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A Third Derivative Estimate for Monge-Ampere Equations with Conic Singularities

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(*Dedicated to Haim Brezis on the occasion of his* 70*th birthday*)

Abstract The author applies the arguments in his PKU Master degree thesis in 1988 to derive a third derivative estimate, and consequently, a $C^{2,\alpha}$-estimate, for complex Monge-Ampere equations in the conic case. This $C^{2,\alpha}$-estimate was used by Jeffres-Mazzeo-Rubinstein in their proof of the existence of Kähler-Einstein metrics with conic singularities.

Keywords Complex, Monge-Ampere, Conic, C^{α}-estimate
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1 Introduction

In this note, we extend a third derivative estimate in [4] to Monge-Ampere equations in the conic case. For the purpose of our application, we will consider the complex Monge-Ampere equations in this note. The same result holds for real Monge-Ampere equations.

Let U be a neighborhood of $(0,\cdots,0)$, and u satisfy

$$\det\Big(\frac{\partial^2 u}{\partial z_i\partial\overline{z}_j}\Big) = \mathrm{e}^F\,|z_1|^{2\beta-2} \tag{1.1}$$

and

$$c_0^{-1}\,\omega_\beta \le \sqrt{-1}\,\partial\overline{\partial}\,u \le c_0\,\omega_\beta,\quad \text{where}\ \ c_0 > 0, \tag{1.2}$$

where $\beta\in(0,1)$ and ω_β is the standard conic flat metric on $\mathbb{C}^n$:

$$\omega_\beta = \sqrt{-1}\Big(\frac{\mathrm{d}z_1\wedge\mathrm{d}\overline{z}_1}{|z_1|^{2-2\beta}} + \sum_{i=2}^n \mathrm{d}z_i\wedge\mathrm{d}\overline{z}_i\Big).$$

Theorem 1.1 *Let V be an open neighborhood of $(0,\cdots,0)$, whose closure is contained in U. Then for any $\alpha<\min\{1,\beta^{-1}-1\}$, there are constants $r_0>0$ and C_α, which may depend on α, c_0, V, $\inf\Delta F$ and $\|F\|_{C^1}$, such that for any $x\in V$ and $0<r<r_0$,*

$$\int_{B_r(x)} |\nabla^3 u|^2\,\omega_\beta^n \le C_\alpha\, r^{2n-2+2\alpha}, \tag{1.3}$$

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where $B_r(x)$ denotes the ball with center x and radius r, ∇ is the covariant derivative, Δ denotes the Laplacian and the norm is taken with respect to ω_β.

It follows from Theorem 1.1 and the standard arguments, e.g., by using the Green function.

Corollary 1.1 *Let u and F be as in Theorem* 1.1. *Then for any $\alpha \in (0, \beta^{-1} - 1)$ and $\alpha < 1$, $\partial\overline{\partial}u$ is C^α-bounded with respect to ω_β.*

This corollary was used by Jeffres-Mazzeo-Rubinstein in their proving the existence of Kähler-Einstein metrics with conic singularities. We refer the readers to Appendix B of [3] for details.

Theorem 1.1 has been known to me for some time. The proof is identical to that in [4]. Its arguments were inspired by Giaquinta-Giusti's work (see [2]) on harmonic maps. For years, I had talked about this approach to the $C^{2,\alpha}$-estimate for complex Monge-Ampere equations in my courses on Kähler-Einstein metrics.

2 The Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by following the arguments in Section 2 of [4]. I will present the proof for complex Monge-Ampere equations in details. In [4], the proof was written for real Monge-Ampere equations, though it also applies to complex Monge-Ampere equations.

Without loss of generality, we may assume $x \in V \cap \{z_1 = 0\}$. In fact, one can apply known estimates (see, e.g., [4]) to u outside the singular set $\{z_1 = 0\}$ of ω_β. If F is smooth on U, we can also apply Calabi's third derivative estimate to u outside $\{z_1 = 0\}$ (see [5]).

