

## Convergence to a Single Wave in the Fisher-KPP Equation\*

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(Dedicated to Haim Brezis, with admiration and respect)

**Abstract** The authors study the large time asymptotics of a solution of the Fisher-KPP reaction-diffusion equation, with an initial condition that is a compact perturbation of a step function. A well-known result of Bramson states that, in the reference frame moving as  $2t - (\frac{3}{2}) \log t + x_\infty$ , the solution of the equation converges as  $t \rightarrow +\infty$  to a translate of the traveling wave corresponding to the minimal speed  $c_* = 2$ . The constant  $x_\infty$  depends on the initial condition  $u(0, x)$ . The proof is elaborate, and based on probabilistic arguments. The purpose of this paper is to provide a simple proof based on PDE arguments.

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### 1 Introduction

We consider the Fisher-KPP equation

$$u_t - u_{xx} = u - u^2, \quad t > 0, x \in \mathbb{R} \quad (1.1)$$

with an initial condition  $u(0, x) = u_0(x)$  which is a compact perturbation of a step function, in the sense that there exist  $x_1$  and  $x_2$ , so that  $u_0(x) = 1$  for all  $x \leq x_1$ , and  $u_0(x) = 0$  for all  $x \geq x_2$ .

This equation has a traveling wave solution  $u(t, x) = \phi(x - 2t)$ , moving with the minimal speed  $c_* = 2$ , connecting the stable equilibrium  $u \equiv 1$  to the unstable equilibrium  $u \equiv 0$ :

$$\begin{aligned} -\phi'' - 2\phi' &= \phi - \phi^2, \\ \phi(-\infty) &= 1, \quad \phi(+\infty) = 0. \end{aligned} \quad (1.2)$$

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Each solution  $\phi(\xi)$  of (1.2) is a shift of a fixed profile  $\phi_*(\xi)$ :  $\phi(\xi) = \phi_*(\xi + s)$ , with some fixed  $s \in \mathbb{R}$ . The profile  $\phi_*(\xi)$  satisfies the asymptotics

$$\phi_*(\xi) = (\xi + k)e^{-\xi} + O(e^{-(1+\omega_0)\xi}), \quad \xi \rightarrow +\infty \quad (1.3)$$

with two universal constants  $\omega_0 > 0$ ,  $k \in \mathbb{R}$ .

The large time behaviour of the solutions of this problem has a long history, starting with a striking paper of Fisher [10], which identifies the spreading velocity  $c_* = 2$  via numerical computations and other arguments. In the same year, the pioneering KPP paper [15] proved that the solution of (1.1), starting from a step function:  $u_0(x) = 1$  for  $x \leq 0$ ,  $u_0(x) = 0$  for  $x > 0$ , converges to  $\phi_*$  in the following sense: There is a function

$$\sigma_\infty(t) = 2t + o(t), \quad (1.4)$$

such that

$$\lim_{t \rightarrow +\infty} u(t, x + \sigma_\infty(t)) = \phi_*(x).$$

Fisher has already made an informal argument that the  $o(t)$  in (1.4) is of the order  $O(\log t)$ . An important series of papers by Bramson proves the following result.

**Theorem 1.1** (see [5–6]) *There is a constant  $x_\infty$ , depending on  $u_0$ , such that*

$$\sigma_\infty(t) = 2t - \frac{3}{2} \log t - x_\infty + o(1) \quad \text{as } t \rightarrow +\infty.$$

Theorem 1.1 was proved through elaborate probabilistic arguments. Bramson also gave necessary and sufficient conditions on the decay of the initial data to zero (as  $x \rightarrow +\infty$ ) in order that the solution converges to  $\phi_*(x)$  in some moving frame. Lau [17] also proved those necessary and sufficient conditions (for a more general nonlinear term) using a PDE approach based on the decrease in the number of the intersection points for a pair of solutions of the parabolic Cauchy problem. The asymptotics of  $\sigma_\infty(t)$  were not identified by that approach.

A natural question is to prove Theorem 1.1 with purely PDE arguments. In that spirit, a weaker version, precise up to the  $O(1)$  term (but valid also for a much more difficult case of the periodic in space coefficients), is the main result of [11–12],

$$\sigma(t) = 2t - \frac{3}{2} \log t + O(1) \quad \text{as } t \rightarrow +\infty. \quad (1.5)$$

Here, we will give a simple and robust proof of Theorem 1.1. These ideas are further developed to study the refined asymptotics of the solutions in [21].

The paper is organized as follows. In Section 2, we shortly describe some connections between the Fisher-KPP equation (1.1) and the branching Brownian motion. In Section 3, we explain, in an informal way, the strategy of the proof of the theorem: In a nutshell, the solution is slaved to the dynamics at  $x = O(\sqrt{t})$ . In Sections 4–5, we make the arguments of Section 3 rigorous.

## 2 Probabilistic Links and Some Related Models

The time delay in models of the Fisher-KPP type has been the subject of various recent investigations, both from the PDE and probabilistic points of view. The Fisher-KPP equation

appears in the theory of the branching Brownian motion (BBM for short) (see [19]) as follows. Consider a BBM starting at  $x = 0$  at time  $t = 0$ , with binary branching at rate 1. Let  $X_1(t), \dots, X_{N_t}(t)$  be the descendants of the original particle at time  $t$ , arranged in the increasing order:  $X_1(t) \leq X_2(t) \leq \dots \leq X_{N_t}(t)$ . Then, the probability distribution function of the maximum

$$v(t, x) = \mathbb{P}(X_{N_t}(t) > x),$$

satisfies the Fisher-KPP equation

$$v_t = \frac{1}{2}v_{xx} + v - v^2$$

with the initial data  $v_0(x) = \mathbb{1}_{x \leq 0}$ . Therefore, Theorem 1.1 is about the median location of the maximal particle  $X_{N_t}$ . Building on the work of Lalley and Sellke [16], recent probabilistic analyses (see [1–3, 7–8]) of this particle system have identified a decorated Poisson-type point process which is the limit of the particle distribution “seen from the tip”: There is a random variable  $Z > 0$ , such that the point process defined by the shifted particles  $\{X_1(t) - c(t), \dots, X_{N_t}(t) - c(t)\}$ , with

$$c(t) = 2t - \frac{3}{2} \log t + \log Z,$$

has a well-defined limit process as  $t \rightarrow \infty$ . Furthermore,  $Z$  is the limit of the martingale

$$Z_t = \sum_k (2t - X_k(t))e^{X_k(t) - 2t},$$

and

$$\phi_*(x) = 1 - \mathbb{E}[e^{-Ze^{-x}}] \quad \text{for all } x \in \mathbb{R}.$$

As we have mentioned, the logarithmic term in Theorem 1.1 arises also in inhomogeneous variants of this model. For example, consider the Fisher-KPP equation in a periodic medium

$$u_t - u_{xx} = \mu(x)u - u^2, \tag{2.1}$$

where  $\mu(x)$  is continuous and 1-periodic in  $\mathbb{R}$ , such that the principal periodic eigenvalue of the operator  $-\partial_{xx} - \mu(x)$  is negative. Then there is a minimal speed  $c_* > 0$ , such that for each  $c \geq c_*$ , there is a unique pulsating front  $U_c(t, x)$ , up to a time shift [4, 13]. It was shown in [12] that there is  $s_0 > 0$  such that, if  $u(t, x)$  solves (2.1) with a nonnegative, nonzero, compactly supported initial condition  $u_0(x)$ , and  $0 < s \leq s_0$ , then the  $s$ -level set  $\sigma_s(t)$  of  $u(t, x)$  (here, the largest  $\sigma > 0$  such that  $u(t, \sigma) = s$ ) must satisfy

$$\sigma_s(t) = c_*t - \frac{3}{2\lambda_*} \log t + O(1),$$

where  $\lambda_* > 0$  is the rate of exponential decay (as  $x \rightarrow \infty$ ) of the minimal front  $U_{c_*}$ , which depends on  $\mu(x)$  but not on  $s$  or on  $u_0$ . This implies the convergence of  $u(t, x - \sigma_s(t))$  to a closed subset of the family of minimal fronts. It is an open problem to determine whether convergence to a single front holds, not to mention the rate of this convergence. When  $\mu(x) > 0$  everywhere, the solution  $u$  of the related model

$$u_t - u_{xx} = \mu(x)(u - u^2)$$

may be interpreted in terms of the extremal particle in a BBM with a spatially-varying branching rate (see [12]).

