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Symmetrization for Fractional Elliptic and Parabolic Equations and an Isoperimetric Application*

Yannick SIRE[1] Juan Luis VÁZQUEZ[2] Bruno VOLZONE[3]

(*Dedicated to Haim Brezis, great master of analysis, on his* 70*th birthday*)

Abstract This paper develops further the theory of symmetrization of fractional Laplacian operators contained in recent works of two of the authors. This theory leads to optimal estimates in the form of concentration comparison inequalities for both elliptic and parabolic equations. The authors extend the theory for the so-called restricted fractional Laplacian defined on a bounded domain Ω of $\mathbb{R}^N$ with zero Dirichlet conditions outside of Ω. As an application, an original proof of the corresponding fractional Faber-Krahn inequality is derived. A more classical variational proof of the inequality is also provided.

Keywords Symmetrization, Fractional Laplacian, Nonlocal elliptic and parabolic equations, Faber-Krahn inequality
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1 Introduction

In this paper, we develop further the theory of symmetrization for fractional Laplacian operators initiated in [25, 64–65], both in the elliptic and the parabolic setting, by extending it to a natural version of the fractional Laplacian defined on a bounded domain Ω of $\mathbb{R}^N$ which is known as the restricted fractional Laplacian. This research direction combines classical themes in the study of nonlinear elliptic and parabolic equations, like symmetrization and accretive operators, with the recent interest in nonlocal versions of the diffusion operators, specially the fractional Laplacians. As an application of the obtained comparison results, we derive an original proof of the Faber-Krahn inequality (FKI for short) for such operators defined on the bounded domain Ω.

Before entering into the description of our results, we review in this introduction the necessary information about symmetrization, the elliptic-to-parabolic technique used to generate

Manuscript received June 25, 2015. Revised October 10, 2015.
[1]Johns Hopkins University, Krieger Hall, 3401 N. Charles St, Baltimore, MD, 21218, USA.
E-mail: sire@math.jhu.edu
[2]Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.
E-mail: juanluis.vazquez@uam.es
[3]Dipartimento di Ingegneria, Universit degli Studi di Napoli "Parthenope", 80143 Italia.
E-mail: bruno.volzone@uniparthenope.it
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evolution semigroups, the precise definition of the fractional Laplacian operators and the relation among these topics. This constitutes a sort of review part of this paper.

Definitions of fractional Laplacians on bounded domains When working in the whole space domain $\mathbb{R}^N$, there are several equivalent definitions of the fractional Laplacian operator $(-\Delta)^{\frac{\sigma}{2}}$, $0<\sigma<2$, classical references being [39, 53]. The interest in these operators has a long history in probability since the fractional Laplacian operators of the form $(-\Delta)^{\frac{\sigma}{2}}$ are infinitesimal generators of stable Lévy processes (see [1, 7, 57]). Further motivation and references on the literature are given for instance in [11, 64]. A particular definition that was convenient for symmetrization purposes defines the operator for every given $0<\sigma<2$ as the trace of a suitable Dirichlet-Neumann problem via an extended potential function w that solves an elliptic equation in an upper half-space in $\mathcal{H}^+=\mathbb{R}^N\times(0,\infty)\subset\mathbb{R}^{N+1}$. This is usually called Caffarelli-Silvestre extension (see [18]). It allows to reduce nonlocal problems involving $(-\Delta)^{\frac{\sigma}{2}}$ to suitable local problems (actually, a degenerate-singular elliptic equation), defined in one more space dimension.

When we work on a bounded domain $\Omega\subset\mathbb{R}^N$, things get complicated because there are several options for defining the fractional Laplacian operator $(-\Delta)^{\frac{\sigma}{2}}$. Two of them appear often in the recent literature. In our previous work [65] on symmetrization, we followed one of these approaches to define the fractional Laplacian as the Dirichlet-to-Neumann map, through an extended potential function defined in a cylinder $\mathcal{C}=\Omega\times(0,\infty)\subset\mathbb{R}^{N+1}$, as was proposed in [17, 22]. Zero values are assigned on the lateral boundary of $\mathcal{C}$. We call this operator the spectral version of the fractional Laplacian on Ω. Let us call this operator $\mathcal{L}_1$ ($\mathcal{L}$ stands for the Laplacian). This setting allowed us to derive in [65] the desired symmetrization results, which extend the standard symmetrization theory applied to elliptic and parabolic equations driven by the standard Laplacian operator. But let us recall that there are remarkable restrictions on their validity in the form of conditions on the nonlinearities that are allowed in the equations.

In this paper, we take the second usual approach to define $(-\Delta)^{\frac{\sigma}{2}}$, which seems to be more natural in many applications. It consists in keeping the definition of fractional Laplacian in $\mathbb{R}^N$ but asking it to act on the null-extensions to $\mathbb{R}^N$ of functions $u(x)$ defined in Ω. So in principle, we can use the most common formulation with a hyper-singular kernel

$$(-\Delta)^{\frac{\sigma}{2}}f(x)=c(N,\sigma)\int_{\mathbb{R}^N}\frac{f(x)-f(y)}{|x-y|^{N+\sigma}}\,\mathrm{d}y \tag{1.1}$$

on the condition that $f(y)=0$ for $y\notin\Omega$. Let us call this operator $\mathcal{L}_2$. This option was called the restricted Laplacian on a bounded domain (see [49, 10]), but we prefer the name natural fractional Laplacian with Dirichlet conditions in this paper. The discussion on the relations and differences between the two types of operators on bounded domains is currently being investigated by several authors. Thus, Musina and Nazarov [44] used the name fractional Laplacian with Navier conditions for the spectral version, and fractional Laplacian with Dirichlet conditions for the restricted version.

Here, we want to extend to operator $\mathcal{L}_2$ the symmetrization theory we had developed for $\mathcal{L}_1$ in [25, 64]. This has an independent interest since there are subtle differences between the two operators (see [10, 12]).

Symmetrization Symmetrization is a very ancient geometrical idea that is used nowadays as an efficient tool of obtaining a priori estimates for the solutions of different partial differential equations, notably those of elliptic and parabolic type. Since the topic is so well-known, let us only recall some facts that are relevant here. Symmetrization techniques appear in classical works like [34, 47]. The application of Schwarz symmetrization to obtaining a priori estimates for elliptic problems was already described in [42, 66]. The standard elliptic result refers to the solutions of an equation of the form

$$Lu = f, \quad Lu = -\sum_{i,j} \partial_i (a_{ij} \partial_j u)$$

posed in a bounded domain $\Omega \subseteq \mathbb{R}^N$; the coefficients $\{a_{ij}\}$ are assumed to be bounded, measurable and satisfy the usual ellipticity condition; finally, we take zero Dirichlet boundary conditions on the boundary $\partial\Omega$. The classical analysis introduced by Talenti [54–55] leads to pointwise comparison between the symmetrized version (more precisely the spherical decreasing rearrangement) of the actual solution of the problem $u(x)$ and the radially symmetric solution $v(|x|)$ of some radially symmetric model problem which is posed in a ball with the same volume as Ω. Sharp a priori estimates for the solutions are then derived. Extensions of this method to more general problems or related equations led to a copious literature.

Elliptic approach to parabolic problems For parabolic problems, this pointwise comparison fails and the appropriate concept is comparison of concentrations (see [2–3, 58]). The latter considers the evolution problems of the form

$$\partial_t u = \Delta A(u), \quad u(0) = u_0, \tag{1.2}$$

where A is a monotone increasing real function, and u_0 is a suitably given initial datum which is assumed to be integrable. For simplicity, the problem was posed for $x \in \mathbb{R}^N$, but bounded open sets can be used as spatial domains. The novel idea of the paper is to use the famous Crandall-Liggett Implicit Discretization theorem (see [24]) to reduce the evolution problem to a sequence of nonlinear elliptic problems of the iterative form

$$-h\, \Delta A(u(t_k)) + u(t_k) = u(t_{k-1}), \quad k = 1, 2, \cdots, \tag{1.3}$$

where $t_k = kh$, and $h > 0$ is the time step per iteration. Writing $A(u) = v$, the resulting chain of elliptic problems can be written in the common form

$$h\, Lv + B(v) = f, \quad B = A^{-1}. \tag{1.4}$$

General theory of these equations (see [8]), ensures that the solution map

$$T : f \mapsto u = B(v)$$

is a contraction in some Banach space, which happens to be $L^1(\Omega)$. Note that the constant $h > 0$ is not essential, it can be put to 1 by scaling. In that context, the symmetrization result can be split into two results:

(i) The first one applies to rearranged right-hand sides and solutions. It says that if two r.h.s. functions f_1, f_2, are rearranged and satisfy a concentration comparison of the form $f_1 \prec f_2$, then the same applies to the solutions, in the form $B(v_1) \prec B(v_2)$.[1]

[1]For the definition of the order relation $\prec$, see Section 7.

(ii) The second result aims at comparing the solution v of (1.4) with a non-rearranged function f with the solution $\widetilde{v}$ corresponding to $f^{\#}$, the radially decreasing rearrangement of f. We obtain that $\widetilde{v}$ is a rearranged function and $B(v^{\#}) \prec B(\widetilde{v})$, i. e., $B(v)$ is less concentrated than $B(\widetilde{v})$.

This precise pair of comparison results can be combined to obtain similar results along the whole chain of iterations $u(t_k)$ of the evolution process, if discretized as indicated above. This allows in turn to conclude the symmetrization theorems (concentration comparison and comparison of L^p norms) for the evolution problem (1.2). This approach can be used in many different situations. In particular, it will be used below.