Define $\mathbf{B}_\beta(r)$ to be the domain in $\mathbb{C} \times \mathbb{C}^{n-1}$ consisting of all (w_1, w'), where $w' = (w_2, \cdots, w_n)$, satisfying

$$|w_1|^2 + |w'|^2 \le r^2 \quad \text{and} \quad w_1 = \rho\, \mathrm{e}^{\sqrt{-1}\,\theta} \quad \text{for } \rho \in [0, r] \text{ and } \theta \in [0, 2\pi\beta]. \tag{2.1}$$

There is an $\overline{r} > 0$ such that $w_1 = \beta^{-1} z_1^\beta$, and $w' = (z_2, \cdots, z_n)$ defines a natural map from $\mathbf{B}_\beta(\overline{r})$ into U. This map is isometric from the interior of $\mathbf{B}_\beta(\overline{r})$ onto its image. For convenience, we will use $w_1, \cdots, w_n$ as coordinates. By scaling, we may assume that $\overline{r} = 1$.

In terms of coordinates $w_1, \cdots, w_n$, we have

$$\omega_\beta = \sqrt{-1}\, \mathrm{d}w_i \wedge \mathrm{d}\overline{w}_i$$

and covariant derivatives of u become ordinary derivatives, e.g.,

$$u_{i\overline{j}} = \frac{\partial^2 u}{\partial w_i \partial \overline{w}_j}, \quad u_{i\overline{j}k} = \frac{\partial^3 u}{\partial w_i \partial \overline{w}_j \partial w_k}, \cdots.$$

We will denote $\det(u_{k\overline{l}})\, u^{i\overline{j}}$ by $U^{i\overline{j}}$, where $(u^{i\overline{j}})$ denotes the inverse of $(u_{i\overline{j}})$.

First we recall two elementary facts.

Lemma 2.1 (see [4, Lemma 2.1]) *For each j, $U^{i\overline{j}}{}_i = 0$, where all derivatives are covariant with respect to ω_β.*

Proof Recall the identities

$$U^{i\overline{l}} U^{k\overline{j}} u_{i\overline{j}} = \det(u_{p\overline{q}})\, U^{k\overline{l}}, \quad \text{where } k, l = 1, \cdots, n.$$

Differentiating them in w_k-direction and summing over k, we get

$$2\,\det(u_{p\overline{q}})\,U^{k\overline{l}} + U^{i\overline{l}}\det(u_{p\overline{q}})\,U^{k\overline{j}}u_{k\overline{j}i} = \det(u_{p\overline{q}})_k\,U^{k\overline{l}} + \det(u_{p\overline{q}})\,{U^{k\overline{l}}}_k.$$

Then the lemma follows.

Lemma 2.2 (see [4, Lemma 2.2]) *For any positive* $\lambda_1, \cdots, \lambda_n$, *we have*

$$\Big|\frac{\lambda}{\lambda_i}\,u_{i\overline{i}} - \det(u_{p\overline{q}}) - (n-1)\lambda\Big| \le C\sum_{i,j}|u_{i\overline{j}} - \lambda_i\delta_{ij}|^2, \tag{2.2}$$

where $\lambda = \lambda_1\cdots\lambda_n$, *and* C *is a constant depending only on* λ_i *and* $(u_{i\overline{j}})$.

Proof In [4], (2.2) is proved by the properties of determinants. Here we outline a simpler proof. First by using the homogeneity and positivity of $(u_{i\overline{j}})$, we only need to prove the case when $u_{i\overline{j}} = \delta_{ij}$. Next, if we denote the left-hand side of (2.2) by $f(\Lambda)$, then by a direct computation, $f(I) = 0$, $\frac{\partial f}{\partial\lambda_i}(I) = 0$ for $i = 1, \cdots, n$. Then (2.2) follows from the Taylor expansion of f at I.

