helmholtz equation, which follows from [27, Lemma 2].

Lemma 2.1 *Let $D_1 \subset\subset D_2 \subset\subset D_3$ be smooth bounded open subsets of $\mathbb{R}^d$, T be a diffeomorphism from $D_2 \setminus \overline{D}_1$ onto $D_3 \setminus \overline{D}_2$ such that $T(x) = x$ on ∂D_2, $a \in [L^\infty(D_2 \setminus D_1)]^{d\times d}$, and $\sigma \in L^\infty(D_2 \setminus D_1)$. Let $u \in H^1(D_2 \setminus D_1)$ and set $v = u \circ T^{-1}$. Then*

$$\operatorname{div}(a \nabla u) + \sigma u = f \quad \text{in } D_2 \setminus D_1$$

for some $f \in L^2(D_2 \setminus D_1)$ if and only if

$$\operatorname{div}(T_* a \nabla v) + T_* \sigma v = T_* f \quad \text{in } D_3 \setminus D_2. \tag{2.12}$$

Moreover,

$$v = u \quad \text{and} \quad T_* a \nabla v \cdot \nu = -a \nabla u \cdot \nu \quad \text{on } \partial D_2. \tag{2.13}$$

Here and in what follows, we use the standard notations:

$$T_* a(y) = \frac{DT(x) a(x) DT^T(x)}{J(x)}, \quad T_* \sigma(y) = \frac{\sigma(x)}{J(x)}, \quad T_* f(y) = \frac{f(x)}{J(x)}, \tag{2.14}$$

where $x = T^{-1}(y)$ and $J(x) = |\det DT(x)|$.

We are ready to give [27, Definition 1].

Definition 2.1 (Reflecting Complementary Media) *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ be smooth connected bounded open subsets of $\mathbb{R}^d$. The media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are said to be reflecting complementary, if there exists a diffeomorphism $F : \Omega_2 \setminus \overline{\Omega}_1 \to \Omega_3 \setminus \overline{\Omega}_2$, such that*

$$F_* A(x) = A(x), \quad F_* \Sigma(x) = \Sigma(x) \quad \text{for } x \in \Omega_3 \setminus \overline{\Omega}_2, \tag{2.15}$$

$$F(x) = x \quad \text{on } \partial \Omega_2, \tag{2.16}$$

and the following two conditions hold:

(1) *There exists a diffeomorphism extension of F, which is still denoted by F, from $\Omega_2 \setminus \{x_1\} \to \mathbb{R}^d \setminus \overline{\Omega}_2$ for some $x_1 \in \Omega_1$.*

(2) *There exists a diffeomorphism $G : \mathbb{R}^d \setminus \overline{\Omega}_3 \to \Omega_3 \setminus \{x_1\}$ such that* [3]

$$G(x) = x \quad \text{on } \partial \Omega_3 \tag{2.17}$$

and

$$G \circ F : \Omega_1 \to \Omega_3 \text{ is a diffeomorphism if one sets } G \circ F(x_1) = x_1. \tag{2.18}$$

[3] In (2.16)–(2.17), F and G denote some diffeomorphism extensions of F and G in a neighborhood of $\partial\Omega_2$ and of $\partial\Omega_3$, respectively.

Some comments on the definition are useful. If $k = 0$, then the condition on Σ is irrelevant in Definition 2.1. Condition (2.15) implies that (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are complementary in the "usual sense"[4]. The term "reflecting" in the definition comes from (2.16) and the assumption $\Omega_1 \subset \Omega_2 \subset \Omega_3$. Conditions (2.15)–(2.16) imply that u_0 (the solution to $\delta = 0$ if it exists) and $u_{1,0} := u_0 \circ F$ satisfy the same equation in $\Omega_3 \setminus \Omega_2$ and the same Cauchy data on $\partial\Omega_2$ by Lemma 2.1, hence the reflecting technique in Section 2.1 can be used. Conditions (2.15)–(2.16) are the key assumptions. Conditions (1)–(2) are mild ones. Introducing G makes the analysis more accessible (see also Sections 3–5). In general, it is not easy to verify the condition (2.15). However, given (A, Σ) in $\Omega_3 \setminus \Omega_2$, it is easy to obtain $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$, such that (2.15) holds by choosing an arbitrary diffeomorphism $F : \Omega_2 \setminus \overline{\Omega}_1 \mapsto \Omega_3 \setminus \Omega_2$ and defining (A, Σ) in $\Omega_2 \setminus \Omega_1$ by $(F_*^{-1}A, F_*^{-1}\Sigma)$. This process was repeatedly used in various applications of NIMs (see Sections 3–5).

Remark 2.2 The definition given here simplifies lightly the one introduced in [27], and suffices for various applications of NIMs discussed later.

Here and in what follows in this section, we confine ourselves to the quasi-static regime: $k = 0$. The finite frequency regime ($k > 0$) can be proceeded similarly (see [27]). The following result follows from [27, Theorems 1–2].

Theorem 2.1 *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$ and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to* (2.11),

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega.$$

Assume that A in $\Omega_3 \setminus \Omega_2$ and $-A$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary and $\operatorname{supp} f \cap \Omega_3 = \emptyset$ *for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. We have the following.*

(a) *Case* 1 : *f is compatible* (*see Definition* 2.2). *Then (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $u_0 \in H^1_0(\Omega)$ the unique solution to*

$$\operatorname{div}(s_0 \nabla u_0) = f \quad \text{in } \Omega \tag{2.19}$$

as $\delta \to 0$. Moreover, $u_0 = \widehat{u}$ in $\Omega \setminus \Omega_3$, where $\widehat{u} \in H^1_0(\Omega)$ is the unique solution to

$$\operatorname{div}(\widehat{A} \nabla \widehat{u}) = f \quad \text{in } \Omega, \quad \text{where } \widehat{A} := \begin{cases} A & \text{in } \Omega \setminus \Omega_3, \\ G_* F_* A & \text{in } \Omega_3. \end{cases} \tag{2.20}$$

(b) *Case* 2 : *f is not compatible. We have*

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(\Omega)} = +\infty. \tag{2.21}$$

In the statement of Theorem 2.1, we use the following definition [27, Definition 2].

Definition 2.2 (Compatibility Condition) *Assume that A in $\Omega_3 \setminus \Omega_2$ and $-A$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary. Then $f \in L^2(\Omega)$ with* $\operatorname{supp} f \cap \Omega_3 = \emptyset$ *is said to be compatible if and only if*

$$\exists V \in H^1(\Omega_3 \setminus \overline{\Omega}_2) \text{ satisfies } \begin{cases} \operatorname{div}(A \nabla V) = 0 & \text{in } \Omega_3 \setminus \overline{\Omega}_2, \\ V = \widehat{u}\big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A \nabla V \cdot \nu = A \nabla \widehat{u} \cdot \nu\big|_{\text{ext}} & \text{on } \partial\Omega_3, \end{cases} \tag{2.22}$$

where $\widehat{u}$ is defined in (2.20).

[4]In fact, complementary media had not been defined precisely, the property mentioned here appeared in various known examples.

Here and in what follows, for a smooth bounded open subset $D \subset \mathbb{R}^d$, on ∂D, ν denotes the outward unit normal vector.

The proof of Theorem 2.1 is quite straightforward from the definition of reflecting complementary media and the heuristic argument presented in Subsection 2.1. We first assume that u_0 exists. As in Subsection 2.1, define $u_{1,0} = u_0 \circ F$ in $\mathbb{R}^d \setminus \Omega_3$ and $u_{2,0} = u_{1,0} \circ G$ in Ω_3. Set

$$\widehat{u} := \begin{cases} u_0 & \text{in } \Omega \setminus \Omega_3, \\ u_{2,0} & \text{in } \Omega_3. \end{cases}$$

Then $u_{1,0} = u_0$ in $\Omega_3 \setminus \Omega_2$ and $\widehat{u}$ satisfies $\operatorname{div}(\widehat{A}\nabla \widehat{u}) = f$ by Lemma 2.1. It follows that $V = u_0$ in $\Omega_3 \setminus \overline{\Omega}_2$. Therefore, the compatibility condition holds, and u_0 is uniquely given by

$$u_0 = \begin{cases} \widehat{u} & \text{in } \Omega \setminus \Omega_3, \\ V & \text{in } \Omega_3 \setminus \Omega_2, \\ V \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ \widehat{u} \circ G \circ F & \text{in } \Omega_1. \end{cases} \tag{2.23}$$

Assume that the compatibility condition holds. Define u_0 by (2.23). One can verify that $u_0 \in H^1_0(\Omega)$ is a solution to (2.19) (see [27, Subsection 3.2.2] for the details). To prove that $u_\delta \to u_0$ weakly as $\delta \to 0$ in this case, we proceed as follows. Define $v_\delta = u_\delta - u_0$ in Ω. Then

$$\operatorname{div}(s_\delta A \nabla v_\delta) = \operatorname{div}((s_\delta - s_0) A \nabla u_0) \quad \text{in } \Omega.$$

Multiplying the equation by $\overline{v}_\delta$, the conjugate of v_δ, considering the real part and the imaginary part, we have (see [27, Lemma 1])

$$\|v_\delta\|_{H^1(\Omega)} \le C\delta^{-1} \| \operatorname{div}((s_\delta - s_0) A \nabla u_0)\|_{H^{-1}(\Omega)} \le C$$

for some positive constant C independent of δ. Hence (u_δ) is bounded in $H^1(\Omega)$. Since u_0 is unique, a standard compactness argument yields that $u_\delta \to u_0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.

It remains to prove Case 2. The proof is based on a contradiction argument. Assume that (2.21) does not hold. It follows that $u_{\delta_n} \to u_0$ weakly in $H^1(\Omega)$ for some $(\delta_n) \to 0_+$. Then $V = u_0$ in $\Omega_3 \setminus \Omega_2$ and f is compatible. We have a contradiction.

The compatibility condition is not easy to verify in general. Nevertheless, we have [27, Corollary 2].

Proposition 2.1 *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$. Assume that A in $\Omega_3 \setminus \Omega_2$ and $-A$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$, and $G_* F_* A = A$ in $\Omega_3 \setminus \Omega_2$. Then $f \in L^2(\Omega)$ with* $\operatorname{supp} f \cap \Omega_3 = \emptyset$ *is compatible.*

Proposition 2.1 is a consequence of the fact $V = \widehat{u}$ in $\Omega_3 \setminus \Omega_2$ since $\widehat{A} = A$ in $\Omega_3 \setminus \Omega_2$. Its applications on superlensing and cloaking a source via ALR are given in Sections 3 and 5 respectively. It is clear that the setting of Nicorovici, McPhedran and Milton satisfies Proposition 2.1 with $\Omega_j = B_j$ $(j = 1, 2, 3)$, and F and G are the Kelvin transforms used in the heuristic argument in Subsection 2.1.