Models with temporal variation in the branching process have also been considered. In [9], Fang and Zeitouni studied the extremal particle of such a spatially homogeneous BBM where the branching particles satisfy

$$dX(t) = \sqrt{2}\kappa\left(\frac{t}{T}\right) dB(t)$$

between branching events, rather than following a standard Brownian motion. In terms of PDE, their study corresponds to the model

$$u_t = \kappa^2\left(\frac{t}{T}\right)u_{xx} + f(u), \quad 0 < t < T, \quad x \in \mathbb{R}. \quad (2.2)$$

They proved that if  $\kappa$  is increasing, and  $f$  is of the Fisher-KPP type, the shift is algebraic and not logarithmic in time: There exists  $C > 0$ , such that

$$\frac{T^{\frac{1}{3}}}{C} \leq X(T) - c_{\text{eff}}T \leq CT^{\frac{1}{3}}, \quad c_{\text{eff}} = 2 \int_0^1 \kappa(s) ds.$$

In [20], we proved the asymptotics

$$X(T) = c_{\text{eff}}T - \bar{\nu}T^{\frac{1}{3}} + O(\log T) \quad \text{with } \bar{\nu} = \beta \int_0^1 \kappa(\tau)^{\frac{1}{3}} \dot{\kappa}(\tau)^{\frac{2}{3}} d\tau. \quad (2.3)$$

Here,  $\beta < 0$  is the first zero of the Airy function. Maillard and Zeitouni [18] refined the asymptotics further, proving a logarithmic correction to (2.3), and convergence of  $u(T)$  to a traveling wave.

### 3 Strategy of the Proof of Theorem 1.1

#### 3.1 Why converge to a traveling wave?

We first provide an informal argument for the convergence of the solution of the initial value problem to a traveling wave. Consider the Cauchy problem (1.1), starting at  $t = 1$  for the convenience of the notation

$$u_t - u_{xx} = u - u^2, \quad x \in \mathbb{R}, \quad t > 1, \quad (3.1)$$

and proceed with a standard sequence of changes of variables. We first go into the moving frame

$$x \mapsto x - 2t + \left(\frac{3}{2}\right) \log t,$$

leading to

$$u_t - u_{xx} - \left(2 - \frac{3}{2t}\right)u_x = u - u^2. \quad (3.2)$$

Next, we take out the exponential factor: Set

$$u(t, x) = e^{-x}v(t, x),$$

so that  $v$  satisfies

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x}v^2 = 0, \quad x \in \mathbb{R}, \quad t > 1. \quad (3.3)$$

Observe that for any shift  $x_\infty \in \mathbb{R}$ , the function  $V(x) = e^x \phi(x - x_\infty)$  is a steady solution of

$$V_t - V_{xx} + e^{-x} V^2 = 0.$$

We regard (3.3) as a perturbation of this equation, and expect that  $v(t, x) \rightarrow e^x \phi(x - x_\infty)$  as  $t \rightarrow \infty$  for some  $x_\infty \in \mathbb{R}$ .

### 3.2 The self-similar variables

We note that for  $x \rightarrow +\infty$ , the term  $e^{-x} v^2$  in (3.3) is negligible, while for  $x \rightarrow -\infty$  the same term will create a large absorption and force the solution to be close to zero. For this reason, the linear Dirichlet problem

$$\begin{aligned} z_t - z_{xx} - \frac{3}{2t}(z - z_x) &= 0, \quad x > 0, \\ z(t, 0) &= 0 \end{aligned} \tag{3.4}$$

is a reasonable proxy for (3.3) for  $x \gg 1$ , and, as shown in [11–12], it provides good sub- and super-solutions for  $v(t, x)$ . The main lesson of [11–12] is that everything relevant to the solutions of (3.4) happens at the spatial scale  $x \sim \sqrt{t}$ , and their asymptotics may be unraveled by a self-similar change of variables. Here, we will accept the full nonlinear equation (3.3) and perform directly the self-similar change of variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}} \tag{3.5}$$

followed by a change of the unknown

$$v(\tau, \eta) = e^{\frac{\tau}{2}} w(\tau, \eta).$$

This transforms (3.3) into

$$w_\tau - \frac{\eta}{2} w_\eta - w_{\eta\eta} - w + \frac{3}{2} e^{-\frac{\tau}{2}} w_\eta + e^{\frac{3\tau}{2} - \eta \exp(\frac{\tau}{2})} w^2 = 0, \quad \eta \in \mathbb{R}, \tau > 0. \tag{3.6}$$

This transformation strengthens the reason why the Dirichlet problem (3.4) appears naturally: For

$$\eta \ll -\tau e^{-\frac{\tau}{2}},$$

the last term in the left-hand side of (3.6) becomes exponentially large, which forces  $w$  to be almost 0 in this region. On the other hand, for

$$\eta \gg \tau e^{-\frac{\tau}{2}},$$

this term is very small, so it should not play any role in the dynamics of  $w$  in that region. The transition region has width of the order  $\tau e^{-\frac{\tau}{2}}$ .

### 3.3 The choice of the shift

Also, through this change of variables, we can see how a particular translation of the wave will be chosen. Considering (3.4) in the self-similar variables, one can show (see [11, 14]) that, as  $\tau \rightarrow +\infty$ , we have

$$e^{-\frac{\tau}{2}} z(\tau, \eta) \sim \alpha_\infty \eta e^{-\frac{\eta^2}{4}}, \quad \eta > 0 \tag{3.7}$$

with some  $\alpha_\infty > 0$ . Therefore, taking (3.4) as an approximation to (3.3), we should expect that

$$u(t, x) = e^{-x}v(t, x) \sim e^{-x}z(t, x) \sim e^{-x}e^{\frac{x}{2}}\alpha_\infty\eta e^{-\frac{\eta^2}{4}} = \alpha_\infty x e^{-x} e^{-\frac{x^2}{4t}}, \tag{3.8}$$

at least for  $x$  of the order  $O(\sqrt{t})$ . This determines the unique translation: If we accept that  $u$  converges to a translate  $x_\infty$  of  $\phi_*$ , then for large  $x$  (in the moving frame), we have

$$u(t, x) \sim \phi_*(x - x_\infty) \sim x e^{-x+x_\infty}. \tag{3.9}$$

Comparing this with (3.8), we infer that

$$x_\infty = \log \alpha_\infty.$$

The difficulty with this argument, apart from the justification of the approximation

$$u(t, x) \sim e^{-x}z(t, x),$$

is that each of the asymptotics (3.8)–(3.9) uses different ranges of  $x$ : (3.8) comes from the self-similar variables in the region  $x \sim O(\sqrt{t})$ , while (3.9) assumes  $x$  to be large but finite. However, the self-similar analysis does not tell us at this stage what happens on the scale  $x \sim O(1)$ . Indeed, it is clear from (3.6) that the error in the approximation (3.7) is at least of the order  $O(e^{-\frac{x}{2}})$  (note that the right-hand side in (3.7) is a solution of (3.6) without the last two terms in the left-hand side). On the other hand, the scale  $x \sim O(1)$  corresponds to  $\eta \sim e^{-\frac{x}{2}}$ . Thus, the leading order term and the error in (3.7) are of the same size for  $x \sim O(1)$ , which means that we can not extract information directly from (3.7) on that scale.

To overcome this issue, we proceed in two steps: First we use the self-similar variables to prove stabilization (that is, (3.8) holds) at the spatial scales  $x \sim O(t^\gamma)$  with a small  $\gamma > 0$ , and not just at the diffusive scale  $O(\sqrt{t})$ . This boils down to showing that

$$w(\tau, \eta) \sim \alpha_\infty \eta e^{-\frac{\eta^2}{4}}$$

for the solution to (3.6), even for  $\eta \sim e^{-(\frac{1}{2}-\gamma)\tau}$ . Next, we show that this stabilization is sufficient to ensure the stabilization on the scale  $x \sim O(1)$  and convergence to a unique wave. This is the core of the argument: Everything happening at  $x \sim O(1)$  should be governed by the tail of the solution (the fronts are pulled).