In terms of $w_1, \cdots, w_n$, (1.1) becomes

$$\det\Big(\frac{\partial^2 u}{\partial w_i \partial\overline{w}_j}\Big) = \mathrm{e}^F. \tag{2.3}$$

By a direct computation, we deduce from this

$$u^{i\overline{j}}u_{k\overline{l}i\overline{j}} = u^{i\overline{q}}u^{p\overline{j}}u_{i\overline{j}k}u_{p\overline{q}\overline{l}} + F_{k\overline{l}}. \tag{2.4}$$

This system resembles the one for harmonic maps whose regularity theory were studied extensively in 70s and 80s. The idea of [4] is to apply the arguments, particularly in [2], from the regularity theory for harmonic maps.

The following lemma follows easily from the Sobolev embedding theorem.

Lemma 2.3 *There is a constant* C_β, *which depends on* β, *such that for any smooth function* h *on* $\mathbf{B}_\beta = \mathbf{B}_\beta(1)$ *with boundary condition*

$$h(\rho\,\mathrm{e}^{\sqrt{-1}2\pi\beta}, w') = \mathrm{e}^{\sqrt{-1}2\pi(1-\beta)}h(\rho, w'), \tag{2.5}$$

we have

$$\int_{\mathbf{B}_\beta}|h|^2\,\mathrm{d}w\wedge\mathrm{d}\overline{w} \le C_\beta\Big(\int_{\mathbf{B}_\beta}|\mathrm{d}h|^{\frac{2n}{n+1}}\,\mathrm{d}w\wedge\mathrm{d}\overline{w}\Big)^{\frac{n+1}{n}}. \tag{2.6}$$

Note that C_β blows up when β tends to 1.

Lemma 2.4 (see [4, Lemma 2.3]) *There is some* $q > 2$, *which may depend on* β, $\|u_{i\overline{j}}\|_{L^\infty}$ *and* $\|F_{i\overline{j}}\|_{L^\infty}$, *such that for any* $B_{2r}(y)\subset U$, *we have*

$$\Big(r^{-2n}\int_{B_r(y)}(1+|\nabla\omega|^2)^{\frac{q}{2}}\,\omega_\beta^n\Big)^{\frac{2}{q}} \le C\,r^{-2n}\int_{B_r(y)}(1+|\nabla\omega|^2)\,\omega_\beta^n. \tag{2.7}$$

where $\omega = \sqrt{-1}\,\partial\overline{\partial}\,u$, *and* C *denotes a uniform constant.*[1]

[1] Note that C, c always denote uniform constant though their actual values may vary in different places.

Proof First we assume $y = x$. Define λ_{ij} by

$$\lambda_{i\bar{j}} \,=\, r^{-2n} \int_{B_r(x)} u_{i\bar{j}}\,\omega_\beta^n, \quad i,j \,=\, 1,\cdots,n. \tag{2.8}$$

By using unitary transformations if necessary, we may assume $\lambda_{i\bar{j}} = 0$ for any $i \neq j$ and $i, j \geq 2$. It follows from (1.2) that

$$c_0^{-1}\,\mathbf{I} \,\leq\, (\lambda_{i\bar{j}}) \,\leq\, c_0\,\mathbf{I},$$

where $\mathbf{I}$ denotes the identity matrix.

Choose a cut-off function $\eta : B_r(x) \mapsto \mathbb{R}$ satisfying

$$\eta(z) = 1, \quad \forall z \in B_{\frac{3r}{4}}(x), \quad \eta(z) = 0, \quad \forall z \in B_r(x)\backslash B_{\frac{7r}{8}}(x), \quad |\nabla\eta| \leq \frac{8}{r}, \quad |\nabla^2\eta| \leq \frac{64}{r^2}.$$