3 Superlensing Using Complementary Media

The construction of a superlens using NIMs was first suggested by Veselago [49] for a slab lens. The superlensing property of the slab lens was also studied by Veselago via the ray theory in [49]. Later, the study of cylindrical lenses in the two dimensional quasistatic regime, the Veselago slab, cylindrical lenses and spherical lenses in the finite frequency regime were considered by Nicorovici, McPhedran and Milton in [42], Pendry in [44–45], and Pendry and Ramakrishna in [47], respectively for some constant isotropic objects.

Let us describe how to magnify m-times the region B_{r_0}, for some $r_0 > 0$ and $m > 1$, in the quasistatic regime in which the medium is characterized by a matrix-valued function a using complementary media. This has roots from [28]. The assumption on the geometry of the object by all means imposes no restriction, since any region can be placed in such a ball provided that the radius and the origin are appropriately chosen. The idea suggested in [42, 45, 47] is to put a lens in $B_{r_2} \setminus B_{r_0}$ whose medium is characterized by matrix $-b$ with $\frac{r_2^2}{r_0^2} = m$. Here $b = I$, the identity matrix, in two dimensions and $b = \big(\frac{r_2^2}{|x|^2}\big) I$ in three dimensions. Our class of superlenses is slightly different from the suggestion mentioned above and motivated from the reflecting complementary media in Section 2. Let $\alpha > 1$ and define $F : B_{r_2} \setminus \{0\} \to \mathbb{R}^d \setminus \overline{B}_{r_2}$ by

$$F(x) = \frac{r_2^{\alpha} x}{|x|^{\alpha}}. \tag{3.1}$$

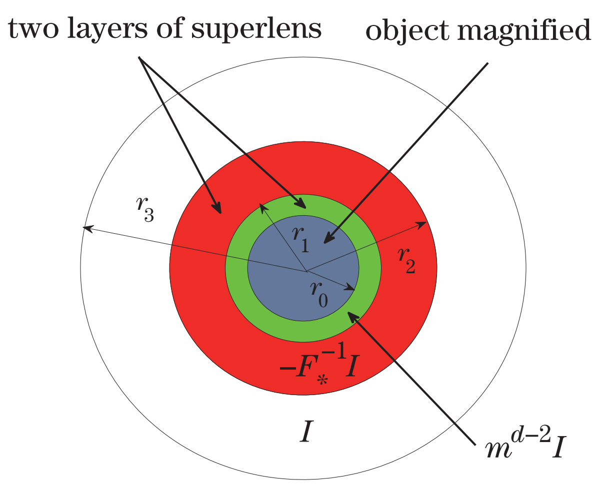

Figure 1 The lensing device contains two parts. The first part $-F_*^{-1}I$ (the red region) in $B_{r_2} \setminus B_{r_1}$ is the complement of I in $B_{r_3} \setminus B_{r_2}$. The second part (the green region) is $m^{d-2}I$ in $B_{r_1} \setminus B_{r_0}$. The magnified region is B_{r_0}.

Our lens contains two parts. The first one is given by

$$-F_*^{-1} I \quad \text{in } B_{r_2} \setminus B_{r_1}, \tag{3.2}$$

and the second one is

$$m^{d-2} I \quad \text{in } B_{r_1} \setminus B_{r_0}. \tag{3.3}$$

Here r_1 and r_2 are such that

$$mr_0 = r_2 \quad \text{and} \quad \frac{r_3}{r_1} = m, \quad \text{where } r_3 := \frac{r_2^\alpha}{r_1^{\alpha-1}}. \tag{3.4}$$

With the loss, the medium is characterized by $s_\delta A$,[5] where

$$A = \begin{cases} F_*^{-1} I & \text{in } B_{r_2} \setminus B_{r_1}, \\ m^{d-2} I & \text{in } B_{r_1} \setminus B_{r_0}, \\ a & \text{in } B_{r_0}, \\ I & \text{otherwise} \end{cases} \quad \text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.5}$$

Let Ω be a smooth open subset of $\mathbb{R}^d$ ($d = 2, 3$) such that $B_{r_3} \subset \Omega$. Given $f \in L^2(\Omega)$, let u_δ, $\widehat{u} \in H^1_0(\Omega)$ be respectively the unique solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega \tag{3.6}$$

and

$$\operatorname{div}(\widehat{A} \nabla \widehat{u}) = f \quad \text{in } \Omega, \qquad \text{where } \widehat{A} = \begin{cases} m^{2-d} a\Big(\dfrac{x}{m}\Big) & \text{in } B_{mr_0}, \\ I & \text{otherwise.} \end{cases} \tag{3.7}$$

We have the following theorem.

Theorem 3.1 *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_3}$. We have*

$$u_\delta \to \widehat{u} \quad \textit{weakly in } H^1(\Omega \setminus B_{r_3}) \textit{ as } \delta \to 0. \tag{3.8}$$

For an observer outside B_{r_3}, the object a in B_{r_0} would act like

$$m^{2-d} a\Big(\frac{x}{m}\Big) \quad \text{in } B_{mr_0}$$

by (3.8). Then one has a superlens whose magnification is m.

We next give some comments on the lens construction and explain how to obtain Theorem 3.1 from Theorem 2.1 and Proposition 2.1. The first part of the lens with $\alpha = 2$ is the same as the known superlens constructions mentioned. Given r_1, one requires that $\frac{r_3}{r_1} = m$ since a superlens of m times magnification is considered as in [42, 45, 47] and the comments for (2.3) (see also (3.7) and Theorem 3.1). Let $G : \mathbb{R}^d \setminus \overline{B}_{r_3} \to B_{r_3} \setminus \{0\}$ be defined by $G(x) = \frac{r_3^\beta x}{|x|^\beta}$, where $\beta = \frac{\alpha}{\alpha - 1}$. Then $G \circ F : B_{r_1} \to B_{r_3}$ satisfies

$$G \circ F(x) = mx \quad \text{in } B_{r_1}. \tag{3.9}$$

This implies, since $A = m^{d-2} I$ in $B_{r_1} \setminus B_{r_0}$ and $r_2 = mr_0$,

$$G_* F_* A = I \quad \text{in } B_{r_3} \setminus B_{r_2}. \tag{3.10}$$

This is the place where the second part of the construction plays its role. From (3.9)–(3.10), one has

$$F_* A = G_* F_* A = A = I \quad \text{in } B_{r_3} \setminus B_{r_2}.$$

[5]In [28], $s_\delta = -1 + i\delta$ in $B_{r_2} \setminus B_{r_1}$; nevertheless this point is not essential.

Theorem 3.1 is now a direct consequence of Theorem 2.1 and Proposition 2.1.

A more delicate analysis implies that (3.8) holds for every $f \in L^2(\Omega)$ with supp $f \cap \overline{B}_{r_2} = \emptyset$ (see Theorems 5.1 and 5.3). Theorem 3.1 can be easily extended for the finite frequency regime using the concept of reflecting complementary media since Theorem 2.1 and Proposition 2.1 also hold in this regime (see [28] for the original proof).

Remark 3.1 In [28], the condition $mr_0 = r_2$ in (3.4) is replaced by $mr_0 \le \sqrt{r_2 r_3}$. Taking the advantage of (3.10), the proof of Theorem 3.1 follows easily from Section 2. The first original proof of superlensing in the acoustic setting was given in [28] in both quasistatic and finite frequency regimes. It is based on the removing singularity technique introduced in [28, 31]. The proof of superlensing in the electromagnetic setting is given in [33] in the same spirit of the proof given here. The removing singularity technique is discussed in the next two sections where cloaking using complementary media and cloaking a source via ALR are dealt with.

This section ends with the following question on the necessity of the second layer in the lens construction.

Open Question 3.1 Does Theorem 3.1 hold where A is given in (3.5) with $r_1 = r_0$?

4 Cloaking Using Complementary Media

Cloaking using complementary media was suggested by Lai et al. in [21]. The idea is to cancel the effect of the object by its complementary medium. The study of this problem faces two difficulties. Firstly, this problem is unstable since the equations describing the phenomenon have sign changing coefficients, hence the ellipticity and the compactness are lost in general. Secondly, the localized resonance might appear.

Let us describe how to cloak the region $B_{2r_2} \setminus B_{r_2}$ for some $r_2 > 0$ in the quasistatic regime in which the medium is characterized by a matrix a using complementary media. The assumption on the cloaked region by all means imposes no restriction, since any bounded set is a subset of such a region provided that the radius and the origin are appropriately chosen. The idea suggested by Lai et al. in [21] (in two dimensions) is to construct a complementary medium in $B_{r_2} \setminus B_{r_1}$ for some $0 < r_1 < r_2$. Inspired by their idea, in [28], we constructed cloaking devices in two and three dimensions and gave the first proof of cloaking using complementary media. Our cloak consists of two parts. The first one, in $B_{r_2} \setminus B_{r_1}$, makes use of reflecting complementary media to cancel the effect of the cloaked region, and the second one (the new one), in B_{r_1}, is to fill the space which "disappears" from the cancellation by the homogeneous medium. For the first part, we modified the strategy in [21]. Instead of $B_{2r_2} \setminus B_{r_2}$, we consider $B_{r_3} \setminus B_{r_2}$ for some $r_3 > 0$ as the cloaked region in which the medium is given by the matrix

$$b = \begin{cases} a & \text{in } B_{2r_2} \setminus B_{r_2}, \\ I & \text{in } B_{r_3} \setminus B_{2r_2}. \end{cases}$$

We will assume that

$$b \in C^1(\overline{B}_{r_3} \setminus B_{r_2}). \tag{4.1}$$

The complementary medium in $B_{r_2} \setminus B_{r_1}$ is given by

$$-(F^{-1})_* b,$$

where $F : B_{r_2} \setminus \overline{B}_{r_1} \to B_{r_3} \setminus \overline{B}_{r_2}$ is the Kelvin transform with respect to ∂B_{r_2}. Concerning the second part, the medium in B_{r_1} is given by

$$\Big(\frac{r_3^2}{r_2^2}\Big)^{d-2} I. \tag{4.2}$$

The reason for this choice is the condition

$$G_* F_* A = I \quad \text{in } B_{r_3}, \tag{4.3}$$

where G is the Kelvin transform with respect to ∂B_{r_3} since the homogeneous medium is filled (see Theorem 2.1 and (2.20)). The cloaking scheme is illustrated in Figure 2.

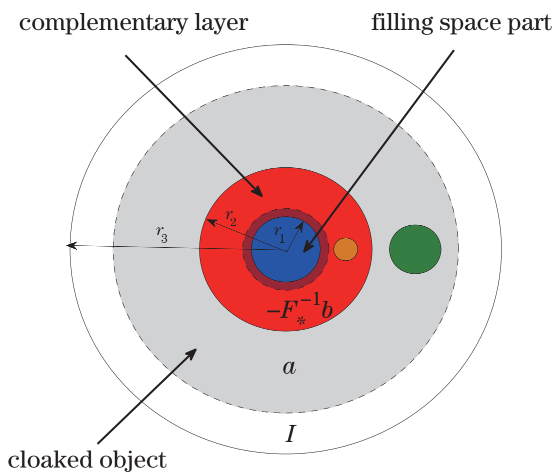

Figure 2 The cloaking device contains two parts. The first part $-F_*^{-1}b$ (the red and orange regions) in $B_{r_2} \setminus B_{r_1}$ is the complement of b which consists of a (grey and blue grey regions) in $B_{2r_2} \setminus B_{r_2}$ and I in $B_{r_3} \setminus B_{r_2}$. The second part (the blue region) $\big(\frac{r_3^2}{r_2^2}\big)^{d-2} I$ is to fill the space which disappears by the cancelation.