Symmetrization for equations with fractional operators The study of elliptic and parabolic equations involving nonlocal operators, usually of fractional type, is currently the subject of great attention. Symmetrization techniques were first applied to PDEs involving fractional Laplacian operators in [25], where the following linear elliptic case is studied:

$$(-\Delta)^{\frac{\sigma}{2}} v = f. \tag{1.5}$$

This paper uses an interesting technique of Steiner symmetrization of the extended problem, based on the Caffarelli-Silvestre extension for the definition of σ-Laplacian operator. In [64] the last two authors of the present paper were able to improve on that progress and combine it with the parabolic ideas of [58] to establish the relevant comparison theorems based on symmetrization for linear and nonlinear parabolic equations. To be specific, they dealt with equations of the form

$$\partial_t u + (-\Delta)^{\frac{\sigma}{2}} A(u) = f, \quad 0 < \sigma < 2\,. \tag{1.6}$$

Following the known theory for the standard Laplacian, the nonlinearity A is an increasing real function such that $A(0) = 0$, and we accept some extra regularity conditions as needed, like A smooth with $A'(u) > 0$ for all $u > 0$. The problem is posed in the whole space $\mathbb{R}^N$. Special attention is paid to cases of the form $A(u) = u^m$ with $m > 0$; the equation is then called the fractional heat equation (FHE for short) when $m = 1$, the fractional porous medium equation (FPME for short) if $m > 1$, and the fractional fast diffusion equation (FFDE for short) if $m < 1$. Let us recall that the linear equation $\partial_t u + (-\Delta)^{\frac{\sigma}{2}} u = 0$ is a model of so-called anomalous diffusion, a much studied topic in physics.

The results of [25, 64] include a comparison of concentrations, in the form $v^{\#} \prec \widetilde{v}$, that parallels the result that holds in the standard Laplacian case; note however that no pointwise comparison is obtained, so the result looks a bit like the parabolic results of the standard theory mentioned above. [64] considered both problems posed in the whole space and on a bounded domain. In the latter case, the spectral fractional Laplacian is always chosen.

2 Outline of Results of the Present Paper

We are interested in considering the application of such symmetrization techniques to linear or nonlinear elliptic and parabolic equations with fractional Laplacian operators posed on a bounded domain, when the natural (i.e., restricted) version of fractional Laplacian is used. We denote the operator by $\mathcal{L}_2$.

Parabolic equations To be specific, we want to treat evolution equations of the form

$$\partial_t u + (-\Delta)^{\frac{\sigma}{2}} A(u) = f, \quad 0 < \sigma < 2. \tag{2.1}$$

We want to consider as nonlinearity A an increasing real function such that $A(0) = 0$, and we may accept some other regularity conditions as needed, like A smooth with $A'(u) > 0$ for all $u > 0$. The problem is posed in Ω, a bounded subset of $\mathbb{R}^N$ with smooth boundary. The parabolic result is developed in Section 5 and has to be compared with the results of papers [64–65]. We focus on the linear case $A(u) = cu$. This is the case that is needed in the isoperimetric application that we study in Section 6.

Elliptic equations The application of the method of implicit time discretization leads to the nonlinear equation of elliptic type

$$h\,(-\Delta)^{\frac{\sigma}{2}} v + B(v) = f \tag{2.2}$$

posed again in the whole space $\Omega \subset \mathbb{R}^N$ with zero Dirichlet boundary conditions; $h > 0$ is a non-essential constant, and the nonlinearity B is the inverse function to the monotone function A that appears in the parabolic equation (1.6). The elliptic results are developed in Sections 3–4 and have to be compared with the results of [25, 64] where the equation is posed either in $\mathbb{R}^N$ or in Ω with operator $\mathcal{L}_1$. Note that the elliptic results we get cover the standard linear case where the term $B(v)$ disappears, and we set $h = 1$.

A geometrical application (the Faber-Krahn inequality) As an application, we use the symmetrization results to prove the Faber Krahn inequality for the fractional Laplacian operator on a bounded domain in both versions considered above. We recall that the FKI is a classical eigenvalue inequality, due separately to [31, 38], based on a conjecture by Rayleigh in 1877, that can be stated as follows.

Let Ω be a bounded domain in $\mathbb{R}^N$, and let B be the ball centered at the origin with $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B)$. Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplacian operator, with zero Dirichlet boundary conditions. Then $\lambda_1(\Omega) \geq \lambda_1(B)$, with equality if and only if $\Omega = B$ almost everywhere.

Remark 2.1 Note that we do not assume any regularity on the bounded domain Ω besides its openness.

This is a classical result in the calculus of variations and proofs can be found in the classical books like [20] (see a recent proof in [15]). The question we want to address here is the following: Will the result also hold for the usual versions of the fractional Laplacian operator $(-\Delta)^{\frac{\sigma}{2}}$ defined on bounded domains of $\mathbb{R}^N$ with zero Dirichlet boundary conditions?

The answer is immediate in the case of the so-called spectral version of the Dirichlet fractional Laplacian, $\mathcal{L}_1$, since its eigenvalues, $\lambda_k(\mathcal{L}_1; \Omega)$, are directly related to those of the standard Laplacian, $\lambda_k((-\Delta); \Omega)$, by the following formula:

$$\lambda_k(\mathcal{L}_1; \Omega) = (\lambda_k((-\Delta); \Omega))^{\frac{\sigma}{2}}. \tag{2.3}$$

However, no simple relation like this one happens for the natural fractional Laplacian with the definition restricted type, $\mathcal{L}_2$.

In Section 6, we use our comparison results to present an original derivation of the fractional FKI. It does not make use of any variational interpretation, but only of some properties of the evolution process. The FKI can also be studied either by probabilistic or variational methods. For completeness, we also present a variational derivation, see more details in the mentioned section. This latter proof is based on the original argument for the FKI for the Laplacian operator and relies on Pólya-Szegö inequalities.

Preliminary material and notation In this paper, we use standard concepts and notations on symmetrization as fixed in [64]. We gather the main facts that we did not present here in the first appendix for the reader's convenience.

3 Elliptic Problem with Lower-Order Term

The case of the natural fractional Laplacian $\mathcal{L}_2$ occupies our attention in this paper. We start our analysis by the following nonlocal elliptic problem with Dirichlet condition:

$$\begin{cases}(-\Delta)^{\frac{\sigma}{2}}v + B(v) = f(x) & \text{in } \Omega,\\ v = 0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases} \tag{3.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $\sigma\in(0,2)$, and f is an integrable function defined in Ω. We are interested in treating the case of a bounded domain Ω with the natural version of the fractional Laplacian. Exceptionally, Ω may be $\mathbb{R}^N$, but this case was treated in [64]. We assume that the nonlinearity is given by a function $B:\mathbb{R}_+\to\mathbb{R}_+$ which is smooth and monotone increasing with $B(0)=0$ and $B'(v)>0$. It is not essential to consider negative values for our main results, but the general theory can be done in that greater generality, just by assuming that B is extended to a function $B:\mathbb{R}_-\to\mathbb{R}_-$ by symmetry, $B(-v)=-B(v)$. Note that we have changed a bit the notation with respect to (2.2) in the introduction, by eliminating the constant $h>0$, but the change is inessential for the comparison results.

For simplicity, all the restrictions of the fractional Laplacian operator will be denoted by $(-\Delta)^{\frac{\sigma}{2}}$. The underlying assumption is that such operator will be restricted to the ground domain of each boundary value problem where it is involved.

The extension method The Caffarelli-Silvestre method can be kept as extension to the whole $\mathcal{H}^+$, with the following important proviso: The extension must act on the null-extensions to $\mathbb{R}^N$ of functions defined in Ω. In view of this discussion, a solution to (2.2) is defined here as the trace of a properly defined Dirichlet-Neumann problem in the following way:

$$\begin{cases}-\operatorname{div}_{x,y}(y^{1-\sigma}\nabla w)=0 & \text{in } \mathcal{H}^+,\\ w(x,0)=0 & \text{for } x\in\mathbb{R}^N\setminus\Omega,\\ -\dfrac{1}{\kappa_\sigma}\lim\limits_{y\to0^+} y^{1-\sigma}\dfrac{\partial w}{\partial y}(x,y)+B(w(x,0))=f(x) & \text{for } x\in\Omega,\end{cases} \tag{3.2}$$

where

$$\mathcal{H}^+ := \mathbb{R}^N\times(0,+\infty),$$

and κ_σ is the constant $\kappa_\sigma := 2^{1-\sigma}\frac{\Gamma(1-\frac{\sigma}{2})}{\Gamma(\frac{\sigma}{2})}$ (see [18]), but such a value is not important here.

3.1 Review of existence, uniqueness and main properties

If (3.2) is solved in an appropriate sense, then the trace of w over Ω, $\mathrm{Tr}_\Omega(w) = w(\cdot,0) =: v$ is said to be a solution to (3.1). Note that the trace of w on the bottom hyperplane $\{y=0\}$ is the extension $\widetilde{v}$ of the function v defined in Ω by assigning the value zero outside of Ω. This is what makes the difference with the case of the spectral Laplacian, where on the contrary the domain of the extended function w is the cylinder $\mathcal{C} = \Omega\times(0,\infty)$ and w takes zero boundary conditions on the lateral boundary, $\Sigma = \partial\Omega\times[0,\infty)$ (see [64]). In accordance with our choice of operator and in view of the iteration process that leads to the solution of the parabolic equations, we need only consider functions f that are restrictions to Ω of functions defined in the whole of $\mathbb{R}^N$.