Using Lemma 2.1 and (2.4), we can deduce

$$\begin{aligned} c\int_{B_r(x)} \eta\,|\nabla\omega|^2\,\omega_\beta^n \,&+\, \int_{B_r(x)} \Big(\sum_{k=1}^n \frac{\lambda}{\lambda_k}\,F_{k\bar{k}}\,\mathrm{e}^F \,-\, U^{i\bar{j}}(\mathrm{e}^F)_{i\bar{j}}\Big)\,\eta\,\omega_\beta^n \\ \leq \int_{B_r(x)} U^{i\bar{j}}\Big(&\sum_{k=1}^n \frac{\lambda}{\lambda_k}\,u_{k\bar{k}} \,-\, \det(u_{p\bar{q}}) \,-\, (n-1)\lambda\Big)\,\eta_{i\bar{j}}\,\omega_\beta^n, \end{aligned}$$

where $\lambda_k = \lambda_{k\bar{k}}$ and $\lambda \,=\, \lambda_1\cdots\lambda_n$.

Using Lemma 2.1, we have

$$\begin{aligned} &\int_{B_r(x)} \Big(\sum_{k=1}^n \frac{\lambda}{\lambda_k}\,F_{k\bar{k}}\,\mathrm{e}^F \,-\, U^{i\bar{j}}(\mathrm{e}^F)_{i\bar{j}}\Big)\,\eta\,\omega_\beta^n \\ &= \int_{B_r(x)} \Big[\mathrm{e}^F\,F \sum_{k=1}^n \frac{\lambda}{\lambda_k}\,\eta_{k\bar{k}} \,-\, \mathrm{e}^F\,U^{i\bar{j}}\,\eta_{i\bar{j}}\Big]\,\omega_\beta^n \\ &\leq r^{-2}\int_{B_r(x)} \Big(F^2 \,+\, \sum_{i,j=1}^n |u_{i\bar{j}} - \lambda_i\,\delta_{ij}|^2\Big)\,\omega_\beta^n \end{aligned}$$

Then by Lemma 2.2, we can deduce from the above that

$$\int_{B_{\frac{3r}{4}}(x)} |\nabla\omega|^2\,\omega_\beta^n \,\leq\, C\,r^{-2}\int_{B_r(x)} \Big(1 \,+\, \sum_{i,j=1}^n |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2\Big)\,\omega_\beta^n.$$

By applying the Sobolev inequality to $u_{1\bar{1}} - \lambda_1, u_{i\bar{j}} - \lambda_i\,\delta_{ij}$ $(i,j \geq 2)$ and Lemma 2.3 to $u_{1\bar{i}}$, $u_{i\bar{1}}$ $(i \geq 2)$ in the above, we get

$$\int_{B_{\frac{3r}{4}}(x)} (1 + |\nabla\omega|^2)\,\omega_\beta^n \,\leq\, C\,r^{-2}\Big(\int_{B_r(x)} (1 + |\nabla\omega|^2)^{\frac{n}{n+1}}\,\omega_\beta^n\Big)^{\frac{n+1}{n}}.$$

This inequality still holds, if we replace $B_r(x)$ by any $B_r(y)$ which is disjoint from the singular set $\{z_1 = 0\}$. This can be proved by using the same arguments, but Lemma 2.3 is not needed. One can easily deduce from this and a covering argument that for any ball $B_{2r}(y) \subset U$,

$$\int_{B_r(y)} (1 + |\nabla\omega|^2)\,\omega_\beta^n \,\leq\, C\,r^{-2}\Big(\int_{B_{2r}(y)} (1 + |\nabla\omega|^2)^{\frac{n}{n+1}}\,\omega_\beta^n\Big)^{\frac{n+1}{n}}.$$

Then (2.7) follows from Gehring's inverse Hölder inequality (see [1]).