In two dimensions, the medium in B_{r_1} is I, as used in [21], while it is not I in three dimensions. With the loss, the medium is characterized by $s_\delta A$ where

$$A = \begin{cases} b & \text{in } B_{r_3} \setminus B_{r_2}, \\ F_*^{-1} b & \text{in } B_{r_2} \setminus B_{r_1}, \\ \Big(\frac{r_3^2}{r_2^2}\Big)^{d-2} I & \text{in } B_{r_1}, \\ I & \text{otherwise} \end{cases} \quad \text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

Given $f \in L^2(\Omega)$, let u_δ, $\widehat{u} \in H_0^1(\Omega)$ be respectively the unique solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega \tag{4.5}$$

and

$$\Delta \widehat{u} = f \quad \text{in } \Omega. \tag{4.6}$$

We established [31, Theorem 1].[6]

[6] In [31], $s_\delta = -1 + i\delta$ in $B_{r_2} \setminus B_{r_1}$; nevertheless this point is not essential.

Theorem 4.1 *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \backslash B_{r_3}$. There exists $\ell > 0$, depending only on r_2, and the ellipticity and the Lipschitz constants of b such that if $r_3 > \ell r_2$ then*

$$u_\delta \to \widehat{u} \quad \text{weakly in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0. \tag{4.7}$$

For an observer outside B_{r_3}, the medium in B_{r_3} looks as the homogeneous one by (4.7). Then one has cloaking.

Proof We only consider the two dimensional case; the proof in three dimensions follows similarly. Multiplying (3.6) by $\overline{u}_\delta$, integrating in Ω, considering first the imaginary part and then the real part, we have

$$\|u_\delta\|^2_{H^1(\Omega)} \le \frac{C}{\delta} \|u_\delta\|_{L^2(\Omega\setminus B_{r_3})} \|f\|_{L^2}. \tag{4.8}$$

Here and in what follows in the proof, C denotes a positive constant independent of δ and f. As in Section 2, define $u_{1,\delta} \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus B_{r_2})$ and $u_{2,\delta} \in H^1(B_{r_3})$ as follows:

$$u_{1,\delta} = u_\delta \circ F^{-1} \quad \text{in } \mathbb{R}^2 \setminus B_{r_2} \quad \text{and} \quad u_{2,\delta} = u_{1,\delta} \circ G^{-1} = u_\delta \circ F^{-1} \circ G^{-1} \quad \text{in } B_{r_3}.$$

Applying Lemma 2.1, we have

$$\operatorname{div}(b\nabla u_\delta) = \operatorname{div}(b\nabla u_{1,\delta}) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}.$$

Fix $\frac{1}{2} < \alpha < 1$. Applying [31, Lemma 1.1] (a three spheres inequality), we have, for ℓ large enough,

$$N(u_{1,\delta} - u_\delta, 3r_2)^2 \le C N(u_{1,\delta} - u_\delta, r_2)^{2\alpha} N(u_{1,\delta} - u_\delta, r_3)^{2(1-\alpha)}, \tag{4.9}$$

where

$$N(v, r) := \|v\|_{H^{\frac{1}{2}}(\partial B_r)} + \|\partial_r v\|_{H^{-\frac{1}{2}}(\partial B_r)}. \tag{4.10}$$

This implies

$$N(u_{1,\delta} - u_\delta, 3r_2)^2 \le C\delta^{2\alpha-1} \|u_\delta\|_{L^2(\Omega\setminus B_{r_3})} \|f\|_{L^2}. \tag{4.11}$$

Applying Lemma 2.1 again, we obtain

$$\Delta u_\delta = \Delta u_{1,\delta} = \Delta u_{2,\delta} = 0 \quad \text{in } B_{r_3} \setminus B_{3r_2}, \tag{4.12}$$

$$u_{2,\delta} = u_{1,\delta} \quad \text{and} \quad \partial_r u_{2,\delta} = (1 + \mathrm{i}\delta)\partial_r u_{1,\delta}|_{\text{int}} \quad \text{on } \partial B_{r_3}. \tag{4.13}$$

From (4.12), one can represent $u_{1,\delta}$, and $u_{2,\delta}$ of the forms

$$u_{1,\delta} = c_0 + d_0 \ln r + \sum_{\ell \ge 1} \sum_{\pm} (c_{\ell,\pm} r^\ell + d_{\ell,\pm} r^{-\ell}) \mathrm{e}^{\pm \mathrm{i}\ell\theta} \quad \text{in } B_{r_3} \setminus B_{3r_2}, \tag{4.14}$$

$$u_{2,\delta} = e_0 + f_0 \ln r + \sum_{\ell \ge 1} \sum_{\pm} (e_{\ell,\pm} r^\ell + f_{\ell,\pm} r^{-\ell}) \mathrm{e}^{\pm \mathrm{i}\ell\theta} \quad \text{in } B_{r_3} \setminus B_{3r_2} \tag{4.15}$$

for $c_0, d_0, e_0, f_0, c_{\ell,\pm}, d_{\ell,\pm}, e_{\ell,\pm}, f_{\ell,\pm} \in \mathbb{C}$ ($\ell \ge 1$). Since, by (4.8),

$$\begin{aligned} &\|u_{1,\delta}\|^2_{H^{\frac{1}{2}}(\partial B_{3r_2})} + \|\partial_r u_{1,\delta}\|^2_{H^{-\frac{1}{2}}(\partial B_{3r_2})} + \|u_{1,\delta}\|^2_{H^{\frac{1}{2}}(\partial B_{r_3})} + \|\partial_r u_{1,\delta}\|^2_{H^{-\frac{1}{2}}(\partial B_{r_3})} \\ &\le \frac{C}{\delta} \|f\|_{L^2} \|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}, \end{aligned}$$

we obtain

$$|c_0|^2 + |d_0|^2 + \sum_{\ell \geq 1} \sum_{\pm} \ell \left(|c_{\ell,\pm}|^2 r_3^{2\ell} + |d_{\ell,\pm}|^2 (3r_2)^{-2\ell} \right) \leq \frac{C}{\delta} \|f\|_{L^2} \|u_\delta\|_{L^2(\Omega \setminus B_{r_3})}. \tag{4.16}$$

From (4.13)–(4.15), a straightforward computation gives

$$\begin{cases} e_0 = c_0 - \mathrm{i}\delta d_0 \ln r_3, \\ f_0 = (1 + \mathrm{i}\delta) d_0 \end{cases} \quad \text{and} \quad \begin{cases} e_{\ell,\pm} = \dfrac{2 + \mathrm{i}\delta}{2} c_{\ell,\pm} - \dfrac{\mathrm{i}\delta}{2} d_{\ell,\pm} r_3^{-2\ell}, \\ f_{\ell,\pm} = -\dfrac{\mathrm{i}\delta}{2} c_{\ell,\pm} r_3^{2\ell} + \dfrac{2 + \mathrm{i}\delta}{2} d_{\ell,\pm} \end{cases} \quad \text{for } \ell \geq 1. \tag{4.17}$$

A combination of (4.14)–(4.15) and (4.17) yields, in $B_{r_3} \setminus B_{3r_2}$,

$$\begin{aligned} u_{1,\delta} - u_{2,\delta} = \mathrm{i}\delta d_0 (\ln r_3 - \ln r) &- \frac{\mathrm{i}\delta}{2} \sum_{\ell \geq 1} \sum_{\pm} (c_{\ell,\pm} - d_{\ell,\pm} r_3^{-2\ell}) r^{\ell} \mathrm{e}^{\pm \mathrm{i}\ell\theta} \\ &+ \frac{\mathrm{i}\delta}{2} \sum_{\ell \geq 1} \sum_{\pm} (c_{\ell,\pm} r_3^{2\ell} - d_{\ell,\pm}) r^{-\ell} \mathrm{e}^{\pm \mathrm{i}\ell\theta}. \end{aligned} \tag{4.18}$$

Set

$$\widehat{u}_\delta = \begin{cases} u_\delta & \text{in } x \in \Omega \setminus B_{r_3}, \\ u_\delta - \widehat{u}_{\delta,\mathrm{rem}} & B_{r_3} \setminus B_{3r_2}, \\ u_{2,\delta} & \text{in } x \in B_{3r_2}, \end{cases} \tag{4.19}$$

where

$$\widehat{u}_{\delta,\mathrm{rem}} = \mathrm{i}\delta d_0 (\ln r_2 - \ln r_3) + \frac{\mathrm{i}\delta}{2} \sum_{\ell \geq 1} \sum_{\pm} (c_{\ell,\pm} r_3^{2\ell} - d_{\ell,\pm}) r^{-\ell} \mathrm{e}^{\pm \mathrm{i}\ell\theta} \quad \text{for } x \in B_{r_3} \setminus B_{r_2}. \tag{4.20}$$

It is clear from the definition of $\widehat{u}_\delta$ that $\widehat{u}_\delta \in H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_1}))$,

$$\mathrm{div}(\widehat{A} \nabla \widehat{u}_\delta) = f \quad \text{in } \Omega \setminus (\partial B_{r_3} \cup \partial B_{3r_2}) \quad \text{and} \quad \widehat{u}_\delta = 0 \quad \text{on } \partial\Omega. \tag{4.21}$$

We claim that

$$\begin{aligned} &\|[\widehat{u}_\delta]\|^2_{H^{\frac{1}{2}}(\partial B_{r_2})} + \left\| \left[\widehat{A} \nabla \widehat{u}_\delta \cdot \frac{x}{|x|} \right] \right\|^2_{H^{-\frac{1}{2}}(\partial B_{r_2})} \|[\widehat{u}_\delta]\|^2_{H^{\frac{1}{2}}(\partial B_{r_3})} + \left\| \left[\widehat{A} \nabla \widehat{u}_\delta \cdot \frac{x}{|x|} \right] \right\|^2_{H^{-\frac{1}{2}}(\partial B_{r_3})} \\ &\leq C\delta \|\widehat{u}_\delta\|_{L^2(\Omega \setminus B_{r_3})} \|f\|_{L^2}. \end{aligned} \tag{4.22}$$

Here and in what follows $[\cdot]$ denotes the jump of a quantity across the boundary. Admitting this, we derive from (4.21) that

$$\|\widehat{u}_\delta\|_{H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{3r_2}))} \leq C \|f\|_{L^2},$$

as δ is small. Without loss of generality, one may assume that $\widehat{u}_\delta \to u$ weakly in $H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_1}))$. It is clear that $u \in H^1_0(\Omega)$ and $\mathrm{div}(\widehat{A} \nabla u) = f$ in Ω and hence $u = \widehat{u}$. Since the limit is unique, the convergence holds for the whole family $(\widehat{u}_\delta)$. The conclusion follows in two dimensions.