In order to make this more precise, we introduce the concept of weak solution to problem (3.2). It is convenient to define the weighted energy space

$$X^{\frac{\sigma}{2}}(\mathcal{H}^+) = \left\{ w \in H^1_{\mathrm{loc}}(\mathcal{H}^+) : \int_{\mathcal{H}^+} y^{1-\sigma} |\nabla_{x,y} w(x,y)|^2 \,\mathrm{d}x\mathrm{d}y < \infty \right\},$$

equipped with the norm

$$\|w\|_{X^{\frac{\sigma}{2}}(\mathcal{H}^+)} := \left(\int_{\mathcal{H}^+} y^{1-\sigma}\, |\nabla w(x,y)|^2 \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \tag{3.3}$$

For an open set E of $\mathbb{R}^N$, we denote by $H^{\frac{\sigma}{2}}(E)$ the classical fractional Sobolev space of order $\frac{\sigma}{2}$ over E. We recall that for any $u \in H^{\frac{\sigma}{2}}(\mathbb{R}^N)$, there exists a unique σ-harmonic extension $w \in X^{\frac{\sigma}{2}}(\mathcal{H}^+)$ of u to the half space $\mathcal{H}^+$, namely w solves

$$\begin{cases} -\,\mathrm{div}_{x,y}(y^{1-\sigma}\nabla w) = 0 & \text{in } \mathcal{H}^+, \\ w(x,0) = u(x) & \text{for } x\in\mathbb{R}^N. \end{cases} \tag{3.4}$$

Then we write $w = \mathrm{Ext}_{\mathcal{H}^+}(u)$. Moreover, for suitable functions u, we have

$$(-\Delta)^{\frac{\sigma}{2}} u = -\frac{1}{\kappa_\sigma} \lim_{y\to 0^+} y^{1-\sigma}\,\frac{\partial w}{\partial y}(x,y).$$

Now, in order to give a proper meaning of solution to a problem like (3.2) in a bounded domain Ω, we define the space of all functions in $X^{\frac{\sigma}{2}}(\mathcal{H}^+)$ whose traces over $\mathbb{R}^N$ vanish outside of Ω, namely,

$$X^{\frac{\sigma}{2}}_{\Omega}(\mathcal{H}^+) = \{ w \in X^{\frac{\sigma}{2}}(\mathcal{H}^+) : w|_{\mathbb{R}^N\times\{0\}} \equiv 0 \ \text{ in } \ \mathbb{R}^N\setminus\Omega \}. \tag{3.5}$$

The domain of the natural fractional Laplacian $(-\Delta)^{\frac{\sigma}{2}}$ is the space $\mathcal{H}(\Omega)$ defined by

$$\mathcal{H}(\Omega) = \begin{cases} H^{\frac{\sigma}{2}}(\Omega), & \text{if } 0<\sigma<1, \\ H^{\frac{1}{2}}_{00}(\Omega), & \text{if } \sigma = 1, \\ H^{\frac{\sigma}{2}}_{0}(\Omega), & \text{if } 1<\sigma\le 2, \end{cases} \tag{3.6}$$

where $H^{\frac{\sigma}{2}}(\Omega)$ and $H^{\frac{\sigma}{2}}_0(\Omega)$ are usual fractional Sobolev spaces (see [40]), and

$$H^{\frac{1}{2}}_{00}(\Omega) = \left\{ u \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(x)}{d^2(x)}\mathrm{d}x < \infty \right\}$$

with $d(x) = \operatorname{dist}(x, \partial\Omega)$. It turns out that

$$\mathcal{H}(\Omega) = \{w|_{\Omega\times\{0\}} : w \in X_\Omega^{\frac{\sigma}{2}}(\mathcal{H}^+)\}$$

(see [10] for a detailed account on this question). Then we provide the following definition.

Definition 3.1 *Let Ω be an open bounded set of $\mathbb{R}^N$ and $f \in L^1(\Omega)$. We say that $w \in X_\Omega^{\frac{\sigma}{2}}(\mathcal{H}^+)$ is a weak solution to* (3.2) *if* $\operatorname{Tr}_\Omega(B(w)) =: B(w(x,0)) \in L^1(\Omega)$ *and*

$$\int_{\mathcal{H}^+} y^{1-\sigma}\nabla_{x,y}w\cdot\nabla_{x,y}\varphi\,\mathrm{d}x\,\mathrm{d}y + \kappa_\sigma\int_\Omega B(w(x,0))\,\varphi(x,0)\mathrm{d}x = \kappa_\sigma\int_\Omega f(x)\,\varphi(x,0)\mathrm{d}x \tag{3.7}$$

for all the test functions $\varphi \in C^1(\overline{\mathcal{H}^+})$ such that $\operatorname{Tr}_{\mathbb{R}^N}(\varphi) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

If w is a solution to the "extended problem" (3.2), then the trace function $v = \operatorname{Tr}_\Omega(w)$ will be called a weak solution to (3.1).

Remark 3.1 It is clear that if $B(t) = ct$ for all $t \geq 0$ for some $c \geq 0$, then (3.2) becomes linear and the function $v = \operatorname{Tr}_\Omega(w)$ belongs to the space $\mathcal{H}(\Omega)$.

Concerning existence of solutions, their smoothness and L^1 contraction properties, we excerpt some known results from [10, 26–27] which can be extended for our more general nonlinearity B. For the regularity, the reader may consult [16, 18, 50–52].

Theorem 3.1 *For any $f \in L^\infty(\Omega)$, there exists a unique weak solution $w \in X_0^{\frac{\sigma}{2}}(\mathcal{C}_\Omega)$ to* (3.2)*, such that $\operatorname{Tr}_\Omega(B(w)) \in L^\infty(\Omega)$. Moreover, we have the following:*

(i) *Regularity. We have $w \in C^\alpha(\mathcal{C}_\Omega)$ for every $\alpha < \sigma$ if $\sigma \leq 1$ (resp. $w \in C^{1,\alpha}(\mathcal{C}_\Omega)$ for every $\alpha < \sigma - 1$ if $\sigma > 1$). Arguing as in* [17], *the higher regularity of w depends easily on the higher regularity of f and B.*

(ii) L^1 *contraction. If $w, \widetilde{w}$ are the solutions to* (3.2) *corresponding to data $f, \widetilde{f}$, the following L^1 contraction property holds:*

$$\int_\Omega [B(w(x,0)) - B(\widetilde{w}(x,0))]_+\mathrm{d}x \leq \int_\Omega [f(x) - \widetilde{f}(x)]_+\mathrm{d}x. \tag{3.8}$$

In particular, we have that $w \geq 0$ in $\overline{\mathcal{C}}_\Omega$ whenever $f \geq 0$ on Ω. Furthermore, if we put $u := B(w(\cdot,0))$, then for all $p \in [1,\infty]$, we have

$$\|u\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

(iii) *For $f \in L^1(\Omega)$, the weak solution is obtained as the limit of the solutions of approximate problems with $f_n \in L^1(\Omega) \cap L^\infty(\Omega)$, $f_n \to f$ in L^1. Then the sequence $\{B(w_n(x,0))\}_n$ also converges in L^1 to some $B(w(x,0))$, and $\|B(w(x,0))\|_1 \leq \|f\|_1$, and hence v_n is uniformly bounded in L^p for all small p. Property* (ii) *holds for such limit solutions.*

Remark 3.2 (On Some Nonhomogeneous Boundary Value Problems) Let $\varepsilon > 0$. For our arguments it is essential to consider problems with nonhomogeneous boundary values of the type

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}}v + B(v) = f(x) & \text{in } \Omega,\\ v = \varepsilon & \text{in } \mathbb{R}^N \setminus \Omega\,. \end{cases} \tag{3.9}$$

In order to ensure the existence of a solution to this problem, we associate it with the following nonhomogeneous extension problem:

$$\begin{cases} -\operatorname{div}_{x,y}(y^{1-\sigma}\nabla w) = 0 & \text{in } \mathcal{H}^+, \\ w(x,0) = \varepsilon & \text{for } x \in \mathbb{R}^N \setminus \Omega, \\ -\dfrac{1}{\kappa_\sigma} \lim\limits_{y\to 0^+} y^{1-\sigma} \dfrac{\partial w}{\partial y}(x,y) + B(w(x,0)) = f(x) & \text{for } x \in \Omega. \end{cases} \tag{3.10}$$

If $f \in L^1(\Omega)$, we say that $w \in X^{\frac{\sigma}{2}}(\mathcal{H}^+)$ is a weak solution to (3.10) if $w - \varepsilon \in X_\Omega^{\frac{\sigma}{2}}(\mathcal{H}^+)$ and w satisfies (3.7). In such case, we say that $v = \mathrm{Tr}_\Omega w$ is a weak solution to (3.9). In particular, if $B(t) = ct$ for all $t \geq 0$ and some $c > 0$, and $\overline{v}$ is the solution to the following linear problem with homogeneous boundary data:

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}}\overline{v} + c\overline{v} = f(x) - c\varepsilon & \text{in } \Omega, \\ \overline{v} = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $v = \overline{v} + \varepsilon$ is the unique solution to the linear, nonhomogeneous problem (3.9).

Using the variational formulation to (3.10), it is easy to prove that if $\varepsilon > 0$, v_ε is the unique weak solution to (3.9) and w_ε is its extension solving (3.10), then

$$\begin{aligned} & w_\varepsilon \to w && \text{in } L^1(\mathcal{H}^+), \\ & \mathrm{Tr}_{\mathbb{R}^N} w_\varepsilon \to \mathrm{Tr}_{\mathbb{R}^N} w && \text{in } L^1(\mathbb{R}^N), \end{aligned}$$

where v and w solve (3.1) and (3.2), respectively.

We warn the reader that the solutions of all the Dirichlet problems throughout this paper are identified with their extension on the whole $\mathbb{R}^N$, whose values out of Ω clearly depend on the boundary conditions considered.