Lemma 2.5 (see [4, Lemma 2.4]) *For any $y \in V$ and $B_{4r}(y) \subset U$ and $\sigma < r$, we have*

$$\begin{aligned}&\int_{B_\sigma(y)} |\nabla \omega|^2 \, \omega_\beta^n \, - \, c\, r^{2n} \\ &\leq C\left[\left(\frac{\sigma}{r}\right)^{2n-4+2\beta^{-1}} + \left(r^{2-2n}\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\right]\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n.\end{aligned} \tag{2.9}$$

Proof Define $v = \sqrt{-1}\, v_{i\bar{i}}\, dw_i \wedge d\overline{w}_i$ by solving

$$\sum_{i=1}^n \frac{\lambda}{\lambda_i}\, v_{i\bar{i}} = 0 \quad \text{on } B_r(y), \quad v = \omega \quad \text{on } \partial B_r(y). \tag{2.10}$$

Multiplying (2.10) by $\widehat{w} = \omega - v$, we get

$$\int_{B_r(y)} \sum_{i=1}^n \frac{\lambda}{\lambda_i}\left|\frac{\partial v}{\partial w_i}\right|^2 \omega_\beta^n \leq \int_{B_r(y)} \sum_{i=1}^n \frac{\lambda}{\lambda_i}\left|\frac{\partial \omega}{\partial w_i}\right|^2 \omega_\beta^n.$$

It follows that

$$\int_{B_r(y)} |\nabla v|^2\, \omega_\beta^n \,\leq\, C \int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n. \tag{2.11}$$

Multiplying (2.4) by $\widehat{w}$ and integrating by parts, we have

$$\int_{B_r(y)} u^{p\bar{q}} u_{i\bar{k}p}\widehat{w}_{k\bar{i}\bar{q}}\,\omega_\beta^n \,\leq\, \int_{B_r(y)} (\widehat{w}^{k\bar{l}} u^{i\bar{q}} u^{p\bar{j}} u_{i\bar{j}k} u_{p\bar{q}\bar{l}} \,-\, \widehat{w}_i F_{\bar{i}})\,\omega_\beta^n.$$

Using the assumption that ∇F is bounded, we can easily deduce from this

$$\int_{B_r(y)} |\nabla\widehat{w}|^2\,\omega_\beta^n \,\leq\, c\left(r^{2n} \,+\, \int_{B_r(y)} (|\widehat{w}| \,+\, |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2)\,|\nabla\omega|^2\,\omega_\beta^n\right). \tag{2.12}$$

By Lemma 2.4 and the Poincare inequality, we have

$$\begin{aligned}&\int_{B_r(y)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2\,|\nabla\omega|^2\,\omega_\beta^n \\ &\leq r^{2n}\left(r^{-2n}\int_{B_r(y)} |\nabla\omega|^q\,\omega_\beta^n\right)^{\frac{2}{q}}\left(r^{-2n}\int_{B_r(y)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^{\frac{2q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}} \\ &\leq C\left(r^{-2n}\int_{B_r(y)} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(y)} (1 + |\nabla\omega|^2)\,\omega_\beta^n \\ &\leq C\left(r^{2-2n}\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(y)} (1 + |\nabla\omega|^2)\,\omega_\beta^n.\end{aligned} \tag{2.13}$$

Without loss of generality, we may assume that $q \geq 2(q-2)$. Since $\widehat{w}$ vanishes on $\partial B_r(y)$, its

L^2-norm is controlled by the L^2-norm of $|\nabla\widehat{w}|$ and consequently, of $|\nabla\omega|$. Then we have

$$\begin{aligned}
&\int_{B_r(y)} |\widehat{w}|\,|\nabla\omega|^2\,\omega_\beta^n \\
&\le r^{2n}\left(r^{-2n}\int_{B_r(y)}|\nabla\omega|^q\,\omega_\beta^n\right)^{\frac{2}{q}}\left(r^{-2n}\int_{B_r(y)}|\widehat{w}|^{\frac{q}{q-2}}\,\omega_\beta^n\right)^{\frac{q-2}{q}} \\
&\le C'\left(r^{-2n}\int_{B_r(y)}|\widehat{w}|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(y)}(1+|\nabla\omega|^2)\,\omega_\beta^n \\
&\le C\left(r^{2-2n}\int_{B_r(y)}|\nabla\widehat{w}|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(y)}(1+|\nabla\omega|^2)\,\omega_\beta^n \\
&\le C\left(r^{2-2n}\int_{B_r(y)}|\nabla\omega|^2\,\omega_\beta^n\right)^{\frac{q-2}{q}}\int_{B_r(y)}(1+|\nabla\omega|^2)\,\omega_\beta^n. \qquad (2.14)
\end{aligned}$$

Next, we recall a simple lemma which can be proved by standard methods.