We conclude this section with some remarks about the generality of the argument. Although we assume, for simplicity, that the reaction term in (1.1) is quadratic, our proof also works for a more general reaction term. Specifically, the function  $u - u^2$  in (1.1) may be replaced by a  $C^2$  function  $f : [0, 1] \rightarrow \mathbb{R}$  satisfying  $f(0) = 0 = f(1)$ ,  $f'(0) > 0$ ,  $f'(1) < 0$ , and  $f'(s) \leq f'(0)$  for all  $s \in [0, 1]$ . In particular, these assumptions imply that there is  $C > 0$ , such that  $0 \leq f'(0)s - f(s) \leq Cs^2$  for all  $s \in [0, 1]$ . Without loss of generality, we may suppose that  $f'(0) = 1$ . Then, if  $g(u) = u - f(u)$ , the equation (3.3) for  $v$  becomes

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^x g(e^{-x}v) = 0, \quad x \in \mathbb{R}, \quad t > 1,$$

and the equation (3.6) for  $w$  becomes

$$w_\tau - \frac{\eta}{2}w_\eta - w_{\eta\eta} - w + \frac{3}{2}e^{-\frac{\tau}{2}}w_\eta + e^{\frac{\tau}{2}+\eta\exp(\frac{\tau}{2})}g(e^{\frac{\tau}{2}-\eta\exp(\frac{\tau}{2})}w) = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0,$$

where  $0 \leq g(s) \leq Cs^2$  and  $g'(s) \geq 0$ . Then all of the arguments below (and in [11]) work in this more general setting. Finally, the arguments also apply to fronts arising from compactly supported initial data  $u_0 \geq 0$  (not just perturbations of the step-function). In that case, one obtains two fronts propagating in opposite directions. Combined with [11], our arguments here imply that Theorem 1.1 holds for both fronts. That is, the fronts moving to  $\pm\infty$  are at positions  $\sigma_\infty^\pm(t)$  with

$$\sigma_\infty^\pm(t) = \pm 2t \mp \frac{3}{2} \log t + x_\infty^\pm + o(1),$$

where the shifts  $x_\infty^+$  and  $x_\infty^-$  may differ and depend on the initial data.

### 4 Convergence to a Single Wave as a Consequence of the Diffusive Scale Convergence

The proof of Theorem 1.1 relies on the following two lemmas. The first is a consequence of [11].

**Lemma 4.1** *The solution of (3.2) with  $u(1, x) = u_0(x)$  satisfies*

$$\lim_{x \rightarrow -\infty} u(t, x) = 1, \quad \lim_{x \rightarrow +\infty} u(t, x) = 0, \tag{4.1}$$

both uniformly in  $t > 1$ .

The main new step is to establish the following.

**Lemma 4.2** *There exists a constant  $\alpha_\infty > 0$  with the following property. For any  $\gamma > 0$  and all  $\varepsilon > 0$ , we can find  $T_\varepsilon$ , so that for all  $t > T_\varepsilon$  we have*

$$|u(t, x_\gamma) - \alpha_\infty x_\gamma e^{-x_\gamma} e^{-\frac{x_\gamma^2}{4t}}| \leq \varepsilon x_\gamma e^{-x_\gamma} e^{-\frac{x_\gamma^2}{6t}} \tag{4.2}$$

with  $x_\gamma = t^\gamma$ .

We postpone the proof of this lemma for the moment, and show how it is used. A consequence of Lemma 4.2 is that the problem for the moment is to understand, for a given  $\alpha > 0$ , the behavior of the solutions of

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t} - \frac{\partial^2 u_\alpha}{\partial x^2} - \left(2 - \frac{3}{2t}\right) \frac{\partial u_\alpha}{\partial x} - u_\alpha + u_\alpha^2 &= 0, \quad x \leq x_\gamma(t), \\ u_\alpha(t, t^\gamma) &= \alpha t^\gamma e^{-t^\gamma - \frac{t^{2\gamma-1}}{4}} \end{aligned} \tag{4.3}$$

for  $t > T_\varepsilon$ , with the initial condition  $u_\alpha(T_\varepsilon, x) = u(T_\varepsilon, x)$ . In particular, we will show that  $u_{\alpha_\infty \pm \varepsilon}(t, x)$  converge, as  $t \rightarrow +\infty$ , to a pair of steady solutions, separated only by an order  $O(\varepsilon)$ -translation. Note that the function  $v(t, x) = e^x u_\alpha(t, x)$  solves

$$\begin{aligned} v_t - v_{xx} + \frac{3}{2t}(v_x - v) + e^{-x} v^2 &= 0, \quad x \leq t^\gamma, \\ v(t, t^\gamma) &= \alpha t^\gamma e^{-\frac{t^{2\gamma-1}}{4}}. \end{aligned} \tag{4.4}$$

Since we anticipate that the tail is going to dictate the behavior of  $u_\alpha$ , we choose the translate of the wave that matches exactly the behavior of  $u_\alpha(t, x)$  at the boundary  $x = t^\gamma$ : Set

$$\psi(t, x) = e^x \phi_*(x + \zeta(t)). \tag{4.5}$$

Recall that  $\phi_*(x)$  is the traveling wave profile. We look for a function  $\zeta(t)$  in (4.5) such that

$$\psi(t, t^\gamma) = v(t, t^\gamma). \tag{4.6}$$

In view of the expansion (1.3), we should have, with some  $\omega_0 > 0$ ,

$$e^{-\zeta(t)}(t^\gamma + \zeta(t) + k) + O(e^{-\omega_0 t^\gamma}) = \alpha t^\gamma e^{-\frac{1}{4t^{1-2\gamma}}},$$

which implies, for  $\gamma \in (0, \frac{1}{2})$ ,

$$\zeta(t) = -\log \alpha - (\log \alpha - k)t^{-\gamma} + O(t^{-2\gamma}),$$

and thus

$$|\dot{\zeta}(t)| \leq \frac{C}{t^{1+\gamma}}.$$

The equation for the function  $\psi$  is

$$\psi_t - \psi_{xx} + \frac{3}{2t}(\psi_x - \psi) + e^{-x}\psi^2 = -\dot{\zeta}\psi + \dot{\zeta}\psi_x + \frac{3}{2t}(\psi_x - \psi) = O\left(\frac{x}{t}\right) = O(t^{-1+\gamma}), \quad |x| < t^\gamma.$$

In addition, the left-hand side above is exponentially small for  $x < -t^\gamma$  because of the exponential factor in (4.5). Hence, the difference  $s(t, x) = v(t, x) - \psi(t, x)$  satisfies

$$\begin{aligned} s_t - s_{xx} + \frac{3}{2t}(s_x - s) + e^{-x}(v + \psi)s &= O(t^{-1+\gamma}), \quad |x| \leq t^\gamma, \\ s(t, -t^\gamma) &= O(e^{-t^\gamma}), \quad s(t, t^\gamma) = 0. \end{aligned} \tag{4.7}$$

**Proposition 4.1** *For  $\gamma \in (0, \frac{1}{3})$ , we have*

$$\lim_{t \rightarrow +\infty} \sup_{|x| \leq t^\gamma} |s(t, x)| = 0. \tag{4.8}$$

**Proof** The issue is whether the Dirichlet boundary conditions would be stronger than the force in the right side of (4.7). Since the principal Dirichlet eigenvalue for the Laplacian in  $(-t^\gamma, t^\gamma)$  is  $\frac{\pi^2}{4t^{2\gamma}}$ , investigating (4.7) is, heuristically, equivalent to solving the ODE

$$f'(t) + (1 - 2\gamma)t^{-2\gamma}f = \frac{1}{t^{1-\gamma}}. \tag{4.9}$$

The coefficient  $(1 - 2\gamma)$  is chosen simply for convenience and can be replaced by another constant. The solution of (4.9) is

$$f(t) = f(1)e^{(-t^{-2\gamma+1}+1)} + \int_1^t s^{\gamma-1}e^{(-t^{-2\gamma+1}+s^{-2\gamma+1})} ds.$$