It remains to prove (4.22). In fact, it is a consequence of (4.8), (4.11), (4.16), and the definition of $\widehat{u}_\delta$. The proof is complete.

Remark 4.1 In the proof, we remove $\widehat{u}_{\delta,\mathrm{rem}}$, the singular part of $\widehat{u}_\delta$, from $\widehat{u}_\delta$ in $B_{r_3} \setminus B_{r_2}$. The function $\widehat{u}_{\delta,\mathrm{rem}}$ becomes more and more singular as r is smaller and smaller and behaves smoothly for large r. This is the idea of the removing localized singularity technique which was introduced in [28, 31]. The removing term $\widehat{u}_{\delta,\mathrm{rem}}$ is in the same spirit with the removing term of "infinite" energy in the theory of Ginzburg-Landau equations proposed by Bethuel, Brezis, and Hélein in [4] and was inspired by their work. In [4], after removing the infinite energy term, one obtained the renormalized energy introduced there. Here, after removing the bad term, the gluing function $\widehat{u}_\delta$ satisfies a standard elliptic equation which characterizes the reflecting medium.

Remark 4.2 Theorem 4.1 hods if A in B_{r_2} is chosen such that $F_*A = b$ in $B_{r_3} \setminus B_{r_2}$ and $G_*F_*A = I$ in B_{r_3} for some F and G as in Definition 2.1 with $\Omega_j = B_{r_j}$ for $j = 1, 2, 3$. This means

$$A = \begin{cases} F_*^{-1} b & \text{in } B_{r_2} \setminus B_{r_1}, \\ F_*^{-1} G^{-1} * I & \text{in } B_{r_1} \end{cases} \tag{4.23}$$

for such a pair (F, G). In particular, Theorem 4.1 hods if A in B_{r_2} is given in (4.4) in which $F(x) := r_2^\alpha \frac{x}{|x|}^\alpha$ for some $\alpha > 1$ (see also Section 3).

Remark 4.3 The method presented here was extended for the Helmholtz equation in [38] in joint work with Nguyen. To this end, we established new type of three spheres inequalities for the Helmholtz equations in which no condition of the smallness of radii is imposed.

We have the following question on the necessity of the layer I in $B_{r_3} \setminus B_{2r_2}$.

Open Question 4.1 Does Theorem 4.1 hold where A is given in (3.5) with $r_3 = 2r_2$?

Remark 4.4 Cloaking can also be achieved via transformation optics or changes of variables. Resonance might also appear in this context (see [19, 25–26, 39]). It is shown in [26] that in the resonance case cloaking might not be achieved and the field inside the cloaked region can depend on the field outside. Cloaking can also be achieved in the time regime via change of variables (see [40–41]).

5 Cloaking a Source via Anomalous Localized Resonance

Cloaking a source via ALR was discovered by Milton and Nicorovici in [23] for constant radial symmetric plasmonic structures in the two dimensional quasi-static regime. Their work has its roots from [42] (see also [22]) where the localized resonance was observed and established for such a setting. More precisely, in [23], the authors studied the setting (2.1)–(2.2). They showed that a dipole is not seen by an observer away from the core-shell structure hence it is cloaked, if and only if the dipole is within distance $r_* := \sqrt{\frac{r_2^3}{r_1}}$ of the shell; moreover, the power $E_\delta(u_\delta)$ of the field u_δ, which is roughly speaking $\delta \|u_\delta\|_{H^1}^2$, blows up. Two key features of this phenomenon are as follows:

(1) The localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss goes to 0.

(2) The connection between the localized resonance and the blow up of the power as the loss goes to 0.

Their work has attracted many investigations, see [2–3, 11–14, 18, 24, 37] in which special structures were considered due to the use of the separation of variables or the blow up of the power was investigated. An important class of NIMs in which the localized resonance might appear is the class of reflecting complementary media in Section 2 (see also Sections 3–4). Nevertheless, the complementary property is not enough to ensure that cloaking a source via ALR takes place, and there is no connection between the blow up of the power and the localized resonance in general as discussed in [37] (joint work with Nguyen).

In [29–30], we investigated CALR for a source for a subclass of complementary media called the class of doubly complementary media for a core-shell structure. Let $d = 2, 3$, and Ω be a smooth open bounded subset of $\mathbb{R}^d$, and let $0 < r_1 < r_2$ be such that $B_{r_2} \subset\subset \Omega$. Set, for $\delta \geq 0$,[7]

$$s_\delta := \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases} \tag{5.1}$$

Let A be a symmetric uniformly elliptic matrix-valued function defined in Ω. The definition of doubly complementary media [30, Definition 1.2] is as follows.

Definition 5.1 *The medium $s_0 A$ is said to be doubly complementary if for some $r_3 > 0$ with $B_{r_3} \subset\subset \Omega$, A in $B_{r_3} \setminus B_{r_2}$ and $-A$ in $B_{r_2} \setminus B_{r_1}$ are reflecting complementary, and*

$$F_* A = G_* F_* A = A \quad \text{in } B_{r_3} \setminus B_{r_2} \tag{5.2}$$

for some F and G coming from Definition 2.1 *with $\Omega_j = B_{r_j}$ for $j = 1, 2, 3$.*

Remark 5.1 Roughly speaking, the shell $B_{r_2} \setminus B_{r_1}$ is not only reflecting complementary to a part of the matrix but also to a part of the core. Indeed, $-A$ in $B_{r_2} \setminus B_{r_1}$ is not only complementary to A in $B_{r_3} \setminus B_{r_2}$ but also to A in $(G \circ F)^{-1}(B_{r_3} \setminus \overline{B_{r_2}})$ (a subset of B_{r_1}) (see Figure 3).

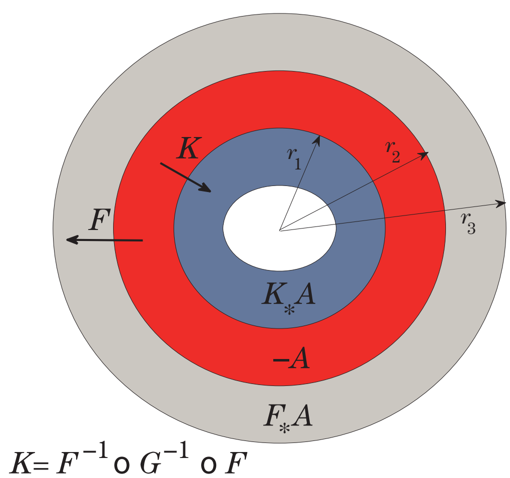

Figure 3 $s_0 A$ is doubly complementary: $-A$ in $B_{r_2} \setminus B_{r_1}$ (the red region) is complementary to $A = F_* A$ in $B_{r_3} \setminus B_{r_2}$ (the grey region) and $A = K_* A$ with $K = F^{-1} \circ G^{-1} \circ F$ in $K(B_{r_2} \setminus B_{r_1})$ (the blue grey region).

[7] In [30], $s_\delta = -1 + i\delta$ in $B_{r_2} \setminus B_{r_1}$; nevertheless this point is not essential.

Let $f \in L^2(\Omega)$ with $\operatorname{supp} f \cap B_{r_2} = \emptyset$ and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega. \tag{5.3}$$

The power $E_\delta(u_\delta)$ is defined by (see, e.g., [23])

$$E_\delta(u_\delta) = \delta \int_{B_{r_2} \setminus B_{r_1}} |\nabla u_\delta|^2.$$

Using the fact that $u_\delta = 0$ on $\partial \Omega$, one has [8]

$$\int_\Omega (|\nabla u_\delta|^2 + |u_\delta|^2) \le C\Big(\int_{B_{r_2} \setminus B_{r_1}} |\nabla u_\delta|^2 + \|f\|^2_{L^2}\Big) \tag{5.4}$$

for some positive constants C independent of f and $\delta \in (0,1)$. Let $v_\delta \in H^1_0(\Omega)$ be the unique solution to

$$\operatorname{div}(s_\delta A \nabla v_\delta) = f_\delta \quad \text{in } \Omega. \tag{5.5}$$

Here $f_\delta = c_\delta f$ and c_δ is the normalization constant such that

$$\delta \int_{B_{r_2} \setminus B_{r_1}} |\nabla v_\delta|^2 = 1. \tag{5.6}$$

In what follows in this section, we assume that

$$A \in [C^3(\overline{B_{r_3} \setminus B_{r_2}})]^{d \times d}. \tag{5.7}$$

The equivalence between the blow up of the power and cloaking a source via ALR for doubly complementary media can be derived from the following result [30, Proposition 4.1].

Theorem 5.1 *Let $d = 2, 3$, let $(\delta_n) \to 0$, $(g_n) \subset L^2(\Omega)$ with $\operatorname{supp} g_n \subset \Omega \setminus B_{r_2}$, and let $v_n \in H^1_0(\Omega)$ be the unique solution to*

$$\operatorname{div}(s_{\delta_n} A \nabla v_n) = g_n \quad \text{in } \Omega.$$

Assume that $s_0 A$ is doubly complementary, $g_n \to g$ weakly in $L^2(\Omega)$ for some $g \in L^2(\Omega)$, and

$$\lim_{n \to \infty} \delta_n \|\nabla v_n\|_{L^2(B_{r_2} \setminus B_{r_1})} = 0. \tag{5.8}$$

Then $v_n \to \widehat{v}$ weakly in $H^1(\Omega \setminus B_{r_3})$ where $\widehat{v} \in H^1_0(\Omega)$ is the unique solution to

$$\operatorname{div}(\widehat{A} \nabla \widehat{v}) = g \quad \text{in } \Omega.$$

Here, as usual,

$$\widehat{A} := \begin{cases} A, & \text{if } x \in \Omega \setminus B_{r_3}, \\ G_* F_* A, & \text{if } x \in B_{r_3}. \end{cases}$$

The equivalence between the blow up of the power and the cloaking a source via ALR can be obtained from Theorem 5.1 as follows. Suppose that the power blows up, i.e.,

$$\lim_{n \to \infty} \delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2} \setminus B_{r_1})} = +\infty.$$

[8] One way to obtain this inequality is to multiply (5.3) by $\overline{u}_\delta$, integrate on Ω, and consider the real part.

Then, by Theorem 5.1, $v_{\delta_n} \to 0$ in $\Omega \setminus B_{r_3}$. The source $\alpha_{\delta_n} f$ is not seen by observers far away from the shell: The source is cloaked. Note that the localized resonance happens in this case since (5.6) takes place. If the power of u_{δ_n} remains bounded, then $u_{\delta_n} \to \widehat{u}$ weakly in $H^1(\Omega \setminus B_{r_3})$, where $\widehat{u} \in H^1_0(\Omega)$ is the unique solution to $\operatorname{div}(\widehat{A}\nabla \widehat{u}) = f$ in Ω, the source is not cloaked.