Lemma 2.6 *Let h be any harmonic function on $\mathbf{B}_\beta=\mathbf{B}_\beta(1)$, such that*

$$h(\rho e^{\sqrt{-1}2\pi\beta},w') \,=\, e^{\sqrt{-1}2\pi(1-\beta)}h(\rho,w'),\quad \int_{\mathbf{B}_\beta}|dh|^2\,dw\wedge d\overline{w}<\infty. \qquad (2.15)$$

Then for any $r<1$,

$$\int_{\mathbf{B}_\beta(r)}|dh|^2\,dw\wedge d\overline{w}\,\le\, Cr^{2n-4+2\beta^{-1}}\int_{\mathbf{B}_\beta}|dh|^2\,dw\wedge d\overline{w}. \qquad (2.16)$$

Note that the exponent $2n-4+2\beta^{-1}$ is sharp, since we have the harmonic function $w_1^{\frac{1-\beta}{\beta}}$ on $\mathbf{B}_\beta$.

In fact, we only need a weaker version of Lemma 2.6 in the subsequent arguments: In addition to the assumption (2.15), we may further assume

$$\frac{\partial h}{\partial z_1}(\rho e^{\sqrt{-1}2\pi\beta},w') \,=\, e^{\sqrt{-1}2\pi(1-\beta)}\frac{\partial h}{\partial z_1}(\rho,w').$$

Remark 2.1 If we replace (2.15) by

$$h(\rho e^{\sqrt{-1}2\pi\beta},w') \,=\, h(\rho,w'),\quad \int_{bB_\beta}|dh|^2\,dw\wedge d\overline{w}<\infty, \qquad (2.17)$$

then we have a better estimate

$$\int_{\mathbf{B}_\beta(r)}|dh|^2\,dw\wedge d\overline{w}\,\le\, Cr^{2n}\int_{\mathbf{B}_\beta}|dh|^2\,dw\wedge d\overline{w}.$$

Now we apply Lemma 2.6 to our v above. If we write $v=\sqrt{-1}\,v_{i\bar{j}}\,dw_i\wedge d\overline{w}_j$, then each coefficient $v_{i\bar{j}}$ is a harmonic function in either Lemma 2.6 or Remark 2.1 with respect to the coordinates $\widetilde{w}_i=\sqrt{\frac{\lambda_i}{\lambda}}w_i$. Thus, by Lemma 2.6 and a standard scaling trick, we can deduce for $\sigma<r$,

$$\int_{B_\sigma(y)}|\nabla v|^2\,\omega_\beta^n\,\le\, C\left(\frac{\sigma}{r}\right)^{2n-4+2\beta^{-1}}\int_{B_r(y)}|\nabla v|^2\,\omega_\beta^n. \qquad (2.18)$$

We observe

$$\int_{B_\sigma(y)} |\nabla\omega|^2\,\omega_\beta^n \le 2\int_{B_\sigma(y)} (|\nabla\widehat{w}|^2 + |\nabla v|^2)\,\omega_\beta^n$$
$$\le 2\int_{B_r(y)} |\nabla\widehat{w}|^2\,\omega_\beta^n + 2\int_{B_\sigma(y)} |\nabla v|^2\,\omega_\beta^n.$$

Hence, by (2.11) and (2.18), we get

$$\int_{B_\sigma(y)} |\nabla\omega|^2\,\omega_\beta^n \le \int_{B_r(y)} \Big(2\,|\nabla\widehat{w}|^2 + c\Big(\frac{\sigma}{r}\Big)^{2n-4+2\beta^{-1}} |\nabla\omega|^2\Big)\,\omega_\beta^n. \tag{2.19}$$

Clearly, (2.9) follows from (2.12)–(2.14) and (2.19).