4 Concentration Comparison for the Extended Problem

Let us address the comparison issue. From now on, we always assume that the right-hand side f is nonnegative. Our goal here is to compare the solution v to (3.1) with the solution V to the problem

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} V + B(V) = f^\#(x) & \text{in } \Omega^\#, \\ V = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\#. \end{cases} \tag{4.1}$$

A reasonable way to do that is to compare the solution w to (3.2) with the solution ψ to the problem

$$\begin{cases} -\operatorname{div}_{x,y}(y^{1-\sigma}\nabla \psi) = 0 & \text{in } \mathcal{H}^+, \\ \psi(x,0) = 0 & \text{for } x \in \mathbb{R}^N \setminus \Omega^\#, \\ -\dfrac{1}{\kappa_\sigma} \lim\limits_{y\to 0^+} y^{1-\sigma} \dfrac{\partial \psi}{\partial y}(x,y) + B(\psi(x,0)) = f^\#(x) & \text{in } \Omega^\#, \end{cases} \tag{4.2}$$

where $\psi(x,0)=V(x)$. According to [25], using the change of variables $z=\left(\frac{y}{\sigma}\right)^{\sigma}$, problems (3.2) and (4.2) become

$$\begin{cases} -z^{\nu}\dfrac{\partial^2 w}{\partial z^2}-\Delta_x w=0 & \text{in } \mathcal{H}^+,\\ w(x,0)=0 & \text{for } x\in\mathbb{R}^N\setminus\Omega^{\#},\\ -\dfrac{\partial w}{\partial z}(x,0)=\sigma^{\sigma-1}\kappa_\sigma(f(x)-B(w(x,0))) & \text{in } \Omega \end{cases} \tag{4.3}$$

and

$$\begin{cases} -z^{\nu}\dfrac{\partial^2 \psi}{\partial z^2}-\Delta_x \psi=0 & \text{in } \mathcal{H}^+,\\ \psi=0 & \text{for } x\in\mathbb{R}^N\setminus\Omega^{\#},\\ -\dfrac{\partial \psi}{\partial z}(x,0)=\sigma^{\sigma-1}\kappa_\sigma(f^{\#}(x)-B(\psi(x,0))) & \text{in } \Omega^{\#}, \end{cases} \tag{4.4}$$

respectively, where $\nu:=2\frac{\sigma-1}{\sigma}$. Then, the problem reduces to prove the concentration comparison between the solutions $w(x,z)$ and $\psi(x,z)$ to (4.3) and (4.4), respectively. We now introduce the function

$$Z(s,z)=\int_0^s (w^*(\tau,z)-\psi^*(\tau,z))\mathrm{d}\tau. \tag{4.5}$$

Using standard symmetrization tools (see [25]), we get the differential inequality

$$-(z^{\nu}Z_{zz}+p(s)Z_{ss})\le 0 \tag{4.6}$$

for a.e. $(s,z)\in(0,+\infty)\times(0,+\infty)$, where $p(s)=N^2\omega_N^{\frac{2}{N}}s^{2-\frac{2}{N}}$. Obviously, we have

$$Z(0,y)=0. \tag{4.7}$$

A crucial point in our arguments below is played by the derivative of Z with respect to z. Due to the boundary conditions contained in (4.3)–(4.4), we have for $0<s<|\Omega|$,

$$Z_z(s,0)\ge\theta_\sigma\int_0^s (B(w^*(\tau,0))-B(\psi^*(\tau,0)))\,\mathrm{d}\tau, \tag{4.8}$$

where $\theta_\sigma:=\sigma^{\sigma-1}\kappa_\sigma$. On the other hand, for $s\ge|\Omega|$, we have $\left(\frac{\partial Z}{\partial s}\right)(s,0)=0$. This is the novelty in the argument with respect to the spectral case treated before, where the boundary condition

$$\frac{\partial Z}{\partial s}(|\Omega|,z)=0,\quad \forall z\ge 0$$

made the real difference as follows: The boundary conditions are imposed on $y=0$ and are of two types. Contradiction must be obtained after taking into account the two possibilities.

4.1 Comparison result for the elliptic problem

We are going to obtain a comparison result for some linear and nonlinear B. Actually the nonlinearities considered here allow to get a result which is weaker than the one for the linear problem, i.e., when $B(t)=ct$ for some $c\ge 0$, which is the only case that we are going to need in addressing the Faber-Krahn inequalities. We also point out that, in order to reach our goal, we use a lifting-type argument of the symmetrized problem (4.1).

Theorem 4.1 *Let v be the nonnegative solution to* (3.1) *posed in a bounded domain with zero Dirichlet boundary condition, nonnegative data $f \in L^1(\Omega)$ and B being smooth, concave, strictly increasing on $\mathbb{R}_+$ and such that $B(0) = 0$. If V is the solution of the corresponding symmetrized problem* (4.1)*, we have*

$$v^{\#}(x) \prec V(x). \tag{4.9}$$

The same is true if $\Omega = \mathbb{R}^N$.

Proof In this case, we pose the problem first in a bounded domain Ω of $\mathbb{R}^N$ with smooth boundary. We also assume that f is smooth, bounded and compactly supported in Ω, since the comparison result for general data can be obtained later by approximation, using the L^1 dependence on the map $f \mapsto v$. Let us choose $\varepsilon > 0$, $f_\varepsilon = f + B(\varepsilon)$, and let us consider the solution V_ε to the following problem:

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} V_\varepsilon + B(V_\varepsilon) = f_\varepsilon^{\#}(x) & \text{in } \Omega^{\#}, \\ V_\varepsilon = \varepsilon & \text{on } \mathbb{R}^N \setminus \Omega^{\#}, \end{cases}$$

where $f_\varepsilon^{\#} = f^{\#} + B(\varepsilon)$. By virtue of Remark 3.2, we have $V_\varepsilon = \psi_\varepsilon(x, 0)$ for all $x \in \mathbb{R}^N$, where ψ_ε is the solution to the following problem:

$$\begin{cases} -\operatorname{div}_{x,y}(y^{1-\sigma} \nabla \psi_\varepsilon) = 0 & \text{in } \mathcal{H}^+, \\ \psi_\varepsilon(x, 0) = \varepsilon & \text{for } x \in \mathbb{R}^N \setminus \Omega^{\#}, \\ -\dfrac{1}{\kappa_\sigma} \lim\limits_{y \to 0^+} y^{1-\sigma} \dfrac{\partial \psi_\varepsilon}{\partial y}(x, y) + B(\psi_\varepsilon(x, 0)) = f^{\#}(x) + B(\varepsilon) & \text{in } \Omega^{\#}, \end{cases}$$

which can be reduced to the following problem:

$$\begin{cases} -z^{\nu} \dfrac{\partial^2 \psi_\varepsilon}{\partial z^2} - \Delta_x \psi_\varepsilon = 0 & \text{in } \mathcal{H}^+, \\ \psi_\varepsilon(x, 0) = \varepsilon & \text{for } x \in \mathbb{R}^N \setminus \Omega^{\#}, \\ -\dfrac{\partial \psi_\varepsilon}{\partial z}(x, 0) = \sigma^{\sigma-1} \kappa_\sigma (f^{\#}(x) + B(\varepsilon) - B(\psi_\varepsilon(x, 0))) & \text{in } \Omega^{\#}. \end{cases} \tag{4.10}$$

Setting

$$Z_\varepsilon(s, z) = \int_0^s (w^*(\tau, z) - \psi_\varepsilon^*(\tau, z)) \mathrm{d}\tau,$$

we prove that

$$Z_\varepsilon(s, z) \le 0. \tag{4.11}$$

To this aim, we first observe that Z_ε satisfies (4.6). In particular, a big role is played by the property of the solution ψ_ε: The level sets $\{x \in \mathbb{R}^N : \psi_\varepsilon(x, z) > t\}$ are bounded, because they are balls centered at the origin. Now we set

$$Y_\varepsilon(s, z) = \int_0^s (B(w^*(\tau, z)) - B(\psi_\varepsilon^*(\tau, z))) \, \mathrm{d}\tau.$$

From (4.8), we obtain

$$\frac{\partial Z_\varepsilon}{\partial z}(s, 0) \ge \theta_\sigma Y_\varepsilon(s, 0) + \theta_\sigma B(\varepsilon). \tag{4.12}$$

By the strong maximum principle applied to (4.6), which is satisfied by Z_ε, a positive maximum of Z_ε cannot be achieved at an interior point, hence it must be achieved either as $s \to \infty$ or at a boundary point $(s_0, 0)$ for some $s_0 > 0$. The first option cannot hold, since $w^*(s,z) \to 0$ while $\psi_\varepsilon(s,z) \to \varepsilon$ as $s \to \infty$. As for the second, we see that for $s \geq |\Omega|$, we have

$$\frac{\partial Z_\varepsilon}{\partial s}(s,0) = v^*(s) - V_\varepsilon^*(s) = -\varepsilon < 0,$$

the function $Z_\varepsilon(s,0)$ is strictly decreasing in $[|\Omega|, \infty)$, then it must happen that $s_0 \in (0, |\Omega|]$. Arguing as in [64], using (4.12), we can write

$$\begin{aligned}\frac{\partial Z_\varepsilon}{\partial z}(s_0,0) &> \theta_\sigma \int_0^{s_0} B'(v^*(s)) \frac{\partial Z_\varepsilon}{\partial s}(s,0)\mathrm{d}s \\ &= \theta_\sigma \Big[g(0) Z_\varepsilon(s_0,0) + \int_0^{s_0} [Z_\varepsilon(s_0,0) - Z_\varepsilon(s,0)]\mathrm{d}g(s) \Big] > 0, \end{aligned} \tag{4.13}$$

where $g(s) := B'(v^*(s))$, which is impossible due to Hopf's boundary maximum principle.

Finally, by (4.11), we have, for $s \geq 0$,

$$\int_0^s v^*(\tau)\mathrm{d}\tau \leq \int_0^s V_\varepsilon^*(\tau)\mathrm{d}\tau,$$

and thus as $\varepsilon \to 0$,

$$\int_0^s v^*(\tau)\mathrm{d}\tau \leq \int_0^s V^*(\tau)\mathrm{d}\tau.$$

Then the result follows. The case $\Omega = \mathbb{R}^N$ was solved in [64, Theorem 3.2], to which the interested reader can refer.

Remark 4.1 If we consider the linear case, i.e., when $B(t) = ct$, for some $c \geq 0$, the proof of Theorem 4.1 can be simplified, because $Z_\varepsilon = cY_\varepsilon$. Moreover, $\psi_\varepsilon = \psi + \varepsilon$, where ψ solves (4.4), and $V_\varepsilon = V + \varepsilon$.