Note that  $f(t)$  tends to 0 as  $t \rightarrow +\infty$  a little faster than  $t^{3\gamma-1}$  as soon as  $\gamma < \frac{1}{3}$ , so the analog of (4.8) holds for the solutions of (4.9). With this idea in mind, we are going to look for a super-solution of (4.7), in the form

$$\bar{s}(t, x) = t^{-\lambda} \cos\left(\frac{x}{t^{\gamma+\varepsilon}}\right), \tag{4.10}$$

where  $\lambda, \gamma$  and  $\varepsilon$  will be chosen to be small enough. We now set  $T_\varepsilon = 1$  for convenience. We have, for  $|x| \leq t^\gamma$ ,

$$\begin{aligned} \bar{s}(t, x) &\sim t^{-\lambda}, \quad -\bar{s}_{xx} = t^{-(2\gamma+2\varepsilon)}\bar{s}(t, x), \\ \bar{s}_t &= -\frac{\lambda}{t}\bar{s} + g(t, x), \quad |g(t, x)| \leq \frac{C|x|}{t^{\lambda+\gamma+\varepsilon+1}} \leq \frac{C}{t^{1+\varepsilon}}\bar{s}(t, x) \end{aligned} \tag{4.11}$$

and

$$\left| \frac{3}{2t}(\bar{s}_x - \bar{s})(t, x) \right| \leq Ct^{-1}\bar{s}(t, x). \tag{4.12}$$

Gathering (4.11) and (4.12), we infer the existence of  $q > 0$ , such that, for  $t$  large enough,

$$\left( \partial_t - \partial_{xx} + \frac{3}{2t}(\partial_x - 1) \right) \bar{s}(t, x) \geq qt^{-(2\gamma+2\varepsilon)}\bar{s}(t, x) \geq \frac{q}{2}t^{-(2\gamma+2\varepsilon+\lambda)} \geq O\left(\frac{1}{t^{1-\gamma}}\right),$$

as soon as  $\varepsilon$  and  $\lambda$  are small enough, since  $\gamma \in (0, \frac{1}{3})$ . Because the right-hand side of (4.7) does not depend on  $\bar{s}$ , the inequality extends to all  $t \geq 1$  by replacing  $\bar{s}$  by  $A\bar{s}$ , with  $A$  large enough, and (4.8) follows.

Let us note that the term  $e^{-x}(v + \psi)$  in (4.7), which results from the quadratic structure of the nonlinearity, is positive. For a more general nonlinearity  $f(u)$  replacing  $u - u^2$ , the monotonicity of  $g(u) = uf'(0) - f(u)$  may be used in an analogous way.

**4.1 Proof of Theorem 1.1**

We are now ready to prove the theorem. Fix  $\gamma \in (0, \frac{1}{3})$ , as required by Proposition 4.1. Given  $\varepsilon > 0$ , take  $T_\varepsilon$  as in Lemma 4.2. Let  $u_\alpha(t, x)$  be the solution of (4.3) for  $t > T_\varepsilon$ , and the initial condition  $u_\alpha(T_\varepsilon, x) = u(T_\varepsilon, x)$ . Here,  $u(t, x)$  is the solution of the original problem (3.2). Taking  $T_\varepsilon$  larger, if necessary, we may assume that  $e^{-\frac{x^2}{4t}} \geq \frac{1}{2}$  for  $t \geq T_\varepsilon$ . It follows from Lemma 4.2 that for any  $t \geq T_\varepsilon$ , we have

$$u_{\alpha_\infty - 2\varepsilon}(t, x) \leq u(t, x) \leq u_{\alpha_\infty + 2\varepsilon}(t, x)$$

for all  $x \leq t^\gamma$ . From Proposition 4.1, we have

$$e^x [u_{\alpha_\infty \pm 2\varepsilon}(t, x) - \phi_*(x + \zeta_\pm(t))] = o(1) \quad \text{as } t \rightarrow +\infty, \tag{4.13}$$

uniformly in  $x \in (-t^\gamma, t^\gamma)$  with

$$\zeta_\pm(t) = -\log(\alpha_\infty \pm 2\varepsilon) - (\log(\alpha_\infty \pm 2\varepsilon) - k)t^{-\gamma} + O(t^{-2\gamma}).$$

Because  $\varepsilon > 0$  is arbitrary, we have

$$\lim_{t \rightarrow +\infty} (u(t, x) - \phi_*(x + x_\infty)) = 0$$

with  $x_\infty = -\log \alpha_\infty$ , uniformly on compact sets. Together with Lemma 4.1, this concludes the proof of Theorem 1.1.

**5 The Diffusive Scale  $x \sim O(\sqrt{t})$  and the Proof of Lemma 4.2**

Our analysis starts with (3.6), which we write as

$$w_\tau + Lw + \frac{3}{2}e^{-\frac{\tau}{2}}w_\eta + e^{\frac{3\tau}{2} - \eta \exp(\frac{\tau}{2})}w^2 = 0, \quad \eta \in \mathbb{R}, \tau > 0. \tag{5.1}$$

Here, the operator  $L$  is defined as

$$Lv = -v_{\eta\eta} - \frac{\eta}{2}v_\eta - v. \tag{5.2}$$

Its principal eigenfunction on the half-line  $\eta > 0$  with the Dirichlet boundary condition at  $\eta = 0$  is

$$\phi_0(\eta) = \frac{\eta}{2}e^{-\frac{\eta^2}{4}}$$

as  $L\phi_0 = 0$ . The operator  $L$  has a discrete spectrum in  $L^2(\mathbb{R}_+)$ , weighted by  $e^{-\frac{\eta^2}{8}}$ , its non-zero eigenvalues are  $\lambda_k = k \geq 1$ , and the corresponding eigenfunctions are related via

$$\phi_{k+1} = \phi_k''.$$

The principal eigenfunction of the adjoint operator

$$L^*\psi = -\psi_{\eta\eta} + \frac{1}{2}\partial_\eta(\eta\psi) - \psi$$

is  $\psi_0(\eta) = \eta$ . Thus, the solution of the unperturbed version of (5.1) on a half-line

$$p_\tau + Lp = 0, \quad \eta > 0, \quad p(\tau, 0) = 0 \tag{5.3}$$

satisfies

$$p(\tau, \eta) = \eta \frac{e^{-\frac{\eta^2}{4}}}{2\sqrt{\pi}} \int_0^{+\infty} \xi v_0(\xi) d\xi + O(e^{-\tau})e^{-\frac{\eta^2}{6}} \quad \text{as } \tau \rightarrow +\infty, \tag{5.4}$$

and our task is to generalize this asymptotics to the full problem (5.1) on the whole line. The weight  $e^{-\frac{\eta^2}{6}}$  in (5.4) is, of course, by no means optimal. We will prove the following.

**Lemma 5.1** *Let  $w(\tau, \eta)$  be the solution of (3.6) on  $\mathbb{R}$ , with the initial condition  $w(0, \eta) = w_0(\eta)$  such that  $w_0(\eta) = 0$  for all  $\eta > M$ , with some  $M > 0$ , and  $w_0(\eta) = O(e^\eta)$  for  $\eta < 0$ . There exists  $\alpha_\infty > 0$  and a function  $h(\tau)$  such that  $\lim_{\tau \rightarrow +\infty} h(\tau) = 0$ , and such that we have, for any  $\gamma' \in (0, \frac{1}{2})$ ,*

$$w(\tau, \eta) = (\alpha_\infty + h(\tau))\eta_+ e^{-\frac{\eta^2}{4}} + R(\tau, \eta)e^{-\frac{\eta^2}{6}}, \quad \eta \in \mathbb{R} \tag{5.5}$$

with

$$|R(\tau, \eta)| \leq C_{\gamma'} e^{-(\frac{1}{2}-\gamma')\tau},$$

where  $\eta_+ = \max(0, \eta)$ .