We next present the proof in the case $A = I$ in $B_{r_3} \setminus B_{r_2}$ in two dimensions to highlight the use of the removing localized singularity technique. This situation is already non-trivial and the standard separation of variables is out of reach, since A can be arbitrary outside B_{r_3}.

Sketch of the Proof of Theorem 5.1 (Under the additional assumption that $A = I$ in $B_{r_3} \setminus B_{r_2}$ and $d = 2$.) Using (5.4), we derive from (5.8) that

$$\lim_{n \to \infty} \delta_n \|v_n\|_{H^1(\Omega)} = 0. \tag{5.9}$$

Define

$$v_{1,n} = v_n \circ F^{-1} \quad \text{in } \mathbb{R}^2 \setminus B_{r_2} \quad \text{and} \quad v_{2,n} = v_{1,n} \circ G^{-1} \quad \text{in } B_{r_3}.$$

Since $A = I$ in $B_{r_3} \setminus B_{r_2}$, it follows from (5.2) and Lemma 2.1 that

$$\Delta v_{1,n} = \Delta v_{2,n} = 0 \quad \text{in } B_{r_3} \setminus B_{r_2} \tag{5.10}$$

and

$$v_{1,n} = v_{2,n}, \quad \partial_r v_{1,n} = \frac{1}{1 + i\delta_n} \partial_r v_{2,n} \quad \text{on } \partial B_{r_3}. \tag{5.11}$$

From (5.10), $v_{1,n}$ and $v_{2,n}$ can be represented respectively as follows:

$$\begin{aligned} v_{1,n} &= c_0 + d_0 \ln r + \sum_{\ell \geq 1} \sum_{\pm} (c_{\ell,\pm} r^\ell + d_{\ell,\pm} r^{-\ell}) e^{\pm i\ell\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \\ v_{2,n} &= e_0 + f_0 \ln r + \sum_{\ell \geq 1} \sum_{\pm} (e_{\ell,\pm} r^\ell + f_{\ell,\pm} r^{-\ell}) e^{\pm i\ell\theta} \quad \text{in } B_{r_3} \setminus B_{r_2} \end{aligned}$$

for $c_0, d_0, e_0, f_0, c_{\ell,\pm}, d_{\ell,\pm}, e_{\ell,\pm}, f_{\ell,\pm} \in \mathbb{C}$ ($\ell \geq 1$). Using (5.11), as in the proof of Theorem 4.1, there exists $\widehat{v}_{n,\mathrm{rem}} \in H^1(B_{r_3} \setminus B_{r_2})$ such that $\Delta \widehat{v}_{n,\mathrm{rem}} = 0$ in $B_{r_3} \setminus B_{r_2}$, and

$$N(\widehat{v}_{n,\mathrm{rem}}, r_3)^2 + N(v_{2,n} - v_{1,n} - \widehat{v}_{n,\mathrm{rem}}, r_2)^2 \leq C\delta_n \|v_n\|_{L^2(\Omega \setminus B_{r_3})} \|f\|_{L^2},$$

where $N(\cdot, r)$ is defined in (4.10). The conclusion now follows as in the proof of Theorem 4.1. Define

$$\widehat{v}_n = \begin{cases} v_n & \text{in } x \in \Omega \setminus B_{r_3}, \\ v_n - \widehat{v}_{n,\mathrm{rem}} & B_{r_3} \setminus B_{r_2}, \\ v_{2,n} & \text{in } x \in B_{r_2}. \end{cases} \tag{5.12}$$

Then $\widehat{v}_n \to \widehat{v}$ weakly in $H^1(\Omega \setminus (\partial B_{r_2} \cup \partial B_{r_3}))$. The proof is complete.

To develop the approach presented above for a general core-shell structure, in [30], we introduced and implemented the separation of variables technique to solve Cauchy problems in a general shell. The main idea behind the method is to find an appropriate dense set of solutions to the equation

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}$$

to replace $r^{\ell}e^{\pm i\ell\theta}$ and $r^{-\ell}e^{\pm i\ell\theta}$ in the case $A = I$ and $d = 2$. Indeed, from [30, Proposition 4.2], there exist two families $(v_\ell)_{\ell\geq 1}$ and $(w_\ell)_{\ell \geq 1}$ such that

$$\operatorname{div}(A\nabla w_\ell) = \operatorname{div}(A\nabla v_\ell) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{5.13}$$

$$w_\ell = v_\ell \quad \text{on } \partial B_{r_3} \quad \text{and} \quad A\nabla w_\ell \cdot \frac{x}{|x|} = -A\nabla v_\ell \cdot \frac{x}{|x|} \quad \text{on } \partial B_{r_3} \tag{5.14}$$

and

$$\{v_\ell, w_\ell;\ \ell \geq 0\} \text{ is dense in } \{v \in H^1(B_{r_3} \setminus B_{r_2}),\ \operatorname{div}(A\nabla v) = 0\}.$$

Here $v_0 = 1$ in $B_{r_3} \setminus B_{r_2}$ and $w_0 \in H^1(B_{r_3} \setminus B_{r_2})$ is the unique solution to

$$\operatorname{div}(A\nabla w_0) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}, \quad w_0 = 1 \quad \text{on } \partial B_{r_3}, \quad w_0 = 0 \quad \text{on } \partial B_{r_2}.$$

More properties on the behaviors of (w_ℓ) and (v_ℓ) are required in the proof of Theorem 5.1 (see [30, Proposition 4.2] for the details). Properties (5.13)–(5.14) are very suitable for the use of removing localized singularity technique. Due to the lack of the orthogonality of v_ℓ and w_ℓ, the implementation of this technique in the general case is more delicate.

In [30], we also showed that the power blows up if the source is located "near" the shell even for reflecting complementary media (see [30, Theorem 2]).

Theorem 5.2 *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$, and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to*

$$\operatorname{div}(s_\delta A\nabla u_\delta) = f \quad \text{in } \Omega.$$

Assume that A in $B_{\hat{r}_3} \setminus B_{r_2}$ and $-A$ in $B_{r_2} \setminus B_{\hat{r}_1}$ are reflecting complementary for some $r_1 \leq \hat{r}_1 < r_2 < \hat{r}_3$, with $B_{\hat{r}_3} \subset\subset \Omega$. There exists a constant $r_* \in (r_2, \hat{r}_3)$, independent of δ and f such that if there is no $w \in H^1(B_{r_*} \setminus B_{r_2})$ with the properties

$$\operatorname{div}(A\nabla w) = f \quad \text{in } B_{r_*} \setminus B_{r_2}, \quad w = 0 \quad \text{on } \partial B_{r_2}, \quad A\nabla w \cdot \nu = 0 \quad \text{on } \partial B_{r_2}, \tag{5.15}$$

then

$$\limsup_{\delta\to 0} \delta^{\frac{1}{2}} \|\nabla u_\delta\|_{L^2(B_{r_2}\setminus B_{r_1})} = +\infty. \tag{5.16}$$

Assume in addition that $A = I$ in $B_{\hat{r}_3} \setminus B_{r_2}$, then

$$r_* \text{ can be taken by any number less than } \sqrt{\hat{r}_3 r_2}. \tag{5.17}$$

Proof We will sketch the proof in the case $\hat{r}_3 = r_3$ and $A = I$ in $B_{r_3} \setminus B_{r_2}$. The proof in the general case follows by the same approach via three spheres inequalities. The proof is again based on the use of reflection. Define $u_{1,\delta} = u_\delta \circ F^{-1}$ and set $w_\delta = u_\delta - u_{1,\delta}$ in $B_{r_3} \setminus B_{r_2}$. We have

$$\Delta w_\delta = f \quad \text{in } B_{r_3} \setminus B_{r_2}, \quad w_\delta = 0 \quad \text{on } \partial B_{r_2}, \quad \partial_r w_\delta = \frac{i\delta}{1 + i\delta} \partial_r u_\delta \quad \text{on } \partial B_{r_2}.$$

We prove by contradiction that

$$\limsup_{n\to+\infty} \delta_n(\|w_{\delta_n}\|^2_{H^1(B_{r_3}\setminus B_{r_2})} + \|\partial_r u_{\delta_n}\|^2_{H^{-\frac{1}{2}}(\partial B_{r_2})}) = +\infty, \tag{5.18}$$

where $\delta_n = 2^{-n}$. Assume that

$$m := \delta_n(\|w_{\delta_n}\|^2_{H^1(B_{r_3}\setminus B_{r_2})} + \|\partial_r u_{\delta_n}\|^2_{H^{-\frac{1}{2}}(\partial B_{r_2})}) < +\infty. \tag{5.19}$$

We claim that (w_{δ_n}) is a Cauchy sequence in $H^1(B_{r_*} \setminus B_{r_2})$. Indeed, set

$$W_n = w_{\delta_{n+1}} - w_{\delta_n} \quad \text{in } B_{r_3} \setminus B_{r_2}.$$

It follows from (5.19) that

$$\Delta W_n = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}, \quad W_n = 0 \quad \text{on } \partial B_{r_2}, \quad \|\partial_r W_n\|_{H^{-\frac{1}{2}}(\partial B_{r_2})} \le Cm\delta^{-\frac{n}{2}}.$$

In this proof, C denotes a constant independent of n. Using the following standard three spheres inequality:

$$\|\varphi\|_{H^1(B_R)} \le C\|\varphi\|^{\alpha}_{H^1(B_{R_1})}\|\varphi\|^{1-\alpha}_{H^1(B_{R_2})},$$

where $\alpha = \frac{\ln\left(\frac{R_2}{R}\right)}{\ln\left(\frac{R_2}{R_1}\right)}$, if $\Delta\varphi = 0$ in B_{R_2} and $0 < R_1 < R < R_2$, one can prove

$$\|W_n\|_{H^1(B_{r_*}\setminus B_{r_1})} \le Cm2^{-n\frac{2\alpha-1}{2}},$$

where $\alpha = \frac{\ln(\frac{r_3}{r_*})}{\ln(\frac{r_3}{r_2})} > \frac{1}{2}$. Thus (w_n) is a Cauchy sequence in $H^1(B_{r_*} \setminus B_{r_2})$. Let w be the limit of w_n in $H^1(B_{R_*} \setminus B_{R_1})$. Then

$$\Delta w = f \quad \text{in } B_{r_*} \setminus B_{r_2}, \quad w = 0 \quad \text{on } \partial B_{r_2}, \quad \partial_r w = 0 \quad \text{on } \partial B_{r_2}.$$

This contradicts the non-existence of w. Hence (5.18) holds. The proof is complete.

Remark 5.2 Theorem 5.2 implies the result in [18] on the blow up of the power.

Concerning the boundedness of the power, we can establish the following more general result.