Remark 4.2 If B is nonlinear and satisfies the assumptions of Theorem 4.1, the mass concentration comparison (4.9) is not enough to obtain the same result when it is passing to the evolution problem via Crandall-Liggett theorem.

A valuable property we are going to prove now is that the equality in (4.9) implies that the domain Ω of the initial problem (3.1) is actually a ball modulo translation. This kind of property is known to be essential for studying the equality case in the Faber–Krahn inequality, i.e., to prove that the ball is the unique minimizer of the principal eigenvalue of the classical Laplacian over all the sets of fixed Lebesgue measure (see [37]).

Proposition 4.1 (Case of Equality) *Assume that we have an equality sign in* (4.9)*, in the sense that*

$$\int_0^s v^*(\sigma)\mathrm{d}\sigma = \int_0^s V^*(\sigma)\mathrm{d}\sigma$$

for all $s \in [0, |\Omega|]$. *Then* Ω *is a ball, i.e.,* $\Omega = \Omega^\#$ (*up to a translation of the origin*).

Proof Using the same notation as in Theorem 4.1, by the given assumption, we have $Y(s,0) = 0$ for all $s \in [0, |\Omega|]$, and since the extensions are null, this equality holds for every

$s \geq 0$. Then by Hopf's maximum principle and (4.8), we find $Y \equiv 0$, and thus

$$w^*(s,z) = \psi^*(s,z) \quad \text{for all } s, z \geq 0. \tag{4.14}$$

We divide the rest into the following steps.

(i) Here we argue as in [54] or in [43]. Recall that the function $w(\cdot, z)$ is smooth on $\mathbb{R}^N$ for any $z > 0$. Then let us fix $z > 0$, multiply both sides of the equation (4.3) by the test function

$$\varphi_h^z(x) = \begin{cases} 1, & \text{if } w(x,z) \geq t+h, \\ \dfrac{w(x,z)-t}{h}, & \text{if } t < w(x,z) < t+h, \\ 0, & \text{if } w(x,z) \leq t, \end{cases}$$

and integrate over $\mathbb{R}^N$. An integration by parts yields the identity

$$\begin{aligned} &\frac{1}{h}\int_{\{x\in\mathbb{R}^N :\, t<w(x,z)<t+h\}} |\nabla_x w|^2 \mathrm{d}x - z^\nu \int_{\{x\in\mathbb{R}^N :\, w(x,z)>t+h\}} \frac{\partial^2 w}{\partial z^2} \mathrm{d}x \\ &- z^\nu \int_{\{x\in\mathbb{R}^N :\, t<w(x,z)<t+h\}} \frac{\partial^2 w}{\partial z^2}\Big(\frac{w-t}{h}\Big) \mathrm{d}x = 0. \end{aligned}$$

Then, if we let $h \to 0$, we find

$$-\frac{\partial}{\partial t}\int_{\{x\in\mathbb{R}^N :\, w(x,z)>t\}} |\nabla_x w|^2 \,\mathrm{d}x = z^\nu \int_{\{x\in\mathbb{R}^N :\, w(x,z)>t\}} \frac{\partial^2 w}{\partial z^2} \mathrm{d}x.$$

Thus using the second order derivation formula (see Section 7), we get

$$-\frac{\partial}{\partial t}\int_{\{x\in\mathbb{R}^N :\, w(x,z)>t\}} |\nabla_x w|^2 \,\mathrm{d}x \leq z^\nu \int_0^{\mu_w(t,z)} \frac{\partial^2 w^*}{\partial z^2} \mathrm{d}s.$$

Concerning the solution ψ to (4.4), since it is spherically decreasing w.r.t. x, the following equality occurs:

$$-\frac{\partial}{\partial t}\int_{\{x\in\mathbb{R}^N :\, \psi(x,z)>t\}} |\nabla_x \psi|^2 \,\mathrm{d}x = z^\nu \int_0^{\mu_\psi(t,z)} \frac{\partial^2 \psi^*}{\partial z^2} \mathrm{d}s.$$

Using the fact that $w^*(s,z) = \psi^*(s,z)$ (which implies $\mu_w(\cdot,z) = \mu_\psi(\cdot,z)$), we have

$$-\frac{\partial}{\partial t}\int_{\{x\in\mathbb{R}^N : w(x,z)>t\}} |\nabla_x w|^2 \,\mathrm{d}x \leq -\frac{\partial}{\partial t}\int_{\{x\in\mathbb{R}^N :\, \psi(x,z)>t\}} |\nabla_x \psi|^2 \,\mathrm{d}x.$$

Integrating between t and ∞, we find

$$\int_{\{x\in\mathbb{R}^N : w(x,z)>t\}} |\nabla_x w|^2 \,\mathrm{d}x \leq \int_{\{x\in\mathbb{R}^N :\, \psi(x,z)>t\}} |\nabla_x \psi|^2 \,\mathrm{d}x,$$

and then by the Pólya-Szegö inequality and (4.14),

$$\begin{aligned} \int_{\{x\in\mathbb{R}^N : w^\#(x,z)>t\}} |\nabla_x w^\#|^2 \,\mathrm{d}x &\leq \int_{\{x\in\mathbb{R}^N : w(x,z)>t\}} |\nabla_x w|^2 \,\mathrm{d}x \\ &\leq \int_{\{x\in\mathbb{R}^N :\, \psi(x,z)>t\}} |\nabla_x \psi|^2 \,\mathrm{d}x \\ &= \int_{\{x\in\mathbb{R}^N : w^\#(x,z)>t\}} |\nabla_x w^\#|^2 \,\mathrm{d}x\,. \end{aligned} \tag{4.15}$$

We conclude that for every $t > 0$,

$$\int_{\{x \in \mathbb{R}^N : w(x,z) > t\}} |\nabla_x w|^2 \, \mathrm{d}x = \int_{\{x \in \mathbb{R}^N : w^\#(x,z) > t\}} |\nabla_x w^\#|^2 \, \mathrm{d}x \,,$$

which is the equality case in the Pólya-Szegö inequality.

(ii) Now we notice that $w(\cdot, z)$ is analytic on the upper half space as a consequence of its representation formula. Indeed, the Poisson kernel for the extension operator L_σ is given by

$$P(x,z) = c_{\sigma,N} \frac{z^\sigma}{(|x|^2 + z^2)^{N+\sigma}}$$

(see, e.g., [18]), which is an analytic function on the upper half-space $z > 0$. This means that for every $h(x) \in L^1(\mathbb{R}^N)$, the solution $w(x,z)$ of the elliptic equation with data $w(x,0) = h(x)$ is analytic, since it is the convolution of h with P (with respect to the x variable). In fact, once we know that $w \in C^\infty$ in a certain subdomain, it is analytic by classical results on solutions of elliptic equations with analytic coefficients (see for instance [33, 35, 46]).

(iii) Moreover, each level set $\{x \in \mathbb{R}^N : w(x,z) > t\}$ is bounded, because $w(\cdot, z)$ decays to zero as $|x| \to \infty$. We may now use the equality case in the Pólya-Szegö inequality (see [14, 30]) to obtain that $\{x \in \mathbb{R}^N : w(x,z) > t\} = \{x \in \mathbb{R}^N : w^\#(x,z) > t\}$ modulo a translation. Then all the level sets $\{x \in \mathbb{R}^N : w(x,z) > t\}$ are balls. The results also imply that for every fixed $z > 0$ the function $w(\cdot, z)$ is radially symmetric up to translation.

(iv) Finally, we take the limit $z \to 0$, and we conclude that $u(x)$ is also radially symmetric (as a function defined in $\mathbb{R}^N$). This means that the domain Ω, which is the positivity set of u in $\mathbb{R}^N$, must be a ball, i.e., $\Omega = \Omega^\#$, up to a translation of the origin.

The following result can be shown as in the proof of [64, Theorem 3.3]

Theorem 4.2 (Comparison of Concentrations for Radial Problems) *Let v_1, v_2 be two non-negative solutions to* (3.1) *posed in a ball $B_R(0)$, with $R \in (0, +\infty]$ with zero Dirichlet boundary conditions if $R < +\infty$, nonnegative radially symmetric decreasing data $f_1, f_2 \in L^1(B_R(0))$ and $B(t) = ct$ for some $c > 0$ and all $t \geq 0$. Then v_1 and v_2 are rearranged, and*

$$f_1 \prec f_2 \quad \textit{implies} \quad v_1 \prec v_2 \,. \tag{4.16}$$

5 Symmetrization for the Parabolic Problem

The theory of the existence of weak solutions for the initial value problem

$$\partial_t u + (-\Delta)^{\frac{\sigma}{2}} A(u) = f, \quad 0 < \sigma < 2 \tag{5.1}$$

with $A(u) = cu^m$, and all $c, m > 0$, was addressed by the first author and collaborators in [10], and the main properties were obtained. In particular, if we take initial data in $L^1 \cap H^{-s}$, then an H^{-s}-contraction semigroup is generated, and the Crandall-Liggett discretization theorem applies. The construction and properties of the solutions of the evolution problem is thus reduced to an iterated application of the results obtained for the elliptic counterpart in the previous section. This was carefully explained in [64] and is reviewed in Subsection 7.3. Thus, using Theorem 7.2 in Subsection 7.3, we obtain the existence of a unique mild solution to the linear Cauchy-Dirichlet problem on the bounded domain Ω as follows:

$$\begin{cases} u_t + (-\Delta)^{\frac{\sigma}{2}} u = f, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{5.2}$$

obtained as a limit of discrete approximate solutions by the ITD scheme.

Concerning the application of symmetrization techniques to this type of parabolic problems, we can employ Theorems 4.1–4.2 and the arguments in the proof of [64, Theorem 5.3] in order to find the following result.