Once again, the weight  $e^{-\frac{\eta^2}{6}}$  is not optimal. Lemma 4.2 is an immediate consequence of this result. Indeed,

$$u(t, x) = e^{-x} \sqrt{t} w\left(\log t, \frac{x}{\sqrt{t}}\right),$$

hence Lemma 5.1 implies, with  $x_\gamma = t^\gamma$ ,

$$\begin{aligned} e^{x_\gamma} u(t, x_\gamma) - \alpha_\infty x_\gamma e^{-\frac{x_\gamma^2}{4t}} &= \sqrt{t} w\left(\log t, \frac{x_\gamma}{\sqrt{t}}\right) - \alpha_\infty x_\gamma e^{-\frac{x_\gamma^2}{4t}} \\ &= h(\log t) x_\gamma e^{-\frac{x_\gamma^2}{4t}} + \sqrt{t} R\left(\log t, \frac{x_\gamma}{\sqrt{t}}\right) e^{-\frac{x_\gamma^2}{6t}}. \end{aligned} \tag{5.6}$$

We now take  $T_\varepsilon$  so that  $|h(\log t)| < \frac{\varepsilon}{3}$  for all  $t > T_\varepsilon$ . For the second term in the right-hand side of (5.6), we write

$$\left| R\left(\log t, \frac{x_\gamma}{\sqrt{t}}\right) \right| \sqrt{t} e^{-\frac{x_\gamma^2}{6t}} \leq C t^{\gamma'} e^{-\frac{x_\gamma^2}{6t}} \leq \varepsilon x_\gamma e^{-\frac{x_\gamma^2}{6t}} \tag{5.7}$$

for  $t > T_\varepsilon$  sufficiently large, as soon as  $\gamma' < \gamma$ . This proves (4.2). Thus, the proof of Lemma 4.2 reduces to proving Lemma 5.1. We will prove the latter by a construction of an upper and lower barrier for  $w$  with the correct behaviors.

**5.1 The approximate Dirichlet boundary condition**

Let us come back to why the solution of (5.1) must approximately satisfy the Dirichlet boundary condition at  $\eta = 0$ . Recall that  $w$  is related to the solution of the original KPP problem via

$$w(\tau, \eta) = u(e^\tau, \eta e^{\frac{\tau}{2}})e^{-\frac{\tau}{2} + \eta e^{\frac{\tau}{2}}}.$$

The trivial a priori bound  $0 < u(t, x) < 1$  implies that we have

$$0 < w(\tau, \eta) < e^{-\frac{\tau}{2} + \eta e^{\frac{\tau}{2}}}, \quad \eta < 0, \tag{5.8}$$

and, in particular, we have

$$0 < w(\tau, -e^{-(\frac{1}{2}-\gamma)\tau}) \leq e^{-e^{\gamma\tau}}. \tag{5.9}$$

We also have

$$w_\tau(\tau, \eta) = u_t(e^\tau, \eta e^{\frac{\tau}{2}})e^{\frac{\tau}{2} + \eta e^{\frac{\tau}{2}}} + \frac{\eta}{2}u_x(e^\tau, \eta e^{\frac{\tau}{2}})e^{\eta e^{\frac{\tau}{2}}} + \left(\frac{\eta}{2}e^{\frac{\tau}{2}} - \frac{1}{2}\right)u(e^\tau, \eta e^{\frac{\tau}{2}})e^{-\frac{\tau}{2} + \eta e^{\frac{\tau}{2}}},$$

so that

$$\begin{aligned} w_\tau(\tau, -e^{-(\frac{1}{2}-\gamma)\tau}) &= u_t(e^\tau, -e^{\gamma\tau})e^{\frac{\tau}{2} - e^{\gamma\tau}} - \frac{1}{2}e^{-(\frac{1}{2}-\gamma)\tau}u_x(e^\tau, -e^{\gamma\tau})e^{-e^{\gamma\tau}} \\ &\quad - \frac{1}{2}(e^{\gamma\tau} + 1)u(e^\tau, -e^{\gamma\tau})e^{-\frac{\tau}{2} - e^{\gamma\tau}} \\ &= O(e^{-\gamma e^{\gamma\tau}}) \end{aligned} \tag{5.10}$$

for  $\gamma > 0$  sufficiently small. Thus, the solution of (5.1) satisfies

$$\begin{aligned} 0 < w(\tau, -e^{-(\frac{1}{2}-\gamma)\tau}) &\leq e^{-e^{\gamma\tau}}, \\ |w_\tau(\tau, -e^{-(\frac{1}{2}-\gamma)\tau})| &\leq Ce^{-\gamma e^{\gamma\tau}}, \end{aligned} \tag{5.11}$$

which we will use as an approximate Dirichlet boundary condition at  $\eta = 0$ .

**5.2 An upper barrier**

Consider the solution of

$$\begin{aligned} \bar{w}_\tau + L\bar{w} + \frac{3}{2}e^{-\frac{\tau}{2}}\bar{w}_\eta &= 0, \quad \tau > 0, \quad \eta > -e^{-(\frac{1}{2}-\gamma)\tau}, \\ \bar{w}(\tau, -e^{-(\frac{1}{2}-\gamma)\tau}) &= e^{-e^{\gamma\tau}} \end{aligned} \tag{5.12}$$

with a compactly supported initial condition  $\bar{w}_0(\eta) = \bar{w}(0, \eta)$  chosen so that  $\bar{w}_0(\eta) \geq u(1, \eta)e^\eta$ . Here,  $\gamma \in (0, \frac{1}{2})$  should be thought of as a small parameter.

It follows from (5.11) that  $\bar{w}(\tau, \eta)$  is an upper barrier for  $w(\tau, \eta)$ . That is, we have

$$w(\tau, \eta) \leq \bar{w}(\tau, \eta) \quad \text{for all } \tau > 0 \text{ and } \eta > -e^{-(\frac{1}{2}-\gamma)\tau}.$$

It is convenient to make a change of variables

$$\bar{w}(\tau, \eta) = \bar{p}(\tau, \eta + e^{-(\frac{1}{2}-\gamma)\tau}) + e^{-e^{\gamma\tau}} g(\eta + e^{-(\frac{1}{2}-\gamma)\tau}), \tag{5.13}$$

where  $g(\eta)$  is a smooth monotonic function such that  $g(\eta) = 1$  for  $0 \leq \eta < 1$  and  $g(\eta) = 0$  for  $\eta > 2$ . The function  $\bar{p}$  satisfies

$$\bar{p}_\tau + L\bar{p} + \left(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3}{2}e^{-\frac{\tau}{2}}\right)\bar{p}_\eta = G(\tau, \eta)e^{-e^{\gamma\tau}}, \quad \eta > 0, \quad \bar{p}(\tau, 0) = 0 \tag{5.14}$$

for  $\tau > 0$ , with a smooth function  $G(\tau, \eta)$  supported in  $0 \leq \eta \leq 2$ , and the initial condition

$$\bar{p}_0(\eta) = \bar{w}_0(\eta - 1) - e^{-1}g(\eta),$$

which also is compactly supported.

We will allow (5.14) to run for a large time  $T$ , after which time we can treat the right-hand side and the last term in the left-hand side of (5.14) as a small perturbation. A variant of Lemma 2.2 from [11] implies that  $\bar{p}(T, \eta)e^{\frac{\eta^2}{6}} \in L^2(\mathbb{R}_+)$  for all  $T > 0$ , as well as the following estimate.