Theorem 5.3 *Let $d = 2, 3$, $f \in L^2(\Omega)$, and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to* (5.3). *Assume that $s_0 A$ is doubly complementary and* $\operatorname{supp} f \cap B_{r_3} = \emptyset$. *Then*

$$\limsup_{\delta\to 0} \|u_\delta\|_{H^1(\Omega)} < +\infty. \tag{5.20}$$

Assume in addition that $A = I$ in $B_{r_3} \setminus B_{r_2}$. If there exists $w \in H^1(B_{r_0} \setminus B_{r_2})$ for some $r_0 > r_2^\alpha r_3^{1-\alpha}$ with the properties

$$\operatorname{div}(A\nabla w) = f \quad \text{in } B_{r_0} \setminus B_{r_2}, \quad w = 0 \quad \text{on } \partial B_{r_2}, \quad A\nabla w \cdot \nu = 0 \quad \text{on } \partial B_{r_2}$$

for some $0 < \alpha < 1$, then

$$\limsup_{\delta\to 0} \delta^\alpha \|u_\delta\|_{H^1(\Omega)} < +\infty. \tag{5.21}$$

Remark 5.3 The first part (5.20) is from Theorem 2.1. The second part (5.21) with $\alpha = \frac{1}{2}$ is given in [30, Theorem 3]. The proof in this case is based on a kind of removing singularity technique. The proof of the result stated here follows similarly. One just needs to replace the constant ξ_ℓ defined in [30, (3.10)] by $\delta^\alpha \big(\frac{r_3}{r_0}\big)^\ell$. The details are left to the reader.

Using Theorems 5.1 and 5.2, we can construct a cloaking device to cloak a general source concentrate on a manifold of codimension 1 in an arbitrary medium (see [30, Section 5] for the details).

6 Limitting Absorption Principle and Well-Posedness of the Helmholtz Equation with Sign Changing Coefficients

Let $k > 0$ and let A be a (real) uniformly elliptic symmetric matrix defined on $\mathbb{R}^d$ ($d \geq 2$), and Σ be a bounded real function defined on $\mathbb{R}^d$. Assume that [9]

$$\begin{aligned} &A(x) = I \quad \text{in } \mathbb{R}^d \setminus B_{R_0} \quad \text{and} \quad A \text{ is piecewise } C^1, \\ &\Sigma(x) = 1 \quad \text{in } \mathbb{R}^d \setminus B_{R_0} \end{aligned}$$

for some $R_0 > 0$. Let $D \subset\subset B_{R_0}$ be a bounded open subset in $\mathbb{R}^d$ of class C^2. Set, for $\delta \geq 0$,

$$s_\delta(x) = \begin{cases} -1 - i\delta & \text{in } D, \\ 1 & \text{in } \mathbb{R}^d \setminus D. \end{cases}$$

In [32], we studied the well-posedness of the following equation:

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = f \quad \text{in } \mathbb{R}^d, \tag{6.1}$$

under various conditions on A and Σ. To make sure that physics solutions are considered, we also study the limiting absorption principle associated with (6.1), i.e., the convergence of u_δ to u_0 (in an appropriate sense). Here $u_\delta \in H^1(\mathbb{R}^d)$ is the unique solution of the equation

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta + i\delta u_\delta = f \quad \text{in } \mathbb{R}^d. \tag{6.2}$$

Recall that a solution $v \in H^1_{\text{loc}}(\mathbb{R}^d \setminus B_R)$ of the equation

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^d \setminus B_R$$

for some $R > 0$, is said to satisfy the outgoing condition if

$$\partial_r v - ikv = o(r^{-\frac{d-1}{2}}) \quad \text{as } r = |x| \to +\infty.$$

Denote

$$\Gamma = \partial D,$$

and, for $\tau > 0$, set

$$D_\tau = \big\{x \in D;\ \operatorname{dist}(x, \Gamma) < \tau\big\} \quad \text{and} \quad D_{-\tau} = \big\{x \in \mathbb{R}^d \setminus \overline{D};\ \operatorname{dist}(x, \Gamma) < \tau\big\}. \tag{6.3}$$

The well-posedness of the Helmholtz equation with sign changing coefficients was first established by Costabel and Stephan in [15]. They proved, by the integral method, that (6.1)

[9] The smoothness assumption of A is mainly required for the use of the unique continuation principle. It can be omitted in two dimensions.

is well-posed if $A = I$ in $\mathbb{R}^d \setminus D$ and $A = \lambda I$ in D provided that λ is positive constant not equal to 1. Later, Ola in [43] proved, using the integral method and the pseudo-differential operators theory, that (6.1) is well-posed in three and higher dimensions if the interface Γ is strictly convex and connected even though $\lambda = 1$, i.e., $A = I$ in $\mathbb{R}^d$. His result was extended for the case where Γ has two connected components by Kettunen, Lassas and Ola in [17]. Recently, the well-posedness was extensively studied by Bonnet-Ben Dhia, Ciarlet, and their coauthors in [5–9] by the T-coercivity approach. This approach was introduced by Bonnet-Ben Dhia, Ciarlet, and Zwölf in [9] and is related to the (Banach-Necas-Babuska) inf-sup condition. The sharpest condition for the acoustic setting in this direction, obtained by Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5], is that (6.1) is well-posed in the Fredholm sense (this means that compactness holds), if A is isotropic, i.e., $A = aI$ for some positive function a, and the contrast of a is not 1 on each connected component of Γ.

The starting point in [32] is to use reflections to obtain Cauchy's problems from the Helmholtz equations with sign changing coefficients as previously discussed in various contexts. The use of reflections to study NIMs was also considered by Milton et al. in [24] and by Bonnet-Ben Dhia, Ciarlet, and their coauthors in their T-coercivity approach (see, e.g., [5] and references there in). However, there is a difference between the use of reflections in [24] and [5] and in our work. In [24], the authors used reflections as a change of variables to obtain a new simple setting from an old more complicated one, and hence the analysis of the old problem becomes simpler. In [5], the authors used a standard reflection to build test functions for the inf-sup condition to obtain an a priori estimate for the solution. Our use of reflections is to derive Cauchy problems. This can be done in a very flexible way via a change of variables formula stated in Lemma 2.1 as observed in [27]. The limiting absorption principle and the well-posedness of (6.1) are then based on a priori estimates for these Cauchy problems.

In [32], we introduced three approaches to obtain a priori estimates for the Cauchy problems. The first one follows from a priori estimates for elliptic systems imposing implementing boundary conditions. This is based on the classic work of Agmon, Douglis, and Nirenberg in [1]. Applying their result, we proved [32, Theorem 1].

Theorem 6.1 *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0}$ and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution to* (6.2). *Assume that $A_+ := A\big|_{\mathbb{R}^d \setminus \overline{D}} \in C^1(\overline{D}_{-\tau})$ and $A_- := A\big|_D \in C^1(\overline{D}_\tau)$ for some $\tau > 0$, and $A_+(x), A_-(x)$ satisfy the* (*Cauchy*) *complementing condition with respect to direction $\nu(x)$ for all $x \in \Gamma$. Then*

$$\|u_\delta\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall R > 0. \tag{6.4}$$

Moreover, $u_\delta \to u_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d)$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^d)$, as $\delta \to 0$, where $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d)$ is the unique outgoing solution to (6.1). *Consequently,*

$$\|u_0\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall R > 0. \tag{6.5}$$

Here C_R denotes a positive constant independent of f and δ.

We recall the following definition.

Definition 6.1 (see [1]) *Two constant positive symmetric matrices A_1 and A_2 are said to satisfy the* (*Cauchy*) *complementing condition with respect to direction $e \in \partial B_1$ if and only if for all $\xi \in \mathbb{R}^d_{e,0} \setminus \{0\}$, the only solution $(u_1(x), u_2(x))$ to the form $(e^{i\langle y, \xi\rangle} v_1(t), e^{i\langle y, \xi\rangle} v_2(t))$ with*

$x = y + te$ where $y \in \mathbb{R}^d_{e,0}$ and $t = \langle x, e\rangle$, of the following system:

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) = \operatorname{div}(A_2 \nabla u_2) = 0 & \text{in } \mathbb{R}^d_{e,+}, \\ u_1 = u_2 \quad \text{and} \quad A_1 \nabla u_1 \cdot e = A_2 \nabla u_2 \cdot e & \text{on } \mathbb{R}^d_{e,0}, \end{cases}$$

which is bounded in $\mathbb{R}^d_{e,+}$ is $(0,0)$.

Here and in what follows, for a unit vector e, the following notations are used:

$$\mathbb{R}^d_{e,+} = \{\xi \in \mathbb{R}^d;\ \langle \xi, e\rangle > 0\} \quad \text{and} \quad \mathbb{R}^d_{e,0} = \{\xi \in \mathbb{R}^d;\ \langle \xi, e\rangle = 0\}.$$

$\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$.

An algebraic characterization of the complementing condition is [32, Proposition 1].

Proposition 6.1 *Two constant positive symmetric matrices A_1 and A_2 are said to satisfy the complementing condition with respect to direction $e \in \partial B_1$ if and only if*

$$\langle A_2 e, e\rangle \langle A_2 \xi, \xi\rangle - \langle A_2 e, \xi\rangle^2 \neq \langle A_1 e, e\rangle \langle A_1 \xi, \xi\rangle - \langle A_1 e, \xi\rangle^2, \quad \forall \xi \in \mathcal{P} \setminus \{0\},$$

where

$$\mathcal{P} := \big\{\xi \in \mathbb{R}^d; \langle \xi, e\rangle = 0\big\}.$$

In particular, if $A_2 > A_1$ then A_1 and A_2 satisfy the complementing boundary condition for all $e \in \partial B_1$.

As a consequence of Theorem 6.1 and Proposition 6.1, we obtained [32, Corollary 1].

Corollary 6.1 *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (6.2). Assume that $A_+ := A\big|_{\mathbb{R}^d \setminus \overline{D}} \in C^1(\overline{D}_{-\tau})$ and $A_- := A\big|_D \in C^1(\overline{D}_\tau)$ for some $\tau > 0$, and $A_+(x) > A_-(x)$ or $A_-(x) > A_+(x)$ for all $x \in \Gamma$. Then the conclusion of Theorem 6.1 holds.*

To our knowledge, Corollary 6.1 is new and cannot be obtained by the known approaches mentioned above. It is in the same spirit of the one of Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5]; nevertheless, A_+ and A_- are not assumed to be isotropic here.

We next discuss briefly the proof of Theorem 6.1. We recall the result as follows.