In view of (2.9), we need the following lemma.

Lemma 2.7 (see [4, Lemma 2.5]) *For any $\epsilon_0 > 0$, there is an ℓ depending only on ϵ_0, $\|\Delta u\|_{L^\infty}$ and $\inf\Delta F$ satisfying that for any $\widetilde{r} > 0$ with $B_{\widetilde{r}}(y) \subset U$, there is $r \in [2^{-\ell}\widetilde{r}, \widetilde{r}]$, such that*

$$r^{2-2n}\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n \le \epsilon_0. \tag{2.20}$$

Proof It follows from (2.4) that

$$\Delta'\Delta u = u^{i\overline{j}}u^{p\overline{q}}u_{i\overline{q}k}u_{\overline{j}p\overline{k}} + \Delta F, \tag{2.21}$$

where Δ' denotes the Laplacian of ω.

Let η be a non-negative function on $B_r(y)$ satisfying that $\eta(z) = 1$ for any $z \in B_{r/2}(y)$, $\eta(z) = 0$ for any z near $\partial B_r(y)$ and $r^2\,|\nabla^2\eta| \le 4$. Then

$$\int_{B_r(y)} \eta\,|\nabla\omega|^2\,\omega_\beta^n \le C\,r^{2n} - \int_{B_r(y)} \Delta'\eta\,(M_r - \Delta u)\,\omega^n$$
$$\le C\,r^{2n} - 4r^{-2}\int_{B_r(y)} (M_r - \Delta u)\,\omega^n, \tag{2.22}$$

where $M_r = \sup_{B_r(y)} \Delta u$.

Set $Z = M_r - \Delta u - c\,r^2$. It follows from (2.21) that for a suitable constant $c > 0$, $\Delta' Z \le 0$, i.e., Z is super-harmonic with respect to ω. Since ω is equivalent to ω_β, we can apply the standard Moser iteration to Δ' to get

$$r^{-2n}\int_{B_r(y)} Z\,\omega^n \le C\Big(\inf_{B_{\frac{r}{2}}(y)} Z + r^2\Big).$$

It follows that

$$r^{2-2n}\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n \le C\,(M_r - M_{\frac{r}{2}} + r^2).$$

Hence, if (2.20) does not hold for $r = \widetilde{r}, 2^{-1}\widetilde{r}, \cdots, 2^{-k}\widetilde{r}$, then

$$k\,\epsilon_0 \le C\,(M_{\widetilde{r}} - M_{2^{-k}\widetilde{r}} + 2\,\widetilde{r}^2).$$

This is impossible if k is sufficiently large. So the lemma is proved.

Now we complete the proof of Theorem 1.1. Choose $\xi = \lambda^{2\alpha}$ and $\lambda \in (0,1)$, such that

$$(1+C)\,\lambda^{2(\beta^{-1}-1-\alpha)} \,\leq\, \frac{1}{2}.$$

Next we choose ϵ_0 and r_0 sufficiently small, such that

$$(C\,\epsilon_0^{\frac{q-2}{q}} \,+\, c\,r_0^2)\,\lambda^{2-2n-2\alpha} \,\leq\, \frac{1}{2}.$$

Then we can deduce from (2.9) that for $\sigma = \lambda\, r$ and $r \leq r_0$ satisfying (2.20),

$$\sigma^{2\nu} + \sigma^{2-2n}\int_{B_\sigma(y)} |\nabla\omega|^2\,\omega_\beta^n \,\leq\, \xi\left(r^{2\nu} + r^{2-2n}\int_{B_r(y)} |\nabla\omega|^2\,\omega_\beta^n\right),$$

where $\nu \,=\, \beta^{-1} - 1 > \alpha$. Then (1.3) follows again from this and a standard iteration.

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