**Lemma 5.2** *Consider  $\omega \in (0, \frac{1}{2})$  and  $G(\tau, \eta)$  smooth, bounded, and compactly supported in  $\mathbb{R}_+$ . Let  $p(\tau, \eta)$  solve*

$$|p_\tau + Lp| \leq \varepsilon e^{-\omega\tau} (|p_\eta| + |p| + G(\tau, \eta)), \quad \tau > 0, \quad \eta > 0, \quad p(\tau, 0) = 0 \tag{5.15}$$

with the initial condition  $p_0(\eta)$  such that  $p_0(\eta)e^{\frac{\eta^2}{6}} \in L^2(\mathbb{R}_+)$ . There exists  $\varepsilon_0 > 0$  and  $C > 0$  (depending on  $p_0$ ), such that, for all  $0 < \varepsilon < \varepsilon_0$ , we have

$$\begin{aligned} p(\tau, \eta) = & \eta \left( \frac{e^{-\frac{\eta^2}{4}}}{2\sqrt{\pi}} \left( \int_0^{+\infty} \xi p_0(\xi) d\xi + \varepsilon R_1(\tau, \eta) \right) \right. \\ & \left. + \varepsilon e^{-\omega\tau} R_2(\tau, \eta) e^{-\frac{\eta^2}{6}} + e^{-\tau} R_3(\tau, \eta) e^{-\frac{\eta^2}{6}} \right), \end{aligned} \tag{5.16}$$

where  $\|R_{1,2,3}(\tau, \cdot)\|_{C^3} \leq C$  for all  $\tau > 0$ .

For any  $\varepsilon > 0$ , we may choose  $T$  sufficiently large, and  $\omega \in (0, \frac{1}{2} - \gamma)$  so that

$$|\bar{p}_\tau + L\bar{p}| \leq \varepsilon e^{-\omega(\tau-T)} (|\bar{p}_\eta| + |G(\tau, \eta)|), \quad \tau > T, \quad \eta > 0, \quad p(\tau, 0) = 0. \tag{5.17}$$

This follows from (5.14). Then, applying Lemma 5.2 for  $\tau > T$ , we have

$$\begin{aligned} \bar{p}(\tau, \eta) = & \eta \left( \frac{e^{-\frac{\eta^2}{4}}}{2\sqrt{\pi}} \left( \int_0^{+\infty} \xi \bar{p}(T, \xi) d\xi + \varepsilon R_1(\tau, \eta) \right) \right. \\ & \left. + \varepsilon e^{-(\tau-T)} R_2(\tau, \eta) e^{-\frac{\eta^2}{6}} \right) \\ & + e^{-(\tau-T)} R_3(\tau, \eta) e^{-\frac{\eta^2}{6}}. \end{aligned} \tag{5.18}$$

We claim that with a suitable choice of  $\bar{w}_0$ , the integral term in (5.18) is bounded from below:

$$\int_0^\infty \eta \bar{p}(\tau, \eta) d\eta \geq 1 \quad \text{for all } \tau > 0. \tag{5.19}$$

Indeed, multiplying (5.14) by  $\eta$  and integrating gives

$$\frac{d}{d\tau} \int_0^\infty \eta \bar{p}(\tau, \eta) d\eta = \left( \gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3}{2} e^{-\frac{\tau}{2}} \right) \int_0^\infty \bar{p}(\tau, \eta) d\eta + e^{-e^{\gamma\tau}} \int G(\tau, \eta) \eta d\eta. \tag{5.20}$$

The function  $G(\tau, \eta)$  need not have a sign, hence we do not know that  $\bar{p}(\tau, \eta)$  is positive everywhere. However, it follows from (5.14) that the negative part of  $\bar{p}$  is bounded as

$$\int_0^\infty \bar{p}(\tau, \eta) d\eta \geq -C_0$$

for all  $\tau > 0$ , with the constant  $C_0$  which does not depend on  $\bar{w}_0(\eta)$  on the interval  $[2, \infty)$ . Thus, we deduce from (5.20) that for all  $\tau > 0$ , we have

$$\int_0^\infty \eta \bar{p}(\tau, \eta) d\eta \geq \int_0^\infty \eta \bar{w}_0(\eta) d\eta - C'_0 \tag{5.21}$$

with, once again,  $C'_0$  independent of  $\bar{w}_0$ . Therefore, after possibly increasing  $\bar{w}_0$ , we may ensure that (5.19) holds.

It follows from (5.18)–(5.19) that there exists a sequence  $\tau_n \rightarrow +\infty$ ,  $C > 0$  and a function  $\bar{W}_\infty(\eta)$ , such that

$$C^{-1} \eta e^{-\frac{\eta^2}{4}} \leq \bar{W}_\infty(\eta) \leq C \eta e^{-\frac{\eta^2}{4}} \tag{5.22}$$

and

$$\lim_{n \rightarrow +\infty} e^{\frac{\eta^2}{8}} |\bar{p}(\tau_n, \eta) - \bar{W}_\infty(\eta)| = 0, \tag{5.23}$$

uniformly in  $\eta$  on the half-line  $\eta \geq 0$ . The same bound for the function  $\bar{w}(\tau, \eta)$  itself follows

$$\lim_{n \rightarrow +\infty} e^{\frac{\eta^2}{8}} |\bar{w}(\tau_n, \eta) - \bar{W}_\infty(\eta)| = 0, \tag{5.24}$$

also uniformly in  $\eta$  on the half-line  $\eta \geq 0$ .

### 5.3 A lower barrier

A lower barrier for  $w(\tau, \eta)$  is devised as follows. First, note that the upper barrier for  $w(\tau, \eta)$  we have constructed above implies that

$$e^{\frac{3\tau}{2} - \eta \exp(\frac{\tau}{2})} w(\tau, \eta) \leq C_\gamma e^{-\exp(\frac{\gamma\tau}{2})},$$

as soon as

$$\eta \geq e^{-(\frac{1}{2}-\gamma)\tau}$$

with  $\gamma \in (0, \frac{1}{2})$ , and  $C_\gamma > 0$  is chosen sufficiently large. Thus, a lower barrier  $\underline{w}(\tau, \eta)$  can be defined as the solution of

$$\underline{w}_\tau + L\underline{w} + \frac{3}{2} e^{-\frac{\tau}{2}} \underline{w}_\eta + C_\gamma e^{-\exp(\frac{\gamma\tau}{2})} \underline{w} = 0, \quad \underline{w}(\tau, e^{-(\frac{1}{2}-\gamma)\tau}) = 0, \quad \eta > e^{-(\frac{1}{2}-\gamma)\tau} \tag{5.25}$$

with an initial condition  $\underline{w}_0(\eta) \leq w_0(\eta)$ . This time it is convenient to make the change of variables

$$\underline{w}(\tau, \eta) = \underline{z}(\tau, \eta - e^{-(\frac{1}{2}-\gamma)\tau}),$$

so that

$$\underline{z}_\tau + L\underline{z} + (-\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3}{2}e^{-\frac{\tau}{2}})\underline{z}_\eta + C_\gamma e^{-\exp(\frac{\gamma\tau}{2})}\underline{z} = 0, \quad \eta > 0, \quad \underline{z}(\tau, 0) = 0. \tag{5.26}$$

We could now try to use an abstract stable manifold theorem to prove that

$$\underline{I}(\tau) := \int_0^\infty \eta \underline{z}(\tau, \eta) d\eta \geq c_0 > 0 \quad \text{for all } \tau > 0. \tag{5.27}$$

That is,  $\underline{I}(\tau)$  remains uniformly bounded away from 0. However, to keep this paper self-contained, we give a direct proof of (5.27). We look for a sub-solution to (5.26) in the form

$$\underline{p}(\tau, \eta) = (\zeta(\tau)\phi_0(\eta) - q(\tau)\eta e^{-\frac{\eta^2}{8}})e^{-F(\tau)}, \tag{5.28}$$

where

$$F(\tau) = \int_0^\tau C_\gamma e^{-\exp(\frac{\gamma s}{2})} ds$$

with the functions  $\zeta(\tau)$  and  $q(\tau)$  satisfying

$$\zeta(\tau) \geq \zeta_0 > 0, \quad \dot{\zeta}(\tau) < 0, \quad q(\tau) > 0, \quad q(\tau) = O(e^{-\frac{\tau}{4}}). \tag{5.29}$$

In other words, we wish to devise  $\underline{p}(\tau, \eta)$  as in (5.28)–(5.29), such that

$$\underline{p}(0, \eta) \leq \underline{z}(0, \eta) = w_0(\eta + 1) \tag{5.30}$$

and

$$\mathcal{L}(\tau)\underline{p} \leq 0 \tag{5.31}$$

with

$$\mathcal{L}(\tau)\underline{p} = \underline{p}_\tau + L\underline{p} + \left(-\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3}{2}e^{-\frac{\tau}{2}}\right)\underline{p}_\eta.$$