Lemma 6.1 (see [1]) *Let D be a smooth bounded open subset of $\mathbb{R}^d$, and A_1 and A_2 be two symmetric uniformly elliptic matrices defined in D of class $C^1(\overline{D})$. Let $f_1, f_2 \in L^2(D)$, and let $u_1, u_2 \in H^1(D)$ be such that*

$$-\operatorname{div}(A_1 \nabla u_1) = f_1 \quad \text{and} \quad -\operatorname{div}(A_2 \nabla u_2) = f_2 \quad \text{in } D, \tag{6.6}$$

$$u_1 = u_2 \quad \text{and} \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu \quad \text{on } \partial D. \tag{6.7}$$

Assume that A_1 and A_2 satisfy the (Cauchy) complementing condition with respect to direction $\nu(x)$ for all $x \in \Gamma$. We have

$$\|(u_1, u_2)\|_{H^2(D)} \leq C(\|(f_1, f_2)\|_{L^2(D)} + \|(u_1, u_2)\|_{L^2(D)}). \tag{6.8}$$

We confine ourselves to the case $\delta = 0$ and give the ideas of the proof of (6.5). Note that the uniqueness of u_0 can be obtained as in the standard setting where the coefficients are positive by Rellich's lemma. Define $F : D_{-\tau} \to D_\tau$ as follows:

$$F(x_\Gamma + t\nu(x_\Gamma)) = x_\Gamma - t\nu(x_\Gamma), \quad \forall x_\Gamma \in \Gamma, t \in (-\tau, 0). \tag{6.9}$$

Let v_0 be the reflection of u_0 through Γ by F, i.e., $v_0 = u_0 \circ F^{-1}$ in D_τ. By Lemma 2.1,

$$\operatorname{div}(F_* A \nabla v_0) + k^2 F_* \Sigma v_0 = F_* f \quad \text{in } D_\tau$$

and

$$v_0 - u_0|_D = 0, \quad F_* A \nabla v_0 \cdot \nu - A \nabla u_0|_D \cdot \nu = 0 \quad \text{on } \Gamma.$$

Note that A_+ and A_- satisfy the complementing condition on Γ if and only if $F_* A_-$ and A_+ satisfy the complementing condition on Γ by Proposition 6.1. Applying Lemma 6.1 and using the outgoing condition, one obtains

$$\|u_0\|_{H^1(B_R)} \le C_R (\|u_0\|_{L^2(B_{R_0})} + \|f\|_{L^2}).$$

The conclusion now follows from the uniqueness via a standard compactness argument (see [32, Section 2] for the details).

One can verify that if $F_* A_+ = A_-$ on Γ, then the complementing condition is not satisfied (see Proposition 6.1). To deal with this situation, in [32], we developed a second approach to obtain a priori estimates for the Cauchy problems. This is a variational approach and based on the Dirichlet principle. Using this approach, we established the following theorem (see [32, Theorem 2]).

Theorem 6.2 *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution to (6.2). Assume that there exists a reflection F from $U \setminus D$ onto D_τ for some $\tau > 0$ and for some smooth open set $U \supset\supset D$, i.e., F is diffeomorphism and $F(x) = x$ on Γ, such that*

$$\text{either} \quad A - F_* A \ge c \operatorname{dist}(x, \Gamma)^\alpha I \quad \text{or} \quad F_* A - A \ge c \operatorname{dist}(x, \Gamma)^\alpha I, \tag{6.10}$$

on each connected component of D_τ, for some $c > 0$ and $0 < \alpha < 2$. Set $v_\delta = u_\delta \circ F^{-1}$ in D_τ. Then

$$\|u_\delta\|_{L^2(B_R)} + \|u_\delta - v_\delta\|_{H^1(D_\tau)} + \Big(\int_{D_\tau} \big| \langle (A - F_* A) \nabla u_\delta, \nabla u_\delta \rangle \big| \Big)^{\frac{1}{2}} \le C_R \|f\|_{L^2(\mathbb{R}^d)}. \tag{6.11}$$

Moreover, $u_\delta \to u_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d \setminus \Gamma)$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^d)$ as $\delta \to 0$, where $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{\text{loc}}(\mathbb{R}^d)$ is the unique outgoing solution to the equation (6.1) such that the LHS of (6.12) is finite where $v_0 = u_0 \circ F^{-1}$ in D_τ. Consequently,

$$\|u_0\|_{L^2(B_R)} + \|u_0 - v_0\|_{H^1(D_\tau)} + \Big(\int_{D_\tau} \big| \langle (A - F_* A) \nabla u_0, \nabla u_0 \rangle \big| \Big)^{\frac{1}{2}} \le C_R \|f\|_{L^2}. \tag{6.12}$$

Here C_R denotes a positive constant independent of f and δ.

The unique solution, which is obtained by the limiting absorption principle, might not be in $H^1_{\text{loc}}(\mathbb{R}^d)$ in this case. The definition [32, Definition 2] is as follows.

Definition 6.2 *Let $f \in L^2(\mathbb{R}^d)$ with compact support, and let F be a reflection from $U \setminus D$ to D_τ for some $\tau > 0$ (small) and for some open set $D \subset\subset U$, i.e., F is diffeomorphism and $F(x) = x$ on Γ. A function $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{\text{loc}}(\mathbb{R}^d)$ such that the LHS of (6.12) is finite is said to be a solution to (6.1) if*

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = f \quad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{6.13}$$

$$u_0 - v_0 = 0, \quad (F_* A \nabla v_0 - A \nabla u_0\big|_D) \cdot \nu = 0 \quad \text{on } \Gamma \tag{6.14}$$

and

$$\lim_{t\to 0_+}\int_{\partial D_t\setminus\Gamma}\big(F_*A\nabla v_0\cdot\nu\bar{v}_0 - A\nabla u_0\cdot\nu\bar{u}_0\big) = 0. \tag{6.15}$$

Remark 6.1 Since the LHS of (6.12) is finite, it follows that $u_0 - v_0 \in H^{\frac{1}{2}}(\Gamma)$ and $(F_*A\nabla v_0 - A\nabla u_0\big|_D)\cdot\nu \in H^{-\frac{1}{2}}(\Gamma)$. Hence requirement (6.14) makes sense. It is clear that the definition of weak solutions in Definition 6.2 coincides with the standard definition of weak solutions when $\alpha = 0$ by Lemma 2.1. Requirements in (6.14)–(6.15) can be seen as generalized transmission conditions.

Once the uniqueness is obtained, the stability is based on a compactness argument. The requirement $\alpha < 2$ is required in the compactness argument (see [32, Lemma 7]); we do not know if this condition is necessary. As a consequence of Theorem 6.2, one can prove [32, Corollary 2].

Corollary 6.2 *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution to* (6.2)*. Assume that $A\circ F^{-1}(x)$ or $A(x)$ is isotropic for $x \in D_\tau$, and*

$$\textit{either}\quad A\circ F^{-1}(x) - A(x) \geq cI \quad \textit{or}\quad A(x) - A\circ F^{-1}(x) \geq cI \tag{6.16}$$

in each connected component D_τ for some small $\tau > 0$ and for some $c > 0$, where F is given by (6.9)*. Then*

$$\|u_\delta\|_{H^1(B_R)} \leq C_R\|f\|_{L^2}.$$

Moreover, $u_\delta \to u_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d)$ as $\delta \to 0$, where $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d)$ is the unique outgoing solution to (6.1) *and*

$$\|u_0\|_{H^1(B_R)} \leq C_R\|f\|_{L^2}.$$

Remark 6.2 Applying Corollary 6.2 for the isotropic case, one rediscovers and extends the result obtained by Bonnet Ben-Dhia, Chesnel and Ciarlet in [5].

We next present another consequence of Theorem 6.2 for the case $\alpha = 1$. The following notation is used.

Definition 6.3 *The boundary Γ of D is called strictly convex, if all its connected components are the boundary of strictly convex sets.*

We are ready to present [32, Corollary 2].

Corollary 6.3 *Let $d \geq 3$ and D be of class C^3. Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$ and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of* (6.2)*. Assume that A is constant in each connected component of a neighborhood of Γ and Γ is strictly convex. There exist $c > 0$, $\tau > 0$, a smooth open subset $U \supset\supset D$, a reflection $F : U \to D_\tau$ such that $F_*A - A \geq c\operatorname{dist}(x,\Gamma)I$ or $A - F_*A \geq c\operatorname{dist}(x,\Gamma)I$ on each connected component of D_τ. Then* (6.11) *holds. Moreover, $u_\delta \to u_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d)$ as $\delta \to 0$, where $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d)$, u_0 satisfies* (6.12)*, and u_0 is the unique outgoing solution to* (6.1)*.*

Remark 6.3 The reflection F in Corollary 6.3 is not given by (6.9). Its choice is quite subtle and depends on the fundamental form of Γ (see [32, Proof of Corollary 2]). Corollary 6.3 does not hold in two dimensions. The strict convexity of Γ is necessary in three dimensions. In four or higher dimensions, the strict convexity of Γ can be relaxed (see [32, Remark 10]).

We next return to the proof of Theorem 6.2. An important step in the proof is to obtain estimate for the Cauchy problem. A result of this type is the following lemma which is somehow a replacement of Lemma 6.1 in this context and a special case of [32, Lemma 5]. The proof is based on the Dirichlet principle.

Lemma 6.2 *Let D be a smooth bounded open subset of $\mathbb{R}^d$, and A_1 and A_2 be two symmetric uniformly elliptic matrices defined in D. Let $f_1, f_2 \in L^2(D)$, and let $u_1, u_2 \in H^1(D)$ be such that*

$$-\operatorname{div}(A_1 \nabla u_1) = f_1 \quad \textit{and} \quad -\operatorname{div}(A_2 \nabla u_2) = f_2 \quad \textit{in } D, \tag{6.17}$$

$$u_1 = u_2 \quad \textit{and} \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu \quad \textit{on } \partial D. \tag{6.18}$$

Assume that

$$A_1 \ge A_2 \quad \textit{in } D. \tag{6.19}$$

Then

$$\int_D \langle (A_1 - A_2) \nabla u_1, \nabla u_1 \rangle \le C(\|(f_1, f_2)\|^2_{L^2(D)} + \|(u_1, u_2)\|^2_{L^2(D)}). \tag{6.20}$$

Proof By considering the real part and the imaginary part separately, without loss of generality, one may assume that all functions in Lemma 6.2 are real. Set

$$\mathcal{M} = \|(f_1, f_2)\|^2_{L^2(D)} + \|(u_1, u_2)\|^2_{L^2(D)}.$$

Multiplying the equation of u_j by u_j (for $j = 1, 2$) and integrating on D, we have

$$\int_D \langle A_j \nabla u_j, \nabla u_j \rangle = \int_D f_j u_j + \int_{\partial D} A_j \nabla u_j \cdot \nu \, u_j. \tag{6.21}$$

Using (6.17)–(6.18), we derive from (6.21) that

$$\int_D \langle A_1 \nabla u_1, \nabla u_1 \rangle - \langle A_2 \nabla u_2, \nabla u_2 \rangle \le C\mathcal{M}. \tag{6.22}$$

Here and in what follows, C denotes a positive constant independent of f_j, h, u_j for $j = 1, 2$. By the Dirichlet principle, we have

$$\begin{aligned} &\frac{1}{2} \int_D \langle A_2 \nabla u_2, \nabla u_2 \rangle - \int_D f_2 u_2 - \int_{\partial D} A_2 \nabla u_2 \cdot \nu \, u_2 \\ \le\ &\frac{1}{2} \int_D \langle A_2 \nabla u_1, \nabla u_1 \rangle - \int_D f_2 u_1 - \int_{\partial D} A_2 \nabla u_2 \cdot \nu \, u_1. \end{aligned} \tag{6.23}$$

A combination of (6.17)–(6.18) and (6.23) yields

$$\int_D \langle A_2 \nabla u_2, \nabla u_2 \rangle - \langle A_2 \nabla u_1, \nabla u_1 \rangle \le C\mathcal{M}. \tag{6.24}$$

Adding (6.22) and (6.24), we obtain the conclusion.