Theorem 5.1 (Concentration Comparison) *Let u be the nonnegative mild solution to* (5.2), $0 < \sigma < 2$, *posed in Ω, with initial data $u_0 \in L^1(\Omega)$, and the right-hand side $f \in L^1(\Omega \times (0,\infty))$. Assume that $\overline{u}_0 \in L^1(\Omega^\#)$ is rearranged, $\overline{f}(x,t) \in L^1(\Omega^\# \times (0,\infty))$ is rearranged w.r.t. any $t > 0$, such that*

$$u_0^\# \prec \overline{u}_0$$

and

$$f^\#(\cdot,t) \prec \overline{f}(\cdot,t)$$

for a.e. $t > 0$. Let v be the solution of the evolution problem

$$\begin{cases} v_t + (-\Delta)^{\frac{\sigma}{2}} v = \overline{f}(x,t), & x \in \Omega^\#,\quad t > 0, \\ v = 0, & x \in \mathbb{R}^N \setminus \Omega^\#,\ t > 0, \\ v(x,0) = \overline{u}_0(x), & x \in \Omega^\#. \end{cases} \tag{5.3}$$

Then, for all $t > 0$, we have

$$u^\#(|x|,t) \prec v(|x|,t). \tag{5.4}$$

In particular, we have $\|u(\cdot,t)\|_p \le \|v(\cdot,t)\|_p$ for every $t > 0$ and every $p \in [1,\infty]$.

Remark 5.1 The parabolic result only covers the linear equation, which is much below our original expectations, since the elliptic result covers indeed nonlinear equations. We do not know if the lack of nonlinear results is due to a failure of the expected form of the theorem, as copied from (5.1), or is only due to a lack of technique. We remind the reader that in the case of the problem in the whole space, we are able to prove the nonlinear symmetrization result when A is concave, and to show that the corresponding statement for A convex is false, so that the question is posed about which kind of statement could be true. Such question is broader in the present case.

6 Application: An Original Proof of the Fractional Faber-Krahn Inequality

Here we prove the validity of the Faber-Krahn inequality for the fractional Laplacian operator $\mathcal{L}_2$ defined on bounded domains of $\mathbb{R}^N$ with zero Dirichlet boundary conditions. This operator appears often in theory and applications, and is known under the name of restricted fractional Laplacian, though we can call it the natural fractional Laplacian. Unlike the spectral Laplacian $\mathcal{L}_1$, the spectral sequence $\{\lambda_k(\mathcal{L}_2;\Omega)\}_k$ is not directly related to the sequence of the standard Laplacian. However, it is known that the spectrum is discrete and given by a strictly increasing sequence $\{\lambda_{k,\frac{\sigma}{2}}(\Omega)=\lambda_k(\mathcal{L}_2;\Omega)\}$ (see, e.g., [10]). Our theorem is then stated as follows.

Theorem 6.1 *We have*

$$\lambda_{1,\frac{\sigma}{2}}(\Omega)\geq\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#}) \tag{6.1}$$

with equality if and only if $\Omega=\Omega^{\#}$, *up to translation.*

The proof we present here is completely elementary and uses neither the variational characterization (6.6) nor the nonlocal Pólya-Szegö inequality as in [13], but only the concentration comparison provided by Theorem 5.1 and the asymptotic definition of $\lambda_{1,\frac{\sigma}{2}}(\Omega)$, which can be derived by the decay rate of the parabolic problem $u_t+\mathcal{L}_2u=0$.

Proof of Theorem 6.1 Suppose that $\{\psi_{k,\frac{\sigma}{2},\Omega}(x)\}_k$ are the (L^2 normalized) eigenfunctions of $\mathcal{L}_2$ and let us consider the function

$$u(x,t)=\mathrm{e}^{-\lambda_{1,\frac{\sigma}{2}}(\Omega)\,t}\,\psi_{1,\frac{\sigma}{2},\Omega}(x),$$

solving the problem

$$\begin{cases} u_t+(-\Delta)^{\frac{\sigma}{2}}u=0, & x\in\Omega,\ t>0,\\ u=0, & x\in\mathbb{R}^N\setminus\Omega,\ t>0,\\ u(x,0)=\psi_{1,\frac{\sigma}{2},\Omega}(x), & x\in\Omega. \end{cases} \tag{6.2}$$

By Theorem 5.1 we find, for all $t>0$,

$$\|u(\cdot,t)\|_2\leq\|v(\cdot,t)\|_2,$$

where v solves the problem

$$\begin{cases} v_t+(-\Delta)^{\frac{\sigma}{2}}v=0, & x\in\Omega^{\#},\ \ t>0,\\ v=0, & x\in\mathbb{R}^N\setminus\Omega^{\#},\ t>0,\\ v(x,0)=\psi^{\#}_{1,\frac{\sigma}{2},\Omega}(x), & x\in\Omega^{\#}. \end{cases} \tag{6.3}$$

Since

$$\|u(\cdot,t)\|_2=\mathrm{e}^{-\lambda_{1,\frac{\sigma}{2}}(\Omega)\,t} \tag{6.4}$$

and the expression for v can be given in terms of superposition, namely,

$$v(x,t)=\sum_{k=1}^{\infty}\langle\psi^{\#}_{1,\frac{\sigma}{2},\Omega},\psi_{k,\frac{\sigma}{2},\Omega^{\#}}\rangle_{L^2(\Omega^{\#})}\,\mathrm{e}^{-\lambda_{k,\frac{\sigma}{2}}(\Omega^{\#})\,t}\,\psi_{k,\frac{\sigma}{2},\Omega^{\#}}(x),$$

we have

$$\begin{aligned}\|v(\cdot,t)\|_2^2 &= \sum_{k=1}^{\infty} |\langle \psi^{\#}_{1,\frac{\sigma}{2},\Omega}, \psi_{k,\frac{\sigma}{2},\Omega^{\#}}\rangle|^2_{L^2(\Omega^{\#})}\, \mathrm{e}^{-2\lambda_{k,\frac{\sigma}{2}}(\Omega^{\#})\,t} \\ &\le \mathrm{e}^{-2\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})\,t} \sum_{k=1}^{\infty} |\langle \psi^{\#}_{1,\frac{\sigma}{2},\Omega}, \psi_{k,\frac{\sigma}{2},\Omega^{\#}}\rangle|^2_{L^2(\Omega^{\#})} \\ &= \mathrm{e}^{-2\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})\,t} \|\psi^{\#}_{1,\frac{\sigma}{2},\Omega}\|^2_{L^2(\Omega^{\#})} = \mathrm{e}^{-2\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})\,t}.\end{aligned}$$

This together with (6.4) implies (6.1).

The case of equality Analyzing the last list of inequalities that starts by $\|v(\cdot,t)\|_2^2$, we conclude that in the case where $\lambda_{1,\frac{\sigma}{2}}(\Omega) = \lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})$ we necessarily have

$$\|v(\cdot,t)\|_{L^2(\Omega^{\#})} = \|u(\cdot,t)\|_{L^2(\Omega)} = \mathrm{e}^{-\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})\,t},$$

so that we conclude that the coefficients of the Fourier expansion of $v_0 = v(\cdot,0)$ in terms of eigenfunctions are all zero but the first, in view of the known fact that the first eigenvalue $\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})$ is simple. This means that $v_0(x) = c\psi_{1,\frac{\sigma}{2},\Omega^{\#}}$, with a constant $c > 0$. By normalization, we get

$$\psi^{\#}_{1,\frac{\sigma}{2},\Omega}(x) = \psi_{1,\frac{\sigma}{2},\Omega^{\#}}(x). \tag{6.5}$$

But this is enough to apply the important Proposition 4.1 and obtain $\Omega = \Omega^{\#}$, and the result ends as before.

6.1 A variational proof

A direct proof of the FKI for our operator and similar can be based on the variational interpretation of the first eigenvalue, since it can be written as the minimizer of the Rayleigh quotient, where the local L^2 gradient energy norm is replaced by the Gagliardo seminorm (see Section 7 for some details).

As in the proof of Theorem 6.1, suppose that $\{\psi_{k,\frac{\sigma}{2},\Omega}(x)\}_k$ are the (L^2 normalized) eigenfunctions of $\mathcal{L}_2$. As already mentioned in the introduction, the proof of Theorem 6.1 is a direct consequence of the variational characterization of $\lambda_{1,\frac{\sigma}{2}}(\Omega)$. Indeed, we know by [48] that

$$\lambda_{1,\frac{\sigma}{2}}(\Omega) = \min_{\substack{u\in H^{\frac{\sigma}{2}}(\mathbb{R}^N)\setminus\{0\} \\ u=0 \text{ on } \mathbb{R}^N\setminus\Omega}} \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\sigma}}\,\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega u^2\mathrm{d}x}. \tag{6.6}$$

Then we could use the nonlocal (hence fractional) version of the Pólya-Szegö inequality (see for instance [45]) to see that replacing u with $u^{\#}$ makes the Gagliardo seminorm in (6.6) decrease, therefore (6.1) holds. Furthermore, if equality occurs in (6.1), the minimality of $\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})$ implies that $\psi^{\#}_{1,\frac{\sigma}{2},\Omega}(x)$ is an eigenfunction. But since the eigenvalue $\lambda_{1,\frac{\sigma}{2}}(\Omega^{\#})$ is also simple, by normalization, we have

$$\psi^{\#}_{1,\frac{\sigma}{2},\Omega}(x) = \psi_{1,\frac{\sigma}{2},\Omega^{\#}}(x)$$

which the same result in [45] shows to be possible only when $\Omega = \Omega^{\#}$ and $\psi_{1,\frac{\sigma}{2},\Omega} = \psi^{\#}_{1,\frac{\sigma}{2},\Omega}$ up to translation.

More general version of the FKI Actually, Brasco et al. [13] was able to establish a more general version of the FKI, which applies to a nonlinear variant of the fractional Laplacian, namely the fractional p-Laplacian. The main argument of the general variational proof is the use of a nonlocal Pólya-Szegö inequality, proved in [32], to estimate the first nonlinear eigenvalue.