Notice that the choice of  $F(\tau)$  in (5.28) has eliminated a low order term involving  $C_\gamma e^{-\exp(\gamma\frac{\tau}{2})}$ . For convenience, let us define

$$h(\tau) = -\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3}{2}e^{-\frac{\tau}{2}},$$

which appears in (5.26). Because  $L\phi_0 = 0$  and

$$L(\eta e^{-\frac{\eta^2}{8}}) = \eta L e^{-\frac{\eta^2}{8}} = \left(\frac{\eta^2}{16} - \frac{3}{4}\right)\eta e^{-\frac{\eta^2}{8}},$$

we find that

$$\mathcal{L}(\tau)\underline{p} = \dot{\zeta}\phi_0 + \zeta h(\tau)\phi'_0 - \left(\dot{q} + \left(\frac{\eta^2}{16} - \frac{3}{4}\right)q\right)\eta e^{-\frac{\eta^2}{8}} + q\frac{\eta^2}{4}e^{-\frac{\eta^2}{8}}h(\tau) - qe^{-\frac{\eta^2}{8}}h(\tau).$$

Let us write this as

$$\eta^{-1}e^{\frac{\eta^2}{8}}\mathcal{L}(\tau)\underline{p} = \dot{\zeta}\eta^{-1}\phi_0 e^{\frac{\eta^2}{8}} + \eta^{-1}h(\tau)\left(\zeta e^{\frac{\eta^2}{8}}\phi'_0 + q\left(\frac{\eta^2}{4} - 1\right)\right) - \left(\dot{q} + \left(\frac{\eta^2}{16} - \frac{3}{4}\right)q\right). \tag{5.32}$$

Our goal is to choose  $\zeta(\tau)$  and  $q(\tau)$ , such that (5.29) holds and the right-hand side of (5.32) is non-positive after a certain time  $\tau_0$ , possibly quite large. However, and this is an important point, this time  $\tau_0$  will not depend on the initial condition  $w_0(\eta)$ .

Let us restrict the small parameter  $\gamma$  to the interval  $(0, \frac{1}{4})$ . Observe that if  $\tau_0 > 0$  is sufficiently large, then  $h(\tau) < 0$  and  $|h(\tau)| \leq e^{-\frac{\tau}{4}}$  for all  $\tau \geq \tau_0$ . As  $\phi_0(\eta) = \eta e^{-\frac{\eta^2}{4}}$ , note that in (5.32) both  $\phi'_0(\eta)e^{\frac{\eta^2}{8}}$  and  $\phi_0(\eta)e^{\frac{\eta^2}{8}}$  are bounded functions. In particular, if  $\tau_0$  is large enough then

$$|\phi'_0 e^{\frac{\eta^2}{8}} h(\tau)| \leq e^{-\frac{\tau}{4}}$$

for all  $\tau \geq \tau_0, \eta \geq 0$ .

Note also that for all  $\eta \geq \eta_1 = \sqrt{28}$ , we have

$$\frac{\eta^2}{16} - \frac{3}{4} \geq 1 \quad \text{and} \quad \frac{\eta^2}{4} - 1 \geq 0. \tag{5.33}$$

Therefore, on the interval  $\eta \in [\eta_1, \infty)$  and for  $\tau \geq \tau_0$ , (5.32) is bounded by

$$\eta^{-1} e^{\frac{\eta^2}{8}} \mathcal{L}(\tau) \underline{p} \leq \eta^{-1} h(\tau) \zeta e^{\frac{\eta^2}{8}} \phi'_0 - (\dot{q} + q) \leq \zeta(\tau) e^{-\frac{\tau}{4}} - (\dot{q} + q),$$

assuming  $q(\tau) > 0$  and  $\dot{\zeta} < 0$ . Hence, if  $q(\tau)$  and  $\zeta(\tau)$  are chosen to satisfy the differential inequality

$$\dot{q} + q - e^{-\frac{\tau}{4}} \zeta \geq 0, \quad \tau \geq \tau_0, \tag{5.34}$$

then we will have

$$\mathcal{L}(\tau) \underline{p} \leq 0 \quad \text{for } \tau \geq \tau_0 \text{ and } \eta \geq \eta_1, \tag{5.35}$$

provided that  $\dot{\zeta} \leq 0$ , as presumed in (5.29). Still assuming  $\dot{\zeta} \leq 0$  on  $(\tau_0, +\infty)$ , a sufficient condition for (5.34) to be satisfied is

$$\dot{q} + q \geq e^{-\frac{\tau}{4}} \zeta(\tau_0), \quad \tau \geq \tau_0.$$

Hence, we choose

$$q(\tau) = e^{-(\tau-\tau_0)} + \frac{4}{3} e^{-\frac{\tau}{4}} \zeta(\tau_0). \tag{5.36}$$

Note that  $q(\tau)$  satisfies the assumptions on  $q$  in (5.29).

Let us now deal with the range  $\eta \in [0, \eta_1]$ . The function  $\eta^{-1} \phi_0(\eta)$  is bounded on  $\mathbb{R}$  and it is bounded away from 0 on  $[0, \eta_1]$ . Define

$$\varepsilon_1 = \min_{\eta \in [0, \eta_1]} \eta^{-1} \phi_0(\eta) e^{\frac{\eta^2}{8}} > 0.$$

As  $h(\tau) < 0$  for  $\tau \geq \tau_0$ , on the interval  $[0, \eta_1]$ , we can bound (5.32) by

$$\eta^{-1} e^{\frac{\eta^2}{8}} \mathcal{L}(\tau) \underline{p} \leq \varepsilon_1 \dot{\zeta}(\tau) + \eta^{-1} h(\tau) (\zeta e^{\frac{\eta^2}{8}} \phi'_0 - q) - \left( \dot{q} - \frac{3}{4} q \right). \tag{5.37}$$

For  $\eta \in [1, \eta_1]$ , where  $\eta^{-1} < 1$ , we have

$$\eta^{-1} e^{\frac{\eta^2}{8}} \mathcal{L}(\tau) \underline{p} \leq \varepsilon_1 \dot{\zeta}(\tau) + e^{-\frac{\tau}{4}} (\zeta + q) - \left( \dot{q} - \frac{3}{4} q \right). \tag{5.38}$$

To make this non-positive, we choose  $\zeta$  to satisfy

$$\varepsilon_1 \dot{\zeta}(\tau) \leq \dot{q} - \frac{3}{4} q - e^{-\frac{\tau}{4}} (\zeta + q) = e^{-\frac{\tau}{4}} \zeta(\tau_0) - \frac{7}{4} q(\tau) - e^{-\frac{\tau}{4}} (\zeta(\tau) + q(\tau)), \tag{5.39}$$

where the last equality comes from (5.36). Assuming  $\dot{\zeta} < 0$ , we have  $\zeta(\tau) < \zeta(\tau_0)$ , so a sufficient condition for (5.39) to hold when  $\tau \geq \tau_0$  is simply

$$\varepsilon_1 \dot{\zeta}(\tau) \leq -3q(\tau). \tag{5.40}$$

For  $\eta$  near 0, the dominant term in (5.37) is  $\eta^{-1}h(\tau)(\zeta e^{\frac{\eta^2}{8}}\phi'_0 - q)$ . Define

$$\varepsilon_2 = \min_{\eta \in [0,1]} \phi'_0(\eta)e^{\frac{\eta^2}{8}} > 0.$$

Therefore, if we can arrange that  $\zeta(\tau) > \frac{q(\tau)}{\varepsilon_2}$ , then for  $\eta \in [0, 1]$ , we have  $\zeta e^{\frac{\eta^2}{8}}\phi'_0 - q \geq 0$ , so

$$\eta^{-1}h(\tau)(\zeta e^{\frac{\eta^2}{8}}\phi'_0 - q) \leq 0.$$

In this case,

$$\eta^{-1}e^{\frac{\eta^2}{8}}\mathcal{L}(\tau)\underline{p} \leq \varepsilon_1 \dot{\zeta}(\tau) - \left(\dot{q} - \frac{3}{4}q\right), \tag{5.41}$$

which is non-positive for  $\tau \geq \tau_0$ , due to (5.39). In summary, we will have  $\mathcal{L}(\tau)\underline{p} \leq 0$  in the interval  $\eta \in [0, \eta_1]$  and  $\tau \geq \tau_0$  if  $\zeta$  satisfies (5.40) and  $\zeta(\tau) > \frac{q(\tau)}{\varepsilon_2}$  for  $\tau \geq \tau_0$ . In view of this, we let  $\zeta(\tau)$  have the form

$$\zeta(\tau) = a_2 + a_3 e^{-\frac{\tau-\tau_0}{4}}.$$

Thus, (5.40) holds if

$$-\frac{\varepsilon_1 a_3}{4} e^{-\frac{\tau-\tau_0}{4}} \leq -3q = -3e^{-(\tau-\tau_0)} - 4e^{-\frac{\tau}{4}}(a_2 + a_3), \quad \tau \geq \tau_0.$$