Similar conclusion holds in the case $F_* A = A$ in D_τ under additional assumptions on $\Sigma - F_* \Sigma \gtrsim \operatorname{dist}(\cdot, \Gamma)^\beta$ in D_τ for some $\beta > 0$ (see [32, Theorem 3] for a more general result). The unique solution in this case is not even in $L^2_{\rm loc}(\mathbb{R}^d)$. As far as we know, [32, Theorem 3] is the first result on the limiting absorption principle and the well-posedness of the Helmholtz equations with sign changing coefficients where the conditions on the coefficients contains the zero order term Σ.

Remark 6.4 The results mentioned here showed that the complementary property of media is necessary for the occurrence of the resonance. In [32, Proposition 2], we showed that even in the case $(F_*A, F_*\Sigma) = (A, \Sigma)$ in $B \cap D_\tau$ for some open set B with $B \cap \Gamma \neq \emptyset$, the system is resonant in the following sense: There exists f with $\operatorname{supp} f \subset\subset B_{R_0} \setminus \Gamma$, such that $\limsup_{\delta \to 0} \|u_\delta\|_{L^2(K)} = +\infty$ for some $K \subset\subset B_{R_0} \setminus \Gamma$. This implies the optimality of the results mentioned above. The proof of [32, Proposition 2] is inspired from the one of Theorem 5.2.

Notes added Very recent progress on negative index materials can be found in [34–36]. The answers to Open Questions 3.1 and 4.1 are negative (see [34]).

Acknowledgements The author thanks Professor Haïm Brezis deeply for his encouragement, his guidance, and his support from his Ph.D studies. His joint work with Bethuel and Hélein on the Ginzburg-Landau equation in [4] inspired the author to introduce the removing localized singularity technique to handle the localized resonance which appears naturally in the study of negative index materials. The author also thanks Professor Graeme Milton for useful discussions on the subject.

References

[1] Agmon, S., Douglis, A. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, *Comm. Pure Appl. Math.*, **17**, 1964, 35–92.

[2] Ammari, H., Ciraolo, G., Kang, H., et. al., Anomalous localized resonance using a folded geometry in three dimensions, *Proc. R. Soc. Lond. Ser. A*, **469**, 2013, 20130048.

[3] Ammari, H., Ciraolo, G., Kang, H., et. al., Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, *Arch. Rational Mech. Anal.*, **218**, 2013, 667–692.

[4] Bethuel, F., Brezis, H. and Helein, F., Ginzburg Landau vortices, Progress in Nonlinear Differential Equations and Their Applications., **13**, Birkhäuser, Boston, 1994.

[5] Bonnet-Ben Dhia, A. S., Chesnel, L. and Ciarlet, P., T-coercivity for scalar interface problems between dielectrics and metamaterials, *ESAIM Math. Model. Numer. Anal.*, **46**, 2012, 1363–1387.

[6] Bonnet-Ben Dhia, A. S., Chesnel, L. and Ciarlet, P., T-coercivity for the Maxwell problem with sign-changing coefficients, *Comm. Partial Differential Equations*, **39**, 2014, 1007–1031.

[7] Bonnet-Ben Dhia, A. S., Ciarlet, P. and Zwölf, C. M., Two- and three-field formulations for wave transmission between media with opposite sign dielectric constants, *J. Comput. Appl. Math.*, **204**, 2007, 408–417.

[8] Bonnet-Ben Dhia, A. S., Ciarlet, P. and Zwölf, C. M., A new compactness result for electromagnetic waves. Application to the transmission problem between dielectrics and metamaterials, *Math. Models Methods Appl. Sci.*, **18**, 2008, 1605–1631.

[9] Bonnet-Ben Dhia, A. S., Ciarlet, P. and Zwölf, C. M., Time harmonic wave diffraction problems in materials with sign-shifting coefficients, *J. Comput. Appl. Math.*, **234**, 2010, 1912–1919.

[10] Bouchitté, G. and Felbacq, D., Homogenization near resonances and artificial magnetism from dielectrics, *C. R. Math. Acad. Sci. Paris*, **339**, 2004, 377–382.

[11] Bouchitté, G. and Schweizer, B., Cloaking of small objects by anomalous localized resonance, *Quart. J. Mech. Appl. Math.*, **63**, 2010, 437–463.

[12] Bouchitté, G. and Schweizer, B., Homogenization of Maxwell's equations in a split ring geometry, *Multiscale Model. Simul.*, **8**, 2010, 717–750.

[13] Brezis, H., Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universititex, Springer-Verlag, New York, 2010.

[14] Bruno, O. P. and Lintner, S., Superlens-cloaking of small dielectric bodies in the quasistatic regime, *J. Appl. Phys.*, **102**, 2007, 12452.

[15] Costabel, M. and Stephan, E., A direct boundary integral equation method for transmission problems, *J. Math. Anal. Appl.*, **106**, 1985, 367–413.

[16] Guenneau, S. and Zolla, F., Homogenization of 3d finite chiral photonic crystals, *Physica B: Condens. Matter*, **394**, 2007, 145–147.

[17] Kettunen, H., Lassas, M. and Ola, P., On absence and existence of the anomalous localized resonance without the quasi-static approximation. http://arxiv.org/abs/1406.6224

[18] Kohn, R. V., Lu, J., Schweizer, B. and Weinstein, M. I., A variational perspective on cloaking by anomalous localized resonance, *Comm. Math. Phys.*, **328**, 2014, 1–27.

[19] Kohn, R. V., Onofrei, D., Vogelius, M. S. and Weinstein, M. I., Cloaking via change of variables for the Helmholtz equation, *Comm. Pure Appl. Math.*, **63**, 2010, 973–1016.

[20] Kohn, R. V. and Shipman, S. P., Magnetism and homogenization of microresonators, *Multiscale Model. Simul.*, **7**, 2008, 62–92.

[21] Lai, Y., Chen, H., Zhang, Z. and Chan, C. T., Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell, *Phys. Rev. Lett.*, **102**, 2009, 093901.

[22] Milton, G. W., Nicorovici, N. A., McPhedran, R. C. and Podolskiy, V. A., A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, *Proc. R. Soc. Lond. Ser. A*, **461**, 2005, 3999–4034.

[23] Milton, G. W. and Nicorovici, N. A. P., On the cloaking effects associated with anomalous localized resonance, *Proc. R. Soc. Lond. Ser. A*, **462**, 2006, 3027–3059.

[24] Milton, G. W., Nicorovici, N. P., McPhedran, R. C., et. al., Solutions in folded geometries, and associated cloaking due to anomalous resonance, *New J. Phys.*, **10**, 2008, 115021.

[25] Nguyen, H.-M., Cloaking via change of variables for the Helmholtz Equation in the whole space, *Comm. Pure Appl. Math.*, **63**, 2010, 1505–1524.

[26] Nguyen, H.-M., Approximate cloaking for the Helmholtz equation via transformation optics and consequences for perfect cloaking, *Comm. Pure Appl. Math.*, **65**, 2012, 155–186.

[27] Nguyen, H.-M., Asymptotic behavior of solutions to the Helmholtz equations with sign changing coefficients, *Trans. Amer. Math. Soc.*, **367**, 2015, 6581–6595.

[28] Nguyen, H.-M., Superlensing using complementary media, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32**, 2015, 471–484.

[29] Nguyen, H.-M., Cloaking via anomalous localized resonance, a connection between the localized resonance and the blow up of the power for doubly complementary media, *C. R. Math. Acad. Sci. Paris*, **353**, 2015, 41–46.

[30] Nguyen, H.-M., Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime, *J. Eur. Math. Soc.*, **17**, 2015, 1327–1365.

[31] Nguyen, H.-M., Cloaking using complementary media in the quasistatic regime, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33**, 2016, 1509–1518.

[32] Nguyen, H.-M., Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients, *J. Math. Pures Appl.*, **106**, 2016, 342–374.

[33] Nguyen, H.-M., Reflecting complementary and superlensing using complementary media for electromagnetic waves, submitted.

[34] Nguyen, H.-M., Cloaking an arbitrary object via anomalous localized resonance: the cloak is independent of the object, submitted. https://arxiv.org/abs/1607.06492

[35] Nguyen, H.-M., Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime, submitted. https://arxiv.org/abs/1511.08053

[36] Nguyen, H.-M., Cloaking using complementary for electromagnetic waves, submitted. https://arxiv.org/abs/1701.02339

[37] Nguyen, H.-M. and Nguyen, H. L., Complete resonance and localized resonance in plasmonic structures, *ESAIM: Math. Model. Numer. Anal.*, **49**, 2015, 741–754.

[38] Nguyen, H.-M. and Nguyen, H. L., Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations, *Trans. Amer. Math. Soc., Ser. B*, **2**, 2015, 93–112.

[39] Nguyen, H.-M. and Vogelius, M. S., Full range scattering estimates and their application to cloaking, *Arch. Rational Mech. Anal.*, **203**, 2012, 769–807.

[40] Nguyen, H.-M. and Vogelius, M. S., Approximate cloaking for the wave equation via change of variables, *SIAM J. Math. Anal.*, **44**, 2012, 1894–1924.

[41] Nguyen, H.-M. and Vogelius, M. S., Approximate cloaking for the full wave equation: A study of the Lorentz model, *J. Math. Pures Appl.*, **106**, 2016, 797–836.

[42] Nicorovici, N. A., McPhedran, R. C. and Milton, G. M., Optical and dielectric properties of partially resonant composites, *Phys. Rev. B*, **49**, 1994, 8479–8482.

[43] Ola, P., Remarks on a transmission problem, *J. Math. Anal. Appl.*, **196**, 1995, 639–658.

[44] Pendry, J. B., Negative refraction makes a perfect lens, *Phys. Rev. Lett.*, **85**, 2000, 3966–3969.

[45] Pendry, J. B., Perfect cylindrical lenses, *Optics Express*, **1**, 2003, 755–760.

[46] Pendry, J. B., Holden, A. J., Robbins, D. J. and Stewart, W. J., Magnetism from conductors and enhanced nonlinear phenomena, *IEEE Trans. Microw. Theory Tech.*, **47**, 1999, 2075–2084.

[47] Ramakrishna, S. A. and Pendry, J. B., Spherical perfect lens: Solutions of Maxwell's equations for spherical geometry, *Phys. Rev. B*, **69**, 2004, 115115.

[48] Shelby, R. A., Smith, D. R. and Schultz, S., Experimental Verification of a Negative Index of Refraction, Science, **292**, 2001, 77–79.

[49] Veselago, V. G., The electrodynamics of substances with simultaneously negative values of ε and μ, *Usp. Fiz. Nauk*, **92**, 1964, 517–526.