Probabilistic approach The Faber-Krahn inequality for the fractional Laplacian with Dirichlet data on a bounded domain of $\mathbb{R}^N$ is stated with a hint of the proof based on probabilistic arguments as the last result (see [4, Theorem 5]). This means that the eigenvalues are also characterized in terms of the evolution, in their case of the stochastic process of Levy type. Another proof with probabilistic flavor can be found in [9].

7 Appendices

7.1 On symmetrization

We gather here some basic information on symmetrization that can be useful to read this paper. We follow standard notations used in this literature, and we recall that we presented a more detailed account in [64]. A measurable real function f defined on $\mathbb{R}^N$ is called radially symmetric (radial, for short) if there is a function $\widetilde{f} : [0, \infty) \to \mathbb{R}$ such that $f(x) = \widetilde{f}(|x|)$ for all $x \in \mathbb{R}^N$. We often write $f(x) = f(r)$, $r = |x| \geq 0$ for such functions by abuse of notation. We say that f is rearranged if it is radial, nonnegative and $\widetilde{f}$ is a right-continuous, non-increasing function of $r > 0$. A similar definition can be applied for real functions defined on a ball $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$.

If Ω is an open set of $\mathbb{R}^N$ and f is a real measurable function on Ω, we denote by $|\cdot|$ the N-dimensional Lebesgue measure. We define the distribution function μ_f of f as

$$\mu_f(k) = |\{x \in \Omega : |f(x)| > k\}|, \quad k \geq 0,$$

and the decreasing rearrangement of f as

$$f^*(s) = \sup\{k \geq 0 : \mu_f(k) > s\}, \quad s \in (0, |\Omega|).$$

We may also think of extending f^* as the zero function in $[|\Omega|, \infty)$ if Ω is bounded. From this definition, it turns out that $\mu_{f^*} = \mu_f$ (i.e., f and f^* are equi-distributed) and f^* is exactly the generalized inverse of μ_f. Furthermore, if ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same Lebesgue measure as Ω, we define the function

$$f^\#(x) = f^*(\omega_N |x|^N), \quad x \in \Omega^\#,$$

which is called spherical decreasing rearrangement of f. From this definition, it follows that f is rearranged if and only if $f = f^\#$.

Rearranged functions have a number of interesting properties. Here, we just recall the conservation of the L^p norms (coming from the definition of rearrangements and the classical Cavalieri principle): For all $p \in [1, \infty]$,

$$\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p(|0,\Omega|)} = \|f^\#\|_{L^p(\Omega^\#)},$$

as well as the classical Hardy-Littlewood inequality (see [34])

$$\int_{\Omega} |f(x)g(x)| \mathrm{d}x \leq \int_0^{|\Omega|} f^*(s)g^*(s)\mathrm{d}s = \int_{\Omega^\#} f^\#(x)\, g^\#(x)\, \mathrm{d}x. \tag{7.1}$$

(1) We often deal with two-variable functions of the type

$$f : (x, y) \in \mathcal{C}_\Omega \to f(x, y) \in \mathbb{R} \tag{7.2}$$

defined on the cylinder $\mathcal{C}_\Omega := \Omega \times (0, +\infty)$, and measurable with respect to x. Here Ω can be a bounded domain or $\mathbb{R}^N$. For such functions, it is convenient to define the so-called Steiner symmetrization of $\mathcal{C}_\Omega$ with respect to the variable x, namely the set $\mathcal{C}_\Omega^\# := \Omega^\# \times (0, +\infty)$. Furthermore, we denote by $\mu_f(k, y)$ and $f^*(s, y)$ the distribution function and the decreasing rearrangements of (7.2), respectively, with respect to x for y fixed, and we also define the function

$$f^\#(x, y) = f^*(\omega_N |x|^N, y)$$

which is called the Steiner symmetrization of f, with respect to the line $x = 0$. Clearly, $f^\#$ is a spherically symmetric and decreasing function with respect to x, for any fixed y.

(2) There are some interesting differentiation formulas which turn out to be very useful in our approach. Typically, they are used when one wants to get sharp estimates satisfied by the rearrangement u^* of a solution u to a certain equation, and it becomes crucial to differentiate with respect to the extra variable y (introduced in the extension process that is used in fractional operators) in the form

$$\int_{\{u(x,y)>u^*(s,y)\}} \frac{\partial u}{\partial y}(x, y)\, \mathrm{d}x.$$

We recall here two formulas that have been already used in [25, 64].

Proposition 7.1 *Suppose that* $f \in H^1(0, T; L^2(\Omega))$ *for some* $T > 0$ *and* f *is nonnegative. Then*

$$f^* \in H^1(0, T; L^2(0, |\Omega|)).$$

If $|\{f(x, t) = f^*(s, t)\}| = 0$ *for a.e.* $(s, t) \in (0, |\Omega|) \times (0, T)$*, the following differentiation formula holds:*

$$\int_{f(x,y)>f^*(s,y)} \frac{\partial f}{\partial y}(x, y)\, \mathrm{d}x = \int_0^s \frac{\partial f^*}{\partial y}(\tau, y)\, \mathrm{d}\tau. \tag{7.3}$$

The second-order differentiation formula is as follows.

Proposition 7.2 *Let* f *be nonnegative and* $f \in W^{2,\infty}(\mathcal{C}_\Omega)$*. Then for almost every* $y \in (0, +\infty)$ *the following differentiation formula holds:*

$$\begin{aligned}
&\int_{f(x,y)>f^*(s,y)} \frac{\partial^2 f}{\partial y^2}(x, y)\mathrm{d}x \\
&= \frac{\partial^2}{\partial y^2} \int_0^s f^*(\tau, y)\mathrm{d}\tau - \int_{f(x,y)=f^*(s,y)} \frac{\left(\frac{\partial f}{\partial y}(x, y)\right)^2}{|\nabla_x f|}\, \mathrm{d}\mathcal{H}^{N-1}(x) \\
&\quad + \Big(\int_{f(x,y)=f^*(s,y)} \frac{\frac{\partial f}{\partial y}(x, y)}{|\nabla_x f|}\, \mathrm{d}\mathcal{H}^{N-1}(x)\Big)^2 \Big(\int_{f(x,y)=f^*(s,y)} \frac{1}{|\nabla_x f|}\, \mathrm{d}\mathcal{H}^{N-1}(x)\Big)^{-1}.
\end{aligned}$$

(3) Mass concentration. We provide estimates of the solutions of our elliptic and parabolic problems in terms of their integrals. For that purpose, the following definition, taken from [58], is remarkably useful.

Definition 7.1 *Let $f,g \in L^1_{\rm loc}(\mathbb{R}^N)$ be two radially symmetric functions on $\mathbb{R}^N$. We say that f is less concentrated than g, and we write $f \prec g$ if for all $R>0$, we get*

$$\int_{B_R(0)} f(x)\mathrm{d}x \le \int_{B_R(0)} g(x)\mathrm{d}x.$$

The partial order relationship $\prec$ is called comparison of mass concentrations. Of course, this definition can be suitably adapted if f,g are radially symmetric and locally integrable functions on a ball B_R. Besides, if f and g are locally integrable on a general open set Ω, we say that f is less concentrated than g, and we write again $f \prec g$ simply if $f^{\#} \prec g^{\#}$, but this extended definition has no use if g is not rearranged.

The comparison of mass concentrations enjoys a nice equivalent formulation if f and g are rearranged, whose proof we refer to [21, 34, 59].

Lemma 7.1 *Let $f,g \in L^1(\Omega)$ be two rearranged functions on a ball $\Omega = B_R(0)$. Then $f \prec g$ if and only if for every convex nondecreasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0)=0$ we have*

$$\int_\Omega \Phi(f(x))\,\mathrm{d}x \le \int_\Omega \Phi(g(x))\,\mathrm{d}x. \tag{7.4}$$

This result still holds if $R=\infty$ and $f,g \in L^1_{\rm loc}(\mathbb{R}^N)$ with $g \to 0$ as $|x| \to \infty$.

From this lemma it easily follows that if $f \prec g$ and f,g are rearranged, then

$$\|f\|_{L^p(\Omega)} \le \|g\|_{L^p(\Omega)}, \quad \forall p \in [1,\infty]. \tag{7.5}$$

7.2 On analyticity

In the analyticity argument of Proposition 4.1, we want to apply the results of [35]. Let us put

$$F(x,z,u,u_j,u_{jk}) = z^\nu u_{(N+1)(N+1)} + \sum_{j=1}^N u_{jj},$$

where subindexes indicate partial derivatives. Then, $\frac{\partial F}{\partial u_{jk}} = z^\nu$ for $j=k=N+1$, $\frac{\partial F}{\partial u_{jj}} = 1$ for each $j=1,\cdots,N$ and $\frac{\partial F}{\partial u_{jk}} = 0$ for $j \ne k$. Thus

$$\sum_{j,k=1}^{N+1} \frac{\partial F}{\partial u_{jk}}(x,z,u,u_j,u_{jk})\zeta_j\zeta_k = z^\nu \zeta_{N+1}^2 + |\zeta_1|^2 + \cdots + |\zeta_N|^2 > 0$$

for all $(x,z) \in \mathbb{R}^{N+1}_+ := \Omega$, $\zeta \in \mathbb{R}^{N+1} \setminus (0,0)$. Then the equation

$$z^\nu w_{zz} + \Delta_x w = F(x,z,\nabla_{x,z} w, \nabla^2_{x,z} w) = 0$$

is elliptic in $\Omega = \mathbb{R}^{N+1}_+$. Since a solution w to such equation is C^∞ in $\mathbb{R}^{N+1}_+$ and the function $F(x,z,u,u_j,u_{jk})$ is analytic in $(z,u_{jj}) \in \mathbb{R}_+ \times \mathbb{R}^{N+1}$, we can apply the main theorem in [35] and conclude that $w(x,z)$ is analytic in $\mathbb{R}^{N+1}_+$.