Hence it suffices that

$$\frac{\varepsilon_1 a_3}{4} \geq 3 + 4e^{-\frac{\tau_0}{4}}(a_2 + a_3)$$

holds; this may be achieved with  $a_2, a_3 > 0$  if  $\tau_0$  is large enough. Then we may take  $a_2$  large enough, so that  $\zeta(\tau) > \frac{q(\tau)}{\varepsilon_2}$  also holds for  $\tau \geq \tau_0$ ; this condition translates to

$$a_2 + a_3 e^{-\frac{\tau-\tau_0}{4}} \geq \frac{1}{\varepsilon_2} \left( e^{-(\tau-\tau_0)} + \frac{4}{3} e^{-\frac{\tau}{4}}(a_2 + a_3) \right), \quad \tau \geq \tau_0.$$

This also is attainable with  $a_2 > \frac{1}{\varepsilon_2}$  and  $a_3 > 0$  if  $\tau_0$  is chosen large enough. This completes the construction of the subsolution  $\underline{p}(\tau, \eta)$  in (5.28).

Let us come back to our subsolution  $\underline{z}(\tau, \eta)$ . From the strong maximum principle, we know that  $\underline{z}(\tau_0, \eta) > 0$  and  $\partial_\eta \underline{z}(\tau_0, 0) > 0$ . Hence, there is  $\lambda_0 > 0$ , such that

$$w(\tau_0, \eta) \geq \lambda_0 \underline{p}(\tau_0, \eta),$$

where  $\underline{p}$  is given by (5.28) with  $\zeta$  and  $q$  defined above, and we have for  $\tau \geq \tau_0$ ,

$$\underline{w}(\tau, \eta) \geq \lambda_0 \underline{p}(\tau, \eta).$$

This, by (5.29), bounds the quantity  $\underline{I}(\tau)$  uniformly from below, so that (5.29) holds with a constant  $c_0 > 0$  that depends on the initial condition  $w_0$ .

Therefore, just as in the study of the upper barrier, we obtain the uniform convergence of (possibly a subsequence of)  $\underline{w}(\tau_n, \cdot)$  on the half-line  $\eta \geq e^{-(\frac{1}{2}-\gamma)\tau}$  to a function  $\underline{W}_\infty(\eta)$  which satisfies

$$C^{-1}\eta e^{-\frac{\eta^2}{4}} \leq \underline{W}_\infty(\eta) \leq C\eta e^{-\frac{\eta^2}{4}}, \tag{5.42}$$

and such that

$$\lim_{n \rightarrow +\infty} e^{\frac{\eta^2}{8}} |\underline{w}(\tau_n, \eta) - \underline{W}_\infty(\eta)| = 0, \quad \eta > 0. \tag{5.43}$$

**5.4 Convergence of  $w(\tau, \eta)$ : Proof of Lemma 5.1**

Let  $X$  be the space of bounded uniformly continuous functions  $u(\eta)$ , such that  $e^{\frac{\eta^2}{8}}u(\eta)$  is bounded and uniformly continuous on  $\mathbb{R}_+$ . We deduce from the convergence of the upper and lower barriers for  $w(\tau, \eta)$  (and ensuing uniform bounds for  $w$ ) that there exists a sequence  $\tau_n \rightarrow +\infty$ , such that  $w(\tau_n, \cdot)$  itself converges to a limit  $W_\infty \in X$ , such that  $W_\infty \equiv 0$  on  $\mathbb{R}_-$ , and  $W_\infty(\eta) > 0$  for all  $\eta > 0$ . Our next step is to bootstrap the convergence along a sub-sequence, and show that the limit of  $w(\tau, \eta)$  as  $\tau \rightarrow +\infty$  exists in the space  $X$ . First, observe that the above convergence implies that the shifted functions  $w_n(\tau, \eta) = w(\tau + \tau_n, \eta)$  converge in  $X$ , uniformly on compact time intervals, as  $n \rightarrow +\infty$  to the solution  $w_\infty(\tau, \eta)$  of the linear problem

$$\begin{aligned} (\partial_\tau + L)w_\infty &= 0, \quad \eta > 0, \\ w_\infty(\tau, 0) &= 0, \\ w_\infty(0, \eta) &= W_\infty(\eta). \end{aligned} \tag{5.44}$$

In addition, there exists  $\alpha_\infty > 0$ , such that  $w_\infty(\tau, \eta)$  converges to  $\bar{\psi}(\eta) = \alpha_\infty \eta e^{-\frac{\eta^2}{4}}$ , in the topology of  $X$  as  $\tau \rightarrow +\infty$ . Thus, for any  $\varepsilon > 0$ , we may choose  $T_\varepsilon$  large enough, so that

$$|w_\infty(\tau, \eta) - \alpha_\infty \eta e^{-\frac{\eta^2}{4}}| \leq \varepsilon \eta e^{-\frac{\eta^2}{8}} \quad \text{for all } \tau > T_\varepsilon \text{ and } \eta > 0. \tag{5.45}$$

Given  $T_\varepsilon$ , we can find  $N_\varepsilon$  sufficiently large so that

$$|w(T_\varepsilon + \tau_n, \eta + e^{-(\frac{1}{2}-\gamma)T_\varepsilon}) - w_\infty(T_\varepsilon, \eta)| \leq \varepsilon \eta e^{-\frac{\eta^2}{8}} \quad \text{for all } n > N_\varepsilon. \tag{5.46}$$

In particular, we have

$$\alpha_\infty \eta e^{-\frac{\eta^2}{4}} - 2\varepsilon \eta e^{-\frac{\eta^2}{8}} \leq w(\tau_{N_\varepsilon} + T_\varepsilon, \eta + e^{-(\frac{1}{2}-\gamma)T_\varepsilon}) \leq \alpha_\infty \eta e^{-\frac{\eta^2}{4}} + 2\varepsilon \eta e^{-\frac{\eta^2}{8}}. \tag{5.47}$$

We may now construct the upper and lower barriers for the function  $w(\tau + \tau_{N_\varepsilon} + T_\varepsilon, \eta + e^{-(\frac{1}{2}-\gamma)T_\varepsilon})$ , exactly as we have done before. It follows, once again from Lemma 5.2 applied to these barriers that any limit point  $\phi_\infty$  of  $w(\tau, \cdot)$  in  $X$  as  $\tau \rightarrow +\infty$  satisfies

$$(\alpha_\infty - C\varepsilon)\eta e^{-\frac{\eta^2}{4}} \leq \phi_\infty(\eta) \leq (\alpha_\infty + C\varepsilon)\eta e^{-\frac{\eta^2}{4}}. \tag{5.48}$$

As  $\varepsilon > 0$  is arbitrary, we conclude that  $w(\tau, \eta)$  converges in  $X$  as  $\tau \rightarrow +\infty$  to  $\bar{\psi}(\eta) = \alpha_\infty \eta e^{-\frac{\eta^2}{4}}$ . Taking into account Lemma 5.2 once again, applied to the upper and lower barriers for  $w(\tau, \eta)$  constructed starting from any time  $\tau > 0$ , we have proved Lemma 5.1, which implies Lemma 4.2.

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