7.3 On accretive operators and the semigroup approach

Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be a nonlinear operator defined on a suitable subset of X. Let us consider the problem

$$\begin{cases} u'(t) + \mathcal{A}(u) = f, \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{7.6}$$

where $u_0 \in X$ and $f \in L^1(I; X)$ for some interval I of the real axis. For a wide class of operators, in particular the ones considered in this paper, a very efficient way to approach such problem is to use an implicit time discretization scheme that we describe next. Suppose to be specific that $I = [0, T]$ (but this can be replaced by any interval $[a, b]$ and the procedure is similar). The method consists in taking first a partition of the interval, say, $t_k = kh$ for $k = 0, 1, \cdots, n$ and $h = \frac{T}{n}$, and then solving the system of difference relations

$$\frac{u_{h,k} - u_{h,k-1}}{h} + \mathcal{A}(u_{h,k}) = f_k^{(h)} \tag{7.7}$$

for $k = 0, 1, \cdots, n$, where we pose $u_{h,0} = u_0$. The data set $\{f_k^{(h)} : k = 1, \cdots, n\}$ is supposed to be a suitable discretization of the source term f, corresponding to the time discretization we choose. This process is called implicit time discretization scheme (ITD for short) of the equation $u'(t) + \mathcal{A}(u) = f$. It can be rephrased in the form

$$u_{h,k} = J_h(u_{h,k-1} + hf_k^{(h)}),$$

where the operator $J_\lambda = (I + \lambda\mathcal{A})^{-1}$, $\lambda > 0$ is called the resolvent operator, with I being the identity operator. Therefore, the application of the method needs the operator $\mathcal{A}$ to have a well-defined family of resolvents with good properties. When the ITD is solved, we construct a discrete approximate solution $\{u_{h,k}\}_k$. By piecing together the values $u_{h,k}$, we take a piecewise constant function, $u_h(t)$, typically defined through

$$u_h(t) = u_{h,k}, \quad \text{if } t \in [(k-1)h, kh] \tag{7.8}$$

(or some other interpolation rule, like linear interpolation). Then the main question consists in verifying if such function u_h converges somehow as $h \to 0$ to a solution u (which we hope to be a classical, strong, weak, or other type of solution) to (7.6). To this regard, we first choose a suitable discretization $\{f_k^{(h)}\}$ in time of the source term f, such that the piecewise constant interpolation of this sequence produces a function $f^{(h)}(t)$ (defined by means of (7.8)) verifies the property

$$\|f^{(h)} - f\|_{L^1(0,T;X)} \to 0, \quad \text{as } h \to 0.$$

By means of these discrete approximate solutions, we introduce the following notion of mild solution.

Definition 7.2 *We say that $u \in C((0, T); X)$ is a mild solution to* (7.6) *if it is obtained as uniform limit of the approximate solutions u_h, as $h \to 0$. The initial data are taken in the sense that $u(t)$ is continuous in $t = 0$ and $u(t) \to u_0$ as $t \to 0$. Besides, we say that $u \in C((0, \infty); X)$ is a mild solution to* (7.6) *in* $[0, \infty)$ *if u is a mild solution to the same problem in any compact subinterval $I \subset [0, \infty)$.*

In order to state a positive existence result, we need to restrict the class of operators according to the following definitions.

Definition 7.3 *Let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be a nonlinear, possibly unbounded operator. Let $R_\lambda(\mathcal{A})$ be the range of $I + \lambda\mathcal{A}$, a subset of X.*

(i) *The operator $\mathcal{A}$ is said accretive if for all $\lambda > 0$ the map $I + \lambda\mathcal{A}$ is one-to-one onto $R_\lambda(\mathcal{A}) \subset X$, and the resolvent operator $J_\lambda : R_\lambda(\mathcal{A}) \to X$ is a (non-strict) contraction in the X-norm (i.e., a Lipschitz map with Lipschitz norm 1).*

(ii) *We say that $\mathcal{A}$ satisfies the rank condition if $R_\lambda(\mathcal{A}) \supset \overline{D(\mathcal{A})}$ for all $\lambda > 0$. In particular, the rank condition is satisfied if $R_\lambda(\mathcal{A}) = X$ for all $\lambda > 0$. In this case, if $\mathcal{A}$ is accretive, we say that $\mathcal{A}$ is m-accretive.*

We are now ready to state the desired semigroup generation result, that generalizes the classical result of Hille-Yosida (valid in Hilbert spaces and for linear $\mathcal{A}$) and the variant by Lumer and Phillips (valid in Banach spaces, still for linear $\mathcal{A}$), and provides the existence and uniqueness of mild solutions to problems of the type (7.6) in the case $f \equiv 0$.

Theorem 7.1 (Crandall-Liggett) *Suppose that $\mathcal{A}$ is an accretive operator satisfying the rank condition. Then for all data $u_0 \in \overline{D(\mathcal{A})}$, the limit*

$$S_t(\mathcal{A})u_0 = \lim_{n\to\infty} (J_{\frac{t}{n}}(\mathcal{A}))^n u_0 \tag{7.9}$$

exists uniformly with respect to t, on compact subset of $[0,\infty)$, and $u(t) = S_t(\mathcal{A})u_0 \in C([0,\infty) : X)$. Moreover, the family of operators $\{S_t(\mathcal{A})\}_{t>0}$ is a strongly continuous semigroup of contractions on $\overline{D(\mathcal{A})} \subset X$.

Using a popular notation in the linear framework, we could write $S_t(\mathcal{A})u_0 = \mathrm{e}^{-t\mathcal{A}}u_0$, and because of this analogy formula, (7.9) is called the Crandall-Liggett exponential formula for the nonlinear semigroup generated by $-\mathcal{A}$. The problem with this very general and useful result is that the X-valued function $u(t) = S_t(\mathcal{A})u_0$ solves the equation only in a mild sense, which is not necessarily a strong solution or a weak solution. Though it is known that strong solutions are automatically mild, the correspondence between mild and weak solutions is not always clear. For the FPME, this issue was discussed in detail in [26–27].

In addition, the Crandall-Liggett theorem result can be extended when we consider nontrivial source term f, according to the following result.

Theorem 7.2 *Suppose that $\mathcal{A}$ is m-accretive, $f \in L^1(0,\infty; X)$ and $u_0 \in \overline{D(\mathcal{A})}$. Then the abstract problem (7.6) has a unique mild solution u, obtained as the limit of the discrete approximate solution u_h by ITD scheme described above, as $h \to 0$,*

$$u(t) := \lim_{h\to 0} u_h(t),$$

and the limit is uniform in compact subsets of $[0,\infty)$. Moreover, $u \in C([0,\infty); X)$ and for any couple of solutions u_1, u_2 corresponding to source terms f_1, f_2, we have

$$\|u_1(t) - u_2(t)\|_X \le \|u_1(s) - u_2(s)\|_X + \int_s^t \|f_1(\tau) - f_2(\tau)\|_X \mathrm{d}\tau$$

for all $0 \le s < t$.

There is a wide literature on these topics, starting with the seminal paper by Crandall and Liggett [24] (see also [23] and the general reference [5]). These notes are based on [60, Chapter 10] (see the references therein). The last formula we mentioned introduces the correct concept of uniqueness for the constructed class of solutions. Characterizing the uniqueness of different concepts of solution is a difficult topic already discussed (with positive results) by Bénilan [6].

8 Comments and Extensions

(1) We have only proved results on parabolic comparison based on symmetrization for the linear case. The elliptic results can be applied to nonlinear equations but still have severe restrictions. It is interesting to know how much is true for nonlinear functions B and A in the respective equations. This question was partially addressed and solved for the spectral fractional Laplacian in [64], and the limitations to the generality of the results were also shown to be necessary, the symmetrization result was false for the concave B or the convex A of power type.

More generally, we would like to know if there is an approach that ensures comparison results of some symmetrization type valid for quite general nonlinearities, as it happens in the non-fradtional case (see [59]).

(2) The variable coefficient case. As a future direction, we are interested in the following problem:

$$\begin{cases} \operatorname{div}(\ y^{1-2s} B(x)\nabla w) = 0 & \text{in } \mathcal{H}^+, \\ w = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\ -y^{1-2s}\partial_y w|_{y=0} = f, \end{cases} \tag{8.1}$$

where

$$B(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

Here the matrix $A(x)$ is supposed to be $W^{1,\infty}(\mathbb{R}^N)$ and uniformly elliptic with lower constant $\Lambda > 0$.

It is a well-known fact that the spectral powers of $\operatorname{div}(A(x)\nabla)$, i.e., $(\operatorname{div}(A(x)\nabla))^s$ for $s \in (0,1)$ in a bounded domain Ω can be described as the Dirichlet-to-Neumann operator of a suitable extension in a cylinder $\mathcal{C} = \Omega \times \mathbb{R}^+$ (see for instance [19] for a detailed account). The previous problem (8.1) is a variant of this extension but in the whole $\mathbb{R}^N$. The Dirichlet-to-Neumann operator in this case is not explicitly identified. However, we believe that it is a natural possible extension of the problem we considered in this paper. The idea here is to develop the techniques produced in the present paper to handle variable coefficients, having in mind an isoperimetric inequality. Indeed, an FKI is proven in terms of the first eigenvalue by means of

$$\lambda_1(\Omega) \geq \Lambda \lambda_1(\Omega^{\#}).$$

The aim here is to prove such a result for the following problem: Let $\mathcal{L}_s$ be the Dirichlet-to-Neumann operator associated to (8.1) defined on Ω. It is obvious that $\mathcal{L}_s$ has discrete spectrum $\{\lambda_{k,s}\}_{k=1}^{\infty}$. It is not clear how to use a variational approach to deal with this operator, since it does not seem obvious that this operator is associated to a norm in $\mathbb{R}^N$ satisfying a Pólya-Szegö inequality. However, the parabolic approach developed in the present paper seems promising.

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